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Lyapunov functions for evolution variational inequalities with locally prox-regular sets

Abderrahim Hantoute* and Marc Mazade†

Abstract

This paper is devoted on the one hand to the study of specific properties of an evolution variational inequality, holding in the Hilbert setting. We give on the other hand a general criterion for Lyapunov pairs of this dynamical system and some results on the asymptotic behaviour of the solution.

Key-words: Differential inclusions - Evolution variational inequalities - Proximal normal cone - Uniformly prox-regular set - Hypomonotonicity - Lyapunov pair - Asymptotic behaviour.

Mathematics Subject Classifications (2010) : 34A60 · 49J52 · 49J20

1 Introduction

This work is devoted on the one hand to the study of regularity properties of locally absolutely continuous solutions of the differential inclusion

$$\begin{cases} \dot{x}(t,x_0) \in f(x(t,x_0)) - N^P(C;x(t,x_0)) \text{ a.e. } t \in [0,+\infty[, \\ x(0,x_0) = x_0,\ x_0 \in C, \end{cases} \tag{1}$$

where C is a closed and uniformly prox-regular subset of a Hilbert space H, $N^P(C;\cdot)$ denotes the proximal normal cone. On the other hand we also provide explicit criteria to characterize Lyapunov pairs associated to differential inclusion (1) and asymptotic behaviour results. Following [1, 29], a pair $V, W : H \to \mathbb{R} \cup \{+\infty\}$ forms an a-Lyapunov pair for differential inclusion (1) if for all $x_0 \in C$, $x(\cdot, x_0)$ satisfies

$$e^{at}V(x(t,x_0)) - e^{at}V(x(s,x_0)) + \int_s^t W(x(\tau,x_0))\,d\tau \leq 0 \text{ for all } t \geq s \geq 0 \tag{2}$$

for some $a \geq 0$. As mentioned in [1], the choice of an appropriate function W in (2) can be a key to obtain some stability properties of the solution $x(\cdot, x_0)$. It is the same with

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the weight e^{at} which emphazises the fact that V is decreasing.

When the set C depends on t, differential inclusion (1) is called sweeping process and has been introduced by J.J. Moreau in the 70's (see [25]). In [18], Henry studied planning procedures in mathematical economy given by the differential inclusion

$$\begin{cases} -\dot{x}(t) \in P_{T_C(x(t))}(F(x(t))) \text{ a.e. } t \in [0,T] \\ x(0) = x_0 \in C, \end{cases} \tag{3}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an upper semicontinuous set-valued mapping with nonempty compact convex values, C is a fixed closed convex set, $T_C(\cdot)$ is the tangent cone to C, and $P_{T_C(x(t))}$ denotes the metric projection mapping onto the closed convex set $T_C(x(t))$. B. Cornet [14] considered (3) with a Clarke tangentially regular set C of $\mathbb{R}^n$, and reduced the problem as in [18] to the existence of a solution of

$$\begin{cases} -\dot{x}(t) \in N(C; x(t)) + F(x(t)) \text{ a.e. } t \in [0,T] \\ x(0) = x_0 \in C. \end{cases} \tag{P'}$$

The case of moving sets $C(t)$ in place of the fixed set C has been largely studied, for example, when the sets $C(t)$ are either convex or complements of open convex sets (see [10, 11]), and also for general nonconvex closed sets of $\mathbb{R}^n$ (see [30]). Several other papers dealt later in the Hilbert setting with perturbed sweeping processes under uniform prox-regularity assumptions, as the works of M. Bounkhel and L. Thibault [5], J.F. Edmond and L. Thibault in [16, 17]. More recently, when C is a locally prox-regular set, M. Mazade and L. Thibault showed in the context of a Hilbert space the well-posedness of evolution variational inequality (1) (see [21, 22]).

For many motivations, several authors studied independently Lyapunov pairs (see [7, 8, 9, 20, 26]). In view of developing some criteria for these functions and extend the study in the nonsmooth case, Clarke et al. [13] called these functions "weakly or strongly decrease along the trajectories" and introduced Hamilton Jacobi equations whose solutions are the lsc Lyapunov functions. The study of Lyapunov pairs of a system is also an important tool to provide information on the trajectories on different models (see [3, 4, 6, 28]). It is also used to obtain stability properties of second order differential equations and nonlinear mechanical systems.
When V is smooth enough (that is $\mathcal{C}^1$ for example), by derivating $t \to e^{at}V(x(t,x_0))$, we can show that (V,W) forms an a-Lyapunov pair for (1) if and only if

$$\sup_{y\in H} \sup_{y^*\in\partial\varphi(y)} V'(y)(f(y)-y^*) + aV(y) + W(y) \leq 0 \tag{4}$$

when $\varphi(\cdot)$ is a lsc convex function or $N^P(C;\cdot)$ is replaced in (1) by a maximal monotone operator.

Our aim here is to give an appropriate criteria and extend condition (4) considering the proximal normal cone $N^P(C,\cdot)$ of a uniformly prox-regular set C in differential inclusion (1). As in [1], we obtain a relaxed condition and prove that (V,W) forms an a-Lyapunov pair for (1) if and only if

$$\sup_{\xi\in\partial_P V(y)} \inf_{y^*\in N^P(C;y)} \langle \xi, f(y)-y^*\rangle + aV(y) + W(y) \leq 0 \qquad \sup_{\xi\in\partial_\infty V(y)} \inf_{y^*\in N^P(C;y)} \langle \xi, f(y)-y^*\rangle \leq 0.$$

The paper is organized as follows. In the next section, we recall several notions of nonsmooth and variational analysis, involved throughout all the paper. We also give some specific results about the solution $x(\cdot)$ of (1). More precisely we estimate the derivative $\dot{x}(t)$ and the right derivative of $x(\cdot)$ at 0, denoted by $\dot{x}_+(0)$. Then, we establish in section 3 a general criterion for a-Lyapunov pairs for problem (1). The last section deals with the asymptotic behaviour of the solution.

2 Preliminaries and regularity properties on global trajectories

Let us recall some fundamental definitions. Throughout all the paper, $(H, \langle\cdot,\cdot\rangle)$ stands for a real Hilbert space and $||\cdot|| = \sqrt{\langle\cdot,\cdot\rangle}$ is the associated norm. The open (resp. closed) ball of H centered at $\bar{x}$ with radius ε is denoted by $B(\bar{x},\varepsilon)$ (resp $B[\bar{x},\varepsilon]$). The closed unit ball of H will be denoted by $\mathbb{B}$. For any subset C of H, $\overline{\text{co}}\, C$ stands for the closed convex hull of C. Now let C be a nonempty closed subset of H and $y \in H$. The distance of y to C, denoted by $d_C(y)$ is given by

$$d_C(y) := \inf\{\, ||x-y|| : x \in C\,\}.$$

One defines the (possibly empty) set of nearest points of y in C by

$$\text{Proj}_C(y) := \{\, x \in C : \; d_C(y) = ||y-x||\,\}.$$

When $\text{Proj}_C(y)$ is a singleton, we will write $P_C(y)$ in place of $\text{Proj}_C(y)$ to emphasize this singleton property. If $x \in \text{Proj}_C(y)$, and $s \geq 0$, then the vector $s(y-x)$ is called (see, e.g., [15]) a *proximal normal* to C at x. Sometimes, it will be convenient to write $\text{Proj}\,(C,x)$ in place of $\text{Proj}_C(x)$. The set of all vectors of this form is a cone which is termed the *proximal normal cone* of C at x. It is denoted by $N^P(C;x)$ or $N^P_C(x)$, and $N^P(C;x) = \emptyset$ whenever $x \notin C$. Observing that, for $x \in C$, a nonzero vector $v \in N^P(C;x)$ if and only if that for some $\rho > 0$ one has $x \in \text{Proj}_C(x + \frac{\rho}{\|v\|}v)$, and translating this as $\rho^2 \leq \|x + \frac{\rho}{\|v\|}v - x'\|^2$ for all $x' \in C$, we obtain that the inclusion $v \in N^P(C;x)$ is equivalent to the existence of some real $\sigma \geq 0$ such that

$$\langle v, x'-x\rangle \leq \sigma\|x'-x\|^2 \quad \text{for all } x' \in C. \tag{5}$$

For $x \in C$, inequality (5) can be also localized in the sense that it holds for some $\sigma > 0$ (i.e., $v \in N^P(C;x)$) if and only if there exist some $\gamma \geq 0$ and $\eta > 0$ such that

$$\langle v, x'-x\rangle \leq \gamma\|x'-x\|^2 \quad \text{for all } x' \in C \cap B(x,\eta). \tag{6}$$

