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THE *LOCAL* INDEX THEOREM

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1. ABSTRACT

This article is based on author's talk at the International Conference "Alexandroff Reading", Moscow 21 - 25 May, 2012. The material presented in article is a programme intended to organise the ingredients of the index formula. The first results results obtained in this project were announced at the International Conference on Non-commutative Geometry, Trieste, November 2007. Progress obtained along the path of the project was reported at different conferences in Crakovia (June 2011), "K-Theory, C*-Algebras and Index Theory International Conference", Goettingen (November 2010) and Iasi (September 2011).

The unifying idea behind our program is to *localise* K -theory and the non-commutative geometry basic tools (Hochschild, cyclic homology and co-homology, Connes-Karoubi Chern character) *along the lines of Alexander-Spanier co-homology and homology*. The motivation for the realisation of this programme is four-fold: -1) the classical Atiyah-Singer type index formula is a global statement with *local* controle, -2) the non-localised existing objects are not fine enough to capture sufficient information in the case of Banach algebras, -3) one wants to make so that the Alexander-Spanier (co)-homology becomes a *natural* member of non-commutative geometry tools and -4) the Alexander-Spanier co-homology, with respect to the existing non-commutative geometry tools, has the advantage that it does need extra regularity beyond the ordinary topology. The paper [22] by the author has to be seen in the optics of -4).

Author's publications [23] and [24] represent parts of this programme.

2. INTRODUCTION.

The need to consider *local* homological objects, see [23], [24], comes from many directions. On the one side, -i) the Hochschild and cyclic homology, as well as the topological K -theory of the Banach algebra of bounded operators and various Schatten classes of compact operators on the Hilbert space of L_2 sections on a space X is trivial, see e.g. [4], [9], [11]; on the other side, -ii) although the Alexander-Spanier homology appears naturally in [6], its entrance into the theory does occur dually, in the co-homological context. Working co-homologically, the relevant invariants (in the world of *operators*) are moved from their natural setting to a different context (that of the algebra of *functions* on the base space), which, *in general*, could lead to a different set of invariants.

Keeping in mind that non-commutative geometry [4] is essentially an abstract index theory, it is important to clarify further the foundations of index theory [1]. The index theory has three stages:

-1) *K-theory level.*

The existing *K*-theory, see Connes-Moscovici [6], applied onto the natural short exact sequence of operators, relevant for the index theory, gives very little information beyond the analytical index, see [6] p. 352.

We propose to construct a *refinement* of the usual *K*-theory and replace it by *local K*-theory.

-2) *Cyclic homology level.*

The passage from stage -1) to stage -2) is realised by the Connes-Karoubi Chern character; it takes values in the periodic cyclic (*co*)-*homology* of the algebra of smooth functions on smooth manifolds, see e. g. [6], [4] or in the cyclic co-homology of the algebra of L^{n+} functions on quasi-conformal manifolds, see [8]. Unfortunately, the Connes-Karoubi Chern character with values in the periodic cyclic *homology with arbitrary supports* of the Banach algebra of pseudo-differential operators would be trivial.

In [23] the author constructed a Connes-Karoubi Chern type character with values in the *local* periodic cyclic *homology* complex, further localised at the separable ring $L = \mathbb{C} + \mathbb{C}e$, where e is a scalar idempotent, see [6] p. 353. The presence of the scalar idempotent e , which is not traceable, prohibits the implementation of the Connes-Karoubi Chern character formula in the *homological* context; this is another reason for which Connes-Moscovici prefer to work with cyclic co-homology rather than cyclic homology.

The same paper [23] shows that the information obtained by means of the proposed *local* cyclic *homology* is at least as reach as the information obtained by *co-homological* means.

The paper [24] shows that the *local* continuous Hochschild *homology* of the algebra of Hilbert-Schmidt operators on homogeneous simplicial spaces is naturally isomorphic to the Alexander-Spanier co-homology of the space.

