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# SOLITON RESOLUTION FOR EQUIVARIANT WAVE MAPS TO THE SPHERE

RAPHAËL CÔTE

ABSTRACT. We consider finite energy corotational wave maps with target manifold  $\mathbb{S}^2$ . We prove that for a sequence of times, they decompose as a sum of decoupled harmonic maps in the light cone, and a smooth wave map (in the blow case) or a linear scattering term (in the global case), up to an error which tends to 0 in the energy space.

## 1. INTRODUCTION

**1.1. Statement of the main results.** Let  $(M, g)$  be a Riemannian manifold, and  $\mathbb{R}^{1+d}$  be endowed with the Minkowski metric  $\eta = \text{diag}(-1, 1, \dots, 1)$ . Wave maps  $U : (\mathbb{R}^{1+d}, \eta) \rightarrow (M, g)$  are defined formally as critical points of the Lagrangian

$$\mathcal{L}(U, \partial U) = \frac{1}{2} \int_{\mathbb{R}^{1+d}} \eta^{\alpha\beta} \langle \partial_\alpha U, \partial_\beta U \rangle_g dx dt.$$

In local coordinates, they satisfy the Euler-Lagrange equation

$$(1.1) \quad \begin{cases} \square U^k = -\eta^{\alpha\beta} \Gamma_{ij}^k(U) \partial_\alpha U^i \partial_\beta U^j \\ (U, \partial_t U)|_{t=0} = (U_0, U_1), \end{cases}$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols on  $TM$ .

We refer to the review article [17] and the reference therein for recent developments regarding general wave maps.

We consider the case where  $d = 2$  and  $M$  is a 2 dimensional surface of revolution with metric

$$ds^2 = d\rho^2 + g(\rho)^2 d\theta,$$

where  $(\rho, \theta)$  are the polar coordinates on  $M$ , and  $g \in \mathcal{C}^3(\mathbb{R})$ .

We assume that  $U$  has corotational equivariant symmetry, that is, denoting  $(r, \omega)$  the polar coordinates on  $\mathbb{R}^2$ , it takes the form

$$U(t, r, \omega) = (\psi(t, r), \omega).$$

for some function  $\psi$ . System (1.1) then simplifies to the following equation on  $\psi$  :

$$(WM) \quad \begin{cases} \partial_{tt}\psi - \partial_{rr}\psi - \frac{1}{r}\partial_r\psi + \frac{f(\psi)}{r^2} = 0 & \text{where } f = gg'. \\ (\psi, \partial_t\psi)|_{t=0} = (\psi_0, \psi_1) \end{cases}$$

We say that such a solution  $\vec{\psi} = (\psi, \partial_t\psi)$  to (WM) is a wave map.

We define the energy space  $\mathcal{H} \times L^2$  and similarly the Hilbert space  $H \times L^2$  as follows: given a couple of function  $\vec{\phi} = (\phi_0, \phi_1)$ , and for  $0 \leq r_1 < r_2 \leq \infty$ ,

$$E(\vec{\phi}; r_1, r_2) := \int_{r_1}^{r_2} \left( |\phi_1(t, r)|^2 + |\partial_r \phi_0(t, r)|^2 + \frac{|g(\phi_0(t, r))|^2}{r^2} \right) r dr,$$

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$$\begin{aligned}\|\phi_0\|_{H([r_1, r_2])}^2 &:= \int_{r_1}^{r_2} \left( |\partial_r \phi_0(r)|^2 + \frac{|\phi_0(r)|^2}{r^2} \right) r dr, \\ \|\vec{\phi}\|_{H \times L^2([r_1, r_2])} &= \int_{r_1}^{r_2} \left( |\phi_1(r)|^2 + |\partial_r \phi_0(r)|^2 + \frac{|\phi_0(r)|^2}{r^2} \right) r dr.\end{aligned}$$

We omit  $r_1, r_2$  in the case  $r_1 = 0$  and  $r_2 = \infty$ : the energy is  $E(\vec{\phi}) := E(\vec{\phi}; 0, \infty)$ , and  $\mathcal{H} \times L^2 = \{\vec{\phi} \mid E(\vec{\phi}) < +\infty\}$ .

If  $\vec{\psi} = (\psi(t), \partial_t \psi(t))$  is a finite energy wave map, then at least formally its energy is preserved: for all  $t$  where defined,

$$(1.2) \quad E(\vec{\psi}(t)) = E(\vec{\psi}(0)).$$

(WM) is energy critical in the following sense. Consider the scaling (for  $\lambda > 0$ )

$$\Lambda[\lambda]\vec{\phi}(t, r) := \left( \phi_0 \left( \frac{t}{\lambda}, \frac{r}{\lambda} \right), \frac{1}{\lambda} \phi_1 \left( \frac{t}{\lambda}, \frac{r}{\lambda} \right) \right).$$

Then  $\vec{\psi}$  is a wave map if and only if  $\Lambda[\lambda]\vec{\phi}$  is a wave map, and the energy is scaling invariant:

$$E(\vec{\psi}) = E(\Lambda[\lambda]\vec{\phi}).$$

Notice that the  $H \times L^2$  norm is also scaling invariant.

Recall that if  $\phi \in \mathcal{H}$ , then  $\phi$  is continuous and bounded, and has well defined limits at 0 and  $+\infty$ , which cancel  $g$ : we denote them  $\phi(0)$  and  $\phi(\infty)$ . If  $\vec{\phi}$  is a wave map, these limits do not depend on time.

This motivates the introduction of the set of points where  $g$  vanishes

$$\mathcal{V} := \{\ell \in \mathbb{R} \mid g(\ell) = 0\}.$$

Also, let

$$G(x) := \int_0^x |g(y)| dy.$$

We recall the local well-posedness result in the energy space, due to Shatah and Tahvildar-Zadeh.

**Theorem** ([23]). *Let  $(\psi_0, \psi_1) \in \mathcal{H} \times L^2$ . Then there exists a unique wave map  $\vec{\psi} = (\psi, \partial_t \psi) \in \mathcal{C}(I, \mathcal{H} \times L^2)$  solution to (WM), defined on a maximal interval  $I =: (T^-(\vec{\psi}), T^+(\vec{\psi}))$ , and which preserves the energy (1.2).*

The wave map equation (WM) has been intensively studied as a model for geometric wave equations. It has been long understood that the geometry of the target  $M$ , i.e. the metric  $g$ , plays a crucial role in the long time behavior of wave maps. Let us mention the result by Struwe [25]: a wave map that blows up in finite time must bubble up a harmonic map at blow up time. In particular, if  $M$  does not admit non constant harmonic maps, then any wave map is global in time.

Actual examples of wave maps blowing up in finite time were *constructed* by Rodnianski and Sterbenz [22] and Raphaël and Rodnianski [21] (as a perturbation of the self similar regime), and by Krieger, Schlag and Tataru [19] (with prescribed, polynomial blow up rate).

On a different side, together with Kenig, Lawrie and Schlag [6, 7], we *classified* the asymptotic behavior of wave maps with energy less than 3 times the energy of a harmonic map, for large time or near blow up time.

Our goal in this paper is to obtain a similar classification for wave maps of arbitrarily large energy, that is to relax the bound on the energy. We provide a description of a wave map into decoupled profiles, a so called soliton resolution.

It turns out that these profiles are harmonic maps and linear scattering terms. Recall that a harmonic map is a solution  $Q$  of finite energy of

$$\partial_{rr}Q + \frac{1}{r}\partial_r Q = \frac{f(Q)}{r^2}.$$

(Hence  $(Q, 0)$  is a finite energy stationary wave map). From [4], they are classified as follows: a non constant harmonic map is monotonic, satisfies one of the ODEs

$$r\partial_r Q = g(Q) \quad \text{or} \quad r\partial_r Q = -g(Q),$$

and joins two consecutive points of  $\mathcal{V}$ , that is for some  $\ell, m \in \mathcal{V}$ ,  $\ell < m$ ,

$$\{Q(0), Q(\infty)\} = \{\ell, m\} \quad \text{and} \quad \mathcal{V} \cap (\ell, m) = \emptyset.$$

It has energy  $E(Q) := E(Q, 0) = 2(G(m) - G(\ell))$ . In particular if  $\mathcal{V} = 1$ , there exists no non constant harmonic map (if  $\mathcal{V}$  is empty, there is no finite energy map). On the other hand, given  $\ell \in \mathcal{V}$ , we define the linearized wave map flow around  $\ell$ :

$$(LW_\ell) \quad \partial_{tt}\phi - \partial_{rr}\phi - \frac{1}{r}\partial_r\phi + \frac{g'(\ell)^2}{r^2}\phi = 0.$$

Solutions to this linear wave equation preserve the following  $H \times L^2$  related quantity

$$\|\vec{\phi}(t)\|_{H_\ell \times L^2}^2 := \int \left( |\partial_t\phi(t, r)|^2 + |\partial_r\phi(t, r)|^2 + \frac{g'(\ell)^2\phi(t, r)^2}{r^2} \right) r dr = \|\vec{\phi}(0)\|_{H_\ell \times L^2}^2.$$

We now state the main result of this paper. For this, we make the following assumptions on the metric  $g$ :

- (A1)  $G(x) \rightarrow \pm\infty$  as  $x \rightarrow \pm\infty$ .
- (A2)  $\mathcal{V}$  is discrete,
- (A3) For all  $\ell \in \mathcal{V}$ ,  $g'(\ell) \in \{-1, 1\}$ .

Assumption (A1) prevents the formation of bubbles at infinity, and is a very natural assumption. (A2) is also a natural assumption of non degeneracy of  $g$ , which prevent a decomposition with harmonic maps of arbitrarily small energy.

The physically relevant metrics  $g$  are

- (1)  $g(\rho) = \sin(\rho)$  (wave maps to the sphere  $\mathbb{S}^2$ ), and
- (2)  $g(\rho) = 1 - \rho^2$  (radial 4D Yang-Mills equation).

Hence (A3) allows to handle wave maps to the sphere  $\mathbb{S}^2$ ; however, dealing with the radial 4D Yang-Mills equation requires to relax (A3) to

- (A3') For all  $\ell \in \mathcal{V}$ ,  $g'(\ell) \in \{-2, -1, 1, 2\}$ .

It should be noted that most of the results in this article hold under (A3') instead of (A3). One could even consider the natural condition  $g'(\mathcal{V}) \subset \mathbb{Z} \setminus \{0\}$ , which makes the linearized problem  $(LW_\ell)$  around  $\ell$  to be of wave type (in dimension  $2g'(\ell) + 2$ ). However large  $g'(\ell)$  raise technical issues for the Cauchy problem as noted in [5, Theorem 2], which is restricted to the case (A3'); if these issues could be dealt with, most results here hold under this last condition.

**Theorem 1.1.** *We make assumptions (A1)-(A2)-(A3).*

*Let  $\vec{\psi}(t)$  be a finite energy wave map. Then there exist a sequence of time  $t_n \uparrow T^+(\vec{\psi})$ , an integer  $J \geq 0$ ,  $J$  sequences of scales  $\lambda_{J,n} \ll \dots \ll \lambda_{2,n} \ll \lambda_{1,n}$  and  $J$  harmonic maps  $Q_1, \dots, Q_J$  such that*

$$Q_J(0) = \psi(0), \quad Q_{j+1}(\infty) = Q_j(0) \quad \text{for } j = 1, \dots, J-1,$$

*and that the following holds.*

(1) If  $T^+(\vec{\psi}) = +\infty$ , denote  $\ell = \psi(\infty)$ . Then  $\lambda_{1,n} \ll t_n$  and there exists a solution  $\vec{\phi}_L(t) \in \mathcal{C}(\mathbb{R}, H \times L^2)$  to the linear wave equation  $(\text{LW}_\ell)$  such that

$$(1.3) \quad \vec{\psi}(t_n) = \sum_{j=1}^J (Q_j(\cdot/\lambda_{j,n}) - Q_j(\infty), 0) + (\ell, 0) + \vec{\phi}_L(t_n) + \vec{b}_n,$$

where  $\vec{b}_n \rightarrow 0$  in  $H \times L^2$  as  $t \rightarrow \infty$  and  $Q_1(\infty) = \ell$ .

(2) If  $T^+(\vec{\psi}) < +\infty$ , then  $\lambda_{1,n} \ll T^+(\vec{\psi}) - t_n$  and there exists a function  $\vec{\phi} \in \mathcal{H} \times L^2$  of finite energy such that

$$(1.4) \quad \vec{\psi}(t_n) = \sum_{j=1}^J (Q_j(\cdot/\lambda_{j,n}) - Q_j(\infty), 0) + \vec{\phi} + \vec{b}_n,$$

where  $\vec{b}_n \rightarrow 0$  in  $H \times L^2$  as  $t \rightarrow \infty$ ,  $\phi(0) = \lim_{t \uparrow T^+(\vec{\psi})} \psi(t, T^+(\vec{\psi}) - t)$  and  $Q_1(\infty) = \phi(0)$ .

*Remark 1.* Notice that in the global case,  $E(\vec{\psi}) = \sum_{j=1}^J E(Q_j) + \|\vec{\phi}_L\|_{H_\ell \times L^2}^2$ , and in the blow up case,  $E(\vec{\psi}) = \sum_{j=1}^J E(Q_j) + E(\vec{\phi})$ . This gives a bound on  $J$ .

Also, if  $T^+(\vec{\psi}) = +\infty$ , then  $J \geq \#(\mathcal{V} \cap [\psi(0), \psi(\infty)]) - 1$ . This last number can be made arbitrarily large when  $\mathcal{V}$  is infinite (as for  $g = \sin$ ).

*Remark 2.* The question whether the decomposition holds for all times and not merely for a sequence is open. However there are some cases where it can be proved. For example, when the excess energy of  $\vec{\psi}$  with respect to the energy to connect  $\psi(0)$  to  $\psi(\infty)$  is not enough to bubble more harmonic maps: that is

$$E(\vec{\psi}) < 2|G(\psi(\infty)) - G(\psi(0))| + 4\delta,$$

where  $\delta = \min_{\ell \in \mathcal{V} \setminus \{\psi(0), \psi(\infty)\}} \{|G(\psi(\ell)) - G(\psi(0))|, |G(\psi(\ell)) - G(\psi(\infty))|\}$ . We refer to [6, Proof of Theorem 1.3] for a detailed argument in the case  $J = 1$ .

*Remark 3.* Many possibilities are left open regarding the behavior of the  $\lambda_{j,n}$ : for example, in the global case, one could have  $\lambda_{J,n} \rightarrow 0$  (infinite time blow-up), or  $\lambda_{J,n} \rightarrow +\infty$  (infinite time flattening). Although no such solutions were constructed for (WM), let us refer to [9] in the context of the semilinear wave equation.

Theorem 1.1 is an extension of [6, 7] where only one profile was allowed (i.e  $J = 1$ ) through the bound on the energy. It is in the spirit of the seminal papers by Duyckaerts, Kenig and Merle [10, 11, 12, 13] where large solutions of the radial energy critical (focusing) wave equation in 3D were described. Let us however observe that their analysis did not encompass type I blow up solutions (i.e when  $\limsup_{t \uparrow T^+(\vec{u})} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} = +\infty$ ). This phenomenon does not occur in the wave map case (mainly because the energy is coercive, even if it doesn't bound  $H \times L^2$ ), and we give a description of any wave map, without any further assumption.

**1.2. Outline of the proof.** Let us mention two delicate issues. First, geometry has to be taken into account: the harmonic maps do never belong to  $H \times L^2$ . This means we must derive a procedure to extract them without relying on a linear profile decomposition, as for the wave equation.

Second, the linearized operator of the wave map flow at spatial infinity is of wave type in *even* dimension: most of the delicate linear estimates available in the radial 3D case break down for wave maps, in particular the so-called ‘‘energy channels’’. When  $g'(\ell)$  is odd,  $(\text{LW}_\ell)$  corresponds to a wave equation in  $\mathbb{R}^d$  with  $d \equiv 0 \pmod{4}$ , and the linear estimate obtained in [8] in this case suffices to conclude. When  $g'(\ell)$  is even, then  $d \equiv 2 \pmod{4}$  and the desired linear estimate fails. This is the reason why we must restrict ourselves to (A3) instead of (A3').

The first step in the proof is to choose a sequence of time  $t_n \rightarrow T^+(\vec{\psi})$  on which the space-time kinetic energy inside the light cone vanishes. This is a reformulation that the averaged kinetic energy inside the light cone vanishes, which is a well known result, and is the content of Section 2.1. Section 2.2 focuses on various aspects of the profile decomposition in  $H \times L^2$  to be used later in the paper.

The second step, Section 3, is concerned with sequence of wave maps whose space-time kinetic energy vanishes, and shows, in Theorem 3.3, that up to a subsequence, one can construct a bubble decomposition i.e extract the harmonic maps. This decomposition holds up to an error which tends to 0 in  $L^\infty$ . Notice that this result does not make use of assumption (A3) or (A3'), but only (A1) and (A2).

The bound on the error is insufficient to capture the linear scattering term for example, but it is enough to derive a sharp scattering theorem *below the threshold in  $L^\infty$* . As linear scattering is involved, we do need assumption (A3') here. This result has its own interest: let us state it here, and postpone the proof to Section 4.

For  $\ell \in \mathcal{V}$ , define  $d_\ell$  as the distance of  $\ell$  to the closest (distinct) element in  $V$ :

$$d_\ell = \inf\{|\ell - k| \mid k \in \mathcal{V} \setminus \{\ell\}\}.$$

$d_\ell > 0$  due to assumption (A2).

**Theorem 1.2.** *Let  $\ell \in \mathcal{V}$  and assume (A1)-(A2), and*

$$g'(\ell) \in \{-2, -1, 1, 2\}.$$

*Let  $\vec{\psi}$  be a wave map such that  $\psi(\infty) = \ell$ , and that for some  $c < \delta_\ell$ ,*

$$(1.5) \quad \forall t \in [0, T^+(\vec{\psi})], \quad \|\psi(t) - \ell\|_{L^\infty} \leq c < d_\ell.$$

*Then  $T^+(\vec{\psi}) = +\infty$  and  $\vec{\psi}$  scatters at  $+\infty$ , in the sense*

$$\|\psi - \ell\|_{S_\ell([0, +\infty))} < +\infty.$$

*( $S_\ell$  is an adequate Strichartz space, defined below on (2.4)). It follows that there exists a (unique) solution  $\vec{\phi}_L$  to (LW) $_\ell$  such that*

$$\|\vec{\psi}(t) - (\ell, 0) - \vec{\phi}_L(t)\|_{H \times L^2} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

*Remark 4.* Observe that if  $g$  vanishes in at most one point  $\ell$ , then  $d_\ell = +\infty$ , and so Theorem 1.2 proves that all wave maps are global and *scatter* in this case. This strengthens the global well posedness result by Struwe [25] mentioned above.

As a consequence of Theorem 1.2, we can extract the scattering term (for *all times*, not merely a sequence) in the global case. In an analogous way, we can define the regular part  $\vec{\phi}$  in the blow up case. This is the content of Propositions 5.1 and 5.2 of Section 5. Let us emphasize that this step only requires (A3').

In Section 6, we revisit Theorem 3.3. Under the additional assumption (A3) – crucial but used only on this step, we show that the error term tends to 0 in  $H \times L^2$ . This section is independent of Sections 4 and 5.

Finally, we gather all the previous results together in Section 7 and prove of Theorem 1.1.

## 2. PRELIMINARIES

The purpose of this section is to recall or adapt a few important earlier results, and derive some consequences.

### 2.1. The self-similar region.

2.1.1. *Global wave maps.* Throughout this Subsection, let  $\vec{\psi}$  be a finite energy wave map such that  $T^+(\vec{\psi}) = +\infty$ .

**Proposition 2.1** ([7, Proposition 2.1]). *For all  $\lambda > 0$ ,*

$$\limsup_{t \rightarrow +\infty} E(\vec{\psi}(t); \lambda t, t - A) \rightarrow 0 \quad \text{as } A \rightarrow +\infty.$$

*Proof.* The argument in [7] is done for  $g = \sin$ , after ideas of [3], and extends seamlessly for any non linearity  $g$ .  $\square$

We derive a few consequences from this. One fundamental outcome of Proposition 2.1 is that the kinetic part of the energy vanishes in an averaging sense. More precisely, we have:

**Corollary 2.2** ([7, Corollary 2.2]).

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \int_A^T \int_0^{t-A} |\partial_t \psi(t, r)|^2 r dr dt \rightarrow 0 \quad \text{as } A \rightarrow +\infty.$$

From there, we find a sequence of times for which the condition (3) in Theorem 3.3 holds.

**Corollary 2.3.** *There exists a sequence  $t_n \uparrow +\infty$  such that*

$$\sup_{s, 0 < s \leq t_n/2} \frac{1}{s} \int_{t_n-s}^{t_n+s} \int_0^{t/2} |\partial_t \psi(t, r)|^2 r dr dt \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

*Proof.* Corollary 2.2 shows that

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \int_{2A}^T \int_0^{t-A} |\partial_t \psi(t, r)|^2 r dr dt \rightarrow 0 \quad \text{as } A \rightarrow +\infty,$$

hence, if we let  $f(t) = \int_0^{t/2} |\partial_t \psi(t, r)|^2 r dr$ , and as  $t/2 \leq t - A$  if  $t \geq 2A$ , we have

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \int_A^T f(t) dt \rightarrow 0 \quad \text{as } A \rightarrow +\infty.$$

We now argue by contradiction. Assume that the conclusion is not correct, then this means that for some  $\delta > 0$ ,

$$\liminf_{T \rightarrow +\infty} \sup_{s, 0 \leq s \leq T/2} \frac{1}{s} \int_{T-s}^{T+s} f(t) dt \geq 41\delta.$$

Fix  $A$  and  $T_0 \geq 2A$  be large enough such that for all  $T \geq T_0$ ,

- $\frac{1}{T} \int_A^T f(t) dt \leq \delta$
- there exists  $s(T) \in [0, T/2]$  such that  $\frac{1}{s(T)} \int_{T-s(T)}^{T+s(T)} f(t) dt \geq 40\delta$ .

Consider the sets  $(T - s(T), T + s(T))$  for  $T \in [T_0, 2T_0]$ . Their diameter is bounded by  $2T_0$ , hence Vitaly covering lemma applies: there exist a sequence  $(T^n)_n$  such that the intervals  $(T^n - s(T^n), T^n + s(T^n))$  are disjoint and

$$[T_0, 2T_0] \subset \bigcup_{T \in [T_0, 2T_0]} (T - s(T), T + s(T)) \subset \bigcup_n (T^n - 5s(T^n), T^n + 5s(T^n)).$$

From this last condition, it follows that

$$T_0 \leq 10 \sum_n s(T^n).$$

On the other hand, by definition of the  $s(T^n)$ , we get that

$$\int_{T^n - s(T^n)}^{T^n + s(T^n)} f(t) dt \geq 40\delta s(T^n),$$

and as the intervals under consideration are disjoint, we infer

$$\int_{T_0/2}^{3T_0} f(t) dt \geq \sum_n \int_{T^n - s(T^n)}^{T^n + s(T^n)} f(t) dt \geq 40\delta \sum_n s(T^n) \geq 4\delta T_0.$$

But as  $T_0/2 \geq A$ , we also have

$$\frac{1}{3T_0} \int_{T_0/2}^{3T_0} f(t) dt \leq \delta,$$

and we reached a contradiction.  $\square$

Finally we recall that the  $L^\infty$  norm outside vanishes in the self similar region and outside the light cone.

