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# A ONE-PARAMETER DEFORMATION OF THE FARAHAT-HIGMAN ALGEBRA

JEAN-PAUL BULTEL

ABSTRACT. We show, by introducing an appropriate basis, that a one-parameter family of Hopf algebras introduced by Foissy [Adv. Math. 218 (2008) 136-162] interpolates between the Faà di Bruno algebra and the Farahat-Higman algebra. Its structure constants in this basis are deformation of the top connection coefficients, for which we obtain analogues of Macdonald's formulas.

## 1. INTRODUCTION

The center  $Z_n$  of the group algebra  $\mathbb{Z}[\mathfrak{S}_n]$  of the symmetric group is spanned by conjugacy classes  $C_\mu$ , which are parametrized by partitions  $\mu$  of  $n$ . The *connection coefficients*  $a_{\mu\nu}^\lambda$  are the structure constants

$$(1) \quad C_\mu C_\nu = \sum_{\lambda \vdash n} a_{\mu\nu}^\lambda C_\lambda$$

of  $Z_n$ . These coefficients, whose calculation is in general very hard, have important applications to various enumerative problems or to the calculation of certain matrix integrals [7].

For a partition  $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r > 0)$ , define

$$(2) \quad \bar{\mu} = (\mu_1 - 1, \mu_2 - 1, \dots, \mu_r - 1),$$

the *reduced cycle type* of any permutation of cycle type  $\mu$ . Denoting by  $c_\rho(n)$  the conjugacy class  $C_\mu$  of  $\mathfrak{S}_n$  such that  $\bar{\mu} = \rho$ , we can rewrite (1) in the form

$$(3) \quad c_\mu(n)c_\nu(n) = \sum_{\lambda} a_{\mu\nu}^\lambda(n)c_\lambda(n).$$

It has been proved by Farahat and Higman [4] that the connection coefficients  $a_{\mu\nu}^\lambda(n)$  are polynomial functions of  $n$ , and are independent of  $n$  if  $|\lambda| = |\mu| + |\nu|$ . These are called the *top connection coefficients*.

One may use the top connection coefficients to define an algebra  $R$ , spanned by formal symbols  $c_\mu$  indexed by all partitions, with multiplication rule

$$(4) \quad c_\mu c_\nu = \sum_{|\lambda|=|\mu|+|\nu|} a_{\mu\nu}^\lambda c_\lambda.$$

This is the Farahat-Higman algebra ([4], see also [10, ex. 24 p 131]). It is also proved in [4] that  $R$  is isomorphic to the algebra of symmetric functions  $\Lambda$ . The construction of an explicit isomorphism  $\varphi : \Lambda \rightarrow R$  is more recent, and due to Macdonald [10, ex.

25 p 132] (another proof has been given by Goulden and Jackson [6]). More precisely, Macdonald constructed a basis  $(g_\mu)$  of  $\Lambda$  such that

$$(5) \quad g_\mu g_\nu = \sum_{|\lambda|=|\mu|+|\nu|} a_{\mu\nu}^\lambda g_\lambda,$$

and obtained a recursive formula for the calculation of  $a_{\mu\nu}^\lambda$ .

Murray [12] explicitated the images in  $\Lambda$  of the projections of symmetric functions of Jucys-Murphy elements in  $R$  under this isomorphism. In Section 3, we give a new derivation of his result, and new proofs of various results of Biane [1] and Matsumoto-Novak [11].

It is well-known that the algebra of symmetric functions is a Hopf algebra. Its standard coproduct, denoted here by  $\Delta_0$ , comes from its interpretation as the algebra of polynomial functions on the multiplicative group  $G_0$  of formal power series with constant term 1. It has, however, another Hopf algebra structure, known as the *Faà di Bruno algebra*, coming from its interpretation as the algebra of polynomial functions on the group  $G_1 = \{ta(t) | a \in G_0\}$  of formal diffeomorphisms of the line under composition (see, *e.g.*, [3]). In fact, Macdonald's basis  $g_\mu$  is the dual of the image  $h_\mu = S_1(h_\mu)$  of the basis of complete homogeneous functions  $h_\mu$  by the antipode of the Faà di Bruno algebra  $\mathcal{F}$ .

This suggests to interpret  $g_\mu$  as living in the dual  $\mathcal{F}^*$  of  $\mathcal{F}$ . However, contrary to  $R$ , this dual is not commutative, being the universal enveloping algebra of the Lie algebra  $\mathfrak{g}_1$  of  $G_1$ . Thus, such an interpretation does not make sense *a priori*.

To clarify this situation, we make use of a one-parameter deformation  $\mathcal{F}_\gamma$  of  $\mathcal{F}$  recently discovered by Foissy [5] in his investigation of combinatorial Dyson-Schwinger equations in the Connes-Kreimer algebra. We then obtain for the structure constants of  $\mathcal{F}_\gamma$  in the dual basis of  $S_1(h_\mu)$  a one-parameter deformation of Macdonald's formulas which are recovered for  $\gamma = 0$ .

We follow the conventions of [10]. For the convenience of the reader, the most essential ones are recalled in Section 2.

## 2. NOTATIONS AND BACKGROUND

**2.1. Partitions.** Let  $n$  be a positive integer. A finite sequence of strictly positive integers  $(\lambda_1, \lambda_2, \dots)$  is called a *partition* of  $n$  if  $\lambda_1 \geq \lambda_2 \geq \dots$  and  $\lambda_1 + \lambda_2 + \dots = n$ . We then write  $\lambda \vdash n$ . The  $\lambda_i$  are the *parts* of  $\lambda$ ,  $|\lambda| = n$  is the *weight* of  $\lambda$  and the number  $l(\lambda)$  of parts in  $\lambda$  is the *length* of  $\lambda$ . The *multiplicity*  $m_\lambda(k)$  of  $k$  in  $\lambda$  is the number of parts in  $\lambda$  equal to  $k$ . We set

$$(6) \quad z_\lambda = \prod_{k \geq 1} k^{m_k(\lambda)} m_k(\lambda)!$$

**2.2. The algebra of symmetric functions.** We denote by  $\Lambda$  the algebra of symmetric functions. The bases  $(m_\lambda), (e_\lambda), (h_\lambda), (p_\lambda)$  and  $(s_\lambda)$  are respectively the *monomial*, *elementary*, *complete*, *power sum* and *Schur* symmetric functions. These bases of  $\Lambda$  are parametrized by partitions of all integers. For any basis  $(b_\lambda)$ , we denote by  $b_n$

the symmetric function  $b_{(n)}$ . Denote by  $\langle \cdot, \cdot \rangle$  the usual *scalar product* on  $\Lambda$ , for which  $(s_\lambda)$  is an orthonormal basis. For this scalar product,  $(p_\lambda)$  is an orthogonal basis, and one has

$$(7) \quad \langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\mu$$

The bases  $(m_\lambda)$  and  $(h_\lambda)$  are dual to each other.