-3) *Differential geometry level.*

The original Atiyah-Singer index theory [1] requires smooth structure on the base space. Teleman extended the index formula to Lipschitz manifolds [18]; combining Sullivan's foundational result [17] concerning existence and uniqueness of Lipschitz structures on topological manifolds in dimensions ≤ 5 with the topological cobordism, (following from results due to Thom, Milnor, Kervaire, Novikov, Sullivan and Kirby-Siebenmann (see [13], Sullivan [16]), Teleman [19] proved that the Atiyah-Singer index formula is a *topological statement*. Successively, Donaldson-Sullivan [10] and Connes-Sullivan-Teleman [8] extended the index formula to quasi-conformal analytical structures on topological manifolds.

The work [18], [19], [10] and [8] showed that in order to perform index theory on topological manifolds it is necessary that the topological manifold be endowed with an analytical structure which provides at least first order partial derivatives with some additional property.

The index formula [18], [19], [10] and [8], without invoking topological cobordism, resides inside *non-commutative geometry*.

Looking in retrospect, the original Atiyah-Singer index formula is the *classical limit* of the non-commutative index formula. By classical limit we mean the *restriction* ∇ of non-commutative (co)-homological objects, defined over all powers of the base space, to the main diagonal of the product, see [20]. The possibility to perform restrictions ∇ to diagonals hides the *problem of multiplying distributions*; classical differential geometry lives inside non-commutative geometry on the diagonals of powers of the main space.

The restriction to the diagonal ∇ , in the case of the Atiyah-Singer index formula, requires an additional analytical structure on the topological manifold which provides at least *second order partial derivatives*. The foundational results due to Thom, Milnor, Kervaire, Novikov, Sullivan, Kirby, Siebenmann show that most topological manifolds *do not possess* such a structure; for more references on the subject see Kirby-Siebenmann [13].

For more information regarding the evaluation in usual differential geometry terms of non-commutative homology objects, i.e. the computation of the restriction ∇ , the reader could refer, in addition to [20], [21], [3].

-4) *Classical Atiyah-Singer index formula*. The Atiyah-Singer index theorem [1], [2] regards the topological and analytical indices of *elliptic pseudo-differential operators* on smooth manifolds. The entire work deals the *symbol* of the elliptic operator. The symbol of a pseudo-differential operator lives on the *co-tangent bundle* of the main space. This is, of course, natural within the category of *smooth* manifolds, but loses its naturality when one tries to extend the theory beyond the smooth case, e.g. in the Lipschitz case [18], [19], quasi-conformal manifolds [10], [8], or more generally in non-commutative geometry.

When the co-tangent bundle is not available, or when it would be natural not to use it, it is advisable to push index theory beyond the classical notion of symbol. The reader should note that in this paper the symbol of elliptic operators, given by Definition 3, is the class of the operator modulo compact operators. For this reason, if the operators under consideration were singular integral operators on M , then the symbol of such operators would be extracted from their distributional kernel and would exist along the diagonal of $M \times M$.

The classical Atiyah-Singer index formula evaluates the *differential geometry topological index* -3) lifted on the co-tangent bundle. This is the instance when the Todd class makes its appearance into the index formula. This requires, indeed, significant work. We believe that dissecting the conceptual ingredients of the index formula is important. This is the meaning of the present paper.

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3. ALEXANDER-SPANIER CO-HOMOLOGY AND HOMOLOGY.

Here we recall the basic facts about Alexander-Spanier co-homology, see e.g. [15].

Let X be a topological space and let R be any ring. Let, for any $r \in \mathbb{N}$,

$$(1) \quad C^r(X, R) = \{f \mid f : X^{r+1} \longrightarrow R \text{ be an arbitrary function}\}.$$

Let d be the boundary map

$$(2) \quad d : C^r(X, R) \longrightarrow C^{r+1}(X, R).$$

Let U be an arbitrary neighbourhood of the main diagonal in X^{r+1} ; define

$$(3) \quad C^{r,U}(X, R) = \{f \mid f : X^{r+1} \longrightarrow R \text{ be an arbitrary function, } \text{Support}(f) \subset U\}$$

and, considering the direct system of neighbourhoods of the diagonal U ordered by inclusion, define

$$(4) \quad C^{r,loc}(X, R) = \text{proj} \lim_U C^{r,U}(X, R).$$

One has the following basic result of Alexander-Spanier theory.