**Corollary 2.4** ([7, Corollary 2.3]). *For any  $\lambda > 0$  we have*

$$\|\psi(t) - \psi(\infty)\|_{L^\infty(r \geq \lambda t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

*Proof.* The argument in [7] is done for  $g = \sin$  and can be extended seamlessly for any non linearity  $g$ .  $\square$

2.1.2. *Blow up wave maps.* Throughout this Subsection, let  $\vec{\psi}$  be a finite energy wave map such that  $T^+(\vec{\psi}) < +\infty$ . The results here very similar to those in the global case, and in fact simpler (integration can be done up to the light cone).

**Proposition 2.5** ([24, Lemma 2.2]). *For all  $\lambda \in (0, 1)$ ,*

$$E(\vec{\psi}(t); \lambda(T^+(\vec{\psi}) - t), T^+(\vec{\psi}) - t) \rightarrow 0 \quad \text{as } t \uparrow 1.$$

**Corollary 2.6** ([24, Corollary 2.2]).

$$\frac{1}{T^+(\vec{\psi}) - T} \int_T^{T^+(\vec{\psi})} \int_0^{T^+(\vec{\psi}) - t} |\partial_t \psi(t, r)|^2 r dr dt \rightarrow 0 \quad \text{as } T \rightarrow T^+(\vec{\psi}).$$

**Corollary 2.7.** *There exists a sequence  $t_n \uparrow T^+(\vec{\psi})$  such that*

$$\sup_{s, 0 < s \leq T^+(\vec{\psi}) - t_n} \frac{1}{s} \int_{t_n - s}^{t_n + s} \int_0^{T^+(\vec{\psi}) - t} |\partial_t \psi(t, r)|^2 r dr dt \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

*Proof.* It is very similar to the proof of Corollary 2.3. Let

$$f(t) = \int_0^{T^+(\vec{\psi}) - t} |\partial_t \psi(t, r)|^2 r dr \rightarrow 0.$$

$f(t) \geq 0$  and we know that

$$\frac{1}{T^+(\vec{\psi}) - T} \int_T^{T^+(\vec{\psi})} f(t) dt \rightarrow 0 \quad \text{as } T \rightarrow T^+(\vec{\psi}).$$

Assume that the conclusion fails, for the sake of contradiction: then there exists  $\delta > 0$  and a function  $s$  defined on  $[T^+(\vec{\psi}) - \delta, T^+(\vec{\psi})]$  such that  $0 \leq s(T) \leq T^+(\vec{\psi}) - T$  and

$$(2.1) \quad \frac{1}{s(T)} \int_{T-s(T)}^{T+s(T)} f(t) dt \geq 40\delta.$$



Corollary 2.6 yields  $T_0 < T^+(\vec{\psi})$  such that for all  $T \in [2T_0 - T^+(\vec{\psi}), T^+(\vec{\psi})$ ,

$$(2.2) \quad \frac{1}{T^+(\vec{\psi}) - T} \int_T^{T^+(\vec{\psi})} f(t) dt \leq \delta.$$

We apply Vitali covering lemma to the intervals  $(T - s(T), T + s(T))$  where  $T \in [T_0, T^+(\vec{\psi})$  to find a sequence  $T^n$  such that  $(T^n - s(T^n), T^n + s(T^n))$  are disjoint and

$$[T_0, T^+(\vec{\psi})] \subset \bigcup (T^n - 5s(T^n), T^n + 5s(T^n)).$$

Hence taking the length:

$$T^+(\vec{\psi}) - T_0 \leq 10 \sum_n s(T^n).$$

Therefore, as the interval are disjoint, and using (2.1),

$$\int_{2T_0 - T^+(\vec{\psi})}^{T^+(\vec{\psi})} f(t) dt \geq \sum_n \int_{T^n - s(T^n)}^{T^n + s(T^n)} f(t) dt \geq 4\delta(T^+(\vec{\psi}) - T_0).$$

Now  $T^+(\vec{\psi}) - (2T_0 - 1) = 2(T^+(\vec{\psi}) - T_0)$ , and we reached a contradiction with (2.2).  $\square$

**2.2. Energy concentration on the light cone for  $(LW_\ell)$ .** For the rest of this Subsection, we fix  $\ell \in \mathcal{V}$  and focus on the linear equation  $(LW_\ell)$ .

We define the transformation  $\mathcal{T}$  defined by

$$(\mathcal{T}\phi)(r) = \phi(r)/r^{g'(\ell)}.$$

Then  $\vec{\phi}$  is a solution of  $(LW_\ell)$  if and only if  $\vec{\varphi} = (\mathcal{T}\phi, \mathcal{T}\partial_t\phi)$  solves the radial wave equation in  $2 + 2g'(\ell)$  dimensions:

$$(2.3) \quad \partial_{tt}\varphi - \partial_{rr}\varphi - \frac{1 + 2g'(\ell)}{r} \partial_r\varphi = 0.$$

Observe that

$$\|\phi\|_{H_\ell} = \|\mathcal{T}\phi\|_{\dot{H}^1(r^{1+2g'(\ell)}dr)}.$$

The norms  $H_\ell$  and  $H$  are equivalent. It follows that  $\mathcal{T}$  is a bicontinuous bijective linear map  $L^2(rdr) \rightarrow L^2(r^{1+2g'(\ell)}dr)$  and  $H \rightarrow \dot{H}^1(r^{1+2g'(\ell)}dr)$ .

We start by recalling a result regarding equipartition of energy and concentration of energy on the light cone for linear solutions.

**Proposition 2.8.** *Let  $\vec{\phi}$  be a solution to  $(LW_\ell)$ . Then*

$$\limsup_{t \rightarrow +\infty} \|\vec{\phi}(t)\|_{H \times L^2(|r-t| \geq A)} \rightarrow 0 \quad \text{as } A \rightarrow +\infty.$$

Also,

$$\|\partial_t\phi(t)\|_{L^2}^2 \rightarrow \frac{1}{2} \|\vec{\phi}(t)\|_{H_\ell \times L^2}^2, \quad \|\phi(t)\|_{H_\ell}^2 \rightarrow \frac{1}{2} \|\vec{\phi}(t)\|_{H_\ell \times L^2}^2.$$

*Proof.* The first statement is the content of [8, Theorem 4]. The second is equipartition of the energy, and is classical for the linear wave equation.  $\square$

We will sometimes use, in the context of a profile decomposition, the following weaker form, namely all the energy concentrates on one scale.

**Corollary 2.9.** *Let  $\vec{\phi}$  be a solution to  $(LW_\ell)$ , and  $(t_n, \lambda_n)$  be two sequences with  $\lambda_n > 0$ , and such that*

$$\frac{|t_n|}{\lambda_n} \rightarrow +\infty.$$

Then for any  $c > 1$ , as  $n \rightarrow +\infty$ ,

$$\left\| \left( \phi \left( -\frac{t_n}{\lambda_n}, \frac{r}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}} \partial_t \phi \left( -\frac{t_n}{\lambda_n}, \frac{r}{\lambda_{j,n}} \right) \right) \right\|_{H_\ell \times L^2(\frac{1}{c}t_n \leq r \leq ct_n)} \rightarrow \|\vec{\phi}(0)\|_{H_\ell \times L^2}.$$

Now we recall a result giving some condition so that some energy of a linear solution remains outside of the light cone. It was already crucial in [6, 7]. Here is the only place in the argument where we need to restrict to odd  $g'(\ell)$  (and hence (A3) to have the nonlinear argument run).

**Proposition 2.10** ([8, Theorem 1]). *Assume  $g'(\ell)$  is an odd integer. There exists  $\beta(\ell) > 0$  such that the following holds. Let  $\vec{\phi}$  be a solution to (LW $_\ell$ ), such that*

$$\partial_t \phi(0) = 0.$$

Then for all  $t \in \mathbb{R}$ ,

$$\|\vec{\phi}(t)\|_{H \times L^2(r \geq |t|)}^2 \geq \beta(\ell) \|\vec{\phi}(0)\|_{H \times L^2}^2.$$

*Proof.* As mentioned, this is an easy consequence of [8, Theorem 1] and the remark that follows. We refer to [6, Corollary 2.3] for the complete argument to pass from the linear wave equation to (LW $_\ell$ ).  $\square$

**2.3. Profile decomposition for (LW $_\ell$ ) in  $H \times L^2$ .** Again, we fix  $\ell \in \mathcal{V}$  for the rest of this subsection.

Our goal is to derive a suitable notion of linear profile decomposition in the spirit of [1], adapted to our setting, in particular  $L^\infty$  bounds.

Using transformation  $\mathcal{T}$ , a notion of profile decomposition for the wave equation will immediately translate to a similar decomposition for (LW $_\ell$ ), but we will in fact improve it. We elaborate on this in what follows.

For  $I$  an interval of  $\mathbb{R}$ , we define

$$(2.4) \quad \|\phi\|_{S_\ell(I)} := \left( \int_{t \in I} \int_{r=0}^{\infty} |\phi(t, r)|^{2+3/g'(\ell)} \frac{dr dt}{r^2} \right)^{\frac{1}{2+3/g'(\ell)}}.$$

It is simply the norm of  $\mathcal{T}\phi$  in the Strichartz space  $L_{t,r}^{2+3/g'(\ell)}(r^{1+2g'(\ell)} dr dt)$ , adapted to the  $\dot{H}^1$  critical wave equation in dimension  $2 + 2g'(\ell)$  (we refer to [5, Section 3] for further details).

**Lemma 2.11.** *Let  $\theta(\ell) = \frac{3}{4 + 6/g'(\ell)} \in (0, 1)$ . There exist  $C > 0$  such that for any finite energy solution  $\vec{\gamma}$  to the linear wave equation (LW $_\ell$ ).*

$$\|\gamma\|_{L_t^\infty(\mathbb{R}, L_r^\infty)} \leq C \|\vec{\gamma}(0)\|_{H \times L^2}^{\theta(\ell)} \|\gamma\|_{S_\ell(\mathbb{R})}^{1-\theta(\ell)}.$$

*Proof.* As  $\vec{\gamma}$  is a finite energy solution to the linear equation (LW $_\ell$ ), all terms in the desired estimate are finite.

Denote  $M = \|\gamma\|_{L_t^\infty(\mathbb{R}, L_r^\infty)}$  and  $A = \|\gamma(0)\|_H$ . Notice that  $\|\vec{\gamma}(t)\|_{H \times L^2}$  is bounded below and above by  $\|\vec{\gamma}(0)\|_{H \times L^2}$  (because it is equivalent to the conserved quantity  $\|\vec{\gamma}(t)\|_{H_\ell \times L^2}$ ). It follows that for some  $K$  only depending on  $\ell$ ,  $M \leq KA$ .

As all the functional space under consideration are invariant under scaling and time translation, we can assume that  $|\gamma(0, 1)| \geq 2M/3$ .

Let  $s, t \in \mathbb{R}$ , and  $r \geq q \geq 0$ . Then for any  $\rho \in [q, r]$  we have

$$\begin{aligned} |\gamma(t, r) - \gamma(s, q)|^2 &\leq \left( \int_q^\rho |\partial_r \gamma(s, r')| dr' + \int_{[s,t]} |\partial_t \gamma(t', \rho)| dt' + \int_\rho^r |\partial_r \gamma(t, r')| dr' \right)^2 \\ &\leq 3 \ln \frac{\rho}{q} \int_q^\rho |\partial_r \gamma(s, r')|^2 r' dr' + 3 \frac{|t-s|}{\rho} \int_{[s,t]} |\partial_t \gamma(t', \rho)|^2 \rho dt' \end{aligned}$$

$$\begin{aligned}
& + 3 \ln \frac{r}{\rho} \int_{\rho}^r |\partial_r \gamma(t, r')|^2 r' dr' \\
& \leq 3 \ln \frac{r}{q} \|\gamma(s)\|_H^2 + 3 \frac{|t-s|}{q} \int_{[s,t]} |\partial_t \gamma(t', \rho)|^2 \rho dt'
\end{aligned}$$

After averaging in  $\rho \in [q, r]$ , it transpires

$$\begin{aligned}
|\gamma(t, r) - \gamma(s, q)|^2 & \leq C \ln \frac{r}{q} \|\tilde{\gamma}(0)\|_{H \times L^2}^2 + 3 \frac{|t-s|}{q|r-q|} \int_q^r \int_{[s,t]} |\partial_t \gamma(t', \rho)|^2 \rho dt' d\rho \\
& \leq C \left( \ln \frac{r}{q} + \frac{|t-s|^2}{q|r-q|} \right) \|\tilde{\gamma}(0)\|_{H \times L^2}^2.
\end{aligned}$$

Pick  $q = 1$ ,  $s = 0$ , and fix  $B = \max(K/3C, 2)$ . We deduce, for all  $r \in [1, 1+B]$  and  $|t| \leq \sqrt{B(r-1)}$ ,

$$|\gamma(t, r)| \geq 2M/3 - CA\sqrt{\ln(1+B)} + B \geq 2M/3 - ABC \geq M/3.$$

$B = M/(3AC) \leq 1/3C$  universal constant. Hence

$$\begin{aligned}
\|\gamma\|_{S(\mathbb{R})}^{2+3/g'(\ell)} & \geq \int_1^{1+B} \int_{|t| \leq \sqrt{B(r-1)}} (M/3)^{2+3/g'(\ell)} \frac{dr dt}{r^2} \\
& \geq (M/3)^{2+3/g'(\ell)} \int_1^{1+B} \frac{\sqrt{B(r-1)}}{r^2} dr \geq CB^{3/2} (M/3)^{2+3/g'(\ell)} \\
& \geq \frac{M^{7/2+3/g'(\ell)}}{CA^{3/2}}.
\end{aligned}$$

We can conclude, with  $\theta = \frac{3}{4+6/g'(\ell)}$ ,  $M \leq CA^\theta \|\gamma\|_{S(\mathbb{R})}^{1-\theta}$ .  $\square$

From this, the profile decomposition for  $(LW_\ell)$  takes the following form.

**Theorem 2.12** (Profile decomposition). *Let  $\vec{\psi}_n$  a bounded sequence of  $H \times L^2$ . Then there exists a sequence of scales  $(t_{j,n}, \lambda_{j,n})_n$  and linear profiles  $\vec{V}_{j,L}$  solutions to  $(LW_\ell)$ , such that, up to a subsequence that we still denote  $\vec{\psi}_n$ , we have for all  $J \geq 1$*

$$\vec{\psi}_n(r) = \sum_{j=1}^J \left( V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}} \partial_t V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right) \right) + \vec{\gamma}_{J,n}(0),$$

where  $\vec{\gamma}_{J,n}$  is a solution to  $(LW_\ell)$  which satisfies

$$\limsup_n \|\gamma_{J,n}\|_{S_\ell(\mathbb{R})} + \|\gamma_{J,n}\|_{L_{\ell,r}^\infty} \rightarrow 0 \quad \text{as } J \rightarrow +\infty,$$

and for all  $j$ ,

$$\forall n, t_{j,n} = 0 \quad \text{or} \quad \left( \frac{t_{j,n}}{\lambda_{j,n}} \text{ has a limit which is } +\infty \text{ or } -\infty \right),$$

and if  $j \neq k$ ,

$$\frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}} \rightarrow +\infty, \quad \text{or} \quad \left( \forall n, \lambda_{j,n} = \lambda_{k,n} \text{ and } \frac{|t_{j,n} - t_{k,n}|}{\lambda_{j,n}} \rightarrow +\infty \right).$$

Furthermore, there hold

(1) *Pythagorean expansion (and equipartition) of the  $H_\ell \times L^2$  norm: for all fixed  $J$ , we have as  $n \rightarrow +\infty$*

$$\|\psi_{0,n}\|_{H_\ell}^2 = \sum_{j=1}^J \left\| V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right) \right\|_{H_\ell}^2 + \|\gamma_{J,n}(0, r)\|_{H_\ell}^2 + o_n(1),$$

$$\|\psi_{1,n}\|_{L^2}^2 = \sum_{j=1}^J \left\| \frac{1}{\lambda_{j,n}} \partial_t V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right) \right\|_{L^2}^2 + \|\partial_t \gamma_{J,n}(0, r)\|_{H^\ell}^2 + o_n(1).$$

(2) *Pythagorean expansion of the energy: for all  $J$  fixed, we have as  $n \rightarrow +\infty$*

$$E(\vec{\psi}_n) = \sum_{j=1}^J E \left( V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}} \partial_t V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}} \right) \right) + E(\vec{\gamma}_{J,n}(0)) + o(1).$$

(3)  *$L^\infty$  profile selection:  $\|\psi_n\|_{L^\infty}$  has a limit as  $n \rightarrow +\infty$  and exists*

$$(2.5) \quad \lim_n \|\psi_n\|_{L^\infty} = \sup_{j \in \mathcal{J}_0} \|V_j(0)\|_{L^\infty} \quad \text{where} \quad \mathcal{J}_0 := \{j \geq 1 \mid \forall n, t_{j,n} = 0\}.$$

(with the convention that the sup is 0 if  $\mathcal{J}_0$  is empty).

*Proof.* This is essentially contained in [1, Main Theorem]. We refer to [6, Corollary 2.15] for the profile decomposition in the wave map context and the Pythagorean expansion of the  $H \times L^2$  norm, and to [6, Lemma 2.16] for the Pythagorean expansion of the energy.

The only extra points with respect to the usual profile decomposition are

$$\|\gamma_{J,n}\|_{L_{t,r}^\infty} \rightarrow 0$$

and the  $L^\infty$  profile selection. For the former, the Pythagorean expansion ensures that  $\|\vec{\gamma}_{J,n}(0)\|_{H \times L^2}$  is a bounded sequence, hence this follows from the previous Lemma and

$$\limsup_n \|\gamma_{J,n}\|_{S_\ell(\mathbb{R})} \rightarrow 0.$$

For the latter, let  $r_n$  such that

$$|\psi_n(r_n) - \|\psi_n\|_{L^\infty}| \leq \frac{1}{n}.$$

First assume  $\liminf_n \|\psi_n\|_{L^\infty} > 0$ , and let  $\varepsilon > 0$  such that  $2\varepsilon < \liminf_n \|\psi_n\|_{L^\infty}$ . Choose  $J$  so large that  $\|\gamma_{J,n}\|_{L_{t,r}^\infty} \leq \varepsilon$  for  $n$  large enough. Then consider  $j \leq J$ . If  $j \notin \mathcal{J}_0$ , then  $|t_{j,n}|/\lambda_{j,n} \rightarrow +\infty$  so that  $\|V_j \left( -\frac{t_{j,n}}{\lambda_{j,n}} \right)\|_{L^\infty} \rightarrow 0$ . Hence

$$|\psi_n(r_n)| \leq \sum_{j \leq J, j \in \mathcal{J}_0} \left| V_j \left( 0, \frac{r_n}{\lambda_{j,n}} \right) \right| + o_n(1) + \varepsilon.$$

In particular  $\mathcal{J}_0 \neq \emptyset$ . As  $\left| \ln \frac{\lambda_{j,n}}{\lambda_{k,n}} \right| \rightarrow +\infty$ , we see that

$$\lim_n \left\| \sum_{j \leq J, j \in \mathcal{J}_0} V_j \left( 0, \frac{r_n}{\lambda_{j,n}} \right) \right\|_{L^\infty} = \max\{\|V_j(0)\|_{L^\infty} \mid j \leq J, j \in \mathcal{J}_0\}.$$

This shows that

$$\limsup_n \|\psi_n\|_{L^\infty} \leq \sup\{\|V_j(0)\|_{L^\infty} \mid j \in \mathcal{J}_0\}.$$

For the reverse inequality, first notice that as  $j \rightarrow +\infty$ ,  $\|V_j(0)\|_H \rightarrow 0$  (due to the Pythagorean expansion of the energy), and so  $\|V_j(0)\|_{L^\infty} \rightarrow 0$ . Hence there exists  $j_0$  such that

$$\sup\{\|V_j(0)\|_{L^\infty} \mid j \leq J, j \in \mathcal{J}_0\} = \|V_{j_0}\|_{L^\infty}.$$

Also, as  $V_{j_0}(0)$  is continuous and tend to 0 at 0 and  $+\infty$ , there exists  $r_0 > 0$  such that

$$|V_{j_0}(0, r_0)| = \sup\{\|V_j(0)\|_{L^\infty} \mid j \in \mathcal{J}_0\}.$$

Consider the sequence  $\psi_n(\lambda_{j_0,n}r_0)$ . Then we have the expansion

$$\psi_n(\lambda_{j_0,n}r_0) = V_{j_0}(0, r_0) + \sum_{j \leq J, j \in \mathcal{J}_0 \setminus \{j_0\}} V_j \left( 0, \frac{r_n}{\lambda_{j,n}} \right) + \gamma_{J,n}(0, \lambda_{j_0,n}r_0).$$

Again due to orthogonality of the profiles, we see that

$$\liminf_n \|\psi_n\|_{L^\infty} \geq \liminf_n |\psi_n(\lambda_{j_0,n}r_0)| \rightarrow |V_{j_0}(0, r_0)|.$$

If  $\liminf_n \|\psi_n\|_{L^\infty} = 0$ , then choosing a subsequence such that  $\psi_{\sigma(n)} \rightarrow 0$  in  $L^\infty$ , and arguing as previously, we see that  $\mathcal{J}_0 = \emptyset$ . Therefore  $\|\psi_n - \gamma_{J,n}\|_{L^\infty} \rightarrow 0$  for all  $J$ , and so  $\|\psi_n\|_{L^\infty} \rightarrow 0$ . The desired equality also holds in this case.  $\square$

**Proposition 2.13** (Pythagorean expansion with cut-off, [8, Corollary 8]). *We use the notation of the previous Proposition. Fix  $J \geq 1$  and Let  $0 \leq r_n \leq s_n \leq +\infty$  be two sequences. Then we have the expansion:*

$$\begin{aligned} & \|\vec{\psi}_n\|_{H_\ell \times L^2(r_n \leq r \leq s_n)}^2 \\ &= \sum_{j=1}^J \left\| \left( V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}} \partial_t V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right) \right) \right\|_{H_\ell \times L^2(r_n \leq r \leq s_n)}^2 \\ &+ \|\vec{\gamma}_{J,n}(0, r)\|_{H_\ell(r_n \leq r \leq s_n)}^2 + o_n(1). \end{aligned}$$

*Proof.* This is the content of [8, Corollary 8]. The proof there is done for one sequence i.e.  $r_n = 0$ . To derive the above expansion, it suffices to do the difference between the expansions with cut-off  $r \leq s_n$  and  $r \leq r_n$ .  $\square$

We will use several times the following simple remark.