$$(8) \quad \langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

( $\delta$  is the Kronecker symbol). We denote by  $p_n^\perp$  the *adjoint* of the multiplication by  $p_n$ , in the sense

$$(9) \quad \langle p_n f, g \rangle = \langle f, p_n^\perp(g) \rangle$$

where  $f$  and  $g$  are any two symmetric functions. This operator is a *derivation*. More precisely,

$$(10) \quad p_n^\perp = n \frac{\partial}{\partial p_n}$$

This operator acts on the  $h_n$  as follows

$$(11) \quad p_n^\perp h_N = h_{N-n}$$

for any  $N > n$ , so that

$$(12) \quad p_n^\perp = \sum_{r \geq 0} h_r \frac{\partial}{\partial h_{n+r}}$$

Now, let  $X$  and  $Y$  be two alphabets. For any symmetric function  $f$  and any scalar  $k$ , we identify  $f$  with  $f(X)$ , and we define the algebra morphisms  $f \rightarrow f(X + Y)$ ,  $f \rightarrow f(kX)$  and  $f \rightarrow f(k)$  by

$$(13) \quad p_n(X + Y) = p_n(X) + p_n(Y)$$

$$(14) \quad p_n(kX) = k p_n(X)$$

$$(15) \quad p_n(k) = k$$

The standard Hopf algebra structure of  $\Lambda$  is defined by the coproduct  $\Delta_0$

$$(16) \quad \Delta_0(f) = f(X + Y)$$

where we identify  $f(X + Y)$  with an element of  $\Lambda \otimes \Lambda$  by identifying  $f \otimes g$  with  $f(X)g(Y)$ . The counit  $\epsilon$  and the antipode  $S_0$  are given by

$$(17) \quad \epsilon(f) = f(0)$$

and

$$(18) \quad S_0(f) = f(-X)$$

Denote by  $\mathcal{H}_0$  this Hopf algebra.

$$(19) \quad \mathcal{H}_0 = (\Lambda, \cdot, 1, \Delta_0, \epsilon, S_0)$$

One can give another interpretation of  $\mathcal{H}_0$ . Let

$$(20) \quad G_0 = \{a \mid a(t) = 1 + a_1 t + a_2 t^2 + \dots\} = 1 + t\mathbf{C}[[t]]$$

be the multiplicative group of formal power series with constant term 1, and let  $\mathcal{H}$  be the algebra of polynomial functions on  $G_0$ . Let  $k_n \in \mathcal{H}$  be the map defined by

$$(21) \quad k_n(a) = a_n$$

The standard coproduct for functions on a group is

$$(22) \quad \Delta f(a, b) = f(ab)$$

where  $\Delta f$  is identified with an element of  $\mathcal{H} \otimes \mathcal{H}$ . Then,  $h_n \mapsto k_n$  is an isomorphism of Hopf algebras  $\mathcal{H}_0 \rightarrow \mathcal{H}$ .

**2.3. Other bases of the algebra of symmetric functions.** The formal series

$$(23) \quad u = tH(t) = t + h_1 t^2 + h_2 t^3 + \dots$$

has an inverse for the composition, so that we can rewrite  $t$  in terms of  $u$ , as follows.

$$(24) \quad t = u + h_1^* u^2 + h_2^* u^3 + \dots$$

The  $h_k^*$  are homogeneous symmetric functions of degree  $k$ , and the algebra morphism  $\psi$  from  $\Lambda$  to  $\Lambda$  defined by  $\psi(h_k) = h_k^*$  is an *involution* :

$$(25) \quad \psi^2 = Id_\Lambda$$

The Lagrange inversion formula shows that  $(n+1)h_n^*$  is the coefficient of  $t^n$  in  $H(t)^{-(n+1)}$ , so that

$$(26) \quad h_n^* = \frac{h_n(-(n+1)X)}{n+1}$$

We can define a multiplicative  $\mathbb{Z}$ -basis  $(h_\lambda^*)$  of  $\Lambda$ , by

$$(27) \quad h_\lambda^* = h_{\lambda_1}^* h_{\lambda_2}^* \dots = \psi(h_\lambda)$$

In [10, ex. 24 p 35], Macdonald shows that

$$(28) \quad (n+1)h_n^* = \sum_{\lambda \vdash n} (-1)^{l(\lambda)} \binom{n+l(\lambda)}{n} u_\lambda h_\lambda$$

where

$$(29) \quad u_\lambda = \frac{l(\lambda)}{\prod_{i \geq 1} m_i(\lambda)!}$$

Let  $(g_\lambda)$  be the adjoint basis of  $(h_\lambda^*)$ , that is

$$(30) \quad \langle g_\lambda, h_\mu^* \rangle = \delta_{\lambda\mu}$$

One has

$$(31) \quad g_n = -m_n = -p_n$$

The algebra morphism  $\psi^T$  (adjoint of  $\psi$ ) is also an involution, and it maps  $g_\lambda$  to  $m_\lambda$ . The matrix  $\Psi$  of  $\psi$  is *strictly upper triangular* in the basis  $(h_\lambda)$  and one has

$$(32) \quad \Psi_{\lambda\lambda} = (-1)^{l(\lambda)}$$

**Theorem 2.1** (Macdonald [10]). *The linear map*

$$(33) \quad \Phi : \begin{cases} R & \rightarrow \Lambda \\ c_\lambda & \mapsto g_\lambda \end{cases}$$

*is an isomorphism of algebras.*

**2.4. Top connection coefficients.** Now, let us recall some properties of the top connection coefficients  $a'_{\lambda\mu}$ , which are the structure constants of the Farahat-Higman algebra  $R$ . By Theorem 2.1,

$$(34) \quad g_\lambda g_\mu = \sum_{|\nu|=|\lambda|+|\mu|} a'_{\lambda\mu} g_\nu$$

When  $\mu$  and  $\nu$  are partitions of length 1 ( $\mu = (r)$  and  $\nu = (m)$ ), Macdonald gives in [10, ex. 24 p 131] an explicit formula for  $a'_{\lambda(r)}$ ,  $m = |\lambda| + r$ , that is

$$(35) \quad a'_{\lambda(r)} = \begin{cases} \frac{(m+1)r!}{(r+1-l(\lambda)) \prod_{i>0} m_i(\lambda)!} & \text{if } l(\lambda) \leq r + 1 \\ 0 & \text{otherwise} \end{cases}$$

Moreover, Macdonald gives a recurrence formula, for any partition  $\nu$  with  $|\nu| = |\lambda| + r$

$$(36) \quad a'_{\lambda(r)} = \sum_{(i,\mu)/\mu \cup \nu = \lambda \cup (r)} a'_{\mu(r)}^{(\nu_i)}$$

One can deduce from these formulas that the  $a'_{\lambda(r)}$  are zero except if  $\nu \geq \lambda \cup (r)$ , and that  $a'_{\lambda(r)}^{\lambda \cup (r)} > 0$ . The multiplicative structure of the Farahat-Higman algebra is uniquely determined by (35) and (36).