One also defines the *Mordukhovich limiting normal cone* and the *Clarke normal cone* respectively by

$$N^L(C;x) := \{\, v \in H : \exists v_n \to^w v, v_n \in N^P(C;x_n), x_n \underset{C}{\to} x\,\}$$

and

$$N^C(C;x) := \overline{\text{co}}\, N^L(C;x),$$

where $v_n \to^w v$ means that the sequence $(v_n)_n$ converges weakly to v and $x_n \underset{C}{\to} x$ means that $x_n \to x$ and $x_n \in C$ for all $n \in \mathbb{N}$. It clearly appears in the definition above that $N^L(C;x)$ is the Painlevé-Kuratowski weak sequential outer (or superior) limit of $N^P(C;x')$ as $x' \to x$, where for a set-valued mapping $M : U \rightrightarrows H$ from a topological space U into H the Painlevé-Kuratowski weak sequential outer limit of M at $x \in U$ is the set

$$^{\text{seq}}\operatorname*{Lim\,sup}_{x' \to x} M(x') := \{\, v \in H : \exists v_n \to^w v,\ v_n \in M(x_n),\ x_n \to x \,\}.$$

Like for $N^P(C;x)$, sometimes one writes $N^L_C(x)$ and $N^C_C(x)$ instead of $N^L(C;x)$ and $N^C(C;x)$. It is worth pointing out that for x outside the closed set C one has $N^L(C;x) = N^C(C;x) = \emptyset$, hence (since $0 \in N^P(C;x)$ for all $x \in C$)

$$\operatorname{Dom} N^P(C;\cdot) = \operatorname{Dom} N^L(C;\cdot) = \operatorname{Dom} N^C(C;\cdot) = C,$$

where for a set valued-mapping $M : U \rightrightarrows H$ we denote by $\operatorname{Dom} M$ its (effective) domain, that is, $\operatorname{Dom} M := \{x \in U :\ M(x) \neq \emptyset\}$.

The elements in the Mordukhovich limiting normal cone can also be obtained as weak limits of sequences of Fréchet normal vectors. A vector $v \in H$ is a *Fréchet normal* of C at $x \in C$ whenever for any real $\varepsilon > 0$ there exists some real $\eta > 0$ such that

$$\langle v, x' - x\rangle \leq \varepsilon\|x' - x\| \quad \text{for all } x' \in C \cap B(x,\eta).$$

Denoting by $N^F(C;x)$ or $N^F_C(x)$ the *cone of all Fréchet normals* to C at $x \in C$ and putting $N^F(C;x) = \emptyset$ for $x \notin C$, it is also known that

$$N^L(C;x) = {}^{\text{seq}}\operatorname*{Lim\,sup}_{x' \to x} N^F(C;x') \quad \text{for all } x \in C.$$

Taking (5) into account, we always have

$$N^P(C;x) \subset N^F(C;x) \subset N^L(C;x) \subset N^C(C;x) \quad \text{for all } x \in C. \tag{7}$$

After this definition of proximal normal cone, we recall the definition of local prox-regularity of sets. For a large development this concept, the reader is referred to [27]. In this paper, we will use some results where the quantified viewpoint [22] of the local prox-regularity has been introduced.

Definition 2.1. *For positive real numbers r and α, the closed set C is said to be (r,α)-prox-regular at a point $\bar{x} \in C$ provided that for any $x \in C \cap B(\bar{x},\alpha)$ and any $v \in N^P(C;x)$ such that $||v|| < r$, one has*

$$x = \operatorname{proj}_C(x + v).$$

The set C is r-prox-regular (resp. prox-regular) at $\bar{x}$ when it is (r,α)-prox-regular at $\bar{x}$ for some real $\alpha > 0$ (resp. for some numbers $r > 0$ and $\alpha > 0$). The set C is said to be r-uniformly prox-regular when $\alpha = +\infty$.

It is not difficult to see that the latter (r,α)-prox-regularity property of C at $\bar{x} \in C$ is equivalent to requiring that

$$x \in \operatorname{Proj}_C(x + rv) \quad \text{for all } x \in C \cap B(\bar{x},\alpha) \text{ and } v \in N^P(C;x) \cap \mathbb{B}. \tag{8}$$

When the set C is (r,α)-prox-regular at $\bar{x}$, we have

$$N^P(C;x) = N^F(C;x) = N^L(C;x) = N^C(C;x) \text{ for all } x \in C \cap B(\bar{x},\alpha).$$

The (r,α)-prox-regularity of the set C gives the following hypomonotonicity property of the truncated normal cone.

Proposition 2.2. *Let C be a closed subset of H, and $\bar{x} \in C$. Then, if there exist positive real numbers r and α such that C is (r,α)-prox-regular at $\bar{x}$, then the set-valued mapping $N^P(C,\cdot) \cap \mathbb{B}$ is $\frac{1}{r}$-hypomonotone on $B(\bar{x},\alpha)$.*

We now come to the study of (1). The problem of existence and uniqueness has been obtained in [16], and is also a consequence of the results obtained in [21].

Theorem 2.3. *Let H be a Hilbert space. Assume that C is r-uniformly prox-regular. Let $f : H \to H$ a k-Lipschitz continuous mapping. Then for any $x_0 \in C$, the following evolution variational inequality*

$$\begin{cases} \dot{x}(t) \in f(x(t)) - N^P(C;x(t)) \ \ a.e. \ t \geq 0, \\ x(0) = x_0 \in C \end{cases} \tag{9}$$

has one and only one locally absolutely continuous solution on $[0,+\infty[$. Moreover one has the estimation

$$\|\dot{x}(t) - f(x(t))\| \leq \|f(x(t))\| \ \ a.e. \ t \geq 0. \tag{10}$$

The next lemma is helpful in studying the derivative $\dot{x}(t)$ of the solution of (9).

Lemma 2.4. *Let H be a Hilbert space. Assume that C is r-uniformly prox-regular. Fix $T > 0$ and consider $x_1(\cdot), x_2(\cdot)$ from $[0,T]$ into H be two absolutely continuous mappings satisfying*

$$\dot{x}_i(t) \in f(x_i(t)) - N^P(C;x_i(t)) \quad a.e. \ , i = 1,2. \tag{11}$$

Then for any $s,t \in [0,T]$ with $s \leq t$ one has

$$\begin{aligned} &\|x_1(t) - x_2(t)\| \\ &\leq [\|x_1(s) - x_2(s)\| + \int_s^t \|f(x_1(\tau)) - f(x_2(\tau))\| \, d\tau] \exp(\frac{1}{r}\int_s^t (\|f(x_1(\tau))\| + \|f(x_2(\tau))\|) \, d\tau). \end{aligned} \tag{12}$$

Proof. Denote by $\mathcal{N}$ the Lebesgue-null subset of $[0,T]$ out of which (11) holds for $i = 1,2$. Fix any $t \in [0,T]\backslash\mathcal{N}$. Since C is uniformly prox-regular, according to the hypomonotonicity of $N(C,\cdot) \cap \mathbb{B}$ we get

$$\langle -\dot{x}_1(t) + f(x_1(t)) + \dot{x}_2(t) - f(x_2(t)), x_1(t) - x_2(t)\rangle \geq -\frac{1}{r}(\|f(x_1(t))\| + \|f(x_2(t))\|)\|x_1(t) - x_2(t)\|^2.$$

Hence,

$$\begin{aligned} &\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 = 2\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle \\ &\leq \frac{2}{r}(\|f(x_1(t))\| + \|f(x_2(t))\|)\|x_1(t) - x_2(t)\|^2 + 2\langle f(x_2(t)) - f(x_1(t)), x_1(t) - x_2(t)\rangle. \end{aligned} \tag{13}$$

Observe that the function $\|x_1(\cdot) - x_2(\cdot)\|$ is absolutely continuous on $[0,T]$ and that for almost all $t \in [0,T]$,

$$\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 = 2\|x_1(t) - x_2(t)\|\frac{d}{dt}\|x_1(t) - x_2(t)\|.$$

We then deduce from (13) that for almost every $t \in [0,T]$:

$$\frac{d}{dt}\|x_1(t) - x_2(t)\| \leq \frac{1}{r}(\|f(x_1(t))\| + \|f(x_2(t))\|)\|x_1(t) - x_2(t)\| + \|f(x_1(t)) - f(x_2(t))\|.$$

Applying Gronwall's lemma, one obtains inequality (12) as desired. □

The next lemma is an adaptation of Lemma 1.8 in [23].