**Corollary 2.14.** *Let  $\vec{\psi}_n$  be a bounded sequence in  $H \times L^2$ . Assume furthermore that  $\|\psi_{0,n}\|_{L^\infty} \rightarrow 0$  and  $\|\psi_{1,n}\|_{L^2} \rightarrow 0$ . Denote  $\vec{\psi}_{n,L}$  the linear evolution (to  $(\text{LW}_\ell)$ ) with data  $\psi_n$  at time 0, then*

$$\|\psi_{n,L}\|_{S(\mathbb{R})} \rightarrow 0.$$

*Proof.* It suffices to prove that any profile decomposition has no non-trivial profiles. Consider such a decomposition with profiles  $(\vec{V}_{j,L})_j$  and parameters  $(t_{j,n}, \lambda_{j,n})$ , and remainder  $\vec{\gamma}_{J,n}$ . Recall equipartition of the energy: if  $|t_{j,n}|/\lambda_{j,n} \rightarrow +\infty$ , then

$$\left\| \frac{1}{\lambda_{j,n}} \partial_t V_j \left( -\frac{t_{j,n}}{\lambda_{j,n}} \right) \right\|_{L^2}^2 \rightarrow \frac{1}{2} \|\vec{V}_j\|_{H_\ell \times L^2}^2.$$

Hence, from Pythagorean expansion of the energy, it follows that all non-trivial profile  $V_{j,L}$  must satisfy  $t_{j,n} = 0$  and  $\partial_t V_{j,L}(0) = 0$ . But as  $\|\psi_n\|_{L^\infty} \rightarrow 0$ , equality (2.5) shows that  $\|V_{j,L}(0)\|_{L^\infty} = 0$ , hence  $V_{j,L} = 0$  and  $\vec{V}_{j,L} = 0$ .

Therefore, all profiles  $\vec{V}_{j,L}$  are trivial, and  $\vec{\psi}_n$  is the remainder term  $\vec{\gamma}_{J,n}$  (which does not depend on  $J$ ), which satisfy the desired dispersion property.  $\square$

We recall the notion of nonlinear profile. If  $\vec{V}$  is a solution of the linear equation  $(\text{LW}_\ell)$  and  $T \in \overline{\mathbb{R}}$ , there exists a unique wave map  $U$  solution to  $(\text{WM})$ , defined on a neighborhood of  $T$  and satisfying  $U(0) = U(\infty) = \ell$  and

$$\|\vec{U}(t) - \vec{V}(t) - (\ell, 0)\|_{H \times L^2} \rightarrow 0 \quad \text{as } t \rightarrow T.$$

Notice that if  $T = +\infty$ , then  $U$  is defined on some interval  $[T_0, +\infty)$  and scatters at  $+\infty$ , i.e

$$\|U - \ell\|_{S([T_0, +\infty))} < +\infty.$$

This notion follows from local well posedness [5, Theorem 2]; we refer to [15, 16] for further details.

**Proposition 2.15** (Evolution of the decomposition). *Let  $\vec{\psi}_n$  be a sequence of wave maps such that*

$$(\vec{\psi}_n(0) - (\ell, 0))_n \text{ admits a profile decomposition in the sense of Theorem 2.12,}$$

from which we use the notations.

Denote  $U_j$  nonlinear profiles associated to  $(\vec{V}_j, -\lim_n \frac{t_{j,n}}{\lambda_{j,n}})$ . Let  $t_n$  be such that

$$\forall J \geq 1, \quad \frac{t_n - t_{j,n}}{\lambda_{j,n}} < T^+(U_j), \quad \text{and} \quad \sup_n \|U_j - \ell\|_{S([- \frac{t_{j,n}}{\lambda_{j,n}}, \frac{t_n - t_{j,n}}{\lambda_{j,n}}])} < +\infty.$$

Then for  $n$  large enough,  $T^+(\vec{\psi}_n) > t_n$ , and for all  $t \in [0, t_n]$ ,

$$\begin{aligned} \vec{\psi}_n(t) - (\ell, 0) &= \sum_{j=1}^J \left( \left( U_j \left( \frac{t - t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}} \partial_t U_j \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right) \right) - (\ell, 0) \right) \\ &\quad + \vec{\gamma}_{J,n}(t, r) + \vec{r}_{n,J}(t, r), \end{aligned}$$

where  $\limsup_{n \rightarrow +\infty} \|r_{J,n}\|_{S_\ell([0, t_n])} + \|\vec{r}_{J,n}\|_{L^\infty([0, t_n], H \times L^2)} \rightarrow 0$  as  $J \rightarrow +\infty$ .

*Proof.* It is the translation of [10, Proposition 2.8] via the transformation  $\mathcal{T}$ .  $\square$

Notice that  $\vec{\psi}_n(0) - (\ell, 0)_n$  is a bounded sequence in  $H \times L^2$ , in particular  $\psi_n(0) = \psi_n(\infty) = \ell$ . Let us emphasize that we can not evolve a decomposition where harmonic maps appear: this is a fundamental difference with the semi linear wave equation, where such stationary profile were allowed (they belong to  $\dot{H}^1 \times L^2$ ).

We will often use this result in the following particular case.

**Corollary 2.16.** *Let  $\vec{\psi}_n$  be a sequence of wave maps such that  $\vec{\psi}_n(\infty) = \vec{\psi}_n(0) = \ell$ , and*

- (1)  $\vec{\psi}_n(0) - (\ell, 0)$  is a bounded sequence in  $H \times L^2$ .
- (2) If  $\vec{\psi}_{n,L}$  denotes the linear solution to  $(\text{LW}_\ell)$  with initial data  $\vec{\psi}_n(0) - (\ell, 0)$ , then  $\|\psi_{n,L}\|_{S(\mathbb{R})} \rightarrow 0$ .

Then for  $n$  large enough  $\vec{\psi}_n$  is defined globally on  $\mathbb{R}$  and

$$\sup_{t \in \mathbb{R}} \|\vec{\psi}_n(t) - (\ell, 0) - \vec{\psi}_{n,L}(t)\|_{H \times L^2} \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty.$$

*Remark 5.* Notice we can combine Corollary 2.14 with this last result, as the hypothesis of the latter are the conclusions given by the former.

*Proof.* The second condition means that any profile decomposition is trivial (i.e) does not contain any non trivial profile. Hence with the notation of the previous Proposition,  $\vec{\psi}_n(0) - (\ell, 0) = \vec{\gamma}_{J,n}(0) =: \vec{\psi}_{n,L}(0)$  (does not depend on  $J$ ) and

$$\vec{\psi}_n(t) + (\ell, 0) = \vec{\psi}_{n,L}(t) + r_n(t).$$

To conclude, we argue by contradiction:

- (1) if  $\vec{\psi}_n$  blow-up in finite time, consider  $t_n = T^+(\vec{\psi}_n) + 1$  (and similarly for  $T^-(\vec{\psi}_n)$ ).
- (2) if the convergence does not hold, there exists  $\eta > 0$  and  $t_n$  be such that  $\|r_n(t_n)\|_{H \times L^2} \geq \eta$ .

Each hypothesis contradicts Proposition 2.15.  $\square$

### 3. BUBBLE DECOMPOSITION FOR A SEQUENCE OF WAVE MAPS

Our goal here is to study a sequence of wave maps with vanishing (space-time) kinetic energy. First we prove that at any scale, there is local (strong) limit which is a harmonic map (possibly constant).

**Proposition 3.1** (Profile at any scale). *We assume (A1)-(A2).*

Let  $A > 0$  and  $\vec{\psi}_n$  be wave maps be defined on the time interval  $[-A, A]$ , such that

- (1)  $\vec{\psi}_n$  have uniformly bounded energy  $E(\vec{\psi}_n) \leq E$ .
- (2)  $\psi_n(0)$  is bounded.
- (3) For some sequence  $r_n \rightarrow +\infty$ ,

$$\|\partial_t \psi_n\|_{L^2((-A,A), L^2(r \leq r_n))} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Then  $\vec{\psi}_{\sigma(n)}$ , up to a subsequence  $\sigma(n)$ , converges to some harmonic map  $(Q, 0)$  locally strongly in the following sense: for any  $R > 0$ ,

$$(3.1) \quad \sup_{t \in [-A, A]} \|\vec{\psi}_{\sigma(n)}(t) - (Q, 0)\|_{H \times L^2([1/R, R])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

*Remark 6.* Assume furthermore that for some sequences  $t_n \subset [-A, A]$  and  $r_n \rightarrow r_\infty \in (0, +\infty)$ ,

$$\psi_{\sigma(n)}(t_n, r_n) \rightarrow l.$$

It follows from the convergence that  $Q(r_\infty) = l$ , hence from classification of harmonic maps,  $Q$  is non constant if and only if  $g(l) \neq 0$ .

*Remark 7.* A result of this type was already obtained in [25], but only for the first scale (and in the  $L^2_{\text{loc}}(H \times L^2([0, +\infty)_{\text{loc}}))$  topology), i.e. under the additional assumption that

$$\sup_{t \in [-A, A]} E(\vec{\psi}_n; 0, 1) \leq \delta_0.$$

for some small  $\delta_0 > 0$ .

*Proof.* Up to rescaling, we can assume that  $A = 1$ . As  $E(\vec{\psi}_n)$  and  $\psi_n(0)$  are bounded,  $\vec{\psi}_n$  is bounded in  $\mathcal{C}([-1, 1], (\dot{H}^1 \cap L^\infty) \times L^2)$ . Therefore, up to a subsequence, there exists  $\vec{\psi}_\infty$  such that  $\vec{\psi}_n \rightarrow \vec{\psi}_\infty$  a.e.  $(-1, 1) \times (0, +\infty)$ , and  $\psi_n \rightarrow \psi_\infty$  in  $\mathcal{C}_{\text{loc}}([-1, 1] \times [0, +\infty))$  and

$$\vec{\psi}_n \overset{*}{\rightharpoonup} \vec{\psi}_\infty \quad L^\infty((-1, 1), \dot{H}^1 \times L^2) \text{ star-weakly.}$$

From Fatou Lemma, we deduce that

$$\forall t \in [-1, 1] - a.e., \quad \int \frac{g^2(\psi_\infty(t, r))}{r^2} r dr \leq \liminf_n \int \frac{g^2(\psi_n(t, r))}{r^2} r dr.$$

It follows that  $\vec{\psi}_\infty$  satisfies the wave map equation (WM) in a weak sense, and has finite energy  $E(\vec{\psi}_\infty) \leq \liminf_n E(\vec{\psi}_n)$ . Now, as  $\|\partial_t \psi_n\|_{L^2((-1, 1), L^2(r \leq r_n))} \rightarrow 0$ , then for all  $R \geq 0$ ,

$$\|\partial_t \psi_\infty\|_{L^2((-1, 1), L^2(r \leq R))} = 0,$$

and hence  $\partial_t \psi_\infty = 0$ , that is  $\vec{\psi}_\infty$  is a weak harmonic map:  $r \partial_r (r \partial_r \psi) = f(\psi)$ . It follows by elliptic regularity that  $\vec{\psi}_\infty = (Q, 0)$  a smooth harmonic map.

Let us prove as a first step that there holds strong local convergence in space-time  $L^2_{\text{loc}}((-1, 1), H \times L^2)_{\text{loc}}$ . Let  $K$  be a compact of  $(-1, 1) \times (0, +\infty)$  (in space time). Then

$$(3.2) \quad \|\psi_n(t, r) - \psi_\infty(t, r)\|_{L^2(K, r dr dt)} + \|\partial_t \psi_n(t, r)\|_{L^2(K, r dr dt)} \rightarrow 0$$

due the compact embedding  $\dot{H}^1 \cap L^\infty(rdrdt) \rightarrow L^2(K, drdt/r)$  and to the vanishing of the kinetic energy. Fix now  $\varphi \in \mathcal{D}((-1, 1) \times (0, +\infty))$ . We compute

$$\begin{aligned}
& \int (\partial_r \psi_n - \partial_r \psi_\infty)^2(t, r) \varphi(t, r) r dr dt \\
&= - \int \Delta(\psi_n - \psi_\infty)(t, r) (\psi_n - \psi_\infty)(t, r) \varphi(t, r) r dr dt \\
&\quad - \int \partial_r(\psi_n - \psi_\infty)(t, r) (\psi_n - \psi_\infty)(t, r) \partial_r \varphi(t, r) r dr dt \\
&= - \int \partial_{tt} \psi_n(t, r) (\psi_n - \psi_\infty)(t, r) \varphi(t, r) r dr dt \\
&\quad - \int \frac{f(\psi_n(t, r)) - f(\psi_\infty(t, r))}{2r} (\psi_n - \psi_\infty)(t, r) \varphi(t, r) dr dt \\
&\quad - \int \partial_r(\psi_n - \psi_\infty)(t, r) (\psi_n - \psi_\infty)(t, r) \partial_r \varphi(t, r) r dr dt \\
&= \int |\partial_t \psi_n(t, r)|^2 \varphi(t, r) r dr dt + \int \partial_t \psi_n(t, r) (\psi_n - \psi_\infty)(t, r) \partial_t \varphi(t, r) r dr dt \\
&\quad - \int \frac{f(\psi_n(t, r)) - f(\psi_\infty(t, r))}{2r} (\psi_n - \psi_\infty)(t, r) \varphi(t, r) dr dt \\
&\quad - \int \partial_r(\psi_n - \psi_\infty)(t, r) (\psi_n - \psi_\infty)(t, r) \partial_r \varphi(t, r) r dr dt
\end{aligned}$$

Now, using again the compact embedding  $H_{\text{loc}}(rdrdt) \rightarrow L^2_{\text{loc}}(drdt/r)$ , and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \left| \int \partial_t \psi_n (\psi_n - \psi_\infty)(t, r) \partial_t \varphi(t, r) r dr dt \right| \\
& \leq \|\partial_t \psi_n \sqrt{|\partial_t \varphi|}\|_{L^2} \|(\psi_n - \psi_\infty) \sqrt{|\partial_t \varphi|}\|_{L^2} \rightarrow 0,
\end{aligned}$$

and similarly

$$\begin{aligned}
& \left| \int \frac{f(\psi_n(t, r)) - f(\psi_\infty(t, r))}{2r} (\psi_n(t, r) - \psi_\infty(t, r)) \varphi(t, r) dr dt \right| \\
& \leq \|(f(\psi_n) - f(\psi_\infty)) \varphi / r\|_{L^2} \|(\psi_n - \psi_\infty) \sqrt{\varphi / r}\|_{L^2} \\
& \leq \omega_f(\sup_n \|\psi_n\|_{L^\infty}) \|(\psi_n - \psi_\infty) \sqrt{\varphi / r}\|_{L^2}^2 \rightarrow 0
\end{aligned}$$

(We recall that  $f$  is  $\mathcal{C}^1$ , hence we can define  $\omega_f(A) := \sup_{x, y \in [-A, A]} \frac{|f(x) - f(y)|}{|x - y|}$ ; we already noticed earlier that  $\psi_n$  was a bounded sequence of  $L^\infty_{t, x}$ ). Finally, as  $\|\partial_r(\psi_n - \psi_\infty)\|_{L^2(rdrdt)}$  is bounded,

$$\begin{aligned}
& \left| \int \partial_r(\psi_n - \psi_\infty)(t, r) (\psi_n - \psi_\infty)(t, r) \partial_r \varphi(t, r) r dr dt \right| \\
& \leq \|\partial_r(\psi_n - \psi_\infty)(t, r)\|_{L^2} \|(\psi_n - \psi_\infty)(t, r) \partial_r \varphi(t, r)\|_{L^2} \rightarrow 0.
\end{aligned}$$

This proves that

$$\int \left( (\partial_r \psi_n - \partial_r \psi_\infty)^2(t, r) + \frac{(\psi_n - \psi_\infty)^2(t, r)}{r^2} \right) \varphi(t, r) r dr dt \rightarrow 0.$$

Let us now prove (3.1). Due to the previous convergence, the set

$$\mathcal{P} := \{t \in [-1, 1] \mid \forall R > 0, \quad \|\vec{\psi}_n(t) - (Q, 0)\|_{H \times L^2([1/(2R), R+1])} \rightarrow 0\}$$

is dense in  $[-1, 1]$ .

We now need the following version of uniform continuity of the flow around the harmonic map  $(Q, 0)$ .



**Lemma 3.2.** *Let  $Q$  be a harmonic map,  $T \geq 0$  and  $\varepsilon > 0$ . There exist  $\delta > 0$  such that for all  $0 \leq r_1 \leq 2r_1 < r_2 \leq +\infty$  and wave map  $\vec{\psi}$  such that*

$$\|\vec{\psi}(0) - (Q, 0)\|_{H \times L^2([r_1, r_2])} \leq \delta,$$

*Then for all time  $t \in (T^-(\vec{\psi}), T^+(\vec{\psi}))$  such that  $|t| \leq \min\{T, (r_2 - r_1)/2\}$ ,*

$$\|\vec{\psi}(t) - (Q, 0)\|_{H \times L^2([r_1+|t|, r_2-|t|])} \leq \varepsilon.$$

*Proof.* We postpone it to the Appendix.  $\square$

Fix  $R > \sqrt{2}$  (so that  $2/R < R$ ) and let  $\varepsilon > 0$ .

Define the integer  $K$  to be the integer part of  $2R + 1$  so that  $K \geq 2R$ . Then for all  $k \in \llbracket -K, K - 1 \rrbracket$ , there exist  $t_k \in [k/K, (k + 1)/K]$  such that  $t_k \in \mathcal{S}$ . Let  $\delta > 0$  be provided by the previous Lemma 3.2 with  $T = 1$  (and our previously fixed  $\varepsilon > 0$ ). Define  $N$  be such that

$$\forall k \in \llbracket -K, K - 1 \rrbracket, \forall n \geq N, \quad \|\vec{\psi}_n(t_k) - (Q, 0)\|_{H \times L^2([1/(2R), R+1])} \leq \delta.$$

Let  $t \in [-1, 1]$  and  $n \geq N$ . There exist  $k \in \llbracket -K, K - 1 \rrbracket$  such that  $|t - t_k| \leq 1/K$ . Hence by 3.2 (as  $1/K \leq 1$ ),

$$\|\vec{\psi}_n(t) - (Q, 0)\|_{H \times L^2([1/(2R)+1/K, R+1-1/K])} \leq \varepsilon.$$

Now  $R + 1 - 1/K \geq R$  and  $1/(2R) + 1/K \leq 1/R$ ; hence Lemma 3.2 yields

$$\forall t \in [-1, 1], \quad \|\vec{\psi}_n(t) - (Q, 0)\|_{H \times L^2([1/R, R])} \leq \varepsilon,$$

that is

$$\forall n \geq N, \quad \sup_{t \in [-1, 1]} \|\vec{\psi}_n(t) - (Q, 0)\|_{H \times L^2([1/R, R])} \leq \varepsilon.$$

This is the desired convergence.  $\square$

We now prove a bubble decomposition result, which is the main result of this Section. It will be central in the soliton resolution in both the blow-up and global cases, and that we will also crucially use for the sharp scattering result (Theorem 1.2).

**Theorem 3.3** (Bubble decomposition). *We assume (A1)-(A2).*

*Let  $(\psi_n)_n \in \mathcal{C}([-1, 1], H \times L^2)$  be a sequence of wave maps. Assume that for some  $R > 0$ ,*

- (1) *The  $\vec{\psi}_n$  have uniformly bounded energy  $\sup_n E(\psi_n) = E < +\infty$ .*
- (2) *For some  $\ell \in \mathcal{V}$ ,  $\psi_n(0, R) \rightarrow \ell$  as  $n \rightarrow +\infty$ .*
- (3) *At  $t = 0$ , the  $\partial_t \psi_n$  have vanishing  $L^2$ -norm on  $[0, R]$ :*

$$\sup_{\lambda \leq 1} \frac{1}{\lambda} \int_{-\lambda}^{\lambda} \int_0^R |\partial_t \psi_n(t, r)|^2 r dr dt \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

- (4)  *$\vec{\psi}_n(0)$  has vanishing energy on scale 1:*

$$\forall t \in [-1, 1], \forall r > 0, \quad E(\vec{\psi}_n(t); r, R) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

*Then there exists an integer  $J \geq 0$ ,  $J$  scales  $(\lambda_{j,n})_n$  verifying*

$$0 < \lambda_{J,n} \ll \dots \ll \lambda_{2,n} \ll \lambda_{1,n} \ll 1,$$

*and  $J$  harmonic maps  $Q_j \in \mathcal{H}$  such that, up to a subsequence  $\vec{\psi}_{\sigma(n)}$ ,*

$$\vec{\psi}_{\sigma(n)}(t) - (\ell, 0) = \sum_{j=1}^J (Q_j(\cdot/\lambda_{j,n}) - Q_j(\infty), 0) + \vec{b}_n(t),$$

*where  $\vec{b}_n \in \mathcal{C}([-1, 1], H \times L^2([0, R]))$  satisfies the following convergences. For all  $A > 0$ ,*

(1) (No energy at all scale) Let  $\lambda_n$  be a sequence such that  $0 \leq \lambda_n \leq R/A$ .  
Then

$$\sup_{t \in [-A\lambda_n, A\lambda_n]} \|\vec{b}_n(t)\|_{H \times L^2(\lambda_n/A \leq r \leq A\lambda_n)} \rightarrow 0.$$

(2) (No energy up to the last scale) If  $J \geq 1$ , then

$$\sup_{t \in [-A\lambda_{J,n}, A\lambda_{J,n}]} \|\vec{b}_n(t)\|_{H \times L^2(r \leq A\lambda_{J,n})} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

If  $J = 0$ , then  $\sup_{t \in [-1/2, 1/2]} \|\vec{b}_n(t)\|_{H \times L^2(r \leq R)} \rightarrow 0$ .

(3) Let  $\mathcal{C}_A = \{(t, r) \mid |t| \leq \min(Ar, 1), r \leq R\}$  be a truncated cone. Then  $\sup_{(t,r) \in \mathcal{C}_A} |b_n(t, r)| \rightarrow 0$ .

Also

$$(1) \sum_{j=1}^J E(Q_j, 0) \leq E,$$

$$(2) \text{ For all } 1 \leq j < J, Q_{j+1}(\infty) = Q_j(0), \text{ and } Q_1(\infty) = \ell.$$

*Proof.* We first need to introduce some notation.