**2.5. Jucys-Murphy elements.** Let  $n$  and  $i$  be two integers,  $i \leq n$ . Define

$$(37) \quad \xi_i = \sum_{j<i} (j, i) \in \mathbb{C}[\mathfrak{S}_n],$$

called the *ith Jucys-Murphy element*. It is the sum of all transpositions  $(i, j)$  with  $j < i$ . The Jucys-Murphy elements do not belong to the center  $Z_n$  of  $\mathbb{C}[\mathfrak{S}_n]$ , but they generate a maximal commutative subalgebra of  $\mathbb{C}[\mathfrak{S}_n]$ . Let

$$(38) \quad \Xi_n = \{\xi_1, \xi_2, \dots, \xi_n\}$$

be the alphabet of Jucys-Murphy elements. It is known that the algebra of symmetric functions in  $\Xi_n$  is exactly  $Z_n$  [8]. Hence, since  $(C_\mu)_{\mu \vdash n}$  is a basis of  $Z_n$ , one can rewrite  $f(\Xi_n)$  as

$$(39) \quad f(\Xi_n) = \sum_{\mu \vdash n} k_{f,\mu}(n) C_\mu$$

for any homogeneous symmetric function  $f$ , and this decomposition is unique. We can rewrite this formula in terms of the reduced cycle types

$$(40) \quad f(\Xi_n) = \sum k_{f,\mu}(n) c_\mu(n).$$

Let  $r$  be the degree of  $f$ . When  $|\mu| = r$ ,  $k_{f,\mu}(n)$  does not depend on  $n$ . Let us rewrite it as  $k_{f,\mu}$  and define a new element  $f(\Xi)$  of the Farahat-Higman algebra  $R$ , by considering only maximal terms in the expansion of  $f(\Xi_n)$  for  $n$  large enough.

$$(41) \quad f(\Xi) = \sum_{|\mu|=r} k_{f,\mu} c_\mu$$

Murray [12] has shown that for any  $f$ , the isomorphism  $\Phi$  between  $R$  and  $\Lambda$  maps  $f(\Xi)$  to  $f(-X) = S_0(f)$ .

$$(42) \quad \Phi(f(\Xi)) = f(-X)$$

Since  $\Phi(c_\mu) = g_\mu$ , using the duality between the bases  $(h_\lambda^*)$  and  $(g_\lambda)$ , one has

$$(43) \quad k_{f,\mu} = \langle f(-X), h_\mu^* \rangle$$

### 3. MORE ABOUT MURRAY'S RESULT

**3.1. A new derivation.** Let us give a new proof of (42). Lascoux and Thibon [9] give the formula

$$(44) \quad p_m(\Xi_n) = \sum_{k=1}^{m+1} \sum_{|\kappa|=k, l(\kappa) \leq m-k+2} \phi_{\kappa,m} a_{\kappa,n}$$

where  $a_{\kappa,n}$  is defined by

$$(45) \quad a_{\kappa,n} = \frac{1}{(n-k)!} z_{\kappa \cup 1^{n-k}} C_{\kappa \cup 1^{n-k}}$$

The  $C_\lambda$  are the classical conjugacy classes, and the  $\phi_{\kappa,m}$  are defined from

$$(46) \quad \phi_\kappa(t) = \left. \frac{(1-q^{-1})^{k-1} p_\kappa(q-1)}{k! z_\kappa} \frac{1}{q-1} \right|_{q=e^t}$$

by

$$(47) \quad \phi_\kappa(t) = \sum_{m \geq |\kappa| + l(\kappa) - 2} \phi_{\kappa,m} \frac{t^m}{m!}.$$

From (44),  $C_{(m+1) \cup 1^{n-k}}$  is the only class of modified weight  $m$  which can give a contribution to  $p_m(\Xi_n)$ . Indeed, for  $|\kappa| - l(\kappa) = m$ , if we had  $l(\kappa) > 1$  we would have  $|\kappa| > m+1$ , but the  $\kappa$  satisfying this inequality do not occur in the sum (44), so that the formula

$$(48) \quad p_m(\Xi) = \sum_{|\mu|=m} k_\mu c_\mu$$

becomes

$$(49) \quad p_m(\Xi) = k_m c_m$$

Now we only have to determine the coefficient  $k_m$ . In order to do that, suppose that  $n = m+1$ . In this case we have

$$(50) \quad p_m(\Xi_{m+1}) = \xi_1^m + \xi_2^m + \dots + \xi_{m+1}^m,$$

that is,

$$(51) \quad p_m(\Xi_{m+1}) = 0^m + (1, 2)^m + ((1, 3) + (2, 3))^m + \dots \\ + ((1, m+1) + (2, m+1) + \dots + (m, m+1))^m$$

Only the last term can give a contribution to  $C_{m+1}$ . This term can be rewritten

$$(52) \quad \sum (i_1, m+1)(i_2, m+1) \dots (i_m, m+1)$$

where the sum is over the  $(i_1, i_2, \dots, i_m)$  where the  $i_k$  are integers such that  $1 \leq i_k \leq m$ . In this new sum, only terms with the  $i_k$  all distinct can give a contribution to  $C_{m+1}$ . The number of these terms is  $m!$ , that is, the cardinality of  $C_{m+1}$ , so that the coefficient of  $C_{m+1}$  in  $p_m(\Xi_{m+1})$  is 1. This coefficient does not depend on  $n$  since  $(m+1)$  has a maximal modified weight, so that the coefficient of  $c_m$  in  $p_m(\Xi)$ , that is  $k_m$ , is again 1, so that

$$(53) \quad p_m(\Xi) = c_m.$$

Hence,  $\Phi(p_m(\Xi)) = \Phi(c_m) = g_m = -p_m$ , and

$$(54) \quad \Phi(p_\lambda(\Xi)) = \Phi\left(\prod_i p_{\lambda_i}(\Xi)\right) = \prod_i \Phi(p_{\lambda_i}(\Xi)).$$

From that we have  $\Phi(p_\lambda(\Xi)) = \prod_i (-p_{\lambda_i}) = (-1)^{l(\lambda)} p_\lambda = p_\lambda(-X)$ , and since  $\Lambda$  is spanned by the  $p_\mu$ , this implies (42).