Theorem 1. (*Alexander-Spanier, see [15]*)

- i) *The non-localised complex $\{C^r(X, R), d\}_{r \in \mathbb{N}}$ is acyclic*
- ii) *for $X = \text{simplicial complex}$, the homology of the localised complex $\{C^r(X, R), d\}_{r \in \mathbb{N}}$ is isomorphic to the ordinary singular co-homology of X : $H^*(X, R)$.*

The explanation of the two parts of this theorem is easy: in the Alexander-Spanier construction, $C^r(X, R)$ behaves as the co-homological complex of the simplicial space where *any* r points of the space X become vertices of an allowed r -simplex. The whole space behaves as a simplex. For this reason the homology of this complex is trivial.

In the localised case, $\{C^{r,loc}(X, R), d\}$, the points of X are allowed to become vertices of a simplex only if they are sufficiently close one to each other. This is essentially the definition of the co-homology of a simplicial complex.

4. K -THEORY LEVEL

4.1. Non localised K -theory: Connes-Moscovici Local Index Theorem [6]. In this section we make reference to the Connes-Moscovici [6] constructions of the *local index class* for an elliptic operator.

Consider the 6-terms K -theory groups exact sequence associated to the short exact sequence of Banach algebras of C^* algebras

$$(5) \quad 0 \rightarrow \mathcal{K}_M \rightarrow \mathcal{L}_M \rightarrow C(S^*M) \rightarrow 0;$$

here \mathcal{K}_M is the algebra of compact operators on $L_2(M)$, \mathcal{L}_M is the norm closure in the algebra of bounded operators of the algebra of pseudo-differential operators of order zero and $C(S^*M)$ is the algebra of continuous functions on the unit co-sphere bundle to M .

The associated long exact sequence in K -theory is

$$(6) \quad \begin{array}{ccccc} K_0(\mathcal{K}_M) & \rightarrow & K_0(\mathcal{L}_M) & \rightarrow & K_0(C(S^*M)) \\ & & \uparrow & & \downarrow \\ K_1(C(S^*M)) & \leftarrow & K_1(\mathcal{L}_M) & \leftarrow & K_1(\mathcal{K}_M) \end{array} .$$

The mapping $\partial : K^1(C(S^*M)) \longrightarrow K^0(\mathcal{K}_M)$ is the *connecting* homomorphism. The elements of $K^1(C(S^*M))$ are equivalence classes of invertible matrices over the unit sphere

co-tangent bundle over M , i.e. equivalence classes of symbols $\sigma(A)$ of elliptic operators A on M .

The significant K -theory groups in this case are

$$K_0(\mathcal{K}_M) = \mathbb{Z}, \quad K_1(\mathcal{K}_M) = 0, \quad K_1(C(S^*M)) = K_{comp}^0(T^*(M)).$$

The connecting homomorphism contains little information as it takes values in $K_0(\mathcal{K}_M) \equiv \mathbb{Z}$. $\partial : K_{comp}^0(T^*(M)) \rightarrow \mathbb{Z}$ is the *analytical index* map.

Let $A : L_2(E) \rightarrow L_2(F)$ be an elliptic pseudo-differential operator of order zero with small support about the diagonal, from the vector bundle E to the vector bundle F on the compact smooth manifold M . Let $\sigma(A)$ be its principal symbol. Let B be a pseudo-differential parametrix for the operator A . The parametrix B , having principal symbol $\sigma_{pr}(B) = \sigma(A)^{-1}$, may be chosen so that the operators $S_0 = 1 - BA$ and $S_1 = 1 - AB$ be smoothing operators with the distributional support sufficiently small about the diagonal.