Lemma 2.5. *Let C be a closed set r-uniformly prox-regular. The function $d(\cdot) : H \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$d(x) := \begin{cases} +\infty & \text{if } x \notin C, \\ d(0, f(x) - N^P(C;x)) & \text{otherwise} \end{cases} \tag{14}$$

is lsc at $x \in C$ with respect to the strong topology of H.

Before giving the main results of this section, recall that for a set S of H and $\bar{x} \in S$, the Clarke tangent cone of S at $\bar{x}$ is defined as the Painlevé-Kuratowski limit inferior of the set-differential quotient

$$T(S;\bar{x}) := \operatorname*{Lim\,inf}_{t\downarrow 0;\, u \underset{S}{\to} \bar{x}} \frac{1}{t}(S - u),$$

i.e., a vector $h \in T(S;\bar{x})$ if for any sequence $(x_n)_n$ in S converging to $\bar{x}$ and any sequence $(t_n)_n$ of positive numbers converging to 0 there exists a sequence $(h_n)_n$ in H converging to h such that

$$x_n + t_n h_n \in S \text{ for all } n \in \mathbb{N}.$$

The Clarke tangent cone $T(S;\bar{x})$ is known to be closed and convex (see [12]). The Clarke normal cone $N^C(S,\bar{x})$ of S at $\bar{x}$ can also be seen as the negative polar $(T(S;\bar{x}))^0$ of the Clarke tangent cone, that is,

$$N^C(S;\bar{x}) := \{v \in H : \langle v, h\rangle \leq 0 \quad \forall h \in T(S;\bar{x})\}.$$

Proposition 2.6. *Let C be a closed set r-uniformly prox-regular. Denote by $x : [0,+\infty[\to H$ the unique locally absolutely continuous solution of*

$$\begin{cases} \dot{x}(t) \in f(x(t)) - N^P(C;x(t)) \text{ a.e. } t \in [0,+\infty[, \\ x(0) = x_0 \in C. \end{cases} \tag{15}$$

Then the following properties hold

a) for almost every $t \geq 0$, $\dot{x}(t) = (f(x(t)) - N^P(C;x(t)))^0$;

b) There exists $T > 0$ and $m > 0$ such that for all $t \in [0,T]$,

$$\|(f(x(t)) - N^P(C;x(t)))^0\| \leq ||f(x_0)||e^{mt/2}.$$

Moreover if $f(x_0) \in T(C, x_0)$,

c) the mapping $t \to (f(x(t)) - N^P(C; x(t)))^0$ *is continuous on the right at* 0*;*

d) the mapping $t \to \dot{x}_+(t)$ *is continuous on the right at* 0 *with*

$$\dot{x}_+(0) = (f(x_0) - N^P(C; x_0))^0.$$

Proof. To prove *a*), we set $z(t) := f(x(t))$ and we fix $t_0 \in [0, +\infty[$ such that

$$\dot{x}(t_0) \in z(t_0) - N^P(C, x(t_0)) \tag{16}$$

$$\text{and } \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \|z(r) - z(t_0)\| \, dr = 0. \tag{17}$$

Note that for a.e. $t \geq 0$

$$\|z(t)\| = \|f(x(t))\| \leq \|f(x_0)\| + k\|x(t) - x_0\|$$

so $z \in L^1_{loc}([0, +\infty[, H)$. Almost every point of $[0, +\infty[$ is a Lebesgue-point for z and hence satisfies (17). Then, define $g : [0, +\infty[\to H$ and $y : [0, +\infty[\to H$ by

$$g(t) := z(t_0) - (z(t_0) - N^P(C, x(t_0)))^0 \text{ and } y(t) := x(t_0) \text{ for any } t \in [0, +\infty[.$$

One clearly has

$$\dot{y}(t) \in g(t) - N^P(C, y(t)) \quad \text{ for all } t \in [0, +\infty[.$$

Then, by Lemma 2.4, for any $\varepsilon > 0$, we can write

$$\begin{aligned}
&\|x(t_0 + \varepsilon) - x(t_0)\| \\
&\leq \left(\int_{t_0}^{t_0+\varepsilon} \|z(\tau) - z(t_0) + (z(t_0) - N^P(C, x(t_0)))^0\| \, d\tau \right) * \\
&\qquad \exp(\frac{1}{r} \int_{t_0}^{t_0+\varepsilon} \|z(\tau)\| \, d\tau + \varepsilon \|z(t_0) - (z(t_0) - N^P(C, x(t_0)))^0\|).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|\varepsilon^{-1}[x(t_0 + \varepsilon) - x(t_0)]\| \\
&\leq \left(\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \|z(\tau) - z(t_0)\| \, d\tau + \|(z(t_0) - N^P(C, x(t_0)))^0\| \right) * \\
&\qquad \exp(\frac{1}{r} \int_{t_0}^{t_0+\varepsilon} \|z(\tau)\| \, d\tau + \varepsilon \|z(t_0) - (z(t_0) - N^P(C, x(t_0)))^0\|).
\end{aligned}$$

When $\varepsilon \downarrow 0$, via (17) we get

$$\|\dot{x}(t_0)\| \leq \|(z(t_0) - N^P(C, x(t_0)))^0\|.$$

By virtue of (16), we conclude that

$$\dot{x}(t_0) = (f(x(t_0)) - N^P(C; x(t_0)))^0 \text{ for a.e. } t_0 \geq 0.$$

Proof of b). Following the notations of Theorem 3.1 in [22], we fix $0 < r' < r$. Let $\beta = \|f(x_0)\| + 2r'k$ and $T > 0$ such that $\beta T < \frac{r'}{2}$. Under these assumptions, for any $\lambda > 0$, the family $x_\lambda(\cdot)_{\lambda>0}$ of the solutions of the differential equation

$$\begin{cases} \dot{x}_\lambda(t) = f(x_\lambda(t)) - \frac{1}{2\lambda}\nabla d_C^2(x_\lambda(t)) \quad t \in [0,T], \\ x_\lambda(0) = x_0 \in C, \end{cases} \tag{18}$$

converge uniformly to the solution $x(\cdot)$ of (9) on $[0,T]$ as $\lambda \downarrow 0$. Recall also that (see Proposition 2.5 in [22])

- $\frac{1}{2}\nabla d_C(x) = x - \mathrm{proj}_C(x) \quad$ for all $\; x \in U_C(r) := \{w \in H : d_C(w) < r\}$;
- the mapping $\mathrm{proj}_C(\cdot)$ is $\frac{r}{r-r'}$-Lipschitz continuous on $U_C(r')$.

We set $c := \frac{r}{r-r'}$ and $z_\lambda(t) := f(u_\lambda(t))$. Consider arbitrary $t \in [0,T]$ and $h > 0$ such that $t + h \leq T$. Remark that

$$-\dot{x}_\lambda(t) + z_\lambda(t) \in N(C; \mathrm{proj}_C(u_\lambda(t)))$$

and according to the hypomonotonicity property of $N(C;\cdot)$ (see a) of Proposition 2.1 in [22]) we get

$$\begin{aligned} &\langle -\dot{x}_\lambda(t+h) + z_\lambda(t+h) + \dot{x}_\lambda(t) - z_\lambda(t), \mathrm{proj}_C(x_\lambda(t+h)) - \mathrm{proj}_C(x_\lambda(t))\rangle \\ &\qquad \geq -\frac{\beta}{r}||\mathrm{proj}_C(x_\lambda(t+h)) - \mathrm{proj}_C(x_\lambda(t))||^2. \end{aligned}$$