As  $E(\psi_n) \leq E$  and  $\psi_n(0, R) \rightarrow \ell$ , hence is bounded, there hold an  $L^\infty$  bound on  $\psi_n$ : for some  $K > 0$ ,

$$\forall n, \forall t, \forall r, \quad |\psi_n(t, r)| \leq K.$$

By (A1),  $\mathcal{V} \cap [-K, K]$  is finite. For  $\ell \in \mathcal{V}$ , denote  $\ell^-, \ell^+ \in \mathcal{V}$  the preceding and following points in  $\mathcal{V}$  (respectively), that is  $\ell^- < \ell < \ell^+$  ( $\ell^-, \ell^+ \cap \mathcal{V} = \{\ell\}$ ). Then define

$$\eta_\ell = \min\{\sup\{|g(x)| \mid x \in (\ell, \ell^+)\}, \sup\{|g(x)| \mid x \in (\ell^-, \ell)\}\} > 0,$$

and

$$\delta_0 = \frac{1}{2} \inf\{\eta_\ell \mid \ell \in V \cap [-K, K]\} > 0.$$

Finally, for  $Q \in \mathcal{H}$  a non constant harmonic map such that  $Q(\infty) = \ell$  and normalized in the sense that

$$Q(1) = \frac{1}{2}(Q(0) + Q(\infty)),$$

let  $\varepsilon_\ell > 0$  be such that

$$\forall r \leq \sqrt{\varepsilon_\ell}, \quad |g(Q(r))| \leq \delta_0/2, \quad \text{and} \quad |g(Q(1/r))| \leq \delta_0/2,$$

and

$$\varepsilon_0 = \inf\{\varepsilon_\ell \mid \ell \in V \cap [-K, K]\} > 0.$$

Notice that by definition of  $\varepsilon_0$ , if  $Q$  is a normalized harmonic map such that  $Q(\infty) \in [-K, K]$  and  $r_0$  is such that  $|g(Q(r_0))| \geq \delta_0/2$ , then

$$\text{if } r \leq \varepsilon_0 r_0 \text{ or } r \geq r_0/\varepsilon_0, \quad |g(Q(r))| \leq \delta_0/2.$$

Also notice that due to monotonicity of the energy along light cones, hypothesis (2) and (4) are in fact uniform in  $t \in [-1, 1]$ . More precisely, there holds

$$(2') \text{ for all } r > 0, \sup_{t \in [-1, 1]} \|\psi_n(t) - \ell\|_{L^\infty([r, R])} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

$$(4') \text{ for all } r > 0, \sup_{t \in [-1, 1]} E(\vec{\psi}_n(t); r, R) \rightarrow 0 \text{ as } n \rightarrow +\infty$$

Let us prove (4') first. Let  $r > 0$  and  $\delta > 0$ . For  $r' \leq r/2$  be such that  $1/r' \in \mathbb{N}$ , apply (4) with  $t = kr'$ ,  $k \in \llbracket -1/r'; 1/r' \rrbracket$ . This gives  $N$  such that for all  $n \geq N$  and all  $k \in \llbracket -1/r'; 1/r' \rrbracket$ ,

$$E(\vec{\psi}_n(kr'); r', R) \leq \delta.$$

By monotonicity of the energy, with  $\tau \in [-r', r']$  we get

$$E(\vec{\psi}_n(kr' + \tau); r' + |\tau|, R) \leq E(\vec{\psi}_n(kr'); r', R) \leq \delta.$$

The  $kr' + \tau$  cover all  $[1, 1]$ , hence (4').

We now turn to (2'). Notice that for  $\phi \in H$ ,

$$\|g(\phi)\|_H^2 \leq (1 + \|g'(\phi)\|_{L^\infty}^2)E(\phi, 0),$$

so that for all  $t$  and  $r > 0$ ,

$$\|g(\psi_n(t))\|_{H([r,R])}^2 \leq (1 + \|g'\|_{L^\infty([-K,K])}^2)E(\vec{\psi}_n(t); r, R) \rightarrow 0,$$

where the convergence is uniform in  $t$  due to (4'). Due to Lemma A.2, we deduce that (for all  $0 < r \leq R/2$ )

$$\sup_{t \in [-1,1]} \|g(\psi_n(t))\|_{L^\infty([r,R])} \rightarrow 0.$$

As  $\mathcal{V} \cap [-K, K]$  is finite and  $\psi_n(0, R) \rightarrow \ell \in V$ , it follows from a continuity argument that

$$\forall r > 0, \quad \sup_{t \in [-1,1]} \|\psi_n(t) - \ell\|_{L^\infty([r,R])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

as desired.

*Step 1: Extraction of the profiles, and definition of  $\lambda_{j,n}$  and  $\vec{b}_n$ .*

We recall that an extraction is a function  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  which is (strictly) increasing.

We now define the set of scales  $S$  made of couples  $\lambda := ((r_n)_n, \phi)$  where

$$(r_n)_n \in [0, R]^\mathbb{N}, \quad \text{and } \phi : \mathbb{N} \rightarrow \mathbb{N} \text{ is an extraction,}$$

such that

$$(3.3) \quad \psi_{\phi(n)}(0, r_n) \text{ has a limit } l_\lambda \text{ such that } |g(l_\lambda)| = \delta_0.$$

We denote by  $S_0$  the subset of  $S$  made of scales  $((r_n)_n, \phi)$  such that for some non constant harmonic map  $Q$ , it satisfies furthermore

$$(3.4) \quad \forall A > 0, \quad \|\vec{\psi}_{\phi(n)}(0) - (Q(r_n \cdot), 0)\|_{H \times L^2([r_n/A, Ar_n])} \rightarrow 0.$$

Notice that for all  $r > 0$ ,  $\|\vec{\psi}_n(0) - (\ell, 0)\|_{H \times L^2([r,R])} \rightarrow 0$  due to our hypothesis (4). Hence the pointwise bound gives

$$\forall r > 0, \quad \|g(\psi_n(0))\|_{L^\infty([r,R])} \rightarrow 0.$$

It transpires that for any  $\lambda = ((r_n)_n, \sigma) \in S$ , then  $r_n \rightarrow 0$ . Also, in this case, if we define

$$\phi_n(t, r) = \psi_{\sigma(n)}(r_n t, r_n r),$$

then  $\phi_n$  is a wave map defined for  $t \in [-1/r_n, 1/r_n]$  and for all  $A > 0$ ,

$$\int_{-A}^A \int_0^{R/r_n} |\partial_t \phi_n(t, r)|^2 r dr dt = \frac{1}{r_n} \int_{-Ar_n}^{Ar_n} \int_0^R |\partial_t \phi_{\sigma(n)}(r_n t, r_n r)|^2 r dr dt \rightarrow 0,$$

and

$$g(\phi_n(1)) = \lim_n g(\psi_{\sigma(n)}(r_n)) = g(l_\lambda) \neq 0.$$

It follows from Proposition 3.1 (by rescaling again by a fixed factor  $A$  and using a diagonal argument) that there exists an extraction  $\pi$  and a non constant harmonic map  $Q$ , such that

$$\forall A > 0, \quad \sup_{t \in [-A, A]} \|\vec{\phi}_{\pi(n)}(t) - (Q, 0)\|_{H \times L^2([1/A, A])} \rightarrow 0.$$

In particular, due to continuity of the flow, we have that

$$\forall t \in \mathbb{R}, \quad \lim_{A \rightarrow +\infty} \lim_n E(\vec{\phi}_{\pi(n)}(t); 1/A, A) = E(Q, 0).$$

Unscaling, this can be rewritten as

$$(3.5) \quad \forall A > 0, \quad \sup_{t \in [-Ar_n, Ar_n]} \|\vec{\psi}_{\phi \circ \pi(n)}(t) - (Q(r_{\pi(n)} \cdot), 0)\|_{H \times L^2([r_{\pi(n)}/A, Ar_{\pi(n)}])} \rightarrow 0.$$

Notice that the scale  $\tilde{\lambda} := ((r_{\pi(n)})_n, \phi \circ \pi) \in S_0$ , we say it is adapted to  $\lambda \in S$ , and we call  $Q$  the local limit at scale  $\tilde{\lambda}$ .

We now proceed with the extraction of the profiles. First assume that  $S_0$  is empty. In this case, let us prove that for  $n$  large enough,  $\|g(\psi_n(0))\|_\infty \leq \delta_0$ . Indeed, recall that  $g(\psi_n(0, R)) \rightarrow 0$ . If there exist an extraction  $\phi$  such that for all  $n$ ,  $\|g(\psi_{\sigma(n)}(0))\|_\infty > \delta_0$ , then by continuity of  $g(\psi_{\phi(n)}(0))$ , for all  $n$  there exist  $r_n$  such that  $|g(\psi_{\phi(n)}(0, r_n))| = \delta_0$ . As  $g^{-1}(\{\pm\delta_0\}) \cap [-K, K]$  is compact, up to extracting a subsequence, we can assume that  $g(\psi_{\phi(n)}(0, r_n)) \rightarrow \ell$  where  $|g(\ell)| = \delta_0$ . Hence  $((r_n)_n, \sigma) \in S$  and we saw at the previous paragraph how to construct an adapted scale to it: it follows that  $S_0 \neq \emptyset$ , a contradiction. Hence for  $n \geq N$ ,

$$\|g(\psi_n(0))\|_\infty \leq \delta_0.$$

Then we choose  $J = 0$  and  $\vec{b}_n = \vec{\psi}_n$ .

If  $S_0$  is not empty, we proceed by induction and construct a (finite) sequence of scales  $\lambda_j = (r_{j,n})_n, \sigma_j) \in S_0$  for  $j = 1, \dots, J$ , such that

- (1) There exist an extraction  $\pi_j$  such that  $\sigma_{j+1} = \sigma_j \circ \pi_j$ .
- (2) Orthogonality:  $\frac{r_{j+1,n}}{r_{j,\pi_{j+1}(n)}} \rightarrow 0$  as  $n \rightarrow +\infty$ .
- (3) For  $j \in \llbracket 1, J-1 \rrbracket$  and all  $n \in \mathbb{N}$  and  $r \in (r_{j+1,n}, \varepsilon_0 r_{j,\pi_j(n)})$ , there holds  $|g(\psi_{\sigma_{j+1}(n)}(0, r))| < \delta_0$ .
- (4) For all  $r \in [0, \varepsilon_0 r_{J,\pi_J(n)})$ ,  $|g(\psi_{\sigma_J(n)}(0, r))| < \delta_0$ .

The first two conditions give an order on  $S_0$ , the third one ensures that the scale  $\lambda_j$  and  $\lambda_{j+1}$  are ‘‘consecutive’’, and the fourth one is the stopping condition.

Let  $\phi_1$  be an extraction such that for all  $n \in \mathbb{N}$ ,  $\|g(\psi_{\sigma_1(n)}(0))\|_{L^\infty([0, R])} \geq \delta_0$ . We can furthermore choose  $\phi_1$  such that  $\psi_{\sigma_1(n)}(r_{1,n})$  has a limit  $l_1 \in g^{-1}(\{\pm\delta_0\})$ , where  $r_{1,n}$  is defined as follows:  $r_{1,n}$  is such that  $|g(\psi_n(r_{1,n}))| = \delta_0$  and

$$\forall r \in (r_{1,n}, R], \quad |g(\psi_n(0, r))| < \delta_0.$$

$\phi_1$  and  $r_{1,n}$  are well defined because  $\psi_n$  is continuous,  $g(\psi_n(R)) \rightarrow 0$  and  $S_0$  is not empty.

Then  $(r_{1,n}, \sigma_1) \in S$ , and let  $\pi_1$  be adapted. This yields the first scale

$$\lambda_1 := ((r_{1,\pi_1(n)})_n, \sigma_1 \circ \pi_1) \in S_0,$$

and  $Q_1$  is the local limit at scale  $\lambda_1$ .

Now assume that  $(r_{j,n})_n, \sigma_j)$  is constructed. For some non constant harmonic map  $Q_j$  we have the convergence

$$\forall A > 0, \quad \|\vec{\psi}_{\sigma_j(n)}(0) - (Q_j(r_{j,n}, \cdot), 0)\|_{H \times L^2(r_{j,n}/A, Ar_{j,n})} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

For  $A \geq 2$ , the convergence also holds in  $L^\infty(r_{j,n}/A, Ar_{j,n})$ , due to Lemma A.2. Now, as  $|g(Q_j(\varepsilon_0))| \leq \delta_0/2$ , we see (with  $A = 1/\varepsilon_0$ ) that for large  $n$ ,

$$|g(\psi_{\sigma_j(n)}(0, \varepsilon_0 r_{j,n}))| \leq 2\delta/3.$$

If for  $n$  large enough,  $\sup\{|g(\psi_{\sigma_j(n)}(0, r))| \mid r \in [0, \varepsilon_0 r_{j,n}]\} \leq \delta_0$ , we stop here. Otherwise, we construct  $(\tilde{r}_{j+1,n})_n, \sigma_{j+1})$  as follows. First choose an extraction  $\rho_j$  such that denoting

$$\phi_{j+1} = \sigma_j \circ \rho_j,$$

we have

$$\forall n, \quad \sup_{r \in [0, \varepsilon_0 r_{j,\rho_j(n)}]} |g(\psi_{\phi_{j+1}(n)}(0, r))| > \delta_0.$$

This allows to define  $\tilde{r}_{j+1,n}$  such that  $|g(\psi_{\phi_{j+1}(n)}(0, \tilde{r}_{j+1,n}))| = \delta_0$  and

$$\forall r \in (\tilde{r}_{j+1,n}, \varepsilon_0 r_{j,\rho_j(n)}), \quad |g(\psi_{\phi_{j+1}(n)}(r))| < \delta_0.$$

Extracting further, we can assume without loss of generality that  $\psi_{\phi_{j+1}(n)}(0, \tilde{r}_{j+1,n})$  has a limit  $l_{j+1}$ , i.e.  $(\tilde{r}_{j+1,n})_n, \phi_{j+1} \in S$ . Notice that due to the local convergence

$$\vec{\psi}_{\sigma_j(\pi_j(n))}(0) - (Q_j(r_{j,\rho_j(n)} \cdot), 0)$$

on the scale  $r_{j,n}$ , it follows that

$$\frac{\tilde{r}_{j+1,n}}{r_{j,\rho_j(n)}} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Let finally choose an extraction  $\varpi_j$  adapted such that  $(r_{j+1,n})_n, \sigma_{j+1} \in S_0$ , where  $\sigma_{j+1} := \phi_{j+1} \circ \varpi_j$  and  $r_{j+1,n} := \tilde{r}_{j+1,\varpi_j(n)}$ . Finally let  $Q_{j+1}$  be the local limit at scale  $(r_{j+1,n}, \sigma_{j+1})$ .

Then  $\pi_j := \rho_j \circ \varpi_j$  is an extraction such that

$$\sigma_{j+1} = \sigma_j \circ \pi_j.$$

Also, we see that

$$\frac{r_{j+1,n}}{r_{j,\pi_j(n)}} = \frac{\tilde{r}_{j+1,\varpi_j(n)}}{r_{j,\rho_j(\varpi_j(n))}} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Finally, by the definition of  $\tilde{r}_{j,+1,n}$ , we have

$$\forall r \in (r_{j+1,n}, \varepsilon_0 r_{j,\pi_j(n)}), \quad |g(\psi_{\sigma_{j+1}(n)}(r))| < \delta_0.$$

Hence  $(r_{j+1,n})_n, \sigma_{j+1}$  is as desired.

Notice that from our construction, we have  $Q_{j+1}(\infty) = Q_j(0)$ .

We know claim that this process has to stop after a finite number of steps. Indeed, fix  $J \geq 1$  consider the sequence  $(\psi_{\sigma_J(n)})_n$ . Then for  $1 \leq j \leq J$ , we have for all  $A > 0$ ,

$$E(\vec{\psi}_{\sigma_j(n)}; 1/Ar_{\pi_j \circ \dots \circ \pi_j(n),j}, Ar_{\pi_j \circ \dots \circ \pi_j(n),j}) \rightarrow E(Q_j, 0; 1/A, A).$$

Now for  $j < j'$ , if we denote  $\varpi = \pi_j \circ \dots \circ \pi_{j'-1}$ , and  $\pi = \pi_j \circ \dots \circ \pi_k$

$$\frac{r_{\pi_{j'} \circ \dots \circ \pi_k(n),j'}}{r_{\pi_j \circ \dots \circ \pi_k(n),j}} = \frac{r_{\pi(n),j}}{r_{\varpi \circ \pi(n),j}} \rightarrow 0$$

due to the fact that the scales  $r_{n,j'}$  and  $r_{\varpi(n),j}$  are orthogonal.

Summing this for  $1 \leq j \leq J$ , we thus get

$$E \geq \liminf_n E(\vec{\psi}_{\sigma_k(n)}) \geq \sum_{j=1}^J E(Q_j, 0; 1/A, A).$$

Letting  $A \rightarrow +\infty$ , we get  $E \geq \sum_{j=1}^J E(Q_j, 0)$ . As  $Q_j(0), Q_j(\infty) \in [-K, K]$  for all  $j$ , we have that

$$E(Q_j, 0) \geq E_K := \inf\{G(k) - G(k') \mid k, k' \in V \cap [-K, K], k > k'\}.$$

$E_K > 0$  due to assumption (A2). Hence  $E \geq JE_K$  and this prove that the process has to stop after at most  $E/E_K$  steps.

Thus we have constructed a sequence of  $J$  scales  $(r_{1,n}, \sigma_1), \dots, (r_{J,n}, \sigma_J) \in S_0$  and of non constant harmonic maps  $Q_j$ . We can now define for  $j = 1, \dots, J$ ,

$$\sigma = \sigma_J, \quad \lambda_{j,n} = r_{j,\sigma_{j+1} \circ \dots \circ \sigma_J(n)}$$

It will be convenient to write

$$\lambda_{0,n} = R, \quad \lambda_{J+1,n} = 0.$$

From our construction, we have that for all  $j = 1, \dots, J$ , and  $A > 0$ ,

$$\sup_{t \in [-A\lambda_{j,n}, A\lambda_{j,n}]} \|\vec{\psi}_{\sigma(n)}(t) - Q_j(\cdot/\lambda_{j,n})\|_{H \times L^2(\lambda_{j,n}/A \leq r \leq A\lambda_{j,n})} \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

and

$$\sup_{t \in [-1, 1]} \|\vec{\psi}_{\sigma(n)}(t) - \ell\|_{H \times L^2(\lambda_{0,n}/A \leq r \leq \lambda_{0,n})} \rightarrow 0.$$

Also, for all  $n \in \mathbb{N}$  and  $j = 0, \dots, J$ ,

$$(3.6) \quad \frac{\lambda_{j+1,n}}{\lambda_{j,n}} \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

$$(3.7) \quad \|g(\psi_{\sigma(n)}(0))\|_{L^\infty([\lambda_{j+1,n}/\varepsilon_0, \varepsilon_0\lambda_{j,n}])} \leq \delta_0,$$

Define the error term: for  $t \in [-1, 1]$

$$(3.8) \quad \vec{b}_n(t) = \vec{\psi}_{\sigma(n)}(t) - (\ell, 0) - \sum_{j=1}^J (Q_j(\cdot/\lambda_{j,n}) - Q_j(\infty)x, 0).$$

*Step 2* : Convergence at all scales. Let  $A > 0$  and a sequence  $\lambda_n \subset [0, R/A]$  be given. We now prove conclusion (1), that is

$$(3.9) \quad \sup_{t \in [-A\lambda_n, A\lambda_n]} \|\vec{b}_n(t)\|_{H \times L^2([\lambda_n/A, A\lambda_n])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

This means that  $b_n$  has no profile in the cone  $\mathcal{C}_A$ .

First let us prove

$$(3.10) \quad \|b_n(0)\|_{L^\infty([0, R])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

From assumption (4), for all  $A > 0$ ,

$$\|\vec{b}_n(0)\|_{H \times L^2(R/A \leq r \leq R)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

And it follows that from *Step 1* and an easy induction that for all  $j = 1, \dots, J$ , and  $A > 0$ ,

$$\|\vec{b}_n(0)\|_{H \times L^2(\lambda_{j,n}/A \leq r \leq A\lambda_{j,n})} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Hence due to Lemma A.2,

$$\|b_n(0)\|_{L^\infty(R/A \leq r \leq R)} + \|b_n(0)\|_{L^\infty(r_{j,n}/A \leq r \leq A\lambda_{j,n})} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

We now argue by contradiction. Let  $r_n \in [0, R]$  be such that  $\limsup_n |b_n(0, r_n)| > 0$ . From convergence on the scales  $\lambda_{j,n}$  of  $b_n(0)$ , we have that

$$\forall j = 0, \dots, J, \quad \frac{r_{\rho(n)}}{\lambda_{j,\rho(n)}} + \frac{\lambda_{j,\rho(n)}}{r_{\rho(n)}} \rightarrow +\infty.$$

We can assume that for some extraction  $\rho$ , and for some  $j_0 \in \llbracket 0, J \rrbracket$ ,

$$\lambda_{j_0+1,n} \ll r_{\rho(n)} \ll \lambda_{j_0,n}.$$

For  $j \leq j_0$ ,  $Q_j(r_{\rho(n)}/\lambda_{j,n}) \rightarrow Q_j(0) = Q_{j+1}(\infty)$  and for  $j > j_0$ ,  $Q_j(r_{\rho(n)}/\lambda_{j,n}) \rightarrow Q_j(\infty) = Q_{j-1}(0)$ , hence

$$\sum_{j=1}^J Q_j(r_{\rho(n)}/\lambda_{j,n}) - Q_j(\infty) \rightarrow \sum_{j=1}^{j_0} Q_{j+1}(\infty) - Q_j(\infty) = Q_{j_0}(0) - Q_1(\infty).$$

Now, as  $Q_1(\infty) = \ell$ , we deduce that

$$b_n(0, r_{\rho(n)}) = \psi_{\sigma \circ \rho(n)}(r_n) - Q_{j_0}(0) + o_n(1).$$

Up to extracting further we can assume, (recall  $b_n(0)$  is continuous and  $b_n(0, R) \rightarrow 0$ ),

$$b_{\rho(n)}(0, r_{\rho(n)}) = \varepsilon,$$

where  $\varepsilon \neq 0$  is small so that we also have  $g(\varepsilon + Q_{j_0}(0)) \neq 0$ . It follows that

$$\psi_{\sigma \circ \rho(n)}(0, r_n) = b_{\rho(n)}(0, r_{\rho(n)}) + Q_{j_0}(0) + o_n(1) \rightarrow \varepsilon + Q_{j_0}(0).$$

Arguing as in *Step 1*, (and relying on Proposition 3.1), we deduce that there exist a harmonic map  $Q$  such that  $Q(1) = Q_{j_0}(0) + \varepsilon$  (in particular,  $Q$  is not constant) and an extraction  $\varpi$  such that

$$\forall A > 0, \quad \|\psi_{\sigma \circ \rho \circ \varpi(n)}(0) - Q\|_{H \times L^2([r_{\rho \circ \varpi(n)}/A, Ar_{\rho \circ \varpi(n)}])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

But then convergence also holds in  $L^\infty$  and as  $\|g(Q)\|_{L^\infty} \geq 2\delta_0$ , and we deduce

$$\liminf_{n \rightarrow +\infty} \|g(\psi_{\sigma(n)}(0))\|_{L^\infty([\lambda_{j_0+1,n}, \lambda_{j_0,n}])} \geq 2\delta_0,$$

and we reached a contradiction with (3.7). This proves that for all sequences  $(r_n)_n \subset [0, R]$ ,  $\lim_n b_n(0, r_n) = 0$ , and hence

$$\|b_n(0)\|_{L^\infty([0, R])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

which is (3.10).