**3.2. Examples.** Using his result and simple calculations in  $\Lambda$ , Murray [12] computes coefficients in the expansion of certain symmetric functions of the Jucys-Murphy elements over the  $c_\lambda$ . For example, he shows the following formulas

$$(55) \quad \langle e_k(-X), h_\lambda^* \rangle = 1$$

$$(56) \quad \langle h_k(-X), h_\lambda^* \rangle = \prod_i \text{Cat}_{\lambda_i-1}$$

(where  $\text{Cat}_i$  is the  $i$ th Catalan number) giving the top coefficients in the expansion of  $e_k(\Xi)$  and  $h_k(\Xi)$ . In the same vein, let us give a new proof of a result of Matsumoto and Novak for the monomial functions. Let  $k$  be an integer,  $\lambda \vdash k$  and  $\mu \vdash k$ . We denote by  $L_\mu^\lambda$  the coefficient defined by

$$(57) \quad m_\lambda(\Xi) = \sum_{|\mu|=|\lambda|} L_\mu^\lambda c_\mu$$

Matsumoto and Novak [11] show that

$$(58) \quad L_\mu^\lambda = \sum_{(\lambda^{(1)}, \lambda^{(2)}, \dots) \in \mathfrak{R}(\lambda, \mu)} RC(\lambda^{(1)}) RC(\lambda^{(2)}) \dots$$

where

$$(59) \quad \mathfrak{R}(\lambda, \mu) = \{(\lambda^{(1)}, \lambda^{(2)}, \dots) / \forall i, \lambda^{(i)} \vdash \mu_i \text{ and } \lambda = \lambda^{(1)} \cup \lambda^{(2)} \cup \dots\}$$



and for any partition  $\lambda$  of  $N$ ,

$$(60) \quad RC(\lambda) = \frac{1}{N+1} m_\lambda(N+1) = \frac{|\lambda|!}{(|\lambda| - l(\lambda) + 1)! \prod_{i \geq 1} m_i(\lambda)!}$$

Matsumoto and Novak also give a combinatorial interpretation for  $RC(\lambda)$ . Let us show how to derive (58) from (43). We have

$$(61) \quad \begin{aligned} L_\mu^\lambda &= \langle m_\lambda(-X), h_\mu^* \rangle \\ &= \langle m_\lambda, h_\mu^*(-X) \rangle \\ &= \left\langle m_\lambda, \frac{h_{\mu_1}((\mu_1+1)X) h_{\mu_2}((\mu_2+1)X) \dots}{(\mu_1+1)(\mu_2+1) \dots} \right\rangle \\ &= \frac{1}{\prod_i (\mu_i+1)} \left\langle m_\lambda, \prod_i \sum_{\lambda^{(i)} \vdash \mu_i} m_{\lambda^{(i)}}(\mu_i+1) h_{\lambda^{(i)}} \right\rangle \end{aligned}$$

Expanding the right factor of the scalar product, we obtain nonzero terms only if  $h_{\lambda^{(1)}} h_{\lambda^{(2)}} \dots = h_\lambda$ , since otherwise one has  $\langle m_\lambda, h_{\lambda^{(1)}} h_{\lambda^{(2)}} \dots \rangle = 0$ . Hence,

$$(62) \quad L_\mu^\lambda = \frac{1}{\prod_i (\mu_i+1)} \sum (m_{\lambda^{(1)}}(\mu_1+1)) (m_{\lambda^{(2)}}(\mu_2+1)) \dots \langle m_\lambda, h_\lambda \rangle$$

The sum is over the  $(\lambda^{(1)}, \lambda^{(2)}, \dots)$  such that  $\lambda^{(i)} \vdash \mu_i$  for all  $i$  and  $\bigcup_i \lambda^{(i)} = \lambda$ , that is, over the set  $\mathfrak{R}(\lambda, \mu)$ . Moreover, one has  $\langle m_\lambda, h_\lambda \rangle = 1$ , so that

$$(63) \quad L_\mu^\lambda = \sum_{(\lambda^{(1)}, \lambda^{(2)}, \dots) \in \mathfrak{R}(\lambda, \mu)} \frac{m_{\lambda^{(1)}}(\mu_1+1)}{\mu_1+1} \frac{m_{\lambda^{(2)}}(\mu_2+1)}{\mu_2+1} \dots$$

This is the result of Matsumoto and Novak.

**3.3. Coefficient of the cycle with maximal length in  $p_\lambda(\Xi_n)$ .** We call *modified cycle type* of a product of cycles the sequence  $(l_1 - 1, l_2 - 1, \dots)$ , where the  $l_i$  are the lengths of the factors. For example, the product (13)(234) has modified cycle type  $(2 - 1, 3 - 1) = (1, 2)$ . Biane [1] obtains an explicit formula for the number  $\alpha_\lambda$  of factorisations with modified cycle type  $\lambda$  for a cycle of length  $n + 1$ , where  $n = |\lambda| = \sum_i \lambda_i$ , that is

$$(64) \quad \alpha_\lambda = (n+1)^{l(\lambda)-1}$$

Let us prove this with symmetric functions. We consider in the Farahat-Higman algebra  $R$  the product

$$(65) \quad c^\lambda = c_{\lambda_1} c_{\lambda_2} \dots$$

Since  $R$  is isomorphic to  $\Lambda$ , it is commutative, so that the order of the elements of  $\lambda$  has no importance. Hence, we can assume that  $\lambda$  is a partition. One has

$$(66) \quad \begin{aligned} \Phi(c^\lambda) &= g_{\lambda_1} g_{\lambda_2} \dots = (-p_{\lambda_1})(-p_{\lambda_2}) \dots \\ &= (-1)^{l(\lambda)} p_\lambda = p_\lambda(-X) \end{aligned}$$

Hence, from (42),

$$(67) \quad c^\lambda = p_\lambda(\Xi)$$

Moreover,  $c^\lambda$  is the sum of all cycle products of modified type  $\lambda$ , and  $c_n$  is the sum of all the cycles of length  $n + 1$ . Hence, the total number of cycle products of modified type  $\lambda$  is  $k_\lambda^n \text{card}(c_n)$ , where  $k_\lambda^n$  is the coefficient of  $c_n$  in  $c^\lambda$ . The number  $\alpha_\lambda$  of these factorisations does not depend on the choice of the cycle, so that  $\alpha_\lambda = k_n^\lambda \frac{\text{card}(c_n)}{\text{card}(c_n)} = k_\lambda^n$ , which is also from (67) the coefficient of  $c_n$  in  $p_\lambda(\Xi)$ . Hence, one has

$$(68) \quad \alpha_\lambda = (-1)^{l(\lambda)} \langle p_\lambda, h_n^* \rangle,$$

so that

$$(69) \quad \alpha_\lambda = \frac{1}{n+1} (n+1)^{l(\lambda)} \langle p_\lambda, h_n \rangle = (n+1)^{l(\lambda)-1} \langle p_\lambda, h_n \rangle$$

(since the bases  $(m_\lambda)$  and  $(h_\lambda)$  are dual to each other,  $\langle p_\lambda, h_n \rangle$  is the coefficient of  $m_n$  in  $p_\lambda$ , that is, 1).