With $\mathrm{proj}_C(x_\lambda(s)) = \lambda(\dot{x}_\lambda(s) - z_\lambda(s)) + x_\lambda(s)$ for $s \in \{t, t+h\}$ and (b) of Proposition 2.5 in [22], we get

$$\langle -\dot{x}_\lambda(t+h) + z_\lambda(t+h) + \dot{x}_\lambda(t) - z_\lambda(t), \lambda(\dot{x}_\lambda(t+h) - z_\lambda(t+h)) - \lambda(\dot{x}_\lambda(t) - z_\lambda(t)) + x_\lambda(t+h) - x_\lambda(t)\rangle$$

$$\geq -\frac{\beta c^2}{r}||x_\lambda(t+h) - x_\lambda(t)||^2.$$

Computing the left hand side, we obtain

$$\begin{aligned} &-\lambda||\dot{x}_\lambda(t+h) - z_\lambda(t+h)||^2 - \lambda||\dot{x}_\lambda(t) - z_\lambda(t)||^2 + 2\lambda\langle \dot{x}_\lambda(t+h) - z_\lambda(t+h), \dot{x}_\lambda(t) - z_\lambda(t)\rangle \\ &+\langle z_\lambda(t+h) - z_\lambda(t), x_\lambda(t+h) - x_\lambda(t)\rangle - \langle \dot{x}_\lambda(t+h) - \dot{x}_\lambda(t), x_\lambda(t+h) - x_\lambda(t)\rangle \\ &\quad \geq -\frac{\beta c^2}{r}||x_\lambda(t+h) - x_\lambda(t)||^2. \end{aligned}$$

Hence we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[||x_\lambda(t+h) - x_\lambda(t)||^2] \\ &\leq \frac{\beta c^2}{r}||x_\lambda(t+h) - x_\lambda(t)||^2 - \lambda||\dot{x}_\lambda(t+h) - z_\lambda(t+h)||^2 - \lambda||\dot{x}_\lambda(t) - z_\lambda(t)||^2 \\ &+ 2\lambda\langle \dot{x}_\lambda(t+h) - z_\lambda(t+h), \dot{x}_\lambda(t) - z_\lambda(t)\rangle + \langle z_\lambda(t+h) - z_\lambda(t), x_\lambda(t+h) - x_\lambda(t)\rangle. \end{aligned}$$

Note that

$$-\lambda||\dot{x}_\lambda(t+h) - z_\lambda(t+h)||^2 - \lambda||\dot{x}_\lambda(t) - z_\lambda(t)||^2 + 2\lambda\langle \dot{x}_\lambda(t+h) - z_\lambda(t+h), \dot{x}_\lambda(t) - z_\lambda(t)\rangle \leq 0$$

and since $f(\cdot)$ is k-Lipschitz continuous we can write

$$\begin{aligned}||z_\lambda(t+h) - z_\lambda(t)|| &= ||f(x_\lambda(t+h)) - f(x_\lambda(t))|| \\ &\leq k||x_\lambda(t+h) - x_\lambda(t)||.\end{aligned}$$

Hence we obtain

$$\frac{1}{2}\frac{d}{dt}[||x_\lambda(t+h) - x_\lambda(t)||^2] \leq (\frac{\beta c^2}{r} + k)||x_\lambda(t+h) - x_\lambda(t)||^2.$$

Set $m := \frac{\beta c^2}{r} + k$. According to Gronwall's lemma, we obtain that given any $t, s \in [0, +\infty[$ with $s \leq t$,

$$||x_\lambda(t+h) - x_\lambda(t)||^2 \leq ||x_\lambda(s+h) - x_\lambda(s)||^2 e^{m(t-s)}$$

and hence

$$||h^{-1}(x_\lambda(t+h) - x_\lambda(t))|| \leq ||h^{-1}(x_\lambda(s+h) - x_\lambda(s))||e^{m(t-s)/2},$$

Finally, letting $h \downarrow 0$, we obtain

$$||\dot{x}_\lambda(t)|| \leq ||\dot{x}_\lambda(s)||e^{m(t-s)/2}, \tag{19}$$

that holds for any $s, t \in [0, +\infty[$ with $s \leq t$. Putting $s = 0$ in the last inequality leads to

$$||\dot{x}_\lambda(t)|| \leq ||\dot{x}_\lambda(0)||e^{mt/2} = ||f(x_0)||e^{mt/2}$$

since $\dot{x}_\lambda(0) = f(x_0) - \frac{1}{2\lambda}\nabla d_C^2(x_0) = f(x_0)$ as $x_0 \in C$. It follows that given any $t \in [0, +\infty[$, the sequence $(\dot{x}_\lambda(t))$ is bounded in H. Then one find some subsequence $(\dot{x}_{\lambda_n}(t))_{n\geq 1}$ that converges weakly in H to some element v_t that satisfies

$$||v_t|| \leq ||f(x_0)||e^{mt/2}. \tag{20}$$

According to the r-uniform prox-regularity of C, we have

$$\langle z_{\lambda_n}(t) - \dot{x}_{\lambda_n}(t), x' - \text{proj}_C(x_{\lambda_n}(t))\rangle \leq \frac{\beta}{2r}||x' - \text{proj}_C(x_{\lambda_n}(t))||^2, \text{ for all } x' \in C. \tag{21}$$

As n goes to $+\infty$, $\text{proj}_C(x_{\lambda_n}(t)) \to x(t)$ so

$$\langle f(x(t)) - v_t, x' - x(t)\rangle \leq \frac{\beta}{2r}||x' - x(t)||^2, \text{ for all } x' \in C,$$

hence $v_t \in f(x(t)) - N^P(C; x(t))$ and $\|v_t\| \geq \|(f(x(t)) - N^P(C; x(t)))^0\|$. According to (20) we have

$$||(f(x(t)) - N^P(C; x(t)))^0|| \leq ||f(x_0)||e^{mt/2},$$

which proves b).
To prove c), as $f(x_0) \in T(C, x_0)$, one has $f(x_0) \in [N^C(C, x_0)]^0 = [N^P(C, x_0)]^0$ since C is uniformly prox-regular. So for all $w \in N^P(C; x_0)$, $\langle w, f(x_0)\rangle \leq 0$. Hence

$$\|f(x_0) - w\|^2 = \|f(x_0)\|^2 - 2\langle w, f(x_0)\rangle + \|w\|^2 \geq \|f(x_0)\|^2 \text{ for all } w \in N^P(C, x_0).$$

So $\mathrm{proj}(f(x_0), N^P(C, x_0)) = \{0\}$ and we get

$$f(x_0) = f(x_0) - \mathrm{proj}(f(x_0), N^P(C, x_0)) = (f(x_0) - N^P(C; x_0))^0.$$

So

$$||(f(x(t)) - N^P(C; x(t)))^0|| \leq ||(f(x_0) - N^P(C, x_0))^0|| e^{mt/2}, \tag{22}$$

Thanks to Lemma 2.5 we know on the one hand that for any $s \in [0, +\infty[$, one has

$$||(f(x(s)) - N^P(C; x(s)))^0|| \leq \liminf_{t \downarrow s} ||(f(x(t)) - N^P(C; x(t)))^0|| \tag{23}$$

and on the other hand, by (22)

$$\limsup_{t \downarrow 0} ||(f(x(t)) - N^P(C; x(t)))^0|| \leq ||(f(x_0) - N^P(C, x_0))^0||. \tag{24}$$

As a result, from (23) with $s = 0$ and (24), we deduce that the function $||(f(x(\cdot)) - N^P(C, x(\cdot)))^0||$ is continuous on the right at 0. Given any sequence $(t_n)_{n \geq 1} \subset [0, T]$ with $t_n \to 0$, we can write

$$||(f(x(t_n)) - N^P(C; x(t_n)))^0|| \underset{n \to +\infty}{\to} ||(f(x_0) - N^P(C; x_0))^0|| \tag{25}$$

and then $\sup_{n \geq 1} ||(f(x(t_n)) - N^P(C; x(t_n)))^0|| < +\infty$. Therefore there exists some $\xi \in H$ and a subsequence $((f(x(t_{n_k})) - N^P(C; x(t_{n_k})))^0)_{k \geq 1}$ that converges weakly in H to ξ when $k \to +\infty$. Using the same technique given by (21), one has $\xi \in f(x(t)) - N^P(C, x(t))$. Inequality (25) yields

$$\|\xi\| \leq \liminf_{k \to +\infty} \|(f(x(t_{n_k})) - N^P(C; x(t_{n_k})))^0\| = ||(f(x_0) - N^P(C; x_0))^0||.$$