We now prove (3.9) arguing by contradiction. Up to extracting a subsequence, we can assume without loss of generality the existence of  $A > 0$  and of a sequences  $t_n$  such that  $0 \leq |t_n| \leq A\lambda_n \leq AR$ , and for some  $\rho \geq 0$  and  $\varepsilon > 0$

$$\|b_n(t_n)\|_{H \times L^2(\lambda_n/A \leq r \leq A\lambda_n)} \geq \varepsilon, \quad \text{and} \quad \frac{t_n}{\lambda_n} \rightarrow \rho.$$

Also (2) and (4) show that

$$t_n, \lambda_n \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Similarly, due to (3.5), for all  $j = 0, \dots, J$ ,

$$\frac{\lambda_n}{\lambda_{j,n}} + \frac{\lambda_{j,n}}{\lambda_n} \rightarrow \infty.$$

Up to extracting further we can assume that there is  $j_0 \in \llbracket 0, J \rrbracket$  such that for all  $n$ ,

$$\lambda_{j_0+1,n} \ll \lambda_n \ll \lambda_{j_0,n}.$$

Now consider the wave map

$$\phi_n(t, r) = \psi_{\sigma(n)}(\lambda_n t, \lambda_n r).$$

By inspection,  $\phi_n$  is a finite energy wave map defined for times  $t \in [-A, A]$ , and for  $n$  large enough,

$$(3.11) \quad \left\| \vec{\phi}_n \left( \frac{t_n}{\lambda_n} \right) - (Q_{j_0}(0), 0) \right\|_{H \times L^2(1/A \leq r \leq A)} \geq \varepsilon/2,$$

and

$$\frac{1}{A} \int_{-A}^A \int_0^{R/\lambda_n} |\partial_t \phi_n(t, r)|^2 r dr dt = \frac{1}{A\lambda_n} \int_{-A\lambda_n}^{A\lambda_n} \int_0^R |\partial_t \psi_{\sigma(n)}(t, r)|^2 r dr dt \rightarrow 0.$$

Hence for some harmonic map  $Q$ ,  $\phi_n \rightarrow Q$  in the sense of Proposition 3.1.  $Q$  is not constant due to (3.11). Unscaling, we have

$$\forall R, \quad \sup_{t \in [-A\lambda_n, A\lambda_n]} \|\vec{b}_n(t) - (Q(\cdot/\lambda_n), 0)\|_{H \times L^2([\lambda_n/A, A\lambda_n])} \rightarrow 0$$

But then, for any  $t \in [-A\lambda_n, A\lambda_n]$ ,  $b_n(t, \lambda_n) \rightarrow Q(1) \neq 0$  as  $n \rightarrow +\infty$ : this contradicts (3.10). Hence (3.9) is proved.

Notice that it follows immediately from (3.9) and Lemma A.2 that

$$(3.12) \quad \sup\{|b_n(t, r)| \mid 0 \leq r \leq R, 0 \leq |t| \leq \min\{1, Ar\}\} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

As  $\|b_n(0)\|_{L^\infty([0,R])} \rightarrow 0$ , and  $\psi_{\sigma(n)}(0) \in \mathcal{V}$ , we see that for  $n$  large enough,

$$\psi_{\sigma(n)}(0) = Q_J(0) =: l.$$

Up to dropping the first terms of the sequence, we can assume that this holds for all  $n$ .

Then (3.8) can be rewritten

$$(3.13) \quad \vec{b}_n(t) = \vec{\psi}_n(t) - (l, 0) - \sum_{j=1}^J (Q_j(\cdot/\lambda_{j,n}) - Q_j(0), 0).$$

*Step 3:* (Convergence up to the last scale) We do the proof for  $J \geq 1$ , the proof in the case  $J = 0$  being completely similar. Fix  $A > 0$ , we now prove that

$$(3.14) \quad \sup_{t \in [-A\lambda_{J,n}, A\lambda_{J,n}]} \|\vec{b}_n(t)\|_{H \times L^2([0, A\lambda_{J,n}])} \rightarrow 0.$$

From *Step 2*, we know that there exists  $\alpha_n \uparrow +\infty$  such that

$$\sup_{t \in [-\alpha_n \lambda_{J,n}, \alpha_n \lambda_{J,n}]} \|\vec{b}_n(t)\|_{H \times L^2([\lambda_{J,n}/\alpha_n, \alpha_n \lambda_{J,n}])} \rightarrow 0.$$

Let us first prove that

$$(3.15) \quad E(\vec{\psi}_{\sigma(n)}(0); 0; \alpha_n \lambda_{J,n}) \rightarrow 0.$$

We argue by contradiction. If the above convergence does not hold, there exists  $\varepsilon > 0$  and a subsequence that we still denote  $\vec{\psi}_{\sigma(n)}$  such that for some  $\mu_n \leq \lambda_{J,n}/\alpha_n$ , we have

$$E(\vec{\psi}_{\sigma(n)}(0); 0; 2\mu_n) = \varepsilon > 0.$$

By decreasing  $\varepsilon > 0$  if necessary, we can furthermore assume that

$$(3.16) \quad \forall x \in \mathbb{R}, \quad |G(x) - G(l)| \leq \varepsilon/2 \implies |f(x) - f'(l)(x-l)| \leq \frac{f'(l)|x-l|}{2}.$$

(we recall that  $l := Q_J(0) \in \mathcal{V}$ , so that  $f(l) = 0$  and  $f'(l) = g'(l)^2 > 0$ ). By monotonicity of the energy, we see that for  $t \in [-\mu_n, \mu_n]$ ,

$$E(\vec{\psi}_{\sigma(n)}(t); 0; \mu_n) \leq \varepsilon \quad \text{and} \quad E(\vec{\psi}_{\sigma(n)}(t); 0; 3\mu_n) \geq \varepsilon.$$

Let

$$\vec{u}_n(t, r) = (\psi_{\sigma(n)}(\mu_n t, \mu_n r), \mu_n \partial_t \psi_{\sigma(n)}(\mu_n t, \mu_n r)).$$

Then  $\vec{u}_n(t, r)$  is a wave map defined on the time interval  $[-1, 1]$ , and for all  $t \in [-1, 1]$ ,

$$(3.17) \quad E(\vec{u}_n(t); 0; 1) \leq \varepsilon \quad \text{and} \quad E(\vec{u}_n(t); 0; 3) \geq \varepsilon.$$

The definition of  $\varepsilon$  shows that for all  $t \in [-1, 1]$ ,  $r \in [0, 1]$

$$(3.18) \quad |f(u_n(t, r)) - f'(l)(u_n(t, r) - l)| \leq \frac{f'(l)|u_n(t, r) - l|}{2}.$$

Also the condition (3) yields

$$(3.19) \quad \int_{-1}^1 \int_0^{R/\mu_n} |\partial_t u_n(t, r)|^2 r dr dt \rightarrow 0.$$

This allows to apply Proposition 3.1. As  $\|u_n(0) - Q_J(0)\|_{L^\infty([0, \lambda_{J,n}/(\alpha_n \mu_n)])} \rightarrow 0$ , the local limit is constant, i.e. for all  $B > 0$ ,

$$(3.20) \quad \sup_{t \in [-1, 1]} \|\vec{u}_n(t) - (l, 0)\|_{H \times L^2([1/B, B])} \rightarrow 0.$$



Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be an even cutoff function such that  $\varphi(x) = 1$  for  $|x| \leq 1/2$  and  $\varphi(x) = 0$  for  $|x| \geq 1$ . We can compute

$$\begin{aligned}
& \iint |\partial_r u_n(t, r)|^2 \varphi(r) \varphi(t) r dr dt \\
&= - \iint (u_n(t, r) - l) \partial_{rr} u_n(t, r) \varphi(r) \varphi(t) r dr dt \\
&\quad - \iint (u_n(t, r) - l) \partial_r u_n(t, r) (r \varphi'(r) + \varphi(r)) \varphi(t) dr dt \\
&= - \iint (u_n(t, r) - l) \partial_{tt} u_n(t, r) \varphi(r) \varphi(t) r dr dt \\
&\quad - \iint \frac{(u_n(t, r) - l) f(u_n(t, r))}{r^2} \varphi(r) \varphi(t) r dr dt \\
&\quad - \iint (u_n(t, r) - l) \partial_r u_n(t, r) \varphi'(r) \varphi(t) r dr dt \\
&= \int |\partial_t u_n(t, r)|^2 \varphi(r) \varphi(t) r dr dt - \iint \frac{(u_n(t, r) - l) f(u_n(t, r))}{r^2} \varphi(r) \varphi(t) r dr dt \\
&\quad + \iint (u_n(t, r) - l) \partial_t u_n(t, r) \varphi(r) \varphi'(t) r dr dt \\
&\quad - \iint (u_n(t, r) - l) \partial_r u_n(t, r) \varphi'(r) \varphi(t) r dr dt
\end{aligned}$$

Now from estimate (3.19),

$$\int |\partial_t u_n(t, r)|^2 \varphi(r) \varphi(t) r dr dt = o(1).$$

Combining (3.19) with the Cauchy-Schwarz inequality, we also have

$$\iint (u_n(t, r) - l) \partial_t u_n(t, r) \varphi(r) \varphi'(t) r dr dt = o(1).$$

Also, as  $\varphi'$  has support on  $[1/2, 1]$ , estimate (3.20) and the Cauchy Schwarz inequality, we have

$$\iint (u_n(t, r) - l) \partial_r u_n(t, r) \varphi'(r) \varphi(t) r dr dt = o(1).$$

We now use that  $f'(l) = g'(l)^2 > 0$ . Then it follows from (3.18) that

$$\begin{aligned}
& \iint \frac{(u_n(t, r) - l) f(u_n(t, r))}{r^2} \varphi(r) \varphi(t) r dr dt \\
&\geq \frac{f'(l)}{2} \iint \frac{|u_n(t, r) - l|^2}{r^2} \varphi(r) \varphi(t) r dr dt.
\end{aligned}$$

Hence

$$\begin{aligned}
0 &\leq \iint |\partial_r u_n(t, r)|^2 \varphi(r) \varphi(t) r dr dt \\
&\leq -\frac{f'(l)}{2} \iint \frac{|u_n(t, r) - l|^2}{r^2} \varphi(r) \varphi(t) r dr dt + o(1).
\end{aligned}$$

From this we deduce first that

$$\iint \frac{|u_n(t, r) - l|^2}{r^2} \varphi(r) \varphi(t) r dr dt \rightarrow 0,$$

then

$$\iint |\partial_r u_n(t, r)|^2 \varphi(r) \varphi(t) r dr dt \rightarrow 0.$$

Adding up the last 2 results along with (3.19), we get

$$\int_{-1/2}^{1/2} \|\vec{u}_n(t) - (l, 0)\|_{H \times L^2(r \leq 1/2)}^2 dt \rightarrow 0.$$

Now recalling (3.20), we get

$$\forall B > 0, \quad \int_{-1/2}^{1/2} \|\vec{u}_n(t) - (l, 0)\|_{H \times L^2(r \leq B)}^2 dt \rightarrow 0.$$

This shows that

$$\forall B > 0, \quad \int_{-1/2}^{1/2} E(\vec{u}_n(t); 0, B) dt \rightarrow 0.$$

However, this contradicts the second estimate in (3.17), and from there, estimate (3.15) holds true.

From (3.15) it is now easy to prove (3.14). Let  $A > 0$ . For  $n$  large enough,  $\alpha_n \geq 2A$ . By finite speed of propagation, we deduce that

$$\sup_{t \in [-A\lambda_{j,n}, A\lambda_{j,n}]} E(\vec{\psi}_{\sigma(n)}(t); 0, A\lambda_{j,n}) \rightarrow 0.$$

By coercivity of the energy around  $l = \psi_{\sigma(n)}(0)$ , we deduce

$$\sup_{t \in [-A\lambda_{j,n}, A\lambda_{j,n}]} \|\vec{\psi}_{\sigma(n)}(t) - (l, 0)\|_{H \times L^2([0, A\lambda_{j,n}])} \rightarrow 0.$$

Notice that for all  $1 \leq j < J$ , as  $\lambda_{j,n} \ll \lambda_{j,n}$ , we have

$$\|(Q_j(\cdot/\lambda_{j,n}) - Q_j(0), 0)\|_{H \times L^2([0, A\lambda_{j,n}])} \rightarrow 0.$$

The last two statements and (3.13) yield (3.14).

In the case  $J = 0$ , the same proof shows that

$$\sup_{t \in [-1/2, 1/2]} \|b_n(t)\|_{H \times L^2([0, 1/2])} \rightarrow 0.$$

Now (4') reads: for all  $r \in (0, R)$ ,

$$\sup_{t \in [-1, 1]} \|b_n(t)\|_{H \times L^2([r, R])} \rightarrow 0.$$

We add up these last two statement to conclude the case  $J = 0$ .  $\square$

#### 4. SCATTERING FOR WAVE MAPS BELOW THE $L^\infty$ THRESHOLD

*Proof of Theorem 1.2.* Let  $\vec{\psi}$  be a finite energy wave map, with  $\psi(\infty) = \ell$  and such that it satisfies (1.5). Notice that  $\psi(0) = \ell$  and there exists  $c > 0$  (depending only on  $\delta_0$  and  $\ell$ ) such that

$$(4.1) \quad \forall t \in [0, T^+(\vec{\psi})], \forall r \geq 0, \quad c|\psi(t, r) - \ell| \leq |g(\psi(t, r))| \leq \frac{1}{c}|\psi(t, r) - \ell|.$$

From this point wise bound, we derive that

$$(4.2) \quad \forall t \in [0, T^+(\vec{\psi})], \quad \|\vec{\psi}(t) - (\ell, 0)\|_{H \times L^2} \leq CE(\vec{\psi}).$$

Also notice that  $T^+(\vec{\psi}) = +\infty$ . Assume indeed for the sake of contradiction that  $T^+(\vec{\psi}) < \infty$ . Due to [25], a bubble would form: hence for a sequence of time  $t_n \uparrow T^+(\vec{\psi})$ , and of points  $r_n$ ,  $\psi(t_n, r_n) \rightarrow k$  where  $k$  is such that  $\ell, k$  are two consecutive elements of  $V$ . Thus  $\liminf_n \|\psi(t_n) - \psi(\infty)\|_{L^\infty} \geq |k - \ell| \geq d_\ell > c$ , a contradiction.

We now do an induction on the energy in the spirit of the Kenig Merle concentration compactness argument [15, 16]. Define  $E_c$  to be the supremum of all  $E \geq 0$  such

that all wave maps  $\vec{\psi}$  of energy  $E(\vec{\psi}) \leq E$ , which satisfies  $\psi(\infty) = \ell$  and (1.5), are global and scatters.

Then [5, Theorem 2] shows that  $E_c > 0$  (recall  $|g'(\ell)| \in \{1, 2\}$ ). We now argue by contradiction and assume that  $E_c$  is finite.

*Step 1.* We first construct a critical element, that is a wave map  $\vec{V}$  defined on  $[0, +\infty)$ , that satisfies (1.5), but  $\|V - \ell\|_{S_\ell([0, +\infty))} = +\infty$ .

Let  $\vec{\psi}_n$  be a minimizing sequence of wave maps, i.e.  $\vec{\psi}_n$  satisfies (1.5) (hence  $T^+(\vec{\psi}_n) = +\infty$ ), and

$$E(\vec{\psi}_n) \leq E_c + \frac{1}{n}, \quad \text{and} \quad \|\psi_n - \ell\|_{S_\ell([0, +\infty))} = +\infty.$$

Up to rescaling, we can also assume that for all  $n$

$$E(\vec{\psi}_n(0); 1, +\infty) = E_c/100.$$

Choose a sequence of time  $t_n$  with vanishing  $L^2$  norm of  $\partial_t \psi_n(t_n)$ . More precisely, we claim that for all  $n$ , there exist a sequence of times  $t_{n,m}$  such that

- (1)  $\sup_{0 < s \leq t_{n,m}/2} \frac{1}{s} \int_{t_{n,m}-s}^{t_{n,m}+s} \int_0^{t_{n,m}/4} |\partial_t \psi_n(t, r)|^2 r dr dt \leq 1/m.$
- (2)  $E(\vec{\psi}_n(t_{n,m}); t_{n,m}/m; t_{n,m}(1 - 1/m)) \leq \frac{1}{m},$
- (3)  $\|\psi_n(t_{n,m}) - \ell\|_{L^\infty(r \geq t_{n,m}/m)} \leq \frac{1}{m},$

Indeed Corollary 2.3 provides us with a sequence satisfying the first condition, and then, up to extracting, Proposition 2.1 and Corollary 2.4 allows to satisfy the second and third condition (we emphasize that these last two results hold for any sequence).

Now choose  $t_n = t_{n,n}$ . Then Theorem 3.3 applies to the sequence of wave maps  $\vec{\varphi}_n(t, r) = \vec{\psi}_n(t_n + t_n t, t_n r)$ , with  $R = 1/2$ : indeed, we have by scaling

$$\sup_{0 < \lambda \leq 1} \frac{1}{\lambda} \int_{-\lambda}^{\lambda} \int_0^{1/2} |\partial_t \varphi_n(t, r)|^2 r dr dt \rightarrow 0,$$

and for  $t \in [-1, 1]$ , and  $r > 0$ , we have due to our second condition

$$E(\vec{\varphi}_n(t); r, 1/2) = E(\vec{\psi}_n(t_n); r t_n, t_n/2) \rightarrow 0.$$

Also notice that due to finite speed of propagation, we have

$$(4.3) \quad \forall c > 1, \quad \|\vec{\psi}_n(t_n) - (\ell, 0)\|_{H \times L^2(r \geq c t_n)} \rightarrow 0.$$

We can therefore apply Theorem 3.3. It yields a bubble decomposition which is trivial: it can not contain any harmonic map profile  $Q_j$  due to (1.5), and we see that for all  $A > 0$  and sequence  $0 < \mu_n \ll t_n$ ,

$$(4.4) \quad \|\vec{\psi}_n(t_n) - (\ell, 0)\|_{H \times L^2([\mu_n/A, A\mu_n])} \rightarrow 0, \quad \text{and} \quad \|\psi_n(t_n) - \ell\|_{L^\infty(0, t_n/2)} \rightarrow 0.$$

Condition (2) then translate into absence of profile on scale 1:

$$(4.5) \quad \|\vec{\psi}_n(t_n) - (\ell, 0)\|_{H \times L^2([t_n/n, (1-1/n)t_n])} \rightarrow 0.$$

Using the third condition, we also get

$$(4.6) \quad \|\psi_n(t_n) - \ell\|_{L^\infty} \rightarrow 0.$$

In view of (4.2)  $\vec{\psi}_n(t_n) - (\ell, 0)$  is bounded in  $H \times L^2$ , hence admits (up to a subsequence) a profile decomposition in the sense of Theorem 2.12. Denote  $V_{j,L}$  the linear profiles, and  $(t_{j,n}, \lambda_{j,n})$  the parameters.

We claim that there are no nontrivial profiles  $V_j$  such that  $t_{j,n} = 0$ . Consider indeed such a profile  $V_j$ , for the sake of contradiction. Up to extracting, and changing scale by a fixed factor, it suffices to rule out three cases:

$$1) \lambda_{j,n} \ll t_n, \quad 2) \lambda_{j,n} = t_n, \quad 3) \lambda_{j,n} \gg t_n.$$

In case 1), from (4.4), for any  $A > 1$ , we have

$$\|\vec{\psi}_n(t_n) - (\ell, 0)\|_{H \times L^2(t\lambda_{j,n}/A \leq r \leq A\lambda_{j,n})} \rightarrow 0.$$

Recall the Pythagorean expansion with cut-off Proposition 2.13, it implies in particular that

$$\left\| \left( V_{j,L} \left( 0, \frac{r}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}} \partial_t V_{j,L} \left( 0, \frac{r}{\lambda_{j,n}} \right) \right) \right\|_{H \times L^2(\lambda_{j,n}/A \leq r \leq A\lambda_{j,n})} \rightarrow 0.$$

Unscaling, we get

$$\|\vec{V}_{j,L}(0)\|_{H \times L^2([1/A, A])} \rightarrow 0.$$

As this is true for all  $A > 1$ , we get  $\vec{V}_{j,L}(0) = 0$ .

In case 3), from (4.3) and Proposition 2.13, we similarly get with  $c = 2$

$$\left\| \left( V_{j,L} \left( 0, \frac{r}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}} \partial_t V_{j,L} \left( 0, \frac{r}{\lambda_{j,n}} \right) \right) \right\|_{H \times L^2(r \geq 2t_n)} \rightarrow 0.$$

Unscaling, we deduce

$$\|\vec{V}_{j,L}(0)\|_{H \times L^2(r \geq 2t_n/\lambda_{j,n})} \rightarrow 0.$$

As  $t_n/\lambda_{j,n} \rightarrow 0$ , we get  $\vec{V}_{j,L}(0) = 0$ .

In case 2), from (4.5) and Proposition 2.13, we get

$$\left\| \left( V_{j,L} \left( 0, \frac{r}{t_n} \right), \frac{1}{\lambda_{j,n}} \partial_t V_{j,L} \left( 0, \frac{r}{t_n} \right) \right) \right\|_{H \times L^2(t_n/n \leq r \leq t_n(1-1/n))} \rightarrow 0,$$

which after unsealing yields

$$\|\vec{V}_{j,L}(0)\|_{H \times L^2([1/n, 1-1/n])} \rightarrow 0.$$

Similarly, (4.3) and Proposition 2.13 give after unscaling

$$\|\vec{V}_{j,L}(0)\|_{H \times L^2([c, +\infty))} \rightarrow 0.$$

Hence  $\vec{V}_{j,L}(0) = 0$ .

We ruled out all three cases, and this establish our claim that there is no profile with  $t_{j,n} = 0$ .

Consider now the set  $\mathcal{J}$  of indices  $j$  such that  $\frac{t_{j,n}}{\lambda_{j,n}} \rightarrow +\infty$  and the nonlinear profile  $V_j$  does not scatter at  $+\infty$  (or blows up in finite time). Any of these profile have energy greater or equal to  $E_c > 0$ , hence (by Pythagorean expansion of the energy) there only is a finite number of them.

If there is no such profiles then for all  $j$ ,  $\|V_j\|_{S(t \geq -t_{j,n}/\lambda_{j,n})}$  is bounded (it tends to 0 if  $t_{j,n}/\lambda_{j,n} \rightarrow -\infty$ ). Then Proposition 2.15 shows that there is a uniform bound  $M$  such that, for all sequence  $\tau_n \geq t_n$ ,

$$\|\psi_n - \ell\|_{S_\ell([t_n, \tau_n])} \leq M.$$

This in turn implies that  $\vec{\psi}_n$  scatters at  $+\infty$ : it is a contradiction. Hence  $\mathcal{J} \neq \emptyset$ .

We can assume without loss of generality that  $\mathcal{J}$  is indexed by  $1, \dots, J_0$ .

Among such  $j \leq J_0$ , choose  $\lambda_{j,n}$  slowest, then among such  $j$ , we consider  $j_0$  such that  $t_{j_0,n}$  is lowest, i.e

$$\forall j \leq J_0, \quad \frac{\lambda_{j,n}}{\lambda_{j_0,n}} \rightarrow +\infty \quad \text{or} \quad \left( \lambda_{j,n} = \lambda_{j_0,n} \text{ and } \frac{t_{j_0,n} - t_{j,n}}{\lambda_{j_0,n}} \rightarrow -\infty \right).$$

$\vec{V} := \vec{V}_{j_0}$  will be our critical element. First let us show that  $\vec{V}$  is global and satisfies (1.5). Let  $t \in [0, T^+(\vec{V}))$  and  $r \geq 0$ . Define  $\tau_n = t_{j_0, n} + \lambda_{j_0, n} t$ . Then

$$\|V_{j_0, L}\|_{S_\ell\left(\left[-\frac{t_{j_0, n}}{\lambda_{j_0, n}}, \frac{\tau_n - t_{j_0, n}}{\lambda_{j_0, n}}\right]\right)} \rightarrow \|V_{j_0, L}\|_{S_\ell((-\infty, t])} < +\infty.$$

By construction, for  $j \in \mathcal{J} \setminus \{j_0\}$ ,

$$\frac{\tau_n - t_{j, n}}{\lambda_{j, n}} \rightarrow -\infty \quad \text{hence} \quad \|V_{j, L}\|_{S_\ell\left(\left[-\frac{t_{j, n}}{\lambda_{j, n}}, \frac{\tau_n - t_{j, n}}{\lambda_{j, n}}\right]\right)} \rightarrow 0.$$

For  $j \notin \mathcal{J}$ , notice that  $\tau_n \geq 0$  for  $n$  large enough ( $t_n/\lambda_{j, n} \rightarrow +\infty$ ) and

- if  $\frac{t_{j, n}}{\lambda_{j, n}} \rightarrow +\infty$ ,  $\|V_{j, L}\|_{S_\ell(\mathbb{R})} < +\infty$ , and
- if  $\frac{t_{j, n}}{\lambda_{j, n}} \rightarrow -\infty$ ,  $\|V_{j, L}\|_{S_\ell\left(\left[-\frac{t_{j, n}}{\lambda_{j, n}}, +\infty\right)\right)} \rightarrow 0$ .