**3.4. Generalization.** Here, we give a combinatorial interpretation for the coefficient of  $c_\mu$  in  $p_\lambda(\Xi)$ . Now we have

$$(70) \quad \langle p_\lambda, h_n^* \rangle = (-1)^{l(\lambda)} (n+1)^{l(\lambda)-1}$$

On another hand, since  $p_\lambda(\Xi) = c_{\lambda_1} c_{\lambda_2} \dots$  in  $R$ ,

$$(71) \quad c_{\lambda_1} c_{\lambda_2} \dots = \sum_{|\lambda|=|\mu|} \langle p_\lambda(-X), h_\mu^* \rangle c_\mu,$$

so that  $\langle p_\lambda(-X), h_\mu^* \rangle$  corresponds to the total number of decompositions in an arbitrary product of modified type  $\lambda$  of a product of disjoint cycles with modified type  $\mu$ . In such a decomposition, each  $c_{\mu_i}$  comes from certain  $\prod_i c_{\lambda^{(i)}}$ , where  $\lambda^{(i)} \vdash \mu_i$  is a subpartition of  $\lambda$ , with  $\lambda^{(1)} \cup \lambda^{(2)} \cup \dots = \lambda$ , that is,  $(\lambda^{(1)}, \lambda^{(2)}, \dots) \in \mathfrak{R}(\lambda, \mu)$ . Moreover, each  $(\lambda^{(1)}, \lambda^{(2)}, \dots)$  must be counted  $m(\lambda^{(1)}, \lambda^{(2)}, \dots)$  times, where

$$(72) \quad m(\lambda^{(1)}, \lambda^{(2)}, \dots) = \prod_{j \geq 1} \frac{m_j(\lambda)!}{m_j(\lambda^{(1)})! m_j(\lambda^{(2)})! \dots}$$

For a given  $c_{\mu_i}$ , the number of decompositions of  $c_{\mu_i}$  of a certain type  $\prod_i c_{\lambda^{(i)}}$  corresponds to the coefficient of  $c_{\mu_i}$  in  $p_{\lambda^{(i)}}(\Xi)$ , that is  $\langle p_{\lambda^{(i)}}(-X), h_{\mu_i}^* \rangle$ , so that we have from (70)

$$(73) \quad \langle p_\lambda(-X), h_\mu^* \rangle = \sum_{(\lambda^{(1)}, \lambda^{(2)}, \dots) \in \mathfrak{R}(\lambda, \mu)} m(\lambda^{(1)}, \lambda^{(2)}, \dots) \prod_i (\mu_i + 1)^{l(\lambda^{(i)})-1}$$

Note that it is nonzero only if  $\lambda$  is a refinement of  $\mu$ . Finally, one has

$$(74)$$

$$p_\lambda(\Xi) = \sum_{|\lambda|=|\mu|} \left( \sum_{(\lambda^{(1)}, \lambda^{(2)}, \dots) \in \mathfrak{R}(\lambda, \mu)} \left( \prod_j \frac{m_j(\lambda)!}{m_j(\lambda^{(1)})! m_j(\lambda^{(2)})! \dots} \right) \left( \prod_i (\mu_i + 1)^{l(\lambda^{(i)})-1} \right) \right) c_\mu$$

Note that from (70) we have

$$(75) \quad h_n^* = \sum_{\mu \vdash n} (-1)^{l(\mu)} (n+1)^{l(\mu)-1} \frac{p_\mu}{z_\mu}$$

Expanding  $h_\mu^*$  by means of this expression, since we have for  $\bigcup_i \lambda^{(i)} = \lambda$

$$(76) \quad \prod_{j \geq 1} \frac{m_j(\lambda)!}{m_j(\lambda^{(1)})! m_j(\lambda^{(2)})! \dots} = \frac{z_\lambda}{z_{\lambda^{(1)}} z_{\lambda^{(2)}} \dots}$$

we see that (73) can also be obtained by simple calculations in  $\Lambda$ .

#### 4. THE FAÀ DI BRUNO ALGEBRA AND ITS DEFORMATION

**4.1. The Faà di Bruno algebra.** There is another coproduct on  $\Lambda$ , denoted here by  $\Delta_1$  and defined by

$$(77) \quad \Delta_1 h_n = \sum_{k=0}^n h_k(X) \otimes h_{n-k}((k+1)X)$$

or equivalently

$$(78) \quad \Delta_1 h_n = \sum_{k=0}^n \sum_{\mu \vdash n-k} m_\mu(k+1) h_k \otimes h_\mu$$

This coproduct defines a Hopf algebra with the counit  $\epsilon$  as in  $\mathcal{H}_0$ , and the antipode  $S_1 = \psi$ , that is, the involution mapping  $h_\lambda$  to  $h_\lambda^*$ . The Hopf algebra

$$(79) \quad \mathcal{H}_1 = (\Lambda, \cdot, 1, \Delta_1, \epsilon, S_1)$$

is called the *Faà di Bruno algebra*. It has the following interpretation. Let

$$(80) \quad G_1 = \{\alpha \mid \exists a \in G_0, \forall t, \alpha(t) = ta(t)\} = tG_0 = t + t^2\mathbb{C}[[t]]$$

be the group of formal diffeomorphisms of the line tangent to the identity, and let again  $k_n$  be the linear form

$$(81) \quad k_n : \alpha(t) = t + a_1 t^2 + a_2 t^3 + \dots \mapsto a_n$$

Let  $\Delta$  be the coproduct such that  $\Delta k_n(\alpha, \beta)$  is the coefficient of  $t^{n+1}$  in

$$(82) \quad (\alpha \circ \beta)(t) = \alpha(\beta(t))$$

The bialgebra  $\mathcal{F}$  defined by this coproduct is isomorphic to  $\mathcal{H}_1$  under the correspondence  $k_n \mapsto h_n$ .

**4.2. A deformation of the Faà di Bruno algebra.** Let  $\gamma$  be a real parameter in  $[0, 1]$ , and  $\Delta_\gamma$  be the coproduct on  $\Lambda$  defined by

$$(83) \quad \Delta_\gamma(h_n) = \sum_{k=0}^n h_k \otimes h_{n-k}((k\gamma + 1)X)$$

or equivalently

$$(84) \quad \Delta_\gamma(h_n) = \sum_{k=0}^n \sum_{\mu \vdash n-k} m_\mu(k\gamma + 1) h_k \otimes h_\mu$$

Foissy obtains this coproduct in [5] in his investigation of formal Dyson-Schwinger equations in the Connes-Kreimer Hopf algebra, and shows that for  $\gamma \in ]0, 1]$ , the

resulting bialgebras  $\mathcal{H}_\gamma$  are Hopf algebras, isomorphic to the Faà di Bruno algebra. Obviously,  $\Delta_0$  corresponds to  $\mathcal{H}_0$ .

**4.3. Graded dual of the deformed Faà di Bruno algebra.** Consider now the Hopf algebra  $\mathcal{H}'_\gamma$ , the graded dual of the deformation  $\mathcal{H}_\gamma$  considered in section 3. Denote by  $\star_\gamma$  the product on  $\mathcal{H}'_\gamma$ . For any  $f \in \mathcal{H}'_\gamma$  and any symmetric function  $g$  in  $\mathcal{H}_\gamma$ , denote by  $\langle f, g \rangle$  the action of  $f$  on  $g$ . One has for any  $f \in \mathcal{H}'_\gamma$ ,  $g \in \mathcal{H}'_\gamma$  and  $h \in \Lambda$ , by definition of a dual Hopf algebra,

$$(85) \quad \langle f \star_\gamma g, h \rangle = \langle f \otimes g, \Delta_\gamma(h) \rangle$$

where

$$(86) \quad \langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle$$

Denote by  $(d_\lambda)$ ,  $(q_\lambda)$  and  $(b_\lambda)$  the bases respectively dual to  $(h_\lambda)$ ,  $\left(\frac{p_\lambda}{z_\lambda}\right)$  and  $h_\lambda^*$  in the sense of  $\mathcal{H}'_\gamma$ , that is, for example

$$(87) \quad \langle d_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$$

For any  $\gamma$  in  $[0, 1]$ ,

$$(88) \quad q_n = d_n = -b_n$$

Note that these  $b_n$  generate  $\mathcal{H}'_\gamma$ .  $\mathcal{H}_0$  is *self-dual*, so that in the case  $\gamma = 0$ ,

$$(89) \quad d_\lambda = m_\lambda, \quad q_\lambda = p_\lambda, \quad b_\lambda = g_\lambda$$

When  $\gamma \neq 0$ ,  $\mathcal{H}_\gamma$  is not self-dual and not commutative, since the coproduct  $\Delta_\gamma$  on  $\Lambda$  is not cocommutative.