Necessarily, $\xi = (f(x(t)) - N^P(C; x(t)))^0$. By uniqueness of the weak cluster point of $((f(x(t_n)) - N^P(C; x(t_n)))^0)_{n \geq 1}$, the whole sequence is weakly convergent to $(f(x_0) - N^P(C; x_0))^0$ in H. Since we also have the convergence in (25), the sequence $((f(x(t_n)) - N^P(C; x(t_n)))^0)_{n \geq 1}$ actually converges strongly to $(f(x_0) - N^P(C; x_0))^0$ which proves $b)$, i.e. the right continuity of $(f(x(\cdot)) - N^P(C; x(\cdot)))^0$ at 0 as desired.
Proof of $d)$. By the absolute continuity of $x(\cdot)$ on $[0, T]$, for all $h > 0$ with $h \leq T$ we have

$$x(0 + h) - x(0) = \int_0^h \dot{x}(r)\, dr.$$

Then, from $a)$ it comes

$$h^{-1}[x(0 + h) - x(0)] = h^{-1} \int_0^h (f(x(r)) - N^P(C, x(r)))^0\, dr.$$

The continuity on the right at 0 of the mapping $(N^P(C, x(\cdot)) + f(x(\cdot)))^0$ ensures the continuity on the right at 0 of $\dot{x}_+(\cdot)$ and the existence of the right derivative $\dot{x}_+(0)$ with

$$\dot{x}_+(0) = (f(x_0) - N^P(C; x_0))^0$$

which is assertion $d)$. □

3 A general criterion for Lyapunov functions

Let us recall the definitions of subdifferentials of functions.
Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and let $x \in \operatorname{dom} f$, that is, $f(x) < +\infty$. Each one of the above normal cones (see, e.g., [24]) leads to a subdifferential through the normal cone to the epigraph $\operatorname{epi} f$ of f, where

$$\operatorname{epi} f := \{ (x, \rho) \in H \times \mathbb{R} : \ f(x) \leq \rho \}.$$

So, the *proximal subdifferential*, the *Fréchet subdifferential*, the *Mordukhovich limiting subdifferential*, and the *Clarke subdifferential* are the (possibly empty) subsets of H given by

$$\partial_? f(x) = \{ v \in H : (v, -1) \in N^?\big(\operatorname{epi} f; (x, f(x))\big) \},$$

where "?" stands for P, F, L, C respectively, and the singular subdifferential is given by

$$\partial_\infty f(x) = \{ v \in H : (v, 0) \in N^C\big(\operatorname{epi} f; (x, f(x))\big) \}.$$

By convention anyone of the above subdifferentials of f at a point $x \notin \operatorname{dom} f$ is empty. The proximal and Fréchet subdifferentials have amenable analytical descriptions. Indeed, for $x \in \operatorname{dom} f$ it is known through (6) that $v \in \partial_P f(x)$ if and only if there exist $\gamma \geq 0$ and $\eta > 0$ such that

$$\langle v, x' - x \rangle \leq f(x') - f(x) + \gamma ||x' - x||^2 \quad \text{for all } x' \in B(x, \eta).$$

Analogously, $v \in \partial_F f(x)$ if and only if for any real $\varepsilon > 0$ there exists some $\eta > 0$ such that

$$\langle v, x' - x \rangle \leq f(x') - f(x) + \varepsilon ||x' - x|| \quad \text{for all } x' \in B(x, \eta).$$

In this section, our aim is to study Lyapunov pairs of the solutions of the differential inclusion

$$\begin{cases} \dot{x}(t, x_0) \in f(x(t, x_0)) - N^P(C; x(t, x_0)) \text{ a.e. } t \in [0, +\infty[, \\ x(0, x_0) = x_0, x_0 \in C \end{cases} \tag{26}$$

where the existence and uniqueness on $[0, +\infty[$ is given by Theorem 2.3 when the set C is uniformly prox-regular. In the remainder of the paper,

i) $V : H \to \mathbb{R} \cup \{+\infty\}$ denotes a weakly lower semicontinuous (lsc) function;

ii) $W : H \to \mathbb{R}_+$ is Lipschitz continuous on bounded sets;

iii) C is a r-uniformly prox-regular closed subset of H;

iv) $x(\cdot, x_0)$ will denote the solution of (26).

Recall the definition of a Lyapunov pair, as in [1].

Definition 3.1. *Let $a \geq 0$. (V, W) forms an a-Lyapunov pair for the differential inclusion* (26) *if for all $x_0 \in C \cap \operatorname{dom} V$ we have that*

$$e^{at} V(x(t, x_0)) + \int_0^t W(x(\tau, x_0))\, d\tau \leq V(x_0) \text{ for all } t \geq 0.$$

When $a = 0$, one says that (V, W) is a Lyapunov pair for (26).

The next theorem can be seen as an extension of the characterization of Lyapunov pairs given by Theorem 3.3 in [1].

Theorem 3.2. *Let $a \geq 0$. Suppose that for all $x \in C, f(x) \in T(C,x)$. Fix $\bar{y} \in C$, $\bar{\lambda} \in [-\infty, V(\bar{y})[$ such that*

$$[V > \bar{\lambda}] \cap \operatorname{dom} V \subset C.$$

Then the following assertions are equivalent

i) For all $x_0 \in C \cap \operatorname{dom} V \cap [V > \bar{\lambda}]$,

$$e^{at}V(x(t,x_0)) + \int_0^t W(x(\tau,x_0))\,d\tau \leq V(x_0) \quad \forall t \in [0, \rho(x_0)], \tag{27}$$

where $\rho(x_0)$ is given by

$$\rho(x_0) = \sup \left\{ \begin{array}{l} \nu > 0 : \exists \rho > 0, B(x_0,\rho) \subset [V > \bar{\lambda}] \text{ s.t. } \forall s \in [0,\nu] : \\ \\ 2\|x(s,x_0) - x_0\| < \rho \text{ and } |(e^{-as}-1)V(x_0) - \int_0^s W(x(\tau,x_0))\,d\tau| < \rho \end{array} \right\}. \tag{28}$$

ii) For all $y \in C \cap [V > \bar{\lambda}]$

$$\sup_{\xi \in \partial_P V(y)} \langle \xi, f(y) \rangle + aV(y) + W(y) \leq 0,$$

$$\sup_{\xi \in \partial_\infty V(y)} \langle \xi, f(y) \rangle \leq 0.$$

iii) For all $y \in C \cap [V > \bar{\lambda}]$

$$\sup_{\xi \in \partial_P V(y)} \inf_{y^* \in N^P(C;y)} \langle \xi, f(y) - y^* \rangle + aV(y) + W(y) \leq 0,$$

$$\sup_{\xi \in \partial_\infty V(y)} \inf_{y^* \in N^P(C;y)} \langle \xi, f(y) - y^* \rangle \leq 0.$$

Moreover, when $\bar{\lambda} = -\infty$, (V,W) forms an a-Lyapunov pair for (26).

Proof. We first prove the statement *i)* $\Rightarrow$ *ii)*. Let us fix $y \in C \cap [V > \bar{\lambda}], \xi \in \partial_P V(y) \cup \partial_\infty V(y)$ and denote

$$k(\xi) := \left\{ \begin{array}{l} 1 \text{ if } \xi \in \partial_P V(y), \\ 0 \text{ if } \xi \in \partial_\infty V(y). \end{array} \right.$$

Hence $y \in C \cap [V > \bar{\lambda}] \cap \operatorname{dom} V$ and $(\xi, -k(\xi)) \in N^P(\operatorname{epi} V, (y, V(y)))$. Denote by $x(\cdot, y)$ the solution of the inclusion

$$\left\{ \begin{array}{l} \dot{x}(t,y) \in f(x(t,y)) - N^P(C; x(t,y)) \text{ a.e. } t \in [0, +\infty[, \\ x(0,y) = y. \end{array} \right.$$

By *i)* and recalling that $\rho(y) > 0$, we let $T \in]0, \rho(y)[$ be such that

$$\big(x(t,y), e^{-at}V(y) - e^{-at}\int_0^t W(x(\tau,y))d\tau\big) \in \operatorname{epi} V \quad \text{for all } t \in [0,T].$$

Let us introduce the mappings ℓ and h given by

$$\ell(t) := (e^{-at} - 1)V(y) - e^{-at} \int_0^t W(x(\tau, y))\, d\tau \text{ and } h(t) := k(\xi)\ell(t). \tag{29}$$

Observing that $\lim_{t\downarrow 0} \alpha\|x(t,y) - y\|^2 + \alpha\ell^2(t) = 0$, we fix $\varepsilon > 0$ such that

$$\alpha\|x(t,y) - y\|^2 + \alpha\ell^2(t) \leq \varepsilon \text{ for all } t \in [0,T].$$