The implementation of the procedure defining the connecting homomorphism ∂ leads to the following operators

$$(7) \quad \mathbf{L} = \begin{pmatrix} S_0 & -(1 + S_0)B \\ A & S_1 \end{pmatrix} : L_2(E) \oplus L_2(F) \rightarrow L_2(E) \oplus L_2(F)$$

which is an invertible operator and the idempotent P

$$(8) \quad \mathbf{P} = \mathbf{L} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{L}^{-1}.$$

Let \mathbf{P}_1 , resp. \mathbf{P}_2 be the projection onto the direct summand $L_2(E)$, resp. $L_2(F)$.

A direct computation shows that

$$(9) \quad \mathbf{R} := \mathbf{P} - \mathbf{P}_2 = \begin{pmatrix} S_0^2 & S_0(1 + S_0)B \\ S_1A & -S_1^2 \end{pmatrix}.$$

The element $\mathbf{R} := \mathbf{P} - \mathbf{P}_2 = \partial(\sigma(A)) \in K_0(\mathcal{K}) = \mathbb{Z}$. The *residue* operator \mathbf{R} is a smoothing operator on $L_2(E) \oplus L_2(F)$ with small support about the diagonal.

Let $C^q(M)$ denote the space of Alexander-Spanier cochains of degree q on M consisting of all smooth, *anti-symmetric* real valued functions ϕ defined on M^{q+1} , which have support on a sufficiently small tubular neighbourhood of the diagonal.

Then, for any $[\mathbf{a}] \in K^1(C(S^*M))$, and for any even number q one considers the linear functional

$$(10) \quad \tau_{\mathbf{a}}^q : C^q(M) \rightarrow \mathbb{C}$$

given by the formula

$$(11) \quad \tau_{\mathbf{a}}^q(\phi) = \int_{M^{q+1}} \mathbf{R}(x_0, x_1) \mathbf{R}(x_1, x_2) \dots \mathbf{R}(x_q, x_0) \phi(x_0, x_1, \dots, x_q),$$

where $\mathbf{R}(x_0, x_1)$ is the kernel of the smoothing operator \mathbf{R} defined above.

Using the above construction, Connes and Moscovici [6] produce the *index class homomorphism*

$$(12) \quad \text{Ind} : K^1(C(S^*M)) \otimes_{\mathbb{C}} H_{AS}^{ev}(M) \rightarrow \mathbb{C},$$

where $H_{AS}^{ev}(M)$ denotes Alexander-Spanier cohomology. On the Alexander-Spanier co-chains ϕ it is defined by

$$(13) \quad \text{Ind}_\phi(A) = \text{Ind}(\mathbf{a} \otimes_{\mathbb{C}} \phi) := \tau_{\mathbf{a}}^q(\phi)$$

The functional $\tau_{\mathbf{a}}^q$ is an Alexander-Spanier cycle of degree q over M ; it defines a homology class $[\tau_{\mathbf{a}}^q] \in H_q(M, R)$.

Theorem 2. *Connes-Moscovici [6] Theorem 3.9. Let A be an elliptic pseudo-differential operator on M and let $[\phi] \in H_{comp}^{2q}(M)$. Then*

$$(14) \quad \text{Ind}_{[\phi]} A = \frac{1}{(2\pi i)^q} \frac{q!}{(2q)!} (-1)^{\dim M} \langle \text{Ch}\sigma(A)\tau(M)[\phi], [T^*M] \rangle$$

where $\tau(M) = \text{Todd}(TM) \otimes C$ and $H^*(T^*M)$ is seen as a module over $H_{comp}^*(M)$.

4.2. Localised K -theory. Connes-Moscovici Theorem 2 recovers *locality* (see [6]) by pairing the non-localised K -theory with the Alexander-Spanier cohomology, which is *local*. In fact, [6] p. 353, they state:

”Evidently, the analytical index map does not capture fully the *local* carried by the symbol. It disregards for instance the possibility of localizing at will, around the diagonal, the above construction. By taking advantage of this important feature, we shall construct a pairing of the above projections with arbitrary Alexander-Spanier cocycles on M , which will recapture the stable information carried by the symbols”.