**4.4. Multiplicative structure of  $\mathcal{H}'_\gamma$ .** Now, let  $f$  and  $g$  be two elements of  $\mathcal{H}'_\gamma$ . One has for any partition  $\mu$ ,

$$(90) \quad \begin{aligned} \langle f \star_\gamma g, h_\mu \rangle &= \langle f \otimes g, \Delta_\gamma(h_\mu) \rangle \\ &= \langle f \otimes g, \Delta_\gamma(h_{\mu_1}) \Delta_\gamma(h_{\mu_2}) \dots \rangle \\ &= \left\langle f \otimes g, \prod_i \sum_{k_i+l_i=\mu_i} h_{k_i} \otimes h_{l_i}((\gamma k_i + 1)X) \right\rangle \\ &= \left\langle f \otimes g, \sum_{k_i+l_i=\mu_i} \prod_i h_{k_i} \otimes h_{l_i}((\gamma k_i + 1)X) \right\rangle \\ &= \sum_i \langle f \otimes g, \prod_i h_{k_i} \otimes h_{l_i}((\gamma k_i + 1)X) \rangle \end{aligned}$$

where the parameters of the sum are the same as above. Hence,

$$(91) \quad \begin{aligned} \langle f \star_\gamma g, h_\mu \rangle &= \sum \langle f, h_{(k_1, k_2, \dots)} \rangle \langle g, \prod_i h_{l_i}(\gamma k_i + 1)X \rangle \\ &= \sum \langle f, h_{(k_1, k_2, \dots)} \rangle \langle g, \prod_i \sum_{\rho+l_i} m_\rho(\gamma k_i + 1)h_\rho \rangle \end{aligned}$$

Now let us change the parameters of the sum, considering the partitions  $\rho_i$ , with  $|\rho_i| = l_i$  and  $k_i = \mu_i - |\rho_i|$ . The sum is now over the  $(\rho_1, \rho_2, \dots, \rho)$ , such that for all  $i$ ,  $|\rho_i| \leq \mu_i$ , and  $\rho$  is the union of the  $\rho_i$ . We obtain

$$(92) \quad \langle f \star_\gamma g, h_\mu \rangle = \sum \langle f, h_{(\mu_1-|\rho_1|, \mu_2-|\rho_2|, \dots)} \rangle \langle g, \prod_i m_{\rho_i}(\gamma\mu_i - \gamma|\rho_i| + 1) h_{\rho_i} \rangle,$$

and finally,

$$(93) \quad \langle f \star_\gamma g, h_\mu \rangle = \sum \left( \prod_i m_{\rho_i}(\gamma\mu_i - \gamma|\rho_i| + 1) \right) \langle f, h_{(\mu_1-|\rho_1|, \mu_2-|\rho_2|, \dots)} \rangle \langle g, h_\rho \rangle$$

One can use this formula to expand  $f \star_\gamma g$  on the  $d_\mu$ , where  $f$  and  $g$  are any two elements in  $\mathcal{H}'_\gamma$

**4.5. Action of  $q_n^\perp$  on the  $h_\mu$ .** We define an operator  $q_n^\perp$  on  $\Lambda$  as follows, for  $f \in \mathcal{H}'_\gamma$  and  $g \in \mathcal{H}_\gamma$ .

$$(94) \quad \langle f \star_\gamma q_n, g \rangle = \langle f, q_n^\perp g \rangle$$

The operator  $q_n^\perp$  is a *derivation*, because it is the adjoint of the right multiplication by a *primitive element*:

$$(95) \quad q_n^\perp(fg) = fq_n^\perp(g) + q_n^\perp(f)g$$

We shall need its action on the  $h_\mu$ . Let  $n > 0$  and  $\mu$  be a partition. We can write from (93) :

$$(96) \quad \langle f \star_\gamma q_n, h_\mu \rangle = \sum \left( \prod_i m_{\rho_i}(\gamma\mu_i - \gamma|\rho_i| + 1) \right) \langle f, h_{(\mu_1-|\rho_1|, \mu_2-|\rho_2|, \dots)} \rangle \langle q_n, h_\rho \rangle$$

Since  $q_n = d_n$  for all  $n$ , one has  $\langle q_n, h_\rho \rangle = \delta_{n\rho}$ , so that a term in this sum gives a nonzero contribution only if

$$(97) \quad \rho = (n)$$

In this case we have also  $\langle q_n, h_\rho \rangle = 1$ , and

$$(98) \quad \langle f \star_\gamma q_n, h_\mu \rangle = \sum_{i/\mu_i \geq n} m_n(\gamma\mu_i - \gamma n + 1) \langle f, h_{\mu \setminus \mu_i \cup (\mu_i - n)} \rangle$$

Since  $m_n(\gamma\mu_i - \gamma n + 1) = \gamma(\mu_i - n) + 1$ , one can deduce

$$(99) \quad q_n^\perp h_\mu = \sum_{i/\mu_i \geq n} (\gamma\mu_i - \gamma n + 1) h_{\mu \setminus \mu_i \cup (\mu_i - n)}$$

When  $\mu$  consists only in one part  $N$ , this formula can be rewritten

$$(100) \quad q_n^\perp h_N = (\gamma N - \gamma n + 1) h_{N-n}$$

Since  $q_n^\perp$  is a derivation, one has its action on the multiplicative basis  $(h_\mu)$ , and one can use (100) to fully explicit it. We have  $q_n^\perp = D_n + E_n$ , where  $D_n$  and  $E_n$  are the derivations defined by

$$(101) \quad D_n h_n = h_{N-n}$$

and

$$(102) \quad E_n h_n = \gamma(N - n)h_{N-n},$$

that is,

$$(103) \quad D_n = \sum_{r \geq 0} h_r \frac{\partial}{\partial h_{n+r}}$$

and

$$(104) \quad E_n = \gamma \sum_{r \geq 0} r h_r \frac{\partial}{\partial h_{n+r}}$$

so that

$$(105) \quad q_n^\perp = \sum_{r \geq 0} (1 + \gamma r) h_r \frac{\partial}{\partial h_{n+r}}$$

We can see that this is valid also in the degenerate case  $\gamma = 0$ .