Hence Proposition 2.15 applies, we can evolve the profile decomposition up to  $\tau_n$ :

$$\begin{aligned} \psi_n(\tau_n, \lambda_{j_0, n} r) - \ell &= V(t, r) - \ell + \sum_{j \neq j_0, j \leq J} \left( V_j \left( \frac{\tau_n - t_{j, n}}{\lambda_{j, n}}, \frac{\lambda_{j_0, n}}{\lambda_{j, n}} r \right) - \ell \right) \\ &\quad + \gamma_{J, n}(\tau_n, \lambda_{j_0, n} r) + r_{J, n}(\tau_n, \lambda_{j_0, n} r). \end{aligned}$$

By inspection (arguing as in the proof of equation (2.5)), we deduce that

$$|V(t, r) - \ell| \leq c.$$

It follows that  $\vec{V}_j$  satisfies (1.5); as noticed above, we then have  $T^+(\vec{V}) = +\infty$ . Also, due to the Pythagorean expansion of the energy, we see that

$$E(\vec{V}) \leq E_c.$$

As we chose it so that it does not scatter at  $+\infty$ , we must have  $E(\vec{V}) = E_c$  by definition of the critical energy: therefore  $\vec{V}$  is a critical element.

*Step 2.* We reach a contradiction.

For this, we can repeat the argument of *Step 1* on  $\vec{V}$ : there exist a sequence of times  $t_n \uparrow +\infty$  such that

- (1)  $\sup_{0 < s \leq t_n/2} \frac{1}{s} \int_{t_n-s}^{t_n+s} \int_0^{t_n/4} |\partial_t V(t, r)|^2 r dr dt \rightarrow 0$  as  $n \rightarrow +\infty$ ,
- (2)  $\|V(t_n) - \ell\|_{L^\infty} \rightarrow 0$ ,
- (3)  $\vec{V}(t_n) - (\ell, 0)$  admits a profile decomposition, with profiles  $\vec{U}_{j, L}$ .

Also, as the bubble decomposition yield no bubble, convergence up to the last scale shows that

$$(4.7) \quad \|\vec{V}(t_n) - (\ell, 0)\|_{H \times L^2([0, t_n/2])} \rightarrow 0.$$

Arguing as in *Step 1*, we see that one of the nonlinear profiles  $\vec{U}_{j_0}$  is critical, in particular  $E(\vec{U}_{j_0}) = E_c = E(\vec{V})$ . By Pythagorean expansion of the energy, it follows that there are no other nontrivial linear profiles in the profile decomposition of  $\vec{V}(t_n)$ , and that the dispersion term tends to 0 in  $H \times L^2$ . In short, there holds

$$\vec{V}(t_n, r) - (\ell, 0) = \left( U_{j_0, L} \left( -\frac{t_{j_0, n}}{\lambda_{j_0, n}}, \frac{r}{\lambda_{j_0, n}} \right), \frac{1}{\lambda_{j_0, n}} \partial_t U_{j_0, L} \left( -\frac{t_{j_0, n}}{\lambda_{j_0, n}}, \frac{r}{\lambda_{j_0, n}} \right) \right) + o_n(1),$$

where the  $o_n(1)$  is in  $H \times L^2$ .

Assume  $t_{j_0, n} = 0$ . Observe that  $\vec{V}$  has energy on the light cone: from monotonicity of the energy along light cones, Proposition 2.1 and (4.7), we have

$$\limsup_{n \rightarrow +\infty} \|\vec{V}(t_n) - (\ell, 0)\|_{H \times L^2(|r-t_n| \geq A)} \rightarrow 0 \quad \text{as} \quad A \rightarrow +\infty.$$

In particular,

$$\|\vec{V}(t_n) - (\ell, 0)\|_{H \times L^2(|r-t_n| \geq t_n/2)} \rightarrow 0.$$

Now, as  $\vec{U}_{j_0, L}(0) \neq 0$ , let  $\rho > 0$  such that  $\|\vec{U}_{j_0, L}(0)\|_{H \times L^2([\rho, 2\rho])} =: \alpha_0 > 0$ . Then

$$\left\| \left( U_{j_0, L} \left( 0, \frac{r}{\lambda_{j_0, n}} \right), \frac{1}{\lambda_{j_0, n}} \partial_t U_{j_0, L} \left( 0, \frac{r}{\lambda_{j_0, n}} \right) \right) \right\|_{H \times L^2([\rho \lambda_{j_0, n}, 2\rho \lambda_{j_0, n}])} = \alpha_0 > 0.$$

Comparing with  $\vec{V}(t_n)$  we must have for  $n$  large enough

$$2\rho \lambda_{j_0, n} \geq t_n/2 \quad \text{and} \quad \rho \lambda_{j_0, n} \leq 3t_n/2.$$

Up to extracting, we can furthermore assume that  $\lambda_{j_0, n}/t_n \rightarrow \lambda \in (0, +\infty)$ . But then, unscaling the concentration on  $\vec{V}(t_n)$  we have

$$\limsup_{n \rightarrow +\infty} \|\vec{U}_{j_0, L}(0)\|_{H \times L^2(|r-t_n|/\lambda_{j_0, n} \geq A/\lambda_{j_0, n})} \rightarrow 0 \quad \text{as} \quad A \rightarrow +\infty.$$

But this implies  $\vec{U}_{j_0, L}(0) = 0$ , a contradiction.

Assume now that  $\frac{t_{j, n}}{\lambda_{j, n}} \rightarrow +\infty$ . Then  $\|U_{j_0, L}\|_{S_\ell((-\infty, -t_{j, n}/\lambda_{j, n}])} \rightarrow 0$  so the same holds for the non linear profile:  $\|U_{j_0}\|_{S_\ell((-\infty, -t_{j, n}/\lambda_{j, n}])} \rightarrow 0$ . Then applying Proposition 2.15 backward in time up to time  $t = 0 \leq t_n$ , we get

$$\|V - \ell\|_{S_\ell(0, t_n)} \leq \|U_{j_0} - \ell\|_{S_\ell((-\infty, -t_{j, n}/\lambda_{j, n}])} + o(1) \rightarrow 0.$$

Hence by monotone convergence, we deduce  $\|U\|_{S_\ell([0, +\infty))} = 0$ , and  $V = 0$ , a contradiction.

Assume finally that  $\frac{t_{j, n}}{\lambda_{j, n}} \rightarrow -\infty$ . Then  $\|U_{j_0, L}\|_{S_\ell([-t_{j, n}/\lambda_{j, n}, +\infty))} \rightarrow 0$  so the same holds for the non linear profile:  $\|U_{j_0} - \ell\|_{S_\ell([-t_{j, n}/\lambda_{j, n}, +\infty))} \rightarrow 0$ . Then we can use Proposition 2.15 to get that for any sequence  $\tau_n \geq t_n$ ,

$$\|V - \ell\|_{S_\ell([t_n, \tau_n])} \rightarrow 0.$$

This implies  $\|V - \ell\|_{S_\ell([t_n, +\infty))} \rightarrow 0$ : in particular  $V$  scatters at  $+\infty$ , a contradiction.

We reached a contradiction in all cases, hence  $E_c = +\infty$ .  $\square$

## 5. OUTSIDE THE LIGHT CONE

**Proposition 5.1** (Scattering state). *We assume (A1)-(A2)-(A3').*

*Let  $\vec{\psi}$  be a finite energy wave map such that  $T^+(\vec{\psi}) = +\infty$ . Denote  $\ell = \psi(\infty)$ . There exist a map  $\vec{\phi}_L$  solution to linear problem (LW $_\ell$ ) and an increasing non-negative continuous function  $\alpha(t)$  such that  $\alpha(t) = o(t)$  and*

$$\|\vec{\psi}(t) - (\ell, 0) - \vec{\phi}_L(t)\|_{H \times L^2(r \geq \alpha(t))} \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

*Proof.* First we recall Proposition 2.1, hence it suffices to construct  $\vec{\phi}_L$  such that for all  $A \geq 0$ ,

$$(5.1) \quad \|\vec{\psi}(t) - (\ell, 0) - \vec{\phi}_L(t)\|_{H \times L^2(r \geq t-A)} \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

The proof follows the scheme of [7, Proposition 2.8], except now it is more involved to obtain scattering for the approximations of  $\psi$  around the light cone. We crucially rely on our new scattering result Theorem 1.2.

Let  $t_n \uparrow +\infty$  and define the sequence of wave maps  $\vec{\phi}_n$  with data at time  $t_n$  as a suitable extension in  $\mathcal{H} \times L^2$  of  $\vec{\psi}(t_n)|_{r \geq t_n/2}$ , as in Lemma A.3. Specifically let  $\vec{\phi}_n(t_n) = (\phi_{n,0}, \phi_{n,1})$  where

$$\phi_{n,0}(r) = \begin{cases} 2 \frac{\psi(t_n, t_n/2) - \ell}{t_n} r + \ell & \text{if } r \leq t_n/2, \\ \psi(t_n, r) & \text{if } r \geq t_n/2, \end{cases}$$

$$\phi_{n,1}(r) = \partial_t \psi(t_n, r).$$

(Recall that  $\ell = \psi(\infty)$ ). Then  $\vec{\phi}_n(0) = \vec{\phi}_n(\infty) = \ell$  and from Lemma A.3,

$$(5.2) \quad \|\vec{\phi}_n(t_n) - (\ell, 0)\|_{H \times L^2(r \leq t_n/2)} \rightarrow 0.$$

By construction  $\vec{\psi}$  and  $\vec{\phi}_n$  coincide at time  $t_n$  on  $[t_n/2, +\infty)$ , hence by finite speed of propagation, as long as they are defined at time  $t$ ,

$$(5.3) \quad \forall r \geq t_n/2 + |t - t_n|, \quad \vec{\phi}_n(t, r) = \vec{\psi}(t, r).$$

*Step 1:* Let us show that for  $n$  large enough,

- (1)  $\vec{\phi}_n$  is defined on  $[A_0, +\infty)$  for some  $A_0$  not depending on  $n$ .
- (2)  $\vec{\phi}_n$  scatters at  $+\infty$ .

Choose  $\lambda = 1/2$  in Proposition 2.1, and apply it to  $\vec{\psi}$ :

$$\limsup_{t \rightarrow +\infty} E(\vec{\psi}(t); t/2, t - A) \rightarrow 0 \quad \text{as } A \rightarrow +\infty.$$

We then deduce that

$$\limsup_n E(\vec{\phi}_n(t_n); 0, t_n - A) \rightarrow 0 \quad \text{as } A \rightarrow +\infty.$$

Let  $E_\ell > 0$  be the minimal energy of a non constant harmonic map  $Q$  with  $Q(\infty) = \ell$  (or if  $\ell_\pm \in \mathcal{V} \setminus \{\ell\}$  are the closest elements to  $\ell$ , with  $\ell_- < \ell < \ell_+$ ,  $E_\ell = 2 \min\{G(\ell_+) - G(\ell), G(\ell) - G(\ell_-)\}$ ). Choose  $A_0$  large enough, so that for  $n$  large enough,

$$E(\vec{\psi}(t_n); 0, t_n - A_0) \leq E_\ell/2,$$

By finite speed of propagation, we deduce that for all  $\tau$  such that  $\vec{\phi}_n(t+\tau)$  is defined

$$E(\vec{\phi}_n(t_n + \tau); 0, t_n - A_0 - |\tau|) \leq E(\vec{\psi}(t_n); 0, t_n - A_0) \leq E_\ell/2.$$

We recall the blowup criterion derived in [25]: blow up concentrates in the light cone an energy at least  $E_\ell$ . Hence for  $|\tau| \leq t_n - A_0$ , this blow up criterion shows that  $\vec{\psi}_n(t_n + \tau)$  is well defined: it then suffices to choose  $\tau = A_0 - t_n$ , and  $T^-(\vec{\phi}_n) < A_0$ . Of course we can drop the first terms, so that is holds for all  $n$ .

We now turn to scattering at  $+\infty$ . For this, we will prove that as  $n \rightarrow +\infty$ ,

$$(5.4) \quad \sup_{t \in [t_n, T^+(\phi_n)]} \|\phi_n(t) - \ell\|_{L^\infty} \rightarrow 0.$$

By monotonicity of the energy outside cones, we see that for all  $A \in \mathbb{R}$ , the limit  $\lim_{t \rightarrow +\infty} E(\vec{\psi}(t); t - A, +\infty)$  exists, let us denote it  $\mathcal{E}(A)$ . As the energy density is non-negative,  $\mathcal{E}(A)$  is an increasing function of  $A$ , and is also bounded by  $E(\vec{\psi})$ . Denote  $\mathcal{E} = \lim_{A \rightarrow +\infty} \mathcal{E}(A) \leq E(\vec{\psi})$ . Let us show that

$$(5.5) \quad E(\vec{\psi}(t); t/2, +\infty) \rightarrow \mathcal{E}.$$

Indeed, let  $\varepsilon > 0$ , and choose  $A$  large so that

$$\mathcal{E} - \varepsilon \leq \mathcal{E}(A) \leq \mathcal{E} \quad \text{and} \quad \limsup_{t \rightarrow +\infty} E(\vec{\psi}(t); t/2, t - A) \leq \varepsilon$$

. There exist  $T$  large such that

$$\forall t \geq T, \quad \mathcal{E}(A) \leq E(\vec{\psi}(t); t - A, +\infty) \leq \mathcal{E}(A) + \varepsilon, \quad \text{and} \quad E(\vec{\psi}(t); t/2, t - A) \leq 2\varepsilon.$$

Then for all  $t \geq T$ , we have

$$\mathcal{E} - \varepsilon \leq \mathcal{E}(A) \leq E(\vec{\psi}(t); t - A, +\infty) \leq \mathcal{E}(A) + \varepsilon \leq \mathcal{E} + \varepsilon.$$

Hence for  $t \geq \max\{T, 2A\}$ ,

$$\mathcal{E} - \varepsilon \leq E(\vec{\psi}(t); t - A, +\infty) \leq E(\vec{\psi}(t); t/2, +\infty) \leq E(\vec{\psi}(t); t/2, t - A) + \mathcal{E} + \varepsilon \leq \mathcal{E} + 3\varepsilon.$$

(5.5) follows.

Now, from (5.5) and 2.1, we have

$$(5.6) \quad \limsup_{t \rightarrow +\infty} |E(\vec{\psi}(t); t - A, +\infty) - \mathcal{E}| \rightarrow 0 \quad \text{as } A \rightarrow +\infty.$$

From (5.5) and (5.2), we deduce that

$$(5.7) \quad E(\vec{\phi}_n(t_n)) \rightarrow \mathcal{E} \quad \text{as } n \rightarrow +\infty.$$

We now prove (5.4). Let  $\varepsilon > 0$ . From (5.6), there exist  $A_1$  and  $T_1$  such that

$$\forall t \geq T_1, \quad |E(\vec{\psi}(t); t - A_1, +\infty) - \mathcal{E}| \leq \varepsilon.$$

We also use Corollary 2.4. Let  $T_2$  such that

$$\forall t \geq T_2, \quad \|\psi(t) - \ell\|_{L^\infty(r \geq t/2)} \leq \varepsilon.$$

Define now  $N$  such that  $t_N \geq \max\{T_1, 2A_1, T_2\}$  and for  $n \geq N$ ,

$$|E(\vec{\phi}_n(t_n)) - \mathcal{E}| \leq \varepsilon.$$

Then for  $n \geq N$ , we have  $t_n \geq t_N$  so that if  $t \geq t_n$ , then  $t - A_1 \geq t - t_n/2 \geq t_n/2 + |t - t_n|$ . By (5.3), it transpires

$$\forall t \in [t_n, T^+(\phi_n)), \quad |E(\vec{\phi}_n(t); t - A_1, +\infty) - \mathcal{E}| \leq \varepsilon.$$

By conservation of the energy,

$$\forall t \in [t_n, T^+(\phi_n)), \quad E(\vec{\phi}_n(t); 0, t - A_1) \leq 2\varepsilon.$$

Due to the point wise bound, we get as  $\phi_n(0) = \ell$

$$\forall t \in [t_n, T^+(\phi_n)), \quad \|\phi_n(t) - \ell\|_{L^\infty(r \leq t - A_1)} \leq C\varepsilon.$$

Also if  $t \geq t_n$ , then  $t \geq T_2$  and again by (5.3) and

$$\forall r \geq t - A_1, \quad |\phi_n(t, r) - \ell| = |\psi(t, r) - \ell| \leq \varepsilon.$$

This proves that for  $n \geq N$ ,

$$\forall t \in [t_n, T^+(\vec{\phi}_n)), \quad \|\phi_n(t) - \ell\|_{L^\infty} \leq 2\varepsilon,$$

which is exactly (5.4).

Then Theorem 1.2 applies for all  $n \geq N$ , and shows that  $T^+(\vec{\phi}_n) = +\infty$  and  $\vec{\phi}_n$  scatters at  $+\infty$ . Also notice that (5.4) and (5.7) show that for some  $C > 0$  and for  $n$  large enough,

$$(5.8) \quad \sup_{t \geq t_n} \|\vec{\phi}_n(t) - (\ell, 0)\|_{H \times L^2} \leq C\varepsilon.$$

*Step 2: Construction of  $\vec{\phi}_L$  and end of proof.*

Let  $\vec{\phi}_{n,L}$  be the linear solution of  $(LW_\ell)$  which is the scattering state of  $\vec{\phi}_n$ , that is, for all  $n$ ,

$$(5.9) \quad \|\vec{\phi}_n(t) - (\ell, 0) - \vec{\phi}_{n,L}(t)\|_{H \times L^2} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Recall that the flow of  $(LW_\ell)$  preserves the  $H_\ell \times L^2$  norm. Along with the bound on  $\|\vec{\phi}_n(t)\|_{H \times L^2}$ , this shows that

$$\|\vec{\phi}_{n,L}(0)\|_{H \times L^2} \leq C\varepsilon.$$

Up to extracting, we can assume that  $\vec{\phi}_{n,L}(0)$  has a weak limit  $\vec{\phi}_L(0)$  in  $H \times L^2$ . We define  $\vec{\phi}_L(t)$  as the linear solution of  $(LW_\ell)$  with initial data  $\vec{\phi}_L(0)$  at time 0. Let  $\tau_n$  be such that

$$\|\vec{\phi}_n(\tau_n) - (\ell, 0) - \vec{\phi}_{n,L}(\tau_n)\|_{H \times L^2} \leq \frac{1}{n}.$$



Up to extracting further, we can assume that the sequence  $\vec{\phi}_n(\tau_n) - (\ell, 0) - \vec{\phi}_L(\tau_n)$ , which is bounded in  $H \times L^2$ , admits a profile decomposition in the sense of Theorem 2.12:

$$\begin{aligned} \vec{\phi}_n(\tau_n, r) - (\ell, 0) &= \vec{\phi}_L(\tau_n, r) \\ &+ \sum_{j=2}^J \left( \frac{1}{\lambda_{j,n}^{d-1}} V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}^d} \partial_t V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right) \right) + \gamma_{J,n}(0, r) \end{aligned}$$

Notice that this appears as a profile decomposition for the sequence  $(\vec{\phi}_{n,L}(\tau_n, r))$  with first profile  $\vec{\phi}_L$  and parameter  $t_{1,n} = \tau_n$ ,  $\lambda_{1,n} = 1$ . Indeed, the profile decomposition is constructed of via taking weak limits, that is:

$$\vec{V}_{j,L}(0) \text{ is the weak limit of } \Lambda[\lambda_{j,n}] S_\ell(t_{j,n})(\vec{\phi}_n(\tau_n) - (\ell, 0)),$$

where  $S(t)$  is the linear flow of  $(\text{LW}_\ell)$  and  $\Lambda$  is the scaling operator

$$\Lambda[\mu](\varphi_0, \varphi_1)(t, r) := \left( \varphi_0 \left( \frac{t}{\mu}, \frac{r}{\mu} \right), \frac{1}{\mu} \varphi_1 \left( \frac{t}{\mu}, \frac{r}{\mu} \right) \right).$$

Also  $\vec{\phi}_L(0)$  is the weak limit of  $S(-\tau_n)(\Lambda[1]\vec{\phi}_n(\tau_n) - (\ell, 0)) = S(-\tau_n)(\vec{\phi}_n(\tau_n) - (\ell, 0))$ . Indeed endow  $H \times L^2$  with the natural scalar product derived from the  $H_\ell \times L^2$  norm:

$$\langle \vec{\rho}, \vec{\sigma} \rangle = \int \left( \rho_1 \sigma_1 + \partial_r \rho_0 \partial_r \sigma_0 + g'(\ell)^2 \frac{\rho_0 \sigma_0}{r^2} \right) r dr$$

Then  $S(t)$  is an isometry for  $\langle \cdot, \cdot \rangle$  and if  $\vec{\varphi} \in H \times L^2$ ,

$$\begin{aligned} \langle S(-\tau_n)(\vec{\phi}_n(\tau_n) - (\ell, 0)), \vec{\varphi} \rangle &= \langle \vec{\phi}_n(\tau_n) - (\ell, 0), S(-\tau_n)\vec{\varphi} \rangle \\ &= \langle \vec{\phi}_{n,L}(\tau_n), S(-\tau_n)\vec{\varphi} \rangle + O(1/n) = \langle \vec{\phi}_{n,L}(0), \vec{\varphi} \rangle + O(1/n) \rightarrow \langle \vec{\phi}_L(0), \vec{\varphi} \rangle. \end{aligned}$$

We now proceed to prove (5.1); it suffices to show

$$\forall A \geq 0, \quad \|\vec{\psi}(t) - (\ell, 0) - \vec{\phi}_L(t)\|_{H \times L^2(r \geq t-A)} \rightarrow 0.$$

Let  $A \geq 0$  and  $N$  be such that for  $t_N \geq 2A$ . Notice that in a similar fashion as previously, there hold the following profile decomposition for the sequence  $(\vec{\phi}_n(\tau_n) - (\ell, 0) - \vec{\phi}_{N,L}(\tau_n))$

$$\begin{aligned} \vec{\phi}_n(\tau_n, r) - (\ell, 0) - \vec{\phi}_{N,L}(\tau_n, r) &= \vec{\phi}_L(\tau_n, r) - \vec{\phi}_{N,L}(\tau_n, r) \\ &+ \sum_{j=2}^J \left( \frac{1}{\lambda_{j,n}^{d-1}} V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}^d} \partial_t V_{j,L} \left( -\frac{t_{j,n}}{\lambda_{j,n}}, \frac{r}{\lambda_{j,n}} \right) \right) + \gamma_{J,n}(0, r) \end{aligned}$$

Now recall (5.9) and (5.3), so that

$$(5.10) \quad \|\vec{\psi}(t) - (\ell, 0) - \vec{\phi}_{N,L}(t)\|_{H \times L^2(r \geq t-t_N/2)} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

As  $\phi_n$  and  $\psi$  coincide for  $r \geq t - t_n/2$ , hence for  $r \geq t - t_N/2$ , we get

$$\|\vec{\phi}_n(t) - (\ell, 0) - \vec{\phi}_{N,L}(t)\|_{H \times L^2(r \geq t-t_N/2)} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Using the Pythagorean expansion with cut off (Proposition 2.13), we deduce that for all profiles in the above profile decomposition, the  $H \times L^2(r \geq t - t_n/2)$  (semi-) norm tends to 0, and more specifically for the first profile, we get

$$\|\vec{\phi}_L(\tau_n) - \vec{\phi}_{N,L}(\tau_n)\|_{H \times L^2(r \geq \tau_n - t_N/2)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

As  $\vec{\phi}_L(t) - \vec{\phi}_{N,L}(t)$  is solution to the linear solution  $(\text{LW}_\ell)$ , using monotonicity of the  $H_\ell \times L^2$  norm on outside cones, we deduce

$$\|\vec{\phi}_L(t) - \vec{\phi}_{N,L}(t)\|_{H \times L^2(r \geq t-t_N/2)} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Combined with (5.10), we get

$$\|\vec{\psi}(t) - (\ell, 0) - \vec{\phi}_L(t)\|_{H \times L^2(r \geq t - t_N/2)} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

As  $t_N/2 \geq A$ , this proves (5.1), and the proof is complete.  $\square$

We now turn to the analogous result regarding blow up wave maps.