By the definition of $N^P_{\operatorname{epi} V}(y, V(y))$, there exists $\alpha > 0$ such that for all $t \in [0,T]$

$$\begin{aligned}
&\langle(\xi, -k(\xi)), (x(t,y), e^{-at}V(y) - e^{-at}\int_0^t W(x(\tau,y))\, d\tau) - (y, V(y))\rangle \\
&\leq \alpha\|(x(t,y), e^{-at}V(y) - e^{-at}\int_0^t W(x(\tau,y))\, d\tau) - (y, V(y))\|^2
\end{aligned}$$

$$\begin{aligned}
\langle\xi, x(t,y) - y\rangle - k(\xi)\ell(t) &\leq \alpha\|x(t,y) - y\|^2 + \alpha\ell^2(t) \\
\langle\xi, x(t,y) - y\rangle - h(t) &\leq \alpha\|x(t,y) - y\|^2 + \alpha\ell^2(t) \leq \varepsilon.
\end{aligned}$$

Hence we get

$$-\varepsilon \leq -\alpha\|x(t,y) - y\|^2 - \alpha\ell^2(t) \leq h(t) + \langle\xi, y - x(t,y)\rangle =: \psi(t).$$

Then we obtain

$$-\varepsilon \leq \langle\xi, y - x(t,y)\rangle + h(t).$$

Hence the function ψ defined on $[0,T]$ by

$$\psi(t) := \langle\xi, y - x(t,y)\rangle + h(t)$$

is such that $\psi(t) \geq \psi(0) - \varepsilon$ for all $t \in [0,T]$.
Let $\lambda > 0$. According to Ekeland variational principle, there exists $(t_k)_{k\in\mathbb{N}}$, $t_k \in [0,T]$ such that $t_k \downarrow 0$ and satisfying

$$\bullet\ \psi(t_k) \leq \psi(0); \quad \bullet\ |t_k| \leq \lambda; \quad \bullet\ \frac{d^+\psi}{dt}(t_k) \geq -\varepsilon/\lambda.$$

Fix $\lambda = 1$. Note that $h'(t) = -k(\xi)(aV(y) + W(y))$ for all t. Hence we obtain

$$\begin{aligned}
\frac{d^+\psi}{dt}(t_k) &\geq -\varepsilon \\
\langle\xi, -\dot{x}_+(t_k, y)\rangle - k(\xi)(aV(y) + W(y)) &\geq -\varepsilon
\end{aligned}$$

According to Proposition 2.6, $\dot{x}_+(\cdot, y)$ continuous on the right at 0. When $t_k \downarrow 0$ in the latter inequality we obtain

$$\langle\xi, -\dot{x}_+(0, y)\rangle - k(\xi)(aV(y) + W(y)) \geq -\varepsilon$$

Remark that

$$\begin{aligned}\dot{x}_+(0,y) &= (f(x(0,y)) - N^P(C;x(0,y)))^0 \\ &= (f(y) - N^P(C;y))^0 = f(y) \text{ since } f(y) \in T(C,y)\end{aligned}$$

which entails $ii)$ as desired as ε is arbitrary.
The proof of $ii) \Rightarrow iii)$ is immediate. Let us prove $iii) \Rightarrow i)$. We fix $x_0 \in C \cap \operatorname{dom} V \cap [V > \bar{\lambda}]$. According to the weak lower semicontinuity of V we fix $\rho > 0$ such that

$$B(x_0, \rho) \subset [V > \bar{\lambda}]. \tag{30}$$

Using the continuity of $x(\cdot, x_0)$, we also select $T > 0$ such that

$$\sup_{t\in[0,T]} 2\|x(t,x_0) - x_0\| + |(e^{-at} - 1)V(x_0) - e^{-at}\int_0^t W(x(\tau,x_0))\,d\tau| < \rho. \tag{31}$$

Now we define $h : [0,T] \to \mathbb{R}_+$ as

$$h(t) := \int_0^t W(x(\tau,x_0))\,d\tau$$

and the functions $\gamma : [0,T] \to \mathbb{R}$, $z : [0,T] \to H \times \mathbb{R}$ and $\eta : [0,T] \to \mathbb{R}_+$ given respectively by

$$\gamma(t) := e^{-at}(V(x_0) - h(t)) \quad z(t) := (x(t,x_0), \gamma(t)) \quad \eta(t) := \frac{1}{2}d^2(z(t), \operatorname{epi} V). \tag{32}$$

Observe that η is Lipschitz continuous on every compact interval in $]0,T]$ so that for all $t \in]0,T[$ we obtain

$$\partial_C\eta(t) = d(z(t), \operatorname{epi} V)\partial_C d(z(\cdot), \operatorname{epi} V)(t) \neq \emptyset.$$

If $z(t) \in \operatorname{epi} V$ then $\partial_C\eta(t) = \{0\}$. □

In order to establish assertion $i)$, we shall proceed in steps. First in the following lemma we recall a general estimation of $\partial_C\eta$. The proof is given in detail in [1].

Lemma 3.3. *Let $t \in]0,T]$ be such that $z(t) \notin \operatorname{epi} V$. Then there exists $\alpha > 0$ such that*

$$\partial_C\eta(t) \subset \overline{\operatorname{co}}\left[\bigcup_{(u,\mu)\in\mathcal{E}} \langle z(t) - \begin{pmatrix} u \\ \mu \end{pmatrix}, \begin{pmatrix} -(N^P(C;x(t,x_0)) \cap \alpha\mathbb{B} + f(x(t,x_0))) \\ -a\gamma(t) - e^{-at}W(x(t,x_0)) \end{pmatrix}\rangle\right]$$

where $\mathcal{E} := \operatorname{Proj}(z(t), \operatorname{epi} V) \cap \big(B(x_0,\rho) \times [\gamma(t), \gamma(t) + \rho]\big)$.

Next we improve the formula given in the preceding lemma under assumption $v)$.

Lemma 3.4. *Let $t \in [0,T]$ be given. Then there exists a positive real M such that*

$$\partial_C\eta(t) \subset]-\infty, M\eta(t)] \quad \textit{for all } t \in [0,T]. \tag{33}$$

Proof. First we suppose that $V(x(t,x_0)) > \gamma(t)$ because otherwise $\partial_C\eta(t) = \{0\}$ and the conlusion holds. Then according to Lemma 3.3 we have to show that for every $x^* \in N^P(C; x(t,x_0)) \cap \alpha\mathbb{B}$ and $(u,\mu) \in \mathrm{Proj}(z(t), \mathrm{epi}\, V) \cap \big(B(x_0,\rho) \times [\gamma(t), \gamma(t)+\rho]\big)$ it holds that

$$\langle z(t) - \begin{pmatrix} u \\ \mu \end{pmatrix}, \begin{pmatrix} f(x(t,x_0)) - x^* \\ -a\gamma(t) - e^{-at}W(x(t,x_0)) \end{pmatrix} \rangle \leq (k + L_W)\eta(t). \tag{34}$$

As the vector $(u,\mu) \in \mathrm{Proj}(z(t), \mathrm{epi}\, V)$ one has $z(t) - (u,\mu) \in N^P_{\mathrm{epi}\, V}(u,\mu)$ with $u \in B(x_0,\rho) \cap \mathrm{dom}\, V \subset [V > \bar{\lambda}] \cap \mathrm{dom}\, V \subset C$. Recall that $z(t) = (x(t,x_0) - u, \gamma(t) - \mu)$. Observing that $\gamma(t) - \mu \leq 0$, so either $\gamma(t) - \mu = 0$ and $x(t,x_0) - u \in \partial_\infty V(u)$, or $\gamma(t) - \mu < 0$ and in this case $\dfrac{x(t,x_0) - u}{\mu - \gamma(t)} \in \partial_P V(u)$. Hence according to *iii)* there exists $u^* \in N^P(C; u)$ such that

$$\langle x(t,x_0) - u, f(u) - u^* \rangle \leq (\gamma(t) - u)(aV(u) + W(u)). \tag{35}$$

Thus writing

$$\begin{aligned}
&\langle x(t,x_0) - u, f(x(t,x_0)) - x^* \rangle = \\
&\langle x(t,x_0) - u, f(x(t,x_0)) - f(u) \rangle + \langle x(t,x_0) - u, f(u) - u^* \rangle + \langle x(t,x_0) - u, u^* - x^* \rangle \\
&\leq k\|x(t,x_0) - u\|^2 + (\gamma(t) - \mu)(aV(u) + W(u)) + \langle x(t,x_0) - u, u^* - x^* \rangle.
\end{aligned}$$