**4.6. Action of  $q_n^\perp$  on the  $p_\mu$ .** From (100), we have

$$(106) \quad q_n^\perp h_N = p_n^\perp h_N + \gamma(N - n)h_{N-n}$$

On another hand,

$$(107) \quad (N - n)h_{N-n} = p_1 h_{N-n-1} + p_2 h_{N-n-2} + p_3 h_{N-n-3} + \dots$$

Then,

$$(108) \quad \begin{aligned} q_n^\perp h_N &= p_n^\perp h_N + \gamma p_1 h_{N-n-1} + \gamma p_2 h_{N-n-2} + \dots \\ &= p_n^\perp h_N + \gamma p_1 (p_{n+1}^\perp h_N) + \gamma p_2 (p_{n+2}^\perp h_N) + \dots \\ &= p_n^\perp h_N + \gamma \sum_{r > n} p_{r-n} p_r^\perp h_N \end{aligned}$$

so that

$$(109) \quad q_n^\perp = p_n^\perp + \gamma \sum_{r > n} p_{r-n} p_r^\perp$$

## 5. A DEFORMATION OF THE FARAHAT-HIGMAN ALGEBRA

**5.1. Recurrences for the structure constants of  $\mathcal{H}'_\gamma$ .** Denote by  $a'_{\lambda,\mu}(\gamma)$  the structure constants in the basis  $b_\mu$ .

$$(110) \quad b_\lambda \star_\gamma b_\mu = \sum_\nu a'_{\lambda,\mu}(\gamma) b_\nu$$

When  $\gamma = 0$ , the  $b_\mu$  are identified with the  $g_\mu$ , and then  $a'_{\lambda,\mu}(0)$  coincides with  $a'_{\lambda,\mu}$ , the top connection coefficient.

Since  $b_n = -q_n$ , one has

$$(111) \quad a'_{\lambda,(n)}(\gamma) = \langle b_\lambda \star_\gamma b_n, h_\nu^* \rangle = -\langle b_\lambda \star_\gamma q_n, h_\nu^* \rangle$$

Hence,

$$(112) \quad a_{\lambda,(n)}^{\nu}(\gamma) = -\langle b_{\lambda}, q_n^{\perp} h_{\nu}^* \rangle$$

Since  $q_n^{\perp}$  is a derivation, one can rewrite this as

$$(113) \quad a_{\lambda,(n)}^{\nu}(\gamma) = -\sum_i \langle b_{\lambda}, h_{\nu \setminus (\nu_i)}^* q_n^{\perp}(h_{\nu_i}^*) \rangle$$

The  $i$ th term in this sum corresponds to the coefficient of  $h_{\lambda}^*$  in

$$(114) \quad h_{\nu \setminus (\nu_i)}^* q_n^{\perp}(h_{\nu_i}^*)$$

Hence this term gives a nonnegative contribution only if there exists a partition  $\mu$  such that

$$(115) \quad (\nu \setminus (\nu_i)) \cup \mu = \lambda,$$

that is,

$$(116) \quad \mu \cup \nu = \lambda \cup (\nu_i)$$

Hence, the  $i$ th term is also equal to the coefficient of  $h_{\mu}^*$  in  $q_n^{\perp}(h_{\nu_i}^*)$ , that is,  $a_{\mu(n)}^{\nu_i}(\gamma)$ . Summarizing, we have proved:

**Theorem 5.1.** *The structure constants  $a_{\lambda,(n)}^{\nu}(\gamma)$  satisfy the recursion*

$$(117) \quad a_{\lambda,(n)}^{\nu}(\gamma) = \sum a_{\mu(n)}^{\nu_i}(\gamma)$$

where the sum is over the  $(i, \mu)$  such that

$$(118) \quad \mu \cup \nu = \lambda \cup (\nu_i)$$

This formula is a generalization of (36), which is recovered for  $\gamma = 0$ .

**5.2. Multiplicative structure of the deformed Farahat-Higman algebra.** In the case where  $\nu$  has only one part, there is a closed formula for  $a_{\lambda(r)}^{\nu}(\gamma)$ .

**Theorem 5.2.** *For  $r \in \mathbb{N}$ ,  $N \in \mathbb{N}$  and  $\mu$  a partition,*

$$(119) \quad a_{\lambda,(r)}^{(N)}(\gamma) = \left(1 - \frac{N-r}{N+1}\gamma\right) a_{\lambda,(r)}^{(N)}(0)$$

$a_{\lambda,(r)}^{(N)}(0)$  corresponds to  $a_{\lambda,(r)}^{(N)}$  in formula (35), where  $m$  is replaced by  $N$ .

Together with the recurrence formula (117), this determines completely the multiplicative structure of  $\mathcal{H}'_{\gamma}$ , since it is generated by the  $b_n$ . In order to derive (119), we shall need the following lemmas.

**Lemma 5.3.** *Let  $r$  and  $n$  be two positive integers, with  $r < n$ . Then,*

$$(120) \quad p_r^{\perp} h_n^* = \sum_{\rho \vdash n-r} (-1)^{l(\rho)-1} (n+1)^{l(\rho)} \frac{p_{\rho}}{z_{\rho}}$$

*Proof.* From (75) we have

$$(121) \quad p_r^\perp h_n^* = \sum_{\mu \vdash n} \frac{(-1)^{l(\mu)} (n+1)^{l(\mu)-1}}{z_\mu} p_r^\perp(p_\mu)$$

The action of  $p_r^\perp$  on  $p_\mu$  is

$$(122) \quad p_r^\perp(p_\mu) = r \frac{\partial p_\mu}{\partial p_r} = r m_r(\mu) p_{\mu \setminus (r)}$$

On another hand,

$$(123) \quad z_{\mu \setminus (r)} = \frac{z_\mu m_r(\mu \setminus (r))!}{r m_r(\mu)!}$$

with  $m_r(\mu \setminus (r)) = m_r(\mu) - 1$ , hence  $\frac{m_r(\mu)!}{m_r(\mu \setminus (r))!} = m_r(\mu)$ , so that

$$(124) \quad \frac{z_\mu}{r m_r(\mu)} = z_{\mu \setminus (r)}$$

Hence,

$$(125) \quad \frac{p_r^\perp(p_\mu)}{z_\mu} = \frac{p_{\mu \setminus (r)}}{z_{\mu \setminus (r)}}$$

and

$$(126) \quad p_r^\perp h_n^* = \sum_{\mu \vdash n} (-1)^{l(\mu)} (n+1)^{l(\mu)-1} \frac{p_{\mu \setminus (r)}}{z_{\mu \setminus (r)}}$$

or, equivalently,

$$(127) \quad p_r^\perp h_n^* = \sum_{\rho \vdash n-r} (-1)^{l(\rho \cup (r))} (n+1)^{l(\rho \cup (r))-1} \frac{p_\rho}{z_\rho}$$

Since  $l(\rho \cup (r)) = l(\rho) - 1$ , we deduce the required formula . ■

**Lemma 5.4.** *Let  $r$  and  $N$  be two positive integers, with  $N > r$ . Then,*

$$(128) \quad \sum_{n>r} p_{n-r}(p_n^\perp h_N^*) = -\frac{N-r}{N+1} p_r^\perp h_N^*$$