**Proposition 5.2.** *We assume (A1)-(A2).*

Let  $\vec{\psi}$  be a finite energy wave map that blows up at time  $T^+(\vec{\psi})$ . Then there exist  $\ell := \lim_{t \uparrow T^+(\vec{\psi})} \psi(t, T^+(\vec{\psi}) - t) \in \mathcal{V}$  and a wave map  $\vec{\phi}$  defined on a neighborhood of  $T^+(\vec{\psi})$  such that for  $t < T^+(\vec{\psi})$  and  $t \geq \max\{0, T^-(\vec{\phi})\}$

$$\forall r \geq T^+(\vec{\psi}) - t, \quad \vec{\psi}(t, r) = \vec{\phi}(t, r),$$

and

$$\|\vec{\phi}(t) - (\ell, 0)\|_{H \times L^2([0, T^+(\vec{\psi}) - t])} \rightarrow 0 \quad \text{as } t \uparrow T^+(\vec{\psi}).$$

*Proof. Step 1.* Let us first construct  $\vec{\phi}$ .

*Claim 5.3.*  $\psi(t, T^+(\vec{\psi}) - t)$  has a limit  $\ell \in \mathcal{V}$  as  $t \uparrow T^+(\vec{\psi})$ .

*Proof.* Indeed, we recall Proposition 2.1 with  $\lambda = 1/2$ :

$$E(\vec{\psi}(t); (T^+(\vec{\psi}) - t)/2, T^+(\vec{\psi}) - t) \rightarrow 0 \quad \text{as } t \uparrow T^+(\vec{\psi}).$$

In particular, we get

$$\sup_{r \in [(T^+(\vec{\psi}) - t)/2, T^+(\vec{\psi}) - t]} |G(\psi(t, r)) - G(\psi(t, T^+(\vec{\psi}) - t))| \rightarrow 0.$$

Also recall that for all  $t$ ,  $\|\psi(t)\|_{L^\infty} \leq K$ . As  $G$  is a homeomorphism  $\mathbb{R} \rightarrow \mathbb{R}$  from Assumption (A1)-(A2), we get that  $G^{-1}$  is uniformly continuous on  $[G(-K), G(K)]$  and from there,

$$\sup_{r \in [(T^+(\vec{\psi}) - t)/2, T^+(\vec{\psi}) - t]} |\psi(t, r) - \psi(t, T^+(\vec{\psi}) - t)| \rightarrow 0.$$

Now we use again the vanishing of the energy in the self-similar region:

$$\int_{(T^+(\vec{\psi}) - t)/2}^{T^+(\vec{\psi}) - t} |g(\psi(t, r))|^2 \frac{dr}{r} \rightarrow 0,$$

so that there exist  $r(t) \in [(T^+(\vec{\psi}) - t)/2, T^+(\vec{\psi}) - t]$  such that  $g(\psi(t, r(t))) \rightarrow 0$  as  $t \uparrow T^+(\vec{\psi})$ .

By the previous uniform convergence and continuity of  $g$ , we derive  $g(\psi(t, T^+(\vec{\psi}) - t)) \rightarrow 0$ . Now  $t \mapsto \psi(t, T^+(\vec{\psi}) - t)$  is continuous on  $[0, T^+(\vec{\psi})]$ . As  $\mathcal{V} = g^{-1}(\{0\})$  is discrete, this implies that  $\psi(t, T^+(\vec{\psi}) - t)$  has limit  $\ell \in \mathcal{V}$  as  $t \rightarrow T^+(\vec{\psi})$ , as desired.  $\square$

Now consider any sequence of time  $\tau_n \uparrow T^+(\vec{\psi})$  and the wave maps  $\vec{\phi}_n$  defined at time  $\tau_n$  as follows

$$\phi_n(\tau_n, r) = \begin{cases} \ell + \frac{\psi(\tau_n, T^+(\vec{\psi}) - \tau_n) - \ell}{T^+(\vec{\psi}) - \tau_n} r & \text{if } 0 \leq r \leq T^+(\vec{\psi}) - \tau_n \\ \psi(\tau_n, r) & \text{if } r \geq T^+(\vec{\psi}) - \tau_n \end{cases}$$

$$\partial_t \phi_n(\tau_n, r) = \begin{cases} 0 & \text{if } 0 \leq r \leq T^+(\vec{\psi}) - \tau_n \\ \partial_t \psi(\tau_n, r) & \text{if } r \geq T^+(\vec{\psi}) - \tau_n. \end{cases}$$

Let  $m > n$ . By definition and finite speed of propagation, we have

$$\forall r \geq T^+(\vec{\psi}) - \tau_n - |\tau_n - \tau_m|, \quad \vec{\phi}_n(\tau_m, r) = \vec{\phi}_m(\tau_m, r),$$

hence, using again finite speed of propagation,

$$\forall r \geq 2T^+(\vec{\psi}) - 2\tau_n, \quad \vec{\phi}_n(T^+(\vec{\psi}), r) = \vec{\phi}_m(T^+(\vec{\psi}), r),$$

It is then meaningful to define  $\vec{\phi}$  to be the wave map with initial data at time  $T^+(\vec{\psi})$ :

$$(5.11) \quad \vec{\phi}(T^+(\vec{\psi}), r) = \vec{\phi}_n(T^+(\vec{\psi}), r) \quad \text{for } r \geq 2T^+(\vec{\psi}) - 2\tau_n.$$

*Step 2. Properties of  $\vec{\phi}$ .*

Let  $\mathcal{E}(t) := E(\vec{\psi}(t); T^+(\vec{\psi}) - t, \infty)$ . This is a non-decreasing function, so let us define here

$$\mathcal{E} := \lim_{t \uparrow T^+(\vec{\psi})} \mathcal{E}(t).$$

Of course  $\mathcal{E} \leq E(\vec{\psi})$ . Then by construction of  $\vec{\phi}_n$  and as  $\vec{\phi}_n(\tau_n, T^+(\vec{\psi}) - \tau_n) \rightarrow \ell$ , there holds

$$E(\vec{\phi}_n) \rightarrow \mathcal{E}.$$

Hence by monotone convergence, we deduce that

$$\begin{aligned} E(\vec{\phi}) &= \lim_n E(\vec{\phi}(T^+(\vec{\psi})); T^+(\vec{\psi}) - 2\tau_n, +\infty) \\ &\leq \lim_n E(\vec{\phi}_n(T^+(\vec{\psi}) - \tau_n); T^+(\vec{\psi}) - \tau_n, +\infty) = \mathcal{E}, \end{aligned}$$

that is  $E(\vec{\phi}) \leq \mathcal{E}$ . On the other side,

$$E(\vec{\phi}) = E(\vec{\phi}(T^+(\vec{\psi})); 0, +\infty) \geq \lim_n E(\vec{\phi}_n(T^+(\vec{\psi}) - \tau_n); T^+(\vec{\psi}) - \tau_n, +\infty) = \mathcal{E},$$

and finally we obtain that  $\vec{\phi}$  has finite energy

$$(5.12) \quad E(\vec{\phi}) = \mathcal{E}.$$

By definition of  $\vec{\phi}$  (5.11) and finite speed of propagation, for all  $t \in (T^-(\vec{\phi}), T^+(\vec{\psi}))$  we have:

$$\forall r \geq 3T^+(\vec{\psi}) - t - 2\tau_n, \quad \vec{\phi}(t, r) = \vec{\phi}_n(t, r),$$

which yields for  $t = \tau_n$

$$\forall r \geq 3T^+(\vec{\psi}) - 3\tau_n, \quad \vec{\phi}(\tau_n, r) = \vec{\phi}_n(\tau_n, r) = \vec{\psi}(\tau_n, r),$$

Hence, again by finite speed of propagation, we conclude that for  $t \in (T^-(\vec{\phi}), \tau_n]$

$$\forall r \geq 3T^+(\vec{\psi}) - t - 2\tau_n, \quad \vec{\phi}(t, r) = \vec{\psi}(t, r),$$

Letting  $n \rightarrow +\infty$ , we finally obtain that for  $t \in (T^-(\vec{\phi}), T^+(\vec{\psi}))$ :

$$(5.13) \quad \forall r > T^+(\vec{\psi}) - t, \quad \vec{\phi}(t, r) = \vec{\psi}(t, r).$$

By continuity, for  $t < T^+(\vec{\psi})$ , it also holds for  $r = T^+(\vec{\psi}) - t$ . In particular,

$$\vec{\phi}(t, T^+(\vec{\psi}) - t) \rightarrow \ell \quad \text{as } t \uparrow T^+(\vec{\psi}).$$

Now by definition of  $\mathcal{E}$ , (5.13) implies that

$$E(\vec{\phi}(t, r); T^+(\vec{\psi}) - t, +\infty) \rightarrow \mathcal{E} = E(\vec{\phi}),$$

and by difference

$$E(\vec{\phi}(t, r); 0, T^+(\vec{\psi}) - t) \rightarrow 0.$$

As  $\vec{\phi}(t, T^+(\vec{\psi}) - t) \rightarrow \ell$ , this implies

$$\sup_{r \in [0, T^+(\vec{\psi}) - t]} |\phi(t, r) - \ell| \rightarrow 0 \quad \text{as } t \uparrow T^+(\vec{\psi}).$$

Therefore  $\phi(0) = \ell$  and

$$\|\vec{\phi}(t) - (\ell, 0)\|_{H \times L^2([0, T + (\bar{\psi}) - t])} \rightarrow 0. \quad \square$$

## 6. $H \times L^2$ CONVERGENCE FOR THE BUBBLE DECOMPOSITION

In this Section we consider again a sequence of wave maps, and improve the result of Theorem 3.3 under the extra assumption (A3), that is

$$\forall \ell \in \mathcal{V}, \quad g'(\ell) \in \{-1, 1\}.$$

We show that that the error term in the bubble decomposition does in fact convergence to 0 in  $H \times L^2$ . This is the only step in this paper where we use (A3): it guaranties that Proposition (2.10) holds for the linearized problem  $(LW)_\ell$ . This Section is independent of Section 4 and 5.

**Proposition 6.1.** *We assume (A1)-(A2)-(A3).*

*Let  $\psi_n$  be a sequence of wave maps as in Theorem 3.3, and we use its notations.*

*We recall the existence of harmonic maps  $Q_1, \dots, Q_J$  and scales  $\lambda_{J,n} \ll \dots \ll \lambda_{1,n} \ll 1$  and denote*

$$\vec{b}_n(t, r) := \vec{\psi}_n(t, r) - (\ell, 0) - \sum_{j=1}^J (Q_j(r/\lambda_{j,n}), 0).$$

*Then*

$$\|\vec{b}_n(0)\|_{H \times L^2([0, R])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

*Proof.* By Theorem 3.3, we recall that  $\vec{b}_n$  has vanishing energy at all scale, but we only use this property for the scales  $\lambda_{0,n}, \dots, \lambda_{J,n}$  here. More precisely, we will use the following convergence: there exists an increasing sequence  $\alpha_n \rightarrow +\infty$  (and we also denote  $\lambda_{0,n} = 1$ ), such that

- (1)  $\|\vec{b}_n(t)\|_{L^\infty([- \alpha_n \lambda_{J,n}, \alpha_n \lambda_{J,n}], H \times L^2(r \leq \alpha_n \lambda_{J,n}))} \rightarrow 0,$
- (2)  $\|\vec{b}_n(t)\|_{L^\infty([- \alpha_n \lambda_{j,n}, \alpha_n \lambda_{j,n}], H \times L^2(\lambda_{j,n}/\alpha_n \leq r \leq \alpha_n \lambda_{j,n}))} \rightarrow 0$  for all  $j = 1, \dots, J-1,$
- (3)  $\|\vec{b}_n(t)\|_{L^\infty([-A, A], H \times L^2(r \geq \lambda_{0,n}/\alpha_n))} \rightarrow 0,$
- (4)  $\|\partial_t b_n(0)\|_{L^2(r \leq R)} \rightarrow 0.$
- (5)  $\|b_n(0)\|_{L^\infty(r \leq R)} \rightarrow 0.$
- (6)  $\sup_{t \in [-A, A]} \|\vec{b}_n(t)\|_{H \times L^2(r \leq R)}$  is bounded.

In particular,

$$(6.1) \quad \|b_n(0)\|_{L^\infty([0, R])} \rightarrow 0, \quad \text{and} \quad \|\partial_t b_n(0)\|_{L^2([0, R])} \rightarrow 0.$$

We now argue by contradiction. Assume that  $\|\vec{b}_n(0)\|_{H \times L^2([0, R])} \not\rightarrow 0$  as  $n \rightarrow +\infty$ . Up to extracting, we can assume that for some  $\delta_0 > 0$ ,

$$(6.2) \quad \forall n, \quad \|b_n(0)\|_{H([0, R])} \geq \delta_0.$$

Due to convergences (1)-(2)-(3), and up to extracting further, there exists  $j \in \llbracket 1, \dots, J \rrbracket$  and  $\delta_1 > 0$  such that

$$(6.3) \quad \|\vec{b}_n(0)\|_{H \times L^2([0, \alpha_n \lambda_{j,n}])} \rightarrow 0, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad \|b_n(0)\|_{H([\alpha_n \lambda_{j,n}, \frac{\lambda_{j-1,n}}{\alpha_n}])} \geq \delta_1.$$

Let  $\beta_n \uparrow +\infty$  such that  $\beta_n = o(\alpha_n)$ ,  $\beta_n^2 = o(\lambda_{j-1,n}/\lambda_{j,n})$  and for all  $k = 0, \dots, J$ ,

$$\beta_n = o(\lambda_{k-1,n}/\lambda_{k,n}),$$

as  $n \rightarrow +\infty$ .

We consider two times:

$$\tau_{1,n} := \beta_n \lambda_{j,n}, \quad \tau_{2,n} = \frac{\lambda_{j-1,n}}{\beta_n}.$$

We denote  $m = Q_j(\infty)$ : the linearized flow which will interest is now given by  $(LW_m)$ . Let  $\vec{b}_{n,L}$  be the linear solution to  $(LW_m)$  with initial data at time 0:

$$b_{n,L}(0, r) = \begin{cases} b_n(0, r) & \text{if } r \leq 4\tau_{2,n} \\ \frac{b_n(0, 4\tau_{2,n})}{\tau_{2,n}}(5\tau_{2,n} - r) & \text{if } 4\tau_{2,n} \leq r \leq 5\tau_{2,n} \\ 0 & \text{if } r \geq 5\tau_{2,n}, \end{cases}$$

$$\partial_t b_{n,L}(0) = 0.$$

*Claim 6.2.* We have

$$(6.4) \quad \|\vec{\psi}_n(\tau_{1,n}) - (m, 0) - \vec{b}_{n,L}(\tau_{1,n})\|_{H \times L^2([\tau_{1,n}, 3\tau_{2,n}])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

*Proof.* By definition of  $\vec{b}_n$  and (2), notice that

$$\|\vec{\psi}_n(\tau_{1,n}) - (m, 0) - \vec{b}_n(\tau_{1,n})\|_{H \times L^2([\tau_{1,n}, 3\tau_{2,n}])} \rightarrow 0.$$

Hence we are led to compare  $\vec{b}_n$  and  $\vec{b}_{n,L}$ .

First consider the interval  $[\tau_{1,n}, 2\tau_{1,n}]$ . Due to finite speed of propagation, we have

$$\|\vec{b}_{n,L}(\tau_{1,n})\|_{H_\ell \times L^2([\tau_{1,n}, 2\tau_{1,n}])} \leq \|\vec{b}_{n,L}(0)\|_{H_\ell \times L^2([0, 3\tau_{1,n}])} \rightarrow 0$$

by (6.3). Using (2), we conclude

$$\begin{aligned} & \|\vec{b}_n(\tau_{1,n}) - (m, 0) - \vec{b}_{n,L}(\tau_{1,n})\|_{H \times L^2([\tau_{1,n}, 2\tau_{1,n}])} \\ & \leq \|\vec{b}_n(\tau_{1,n})\|_{H_\ell \times L^2([\tau_{1,n}, 2\tau_{1,n}])} + \|\vec{b}_{n,L}(\tau_{1,n})\|_{H \times L^2([\tau_{1,n}, 2\tau_{1,n}])} \rightarrow 0. \end{aligned}$$

We now work on  $[2\tau_{1,n}, 3\tau_{2,n}]$ . Consider the wave maps  $\vec{\vartheta}_n$  with initial data defined as follows

$$\vartheta_n(0, r) = \begin{cases} m + \frac{\psi_n(0, \tau_{1,n}) - m}{\tau_{1,n}} r & \text{if } 0 \leq r \leq \tau_{1,n} \\ \psi_n(0, r) & \text{if } \tau_{1,n} \leq r \leq 4\tau_{2,n} \\ m + \frac{\psi_n(0, 4\tau_{2,n}) - m}{\tau_{2,n}}(5\tau_{2,n} - r) & \text{if } 4\tau_{2,n} \leq r \leq 5\tau_{2,n} \\ m & \text{if } r \geq 5\tau_{2,n} \end{cases}$$

$$\partial_t \vartheta_n(0, r) = \begin{cases} \partial_t \psi_n(0, r) & \text{if } \tau_{1,n} \leq r \leq 4\tau_{2,n} \\ 0 & \text{otherwise.} \end{cases}$$

$\vec{\vartheta}_n(0)$  is an extension of  $\vec{\psi}_n(0)$  with adequate affine reconnection.  $\vec{\vartheta}_n$  coincide with  $\vec{\psi}_n$  on  $[\tau_{1,n}, 4\tau_{2,n}]$  at time 0, and hence at time  $\tau_{1,n} \leq \tau_{2,n}$  we have

$$(6.5) \quad \forall r \in [2\tau_{1,n}, 3\tau_{2,n}], \quad \vec{\psi}_n(\tau_{1,n}, r) = \vec{\vartheta}_n(\tau_{1,n}, r).$$

Also  $\vec{\vartheta}_n(0) = \vec{\vartheta}_n(\infty) = m$  and

$$\|\vartheta_n - m\|_{L^\infty} \rightarrow 0.$$

Because  $\psi_n(0, \tau_{1,n})$  and  $\psi_n(0, 4\tau_{2,n})$  tend to  $m$ , and with the equivalence of energy and  $H \times L^2$  norm for  $L^\infty$ -small perturbation of  $m \in \mathcal{V}$  (Lemma A.1), we infer that

$$\vec{\vartheta}_n(0) \text{ is bounded in } H \times L^2.$$

And finally, we clearly have

$$\|\partial_t \vartheta_n(0)\|_{L^2} \rightarrow 0.$$

Hence Corollary 2.14 combined with Corollary 2.16 allows to conclude that  $\vec{\vartheta}_n$  is globally defined on  $\mathbb{R}$  and

$$(6.6) \quad \sup_{t \in \mathbb{R}} \|\vec{\vartheta}_n(t) - (m, 0) - \vec{\vartheta}_{n,L}(t)\|_{H \times L^2} \rightarrow 0,$$

where  $\vec{\vartheta}_{n,L}$  is the linear solution to  $(LW_m)$  with initial data  $\vec{\vartheta}_{n,L}(0) = \vec{\vartheta}_n(0) - (m, 0)$  at time 0.

Now notice that  $\vec{\vartheta}_{n,L} - \vec{b}_{n,L}$  is a solution to  $(LW_m)$  which also satisfies

$$\|\vec{\vartheta}_{n,L}(0) - \vec{b}_{n,L}(0)\|_{H_m \times L^2} \rightarrow 0.$$

By conservation of the  $H_m \times L^2$  norm, we deduce that

$$(6.7) \quad \|\vec{\vartheta}_{n,L}(\tau_{1,n}) - \vec{b}_{n,L}(\tau_{1,n})\|_{H_m \times L^2} \rightarrow 0.$$

Combining (6.5), (6.6) and (6.7) yields

$$\|\vec{\psi}_n(\tau_{1,n}) - (m, 0) - \vec{b}_{n,L}(\tau_{1,n})\|_{H \times L^2([\tau_{1,n}, 3\tau_{2,n}])} \rightarrow 0.$$

This proves the claim.  $\square$

We now evolve up to time  $\tau_{2,n}$ .