Set $\bar{\alpha} := \max\{\alpha, \|u^*\|\}$. According to the r-uniform prox-regularity of C, note that

$$\langle x(t,x_0) - u, u^* - x^* \rangle \leq \frac{1}{2r}(\|u^*\| + \|x^*\|)\|x(t,x_0) - u\|^2 \leq \frac{\bar{\alpha}}{r}\|x(t,x_0) - u\|^2,$$

Set $\beta := \frac{\bar{\alpha}}{r}$. Hence

$$\begin{aligned}
&\langle x(t,x_0) - u, f(x(t,x_0)) - x^* \rangle \\
&\leq k\|x(t,x_0) - u\|^2 + (\gamma(t) - \mu)(aV(u) + W(u)) + \beta\|x(t,x_0) - u\|^2.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\langle x(t,x_0) - u, f(x(t,x_0)) - x^* \rangle + a(\mu - \gamma(t))\gamma(t) + (\mu - \gamma(t))e^{-at}W(x(t,x_0)) \\
&\leq (\beta + k)\|x(t,x_0) - u\|^2 - (\mu - \gamma(t))(aV(u) + W(u)) \\
&\quad + a(\mu - \gamma(t))\gamma(t) + (\mu - \gamma(t))e^{-at}W(x(t,x_0)) \\
&= (\beta + k)\|x(t,x_0) - u\|^2 \\
&\quad + a(\mu - \gamma(t))(\gamma(t) - V(u)) + (\mu - \gamma(t))(e^{-at}W(x(t,x_0)) - W(u)) \\
&\leq (\beta + k)\|x(t,x_0) - u\|^2 + a(\mu - \gamma(t))(\gamma(t) - V(u)) + (\mu - \gamma(t))e^{-at}|W(x(t,x_0)) - W(u)| \\
&\leq (\beta + k)\|x(t,x_0) - u\|^2 + 0 + L_W|\mu - \gamma(t)|\|x(t,x_0) - u\| \quad (\text{since } \gamma(t) - V(u) \leq 0) \\
&\leq (\beta + k)\|x(t,x_0) - u\|^2 + \frac{L_W}{2}(\|x(t,x_0) - u\|^2 + |\mu - \gamma(t)|^2) \\
&\leq (2(\beta + k) + L_W)\eta(t).
\end{aligned}$$

Finally in the last lemma we obtain the desired conclusion with $M := 2(\beta + k) + L_W$. □

Lemma 3.5. *Assertion* (1) *holds that is, for all* $t \in [0, \rho(x_0)]$ *we have that*

$$e^{at}V(x(t,x_0)) + \int_0^t W(x(\tau,x_0))\,d\tau \le V(x_0).$$

Proof. We fix $t_1, t_2 \in]0,T[$, $t \in]t_1,t_2[$. By invoking (33) together with Gronwall's lemma we obtain that

$$e^{-Mt}\eta(t) \le e^{-Mt_1}\eta(t_1).$$

Thus by making $t_1 \to 0$, it follows that $d((x(t,x_0), e^{-at}(V(x_0) - h(t))), \operatorname{epi} V) = 0$ which in turn gives us

$$e^{at}V(x(t,x_0)) + \int_0^t W(x(\tau,x_0))\,d\tau \le V(x_0). \tag{36}$$

Recall that

$$\rho(x_0) = \sup \left\{ \begin{array}{l} \nu > 0 : \exists \rho > 0 \text{ with } B(x_0,\rho) \subset [V > \bar{\lambda}] \text{ s.t. } \forall s \in [0,\nu] : \\ \\ 2\|x(s,x_0) - x_0\| < \rho \text{ and } |(e^{-as}-1)V(x_0) - \int_0^s W(x(\tau,x_0))\,d\tau| < \rho \end{array} \right\}.$$

The set defined in the right member is not empty. Indeed, according to the lsc property of V we can find such a ρ (see (30)) and due to the continuity of $x(\cdot,x_0)$, there exists $T > 0$ such that (31) is satisfied.
Let $T < \rho(x_0)$ and fix $t \in]T, \rho(x_0)[$ such that (31) holds. If $\rho(x_0) = +\infty$ then the inequality (36) is valid on $[0,+\infty[$. Otherwise let $t_n \to \rho(x_0)$, (31) being satisfied for each t_n. Then inequality (36) is true for all n, and according to the lsc of V and the continuity of $x(\cdot,x_0)$, passing to the limit through n gives that (36) is true for all $t \in [0,\rho(x_0)]$. Hence *iii)* $\Rightarrow$ *i)* as desired.
When $\bar{\lambda} = -\infty$, we get that $[V > \bar{\lambda}] = H$ and since $x(\cdot)$ is locally absolutely continuous, $\rho(x_0) = +\infty$ so inequality (27) is true for all $t \in [0,+\infty[$, that is, (V,W) is an a-Lyapunov pair for (26). □

4 Asymptotic behaviour of global trajectories

Denote by $x(\cdot,x_0)$ the global solution of (26). The next proposition is an extension of Proposition 4.1 in [23] given when $N^P(C;\cdot)$ is replaced by the subdifferential $\partial_P\varphi(\cdot)$ of a primal lower nice function, $V = \varphi$ and $f = 0$. Here we give these results for any Lyapunov function V (resp. any a-Lyapunov pair (V,W)), according to the definition given in [1].

Proposition 4.1. *Let* $V : H \to \mathbb{R} \cup \{+\infty\}$ *be a Lyapunov function for* (26)*. Suppose in the following that* V *is bounded below on the set* $x([0,+\infty[)$*. Then*

1) $\lim_{t\to+\infty} V(x(t,x_0))$ *exists in* $\mathbb{R}$ *and* $\lim_{t\to+\infty} V(x(t,x_0)) = \inf_{t\ge0} V(x(t,x_0))$*;*

2) *if* (V,W) *is an a-Lyapunov pair for* (26)*, then*

$$\lim_{t\to+\infty}\left(V(x(t,x_0)) + e^{-at}\int_0^t W(x(\tau,x_0))\,d\tau\right) < +\infty.$$

As a consequence,

$$\int_0^{+\infty} W(x(t,x_0))\,dt < +\infty \qquad \textit{when } a = 0.$$

3) *if* $(x_\infty, v_\infty) \in H \times H$ *is a* $(\|\cdot\| \times w)$*-sequential cluster point of* $\{(x(t,x_0), \dot{x}(t,x_0)), t \geq 0\}$ *when* $t \to +\infty$ *then*

$$\lim_{t\to+\infty} V(x(t,x_0)) = V(x_\infty) \text{ and } v_\infty \in f(x_\infty) - N^P(C; x_\infty).$$

4) *In addition, suppose that*

$$a = 0, \quad \lim_{\|x\|\to+\infty} V(x) = \infty \text{ and } W(x) \geq \|f(x)\| \text{ for all } x \in H. \tag{37}$$

Then all the solutions of (26) *are bounded.*

Proof. To prove 1), since V is Lyapunov, $V \circ x(\cdot)$ is nonincreasing on $[0, +\infty[$ and we conclude according to the boundedness assumption on V.
Proof of 2). (V, W) forms an a-Lyapunov pair for (26), so we have

$$e^{at}V(x(t,x_0)) + \int_0^t W(x(\tau,x_0))\, d\tau \leq V(x_0), \text{ for all } t \geq 0. \tag{38}$$

Dividing by e^{at} in (38) yields

$$V(x(t,x_0)) + e^{-at}\int_0^t W(x(\tau,x_0))\, d\tau \leq e^{-at}V(x_0)$$

and when t goes to $+\infty$, this leads to 2).
When $a = 0$, we get that $\int_0^{+\infty} W(x(t,x_0))\, dt < +\infty$ since $\lim_{t\to+\infty} V(x(t,x_0)) \in \mathbb{R}$ according to 1).
Proof of 3). Let $(t_n)_{n\geq1}$, $t_n > 0$, be a sequence such that $t_n \to +\infty$. Set $x_n := x(t_n, x_0)$ and $v_n := \dot{x}(t_n, x_0)$. As a direct consequence of 1), we get that $\lim_{n\to+\infty} V(x_n) = V(x_\infty)$. Further, since f is Lipschitz continuous on H, we have $f(x_n) \underset{n\to+\infty}{\to} f(x_\infty)$. We have also $v_n \in f(x_n) - N^P(C, x_n)$ for all $n \geq 1$. The r-uniform prox-regularity of C yields

$$\langle -v_n + f(x_n), x_n - x' \rangle \leq \frac{1}{2r}(\|v_n\| + \|f(x_n)\|)\|x_n - x'\| \text{ for all } x' \in C.$$