*Proof.* From (120), one has

$$(129) \quad \sum_{n>r} p_{n-r}(p_n^\perp h_N^*) = \sum_{n>r} \sum_{\rho \vdash n-r} (-1)^{l(\rho)-1} (n+1)^{l(\rho)} \frac{p_{\rho \cup (n-r)}}{z_\rho}$$

for  $\mu = \rho \cup (n-r)$  and  $k = n-r$ , we obtain :

$$(130) \quad \sum_{n>r} p_{n-r}(p_n^\perp h_N^*) = \sum_{\mu \vdash N-r} \sum_{k \in \mu} (-1)^{l(\mu)} (N+1)^{l(\mu)-1} \frac{p_\mu}{z_{\mu \setminus (k)}}$$

$$(131) \quad \sum_{n>r} p_{n-r}(p_n^\perp h_N^*) = \sum_{\mu \vdash N-r} (-1)^{l(\mu)} (N+1)^{l(\mu)-1} \left( \sum_{k \in \mu} \frac{1}{z_{\mu \setminus (k)}} \right) p_\mu$$



From (124), one has

$$(132) \quad \sum_{k \in \mu} \frac{1}{z_{\mu \setminus (k)}} = \sum_{k \in \mu} \frac{km_{\mu}(k)}{z_{\mu}}$$

Hence,

$$(133) \quad \sum_{k \in \mu} \frac{1}{z_{\mu \setminus (k)}} = \frac{1}{z_{\mu}} \sum_{k \in \mu} km_{\mu}(k) = \frac{|\mu|}{z_{\mu}}$$

So we can rewrite (131) as

$$(134) \quad \sum_{n > r} p_{n-r} (p_n^{\perp} h_N^*) = \sum_{\mu \vdash N-r} (-1)^{l(\mu)} (N+1)^{l(\mu)-1} \frac{N-r}{z_{\mu}} p_{\mu}$$

$$(135) \quad \sum_{n > r} p_{n-r} (p_n^{\perp} h_N^*) = \frac{N-r}{N+1} \sum_{\mu \vdash N-r} (-1)^{l(\mu)} (N+1)^{l(\mu)} \frac{p_{\mu}}{z_{\mu}}$$

From (120), we get the required formula. ■

*Proof* – (of Theorem 5.2) Since  $b_r = -q_r$  and  $g_r = -p_r$ , one has

$$(136) \quad \langle b_{\lambda} \star_{\gamma} b_r, h_N^* \rangle = -\langle b_{\lambda}, q_r^{\perp} h_N^* \rangle$$

Then, from (109) and lemma 5.3,

$$(137) \quad \begin{aligned} \langle b_{\lambda} \star_{\gamma} b_r, h_N^* \rangle &= -\langle b_{\lambda}, p_r^{\perp} h_N^* \rangle - \gamma \langle b_{\lambda}, \sum_{n > r} p_{n-r} (p_n^{\perp} h_N^*) \rangle \\ &= -\langle g_{\lambda}, p_r^{\perp} h_N^* \rangle + \gamma \frac{N-r}{N+1} \langle b_{\lambda}, p_r^{\perp} h_N^* \rangle \\ &= -\langle g_{\lambda} p_r, h_N^* \rangle + \gamma \frac{N-r}{N+1} \langle g_{\lambda}, p_r^{\perp} h_N^* \rangle \\ &= \langle g_{\lambda} g_r, h_N^* \rangle - \gamma \frac{N-r}{N+1} \langle g_{\lambda} g_r, h_N^* \rangle \\ &= \left( 1 - \gamma \frac{N-r}{N+1} \right) \langle g_{\lambda} g_r, h_N^* \rangle \end{aligned}$$

From that, we deduce (119).

## 6. A DEFORMATION OF THE WITT ALGEBRA

**6.1. A simpler multiplicative formula in  $\mathcal{H}'_{\gamma}$ .** Now let us expand  $q_k \star_{\gamma} q_n$  on the  $q_{\mu}$ . One has

$$(138) \quad q_k \star_{\gamma} q_n = \sum_{\mu} \frac{1}{z_{\mu}} \langle q_k \star_{\gamma} q_n, p_{\mu} \rangle q_{\mu},$$

so that

$$(139) \quad q_k \star_{\gamma} q_n = \sum_{\mu} \frac{1}{z_{\mu}} \langle q_k, q_n^{\perp}(p_{\mu}) \rangle q_{\mu}$$

From (109), deduce that

$$(140) \quad q_n^\perp p_\mu = p_n^\perp p_\mu + \gamma \sum_{r>n} p_{r-n} p_r^\perp p_\mu$$

so that

$$(141) \quad \begin{aligned} q_n^\perp p_\mu &= p_n^\perp p_\mu + \gamma \sum_{r>n} r p_{r-n} \frac{\partial p_\mu}{\partial p_r} \\ &= p_n^\perp p_\mu + \gamma \sum_{r>n} r m_r(\mu) p_{\mu \setminus (r) \cup (r-n)} \end{aligned}$$

Hence,  $q_\mu$  gives to  $q_k \star_\gamma q_n$  a contribution to the factor of  $\gamma$  only if  $\mu = (k+n)$ . In this case we have

$$(142) \quad \langle q_k, q_n^\perp p_{k+n} \rangle = \gamma(k+n)k$$

Hence,

$$(143) \quad q_k \star_\gamma q_n = q_{(k,n)} + k\gamma q_{k+n}$$

this formula also completely determines the multiplicative structure of  $\mathcal{H}'_\gamma$ . In the case  $\gamma = 1$ , we can see that it is coherent with the result of [3].

**6.2. Lie algebra structure corresponding to  $\mathcal{H}'_\gamma$ .** Now, suppose that  $\gamma \neq 0$ . Since  $\mathcal{H}'_\gamma$  is a connected cocommutative Hopf algebra, it is the universal enveloping algebra of a Lie algebra  $\mathcal{L}_\gamma$ . From (143), its bracket is determined by

$$(144) \quad [q_k, q_n]_\gamma = \gamma(k-n)q_{k+n}.$$

Denote by  $d_{n,\gamma}$  the differential operator

$$(145) \quad d_{n,\gamma} = t^{1-n\gamma} \frac{d}{dt}$$

The  $d_{n,\gamma}$  satisfy the same relation (144) as the  $q_n$  of  $\mathcal{L}_\gamma$ , so that  $\mathcal{H}'_\gamma$  can be interpreted as a Lie algebra of differential operators: it is the Lie algebra generated by the  $d_{n,\gamma}$ . In the case  $\gamma = 1$  one has  $d_{n,1} = t^{1-n} \frac{d}{dt}$ . These operators generate  $\mathcal{L}_1$ , that is called the *Witt algebra*. It is known that the universal enveloping algebra of the Witt algebra is the dual of the Fàa di Bruno algebra. Note also that from the commutativity of  $\mathcal{H}'_0$ , (144) is also valid in the degenerate case  $\gamma = 0$ .

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