*Claim 6.3.* We have

$$\|\vec{\psi}_n(\tau_{2,n}) - (m, 0) - \vec{b}_{n,L}(\tau_{2,n})\|_{H \times L^2([\tau_{2,n}, 2\tau_{2,n}])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

*Proof.* Let  $\vec{\omega}_n$  be the wave map with initial data at time  $\tau_{1,n}$  as follows:

$$\varpi_n(\tau_{1,n}, r) = \begin{cases} m + \frac{\psi_n(\tau_{1,n}, \tau_{1,n}) - m}{\tau_{1,n}} r & \text{if } 0 \leq r \leq \tau_{1,n} \\ \psi_n(\tau_{1,n}, r) & \text{if } \tau_{1,n} \leq r \leq 3\tau_{2,n} \\ m + \frac{\psi_n(0, 3\tau_{2,n}) - m}{\tau_{2,n}} (4\tau_{2,n} - r) & \text{if } 3\tau_{2,n} \leq r \leq 4\tau_{2,n} \\ m & \text{if } r \geq 4\tau_{2,n} \end{cases}$$

$$\partial_t \varpi_n(\tau_{1,n}, r) = \begin{cases} \partial_t \psi_n(\tau_{1,n}, r) & \text{if } \tau_{1,n} \leq r \leq 3\tau_{2,n} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that  $E(\vec{\omega}_n(\tau_{1,n}); 0, 3\tau_{1,n}) \rightarrow 0$  (consider separately the intervals  $[0, \tau_{1,n}]$  and  $[\tau_{1,n}, 3\tau_{1,n}]$ ), hence  $\vec{b}_{n,NL}$  is defined at least on the time interval  $(-2\tau_{1,n}, 4\tau_{1,n})$ , and by monotonicity of the energy along cones

$$(6.8) \quad E(\vec{\omega}_n(0); 0, 2\tau_{1,n}) \rightarrow 0.$$

Similarly,  $E(\vec{\omega}_n(\tau_{1,n}); 3\tau_{2,n} - 2\tau_{1,n}, +\infty) \rightarrow 0$  because  $3\tau_{2,n} - 2\tau_{1,n} \geq 2\tau_{2,n}$  (consider separately the intervals  $[3\tau_{2,n} - 2\tau_{1,n}, 3\tau_{2,n}]$ ,  $[3\tau_{2,n}, 4\tau_{2,n}]$ , and  $[4\tau_{2,n}, \infty)$ ). Hence, again by monotonicity of the energy along light cones,

$$(6.9) \quad E(\vec{\omega}_n(0); 3\tau_{2,n} - \tau_{1,n}, \infty) \rightarrow 0.$$

Let us show that

$$(6.10) \quad \|\varpi_n(0) - (m, 0)\|_{L^\infty} + \|\partial_t \varpi_n(0)\|_{L^2} \rightarrow 0.$$

Indeed, by finite speed of propagation,

$$\forall r \in [2\tau_{1,n}, 3\tau_{2,n} - \tau_{1,n}], \quad \vec{\omega}_n(0, r) = \vec{\psi}_n(0, r).$$

Hence the convergence on this interval follows from (6.1) which implies

$$\|\vec{\psi}_n(0) - (m, 0)\|_{H \times L^2([2\tau_{1,n}, 3\tau_{2,n}])} \rightarrow 0.$$

Therefore, (6.10) follows from this combined with (6.8), (6.9) (recall the equivalence of the energy and the  $H \times L^2$  norm under the assumption of small energy, Lemma A.1).

Due to (6.10), we conclude from Corollary 2.14 combined with Corollary 2.16 that  $\vec{\omega}_n$  is global and if we denote  $\vec{\omega}_{n,L}$  the linear solution to  $(LW_m)$  with initial data  $\vec{\omega}_n(0) - (m, 0)$  at time 0 :

$$(6.11) \quad \|\varpi_{n,L}\|_{S(\mathbb{R})} + \sup_{t \in \mathbb{R}} \|\vec{\omega}_n(t) - (m, 0) - \vec{\omega}_{n,L}(t)\|_{H \times L^2} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Notice that at time  $\tau_{1,n}$ , we have

$$\begin{aligned} & \|\vec{b}_{n,L}(\tau_{1,n}) - \vec{\omega}_{n,L}(\tau_{1,n})\|_{H \times L^2([\tau_{1,n}, 3\tau_{2,n}])} \\ & \leq \|\vec{b}_{n,L}(\tau_{1,n}) + (m, 0) - \vec{\psi}_n(\tau_{1,n})\|_{H \times L^2([\tau_{1,n}, 3\tau_{2,n}])} \\ & \quad + \|\vec{\omega}_n(\tau_{1,n}) - (m, 0) - \vec{\omega}_{n,L}(\tau_{1,n})\|_{H \times L^2([\tau_{1,n}, 3\tau_{2,n}])} \\ & \rightarrow 0. \end{aligned}$$

Now  $\vec{b}_{n,L} - \vec{\omega}_{n,L}$  is a solution to  $(LW_m)$ : due to monotonicity of the  $H_m \times L^2$  along cones, and as  $[\tau_{2,n}, 2\tau_{2,n}] \subset [\tau_{2,n}, 3\tau_{2,n} - (\tau_{2,n} - \tau_{1,n})]$ , we deduce

$$\|\vec{b}_{n,L}(\tau_{2,n}) - \vec{\omega}_{n,L}(\tau_{2,n})\|_{H \times L^2([\tau_{2,n}, 2\tau_{2,n}])} \rightarrow 0.$$

Hence, we get from (6.11)

$$\|\vec{\omega}_n(\tau_{2,n}) - (m, 0) - \vec{b}_{n,L}(\tau_{2,n})\|_{H \times L^2([\tau_{2,n}, 2\tau_{2,n}])} \rightarrow 0.$$

To complete the proof, it suffices to notice that  $\vec{\omega}_n$  and  $\vec{\psi}_n$  coincide at time  $\tau_{1,n}$  on the interval  $[\tau_{1,n}, 3\tau_{2,n}]$ , so that at time  $\tau_{2,n}$ , they coincide on the interval  $[\tau_{2,n}, 2\tau_{2,n}]$ .  $\square$

We can now easily reach a contradiction. Indeed from (2)-(3)-(4), equivalence of energy and  $H \times L^2$  for  $L^\infty$  perturbation (Lemma A.1) and the definition of  $\beta_n$ , we have

$$\begin{aligned} & \|\vec{\psi}(\tau_{2,n}) - (m, 0)\|_{H \times L^2([\tau_{2,n}, 2\tau_{2,n}])}^2 \\ & \leq CE(\vec{\psi}(\tau_{2,n}) - (m, 0); \tau_{2,n}, 2\tau_{2,n}) \\ & \leq C \sum_{k=1}^J E((Q_k(\cdot/\lambda_{k,n}), 0); \tau_{2,n}, 2\tau_{2,n}) + o(1) \\ & \leq C \sum_{k=1}^J E\left((Q_k, 0); \frac{\lambda_{j,n}}{\beta_n \lambda_{k,n}}, 2 \frac{\lambda_{j,n}}{\beta_n \lambda_{k,n}}\right) + o(1) \\ (6.12) \quad & \rightarrow 0. \end{aligned}$$

However, due to Proposition 2.10, we have

$$\|\vec{b}_{n,L}(\tau_{2,n})\|_{H \times L^2([\tau_{2,n}, +\infty))} \geq \beta(1) \|b_{n,L}(0)\|_H \geq \beta(1) \delta_1.$$

From the definition of  $\vec{b}_{n,L}(0)$ , we see that

$$\|\vec{b}_{n,L}(0)\|_{H_\ell \times L^2([\tau_{2,n}, +\infty))} \rightarrow 0,$$

hence by monotonicity of the  $H_m \times L^2$  norm along cones, we get

$$\|\vec{b}_{n,L}(\tau_{2,n})\|_{H \times L^2([2\tau_{2,n}, +\infty))} \rightarrow 0.$$

We can then conclude:

$$(6.13) \quad \|\vec{b}_{n,L}(\tau_{2,n})\|_{H \times L^2([\tau_{2,n}, 2\tau_{2,n}])} \geq \beta(1) \delta_1 > 0.$$

Then (6.12) and (6.13) are in contradiction with Claim 6.3.

This shows that  $\|b_n(0)\|_{H([0, R])} \rightarrow 0$ , and so  $\|\vec{b}_n(0)\|_{H \times L^2([0, R])} \rightarrow 0$ , as desired.  $\square$

## 7. PROOF OF THE MAIN THEOREM

**7.1. Global case.** Let  $\vec{\psi}$  be a finite energy wave map such that  $T^+(\vec{\psi}) = +\infty$  and let  $\ell := \psi(\infty) \in \mathcal{V}$ . Then Proposition 5.1 (and Proposition 2.1) provides us with a scattering state  $\vec{\phi}_L$  and  $\alpha(t) = o(t)$  such that:

$$(7.1) \quad \|\vec{\psi}(t) - (\ell, 0) - \vec{\phi}_L(t)\|_{H \times L^2(r \geq \alpha(t))} \rightarrow 0.$$

Recall Proposition 2.8, from which a weak version yields

$$(7.2) \quad \|\vec{\phi}_L(t)\|_{H \times L^2([0, t/2])} \rightarrow 0.$$

On the other side, Corollary 2.3 yields a sequence  $t_n$  such that

$$\sup_{s, 0 < s \leq t_n/4} \frac{1}{s} \int_{t_n-s}^{t_n+s} \int_0^{t/2} |\partial_t \psi(t, r)|^2 r dr dt \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Consider now the sequence of wave maps

$$\vec{\psi}_n(t, r) = (\psi(t_n + t_n t, t_n r), t_n \partial_t \psi(t_n + t_n t, t_n r)),$$

Then for  $R = 1/2$  and  $A = 1/4$ , the sequence  $\vec{\psi}_n$  satisfies the conditions of Theorem 3.3, which we can combine with Proposition 6.1. Hence there exists harmonic maps  $Q_1, \dots, Q_J$  and scales  $\lambda_{J,n} \ll \dots \ll \lambda_{1,n} \ll 1$  such that

$$\left\| \vec{\psi}_n(0) - (\ell, 0) - \sum_{j=1}^J (Q_j(\cdot/\lambda_{j,n}) - Q_j(\infty), 0) \right\|_{H \times L^2([0, 1/2])} \rightarrow 0.$$

Unscaling, and denoting  $\mu_{j,n} = \lambda_{j,n}/t_n$ , we have

$$(7.3) \quad \left\| \vec{\psi}(t_n) - (\ell, 0) - \sum_{j=1}^J (Q_j(\cdot/\mu_{j,n}) - Q_j(\infty), 0) \right\|_{H \times L^2([0, t_n/2])} \rightarrow 0.$$

Also, for all  $j$ , as  $\mu_{j,n} \ll t_n$ ,  $\|Q_j(\cdot/\mu_{j,n}) - Q_j(\infty)\|_{H([t_n/2, \infty])} \rightarrow 0$ . Combining this with (7.1), (7.2) and (7.3) yields

$$\left\| \vec{\psi}(t_n) - (\ell, 0) - \sum_{j=1}^J (Q_j(\cdot/\mu_{j,n}) - Q_j(\infty), 0) - \vec{\phi}_L(t) \right\|_{H \times L^2} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

This concludes the proof in the global case.

**7.2. Blow up case.** Let  $\vec{\psi}$  be a wave map which blow-up in finite time  $T^+(\vec{\psi})$ . Proposition 5.2 show that there is a regular wave map  $\vec{\phi}(t)$  defined on a neighborhood of  $T^+(\vec{\psi})$  such that for  $t < T^+(\vec{\psi})$  (and  $t$  near enough  $T^+(\vec{\psi})$ ) there holds

$$(7.4) \quad \forall r \geq T^+(\vec{\psi}) - t, \quad \vec{\psi}(t, r) = \vec{\phi}(t, r).$$

Also  $\vec{\psi}(t, T^+(\vec{\psi}) - t) \rightarrow \ell \in \mathcal{V}$  and

$$(7.5) \quad \|\vec{\phi}(t) - (\ell, 0)\|_{H \times L^2([0, T^+(\vec{\psi}) - t])} \rightarrow 0 \quad \text{as } t \uparrow T^+(\vec{\psi}).$$

Now Corollary 2.7 provides us with a sequence  $t_n \uparrow T^+(\vec{\psi})$  such that (we denote  $\theta_n = T^+(\vec{\psi}) - t_n$ )

$$\sup_{s, 0 < s \leq \theta_n} \frac{1}{s} \int_{t_n-s}^{t_n+s} \int_0^{1-t} |\partial_t \psi(t, r)|^2 r dr dt \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Then sequence of wave maps

$$\vec{\psi}_n(t, r) := (\psi(t_n + \theta_n t, \theta_n r), \theta_n \partial_t \psi(t_n + \theta_n t, \theta_n r)),$$



satisfies the conditions of Theorem 3.3 with  $R = 1$  (and  $A = 1$ ) which we can combine with Proposition 6.1. Hence there exists harmonic maps  $Q_1, \dots, Q_J$  and scales  $\lambda_{J,n} \ll \dots \ll \lambda_{1,n} \ll 1$  such that

$$\left\| \vec{\psi}_n(0) - (\ell, 0) - \sum_{j=1}^J (Q_j(\cdot/\lambda_{j,n}) - Q_j(\infty), 0) \right\|_{H \times L^2([0,1])} \rightarrow 0.$$

Let us unscale, and denote  $\mu_{j,n} = \lambda_{j,n}/\theta_n$ , and recall (7.5) to deduce

$$(7.6) \quad \left\| \vec{\psi}(t_n) - \vec{\phi}(t_n) - \sum_{j=1}^J (Q_j(\cdot/\mu_{j,n}) - Q_j(\infty), 0) \right\|_{H \times L^2([0,\theta_n])} \rightarrow 0.$$

Hence combining with (7.4), this yields

$$\left\| \vec{\psi}(t_n) - \vec{\phi}(t_n) - \sum_{j=1}^J (Q_j(\cdot/\mu_{j,n}) - Q_j(\infty), 0) \right\|_{H \times L^2} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

This settles the proof of Theorem 1.1 in the blow-up case.

#### APPENDIX A. PROOF OF LEMMA 3.2

**Lemma A.1** (Energy and  $H_\ell$  norm). *Let  $\ell \in \mathcal{V}$ , and  $\psi$  be a function of finite energy on  $[r_1, r_2]$ , with  $0 \leq r_1 < r_2 \leq \infty$ .*

- (1)  $2|G(\psi(r_2)) - G(\psi(r_1))| \leq E((\psi, 0); r_1, r_2)$ .
- (2) *There exists  $\delta_\ell > 0$  and  $C_\ell > 0$  independent of  $\psi$  such that if*

$$\|\psi - \ell\|_{L^\infty([r_1, r_2])} \leq \delta_\ell,$$

*then*

$$\frac{1}{C_\ell} \|\psi - \ell\|_{H([r_1, r_2])}^2 \leq E((\psi, 0); r_1, r_2) \leq C_\ell \|\psi - \ell\|_{H([r_1, r_2])}^2.$$

- (3) *If  $\psi(r_1) = \ell$ , and  $E(\psi, 0; r_1, r_2) \leq \delta'_\ell$  then the hypothesis and the conclusion of (2) above hold.*

*Proof.* These bounds are elementary, based on the Taylor expansion of  $g$  and  $G$  around  $\ell$ . We refer to [4] for the first result, and to [6] for the second and third ones.  $\square$

Our goal is now to prove Lemma 3.2. We start by a claim regarding the control of  $L^\infty$  norm by  $H$  even locally outside 0 or  $\infty$ .

**Lemma A.2.** *There exists  $c > 0$  such that for any  $0 \leq r_1 < r_2 \leq +\infty$  with  $r_2 \geq 2r_1$ , and  $\phi \in H([r_1, r_2])$ , then  $\phi \in \mathcal{C}([r_1, r_2])$  and*

$$\|\phi\|_{L^\infty([r_1, r_2])} \leq c \|\phi\|_{H([r_1, r_2])}.$$

*Proof.* We focus on the case  $0 < r_1$  and  $r_2 < \infty$ , as the other cases are well-known and simpler. We recall that for  $r \geq s$ ,

$$|\phi(r) - \phi(s)| \leq \int_s^r |\partial_r \phi(u)| du \leq \sqrt{\int_s^r |\partial_r \phi(u)|^2 u du} \sqrt{\int_s^r \frac{du}{u}} \leq \|\phi\|_H \sqrt{\ln \frac{r}{s}}.$$

This proves continuity.

Then let  $r_0 \in [r_1, r_2]$  be such that  $\|\phi\|_{L^\infty([r_1, r_2])} = |\phi(r_0)|$ . If  $\|\phi\|_{L^\infty([r_1, r_2])} \leq \|\phi\|_{H([r_1, r_2])}$ , there is nothing to prove. Assume the opposite, that is  $\|\phi\|_{L^\infty([r_1, r_2])} \geq \|\phi\|_{H([r_1, r_2])}$ .

Then if  $r \in [r_1, r_2]$  is such that

$$\frac{|r - r_0|}{r_0} \leq \frac{1}{4} \leq \frac{\|\phi\|_{L^\infty([r_1, r_2])}^2}{4\|\phi\|_{H([r_1, r_2])}^2},$$

then  $\sqrt{\left|\ln \frac{r}{r_0}\right|} \leq \frac{\|\phi\|_{L^\infty([r_1, r_2])}}{2\|\phi\|_{H([r_1, r_2])}}$  and

$$|\phi(r)| \geq |\phi(r_0)| - |\phi(r) - \phi(r_0)| \geq |\phi(r_0)| - \|\phi\|_H \frac{\|\phi\|_{L^\infty}}{2\|\phi\|_{H([r_1, r_2])}} \geq \frac{\|\phi\|_{L^\infty([r_1, r_2])}}{2}.$$

Now as  $\frac{r_2}{r_1} \geq 2$ , then  $\frac{r_0}{r_1}$  or  $\frac{r_2}{r_0} \geq \sqrt{2}$ . Let us assume the latter (the former would be treated accordingly), that is

$$\frac{r_2 - r_0}{r_0} \geq \sqrt{2} - 1 > \frac{1}{4}.$$

Then

$$\begin{aligned} \|\phi\|_{H([r_1, r_2])}^2 &\geq \int_{r_0}^{r_2} \frac{\phi(r)^2}{r} dr \geq \int_{r_0}^{5r_0/4} \frac{\phi(r)^2}{r} dr \geq \frac{1}{4} \|\phi\|_{L^\infty([r_1, r_2])}^2 \int_{r_0}^{5r_0/4} \frac{dr}{r} \\ &\geq \frac{\ln(5/4)}{4} \|\phi\|_{L^\infty([r_1, r_2])}^2. \quad \square \end{aligned}$$

**Lemma A.3** (Extension in  $H$ ). *Let  $c$  as in Lemma A.2, and  $0 \leq r_1 < r_2 \leq +\infty$ . Let  $\phi \in H([r_1, r_2])$ , then there exists  $\psi \in H$  that extends  $\phi$ , that is,*

$$\forall r \in [r_1, r_2], \quad \psi(r) = \phi(r),$$

and  $\|\psi\|_H \leq \|\phi\|_{H([r_1, r_2])} + 3\|\phi\|_{L^\infty([r_1, r_2])}$ .

If  $r_2 \geq 2r_1$ , the previous Lemma ensures  $\|\psi\|_H \leq (3c + 1)\|\phi\|_{H([r_1, r_2])}$ .

*Proof.* Again, we only consider here the case  $0 < r_1 < r_2 < +\infty$  and leave the other simpler cases to the reader. We extend  $\phi$  in the following way: let  $r'_1 = r_1/2$  and  $r'_2 = 2r_2$  and

$$\psi(r) = \begin{cases} 0 & \text{if } r \leq r'_1 \\ \phi(r_1) \frac{r - r'_1}{r_1 - r'_1} & \text{if } r'_1 \leq r \leq r_1 \text{ (affine extension)} \\ \phi(r) & \text{if } r_1 \leq r \leq r_2 \\ \phi(r_2) \frac{r'_2 - r}{r'_2 - r_2} & \text{if } r_2 \leq r \leq r'_2 \text{ (affine extension)} \\ 0 & \text{if } r'_2 \leq r \end{cases}.$$

Then  $\psi$  is continuous, hence we deduce  $\psi \in H$  and

$$\begin{aligned} \int_{r'_1}^{r_1} |\partial_r \psi(r)|^2 r dr &= \phi(r_1)^2 \frac{r_1^2 - r'_1{}^2}{2(r_1 - r'_1)^2} = \frac{3}{2} \phi(r_1)^2, \\ \int_{r'_1}^{r_1} \frac{\psi(r)^2}{r^2} r dr &\leq \phi(r_1)^2 \int_{r'_1}^{r_1} \frac{dr}{r} \leq (\ln 2) \phi(r_1)^2, \\ \int_{r_2}^{r'_2} |\partial_r \psi(r)|^2 r dr &= \frac{3}{2} \phi(r_2)^2, \\ \int_{r_2}^{r'_2} \frac{\psi(r)^2}{r^2} r dr &\leq (\ln 2) \phi(r_2)^2. \end{aligned}$$

From this, we see that

$$\|\psi\|_{H([0, r_1])} + \|\psi\|_{H([r_2, +\infty))} \leq 3\|\phi\|_{L^\infty([r_1, r_2])},$$

hence, if we combine it with the previous Lemma when  $r_2 \geq 2r_1$ , we get

$$\|\psi\|_H \leq \|\phi\|_{H([r_1, r_2])} + 3\|\phi\|_{L^\infty([r_1, r_2])} \leq (3c+1)\|\phi\|_{H([r_1, r_2])}. \quad \square$$

*Proof of Lemma 3.2.* The proof relies on [4, Corollary 2.4], which we recall for the convenience of the reader:

**Lemma A.4** ([4, Corollary 2.4]). *Let  $T \geq 0$  and  $\varepsilon > 0$ . There exist  $\eta > 0$  such that if the wave map  $\vec{\psi}$  satisfies*

$$\|\vec{\psi}(0) - (Q, 0)\|_{H \times L^2} \leq \eta,$$

then  $T^+(\vec{\psi}) > T$  and

$$\forall t \in [0, T], \quad \|\vec{\psi}(t) - (Q, 0)\|_{H \times L^2} \leq \varepsilon.$$

Let  $\delta = \eta/(3c+2)$ , where  $\eta$  is given by Lemma A.4 and  $c$  is as in Lemmas A.2 and A.3.

If we consider the restriction of  $\psi(0) - Q$  to  $[r_1, r_2]$ , Lemma A.3 yields an initial data  $\phi_0$  such that  $\phi_0 - Q \in H$ ,

$$\forall r \in [r_1, r_2], \quad \phi_0(r) = \psi(0, r),$$

and

$$\|\phi_0(0) - Q\|_H \leq C\|\psi_0(0) - Q\|_{H([r_1, r_2])} \leq (3c+1)\delta.$$

Define

$$\phi_1(0, r) = \begin{cases} \partial_t \psi(0, r) & \text{if } r_1 \leq r \leq r_2, \\ 0 & \text{otherwise,} \end{cases}$$

(observe that  $\|\phi_1\|_{L^2} \leq \delta$ ). Let  $\vec{\phi}$  be the wave map with initial data  $(\phi_0, \phi_1)$ : it has finite energy, coincide with  $\vec{\psi}$  on  $[r_1, r_2]$  at time  $t = 0$  and

$$\|\vec{\phi}(0) - (Q, 0)\|_{H \times L^2} \leq (3c+2)\delta \leq \eta.$$

Hence by Lemma A.4,  $\vec{\phi}$  is defined at least up to time  $T$  and if furthermore  $|t| \leq T$ ,

$$\|\vec{\phi}(t) - (Q, 0)\|_{H \times L^2([r_1+|t|, r_2-|t|])} \leq \|\vec{\phi}(t) - (Q, 0)\|_{H \times L^2} \leq \varepsilon.$$

Also, by finite speed of propagation for any  $t \in (T^-(\vec{\psi}), T^+(\vec{\psi}))$  such that  $|t| \leq (r_2 - r_1)/2$ ,

$$\forall r \in [r_1 + |t|, r_2 - |t|], \quad \vec{\phi}(t, r) = \vec{\psi}(t, r).$$

Combining the last two properties proves Lemma 3.2.  $\square$

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