Since $(v_n)_{n\geq1}$ is weakly convergent, one has $M := \sup_{n\geq1} v_n < +\infty$. So when n goes to $+\infty$ in the latter inequality we have

$$\langle -v_\infty + f(x_\infty), x_\infty - x' \rangle \leq \frac{1}{2r}(M + \|f(x_\infty)\|)\|x_\infty - x'\| \text{ for all } x' \in C,$$

i.e. $v_\infty \in f(x_\infty) - N^P(C; x_\infty)$, as desired.
Proof of 4). If $x_0 \in C \cap \operatorname{dom} V$, then $V(x(t,x_0)) \leq V(x_0)$ hence $t \to V(x(t,x_0))$ is bounded above. From (37) it follows that $t \to \|x(t,x_0)\|$ is bounded. The consequence in 2) yields that $W \circ x(\cdot) \in L^1([0, +\infty[, H)$. According to the hypomonotonicity of $N^P(C; \cdot) \cap \mathbb{B}$, for every $x \in C \cap \operatorname{dom} V$ we have the estimation

$$\begin{aligned}
&\langle -\dot{x}(t,x) + f(x(t,x)) + \dot{x}(t,x_0) - f(x(t,x_0)), x(t,x) - x(t,x_0) \rangle \\
&\leq \frac{1}{r}(\|\dot{x}(t,x) - f(x(t,x))\| + \|\dot{x}(t,x_0) - f(x(t,x_0))\|)\|x(t,x) - x(t,x_0)\|^2 \\
&\leq \frac{1}{r}(\|f(x(t,x))\| + \|f(x(t,x_0))\|)\|x(t,x) - x(t,x_0)\|^2 \text{ (due to (10))} \\
&\leq \frac{1}{r}(W(x(t,x)) + W(x(t,x_0)))\|x(t,x) - x(t,x_0)\|^2.
\end{aligned}$$

Then we get

$$\frac{d}{dt}\left(\|x(t,x)-x(t,x_0)\|^2\right)$$
$$\leq \frac{2}{r}[W(x(t,x))+W(x(t,x_0))]\|x(t,x)-x(t,x_0)\|^2+2\langle f(x(t,x_0))-f(x(t,x)),x(t,x)-x(t,x_0)\rangle.$$

For almost every $t\in[0,+\infty[$ one has

$$\frac{d}{dt}\|x(t,x)-x(t,x_0)\|^2=2\|x(t,x)-x(t,x_0)\|\frac{d}{dt}\|x(t,x)-x(t,x_0)\|$$

hence the last inequality becomes

$$\frac{d}{dt}\|x(t,x)-x(t,x_0)\|\leq\frac{2}{r}[W(x(t,x))+W(x(t,x_0))]\|x(t,x)-x(t,x_0)\|+[W(x(t,x))+W(x(t,x_0))].$$

According to Gronwall's lemma, there exists two positive constants α and β such that

$$\|x(t,x)-x(t,x_0)\|\leq\alpha\|x-x_0\|+\beta \text{ a.e. } t\geq 0 \text{ where}$$

$\alpha:=\exp\left(\frac{2}{r}\int_0^{+\infty}(W(x(t,x_0))+W(x(t,x)))\,dt\right)$, $\beta:=\left(2\int_0^{+\infty}(W(x(t,x_0))+W(x(t,x)))\,dt\right)\alpha$.
So for all $x\in C\cap\operatorname{dom}V$,

$$\|x(t,x)\|\leq\|x(t,x)-x(t,x_0)\|+\|x(t,x_0)\|\leq\alpha\|x-x_0\|+\beta+\|x(t,x_0)\| \text{ a.e. } t\geq 0,$$

hence all solutions of (26) are bounded, which is assertion 4), as desired. □
Define

$$\overline{\omega_0}:=\bigcap_{T\geq 0}\mathrm{cl}_{\|\cdot\|}(x([T,+\infty[))$$

as the strong limit set of the trajectory $x(\cdot,x_0)$.
Then the set $\overline{\omega_0}$ is nonempty, connected and compact in $(H,\|\cdot\|)$ whenever $x(\cdot,x_0)$ is strongly relatively compact on $[0,+\infty[$, that is $\mathrm{cl}_{\|\cdot\|}(\{x(t,x_0):t\geq 0\})$ is a compact subset of H with respect to the strong topology. The strong relative compactness is satisfied when the sublevel set $\{V\leq V(x_0)\}$ is compact in $(H,\|\cdot\|)$.
The next proposition brings more information about the asymptotic behaviour of the differential variational inequality (26).

Proposition 4.2. *Let (V,W) be a Lyapunov pair for (26) such that*

i) V is bounded below on H;

ii) $W(x)\geq\|f(x)\|$ for all $x\in H$.

If $x_\infty\in\overline{\omega_0}$, then

$$\lim_{t\to+\infty}V(x(t,x_0))=V(x_\infty)=\inf_{t\geq 0}V(x(t,x_0)) \text{ and } 0\in N^P(C,x_\infty)+f(x_\infty).$$

Inspired by some ideas of [23] to prove Proposition 4.2, we wil need the following lemmas.

Lemma 4.3. *Let $\lambda(\cdot)$ stand for the Lebesgue measure on $\mathbb{R}$ and let $v\in L^1([0,+\infty[,H)$. Then*

$$\lambda(\{t\geq 0:\|v(t)\|\geq\varepsilon\})<+\infty \text{ and } \lambda(\{t\geq 0:\|v(t)\|<\varepsilon\})=+\infty.$$

Lemma 4.4. *Let $(s_n)_{n\in\mathbb{N}} \subset [0,+\infty[$ be any sequence satisfying $s_n \to +\infty$. Then, there exists a strictly increasing mapping $\nu(\cdot) : \mathbb{N} \to \mathbb{N}$ such that:*
$\forall \delta > 0, \exists m(\delta) \in \mathbb{N}, \forall n \geq m(\delta), \exists t_{n,\delta} \geq 0$ *such that*

a) $s_{\nu(n)} - \delta < t_{n,\delta} < s_{\nu(n)} + \delta$,

b) $\dot{x}(t_{n,\delta}, x_0)$ *exists,* $\dot{x}(t_{n,\delta}, x_0) \in f(x(t_{n,\delta}, x_0)) - N^P(C, x(t_{n,\delta}, x_0))$, *and*

c) $\|\dot{x}(t_{n,\delta}, x_0)\| < \delta$.

Proof of Proposition 4.2. Observe that $\dot{x}(\cdot, x_0) \in L^1([0,+\infty[, H)$. Indeed, Proposition 4.1 ensures that

$$\int_0^{+\infty} W(x(t, x_0))\, dt < +\infty.$$

Further, according to (10), for a.e. $t \geq 0$

$$\|\dot{x}(t, x_0)\| \leq \|f(x(t, x_0))\| + \|\dot{x}(t, x_0) - f(x(t, x_0))\|\| \leq 2\|f(x(t, x_0)\| \leq W(x(t, x_0)).$$

Assume that $x_\infty \in \overline{\omega_0}$. Let $(s_n)_{n\geq 1} \subset [0,+\infty[$ be such that

$$s_n \to +\infty \text{ and } x(s_n, x_0) \to x_\infty.$$

According to Proposition 4.1, the function $V \circ x(\cdot)$ admits a limit in $+\infty$ and this limit equals $\inf\limits_{t\geq 0} V(x(t, x_0))$. Hence,

$$\liminf_{n\to+\infty} V((x(s_n, x_0)) = \lim_{t\to+\infty} V(x(t, x_0)) \text{ and } V(x_\infty) = \inf_{t\geq 0} V(x(t, x_0)).$$

Thanks to Lemma 4.4, we find some increasing mapping $\nu : \mathbb{N} \to \mathbb{N}$ and generate a sequence $(\sigma(k))_{k\geq 1}$, $\sigma(k) \underset{k\to+\infty}{\to} +\infty$, with $t'_k := t_{\sigma(k),k^{-1}}$ such that for each $k \geq 1$

$$|t'_k - s_{\nu(\sigma(k))}| < \frac{1}{k}, \dot{x}(t'_k, x_0) \in f(x(t'_k, x_0)) - N^P(C; x(t'_k, x_0)) \text{ and } \|\dot{x}(t'_k, x_0)\| < \frac{1}{k}.$$

Applying Proposition 4.1, we get the second conclusion when k goes to $+\infty$. □

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