The technical reason why the analytical index map in [6] *may not* capture fully the *local* carried by the symbol resides in the fact that the symbol of the operator is used by means of its image through the connecting homomorphism ∂ , which, as said before, contains very little information: $K_0(\mathcal{K}) = \mathbb{Z}$; this step losses most of the local information carried by the symbol.

We propose to construct a *local* K -theory, denoted K_i^{loc} (for $i = 0, 1$, at least) along the main lines of the Alexander-Spanier construction. We expect the new K_*^{loc} -theory to be reach enough to recover the lost information carried by the symbol of elliptic operators.

Let $\Psi^r(M)$ denote the space of pseudo-differential operators of order r on the smooth manifold M . We consider the exact sequence of algebras

$$(15) \quad 0 \rightarrow \Psi^{-1}(M) \xrightarrow{\iota} \Psi^0(M) \xrightarrow{\pi} \Psi^0(M)/\Psi^{-1}(M) \rightarrow 0.$$

$\Psi^{-1}(M)$ is a compact bi-lateral ideal of $\Psi^0(M)$.

The exact sequence (6) should becomes

$$(16) \quad \begin{array}{ccccc} K_0^{loc}(\Psi^{-1}(M)) & \rightarrow & K_0^{loc}(\Psi^0(M)) & \rightarrow & K_0^{loc}(\Psi^0(M)/\Psi^{-1}(M)) \\ & & \uparrow & & \\ K_1^{loc}(\Psi^0(M)/\Psi^{-1}(M)) & \leftarrow & K_1^{loc}(\Psi^0(M)) & \leftarrow & K_1^{loc}(\Psi^{-1}(M)) \end{array} .$$

where

$$(17) \quad \partial^{K,loc} : K_1^{loc}(\Psi^0(M)/\Psi^{-1}(M)) \longrightarrow K_0^{loc}(\Psi^{-1}(M))$$

is the *connecting* homomorphism in the *local* K_* -theory.

The quotient algebra $\Psi^0(M)/\Psi^{-1}(M)$ is already *local*. In this case, by definition,

$$(18) \quad K_i^{loc}(\Psi^0(M)/\Psi^{-1}(M)) := K_i(\Psi^0(M)/\Psi^{-1}(M)).$$

For the same reason, for the algebra $\Psi^0(M)/\Psi^{-1}(M)$, by definition, the Chern character of elements in $K_i^{loc}(\Psi^0(M)/\Psi^{-1}(M))$ is the Connes-Karoubi Chern character of elements of the existing K -theory, see [7], [4], [5], [12].

Definition 3. *Let $A \in \mathbb{M}(\Psi^0(M))$ be a Fredholm operator. Then A is called elliptic operator on M .*

Then $\sigma(A) := \pi(A) \in \mathbb{M}(\Psi^0(M)/\Psi^{-1}(M))$ is an invertible element in $\mathbb{M}(\Psi^0(M)/\Psi^{-1}(M))$. $\sigma(A) \in \mathbb{GL}(\Psi^0(M)/\Psi^{-1}(M))$ is called the symbol of the elliptic operator A .

The reader should note that the symbol of the operator A given by this definition is not the classical symbol of pseudo-differential operators used by Atiyah-Singer [1]; here, the symbol of the operator A does not use the co-tangent bundle $T^(M)$. The symbol of the operator A is an element $[\sigma(A)] \in K_1^{loc}(\Psi^0(M)/\Psi^{-1}(M))$.*

Conjecture 4. *In the local K -theory, the connecting homomorphism $\partial^{K,loc} : K_1^{loc}(\Psi^0(M)/\Psi^{-1}(M)) \longrightarrow K_0^{loc}(\Psi^{-1}(M))$ from (17) is an isomorphism, rationally.*

5. INDEX THEOREM AT THE K_*^{loc} LEVEL.

Given an elliptic operator A of order zero on M , consider its symbol $\sigma(A)$ and the corresponding element $[\sigma(A)] \in K_1^{loc}(\Psi^0(M)/\Psi^{-1}(M))$.

Definition 5. *The K^{loc} topological index class of A is by definition*

$$(19) \quad (T^K.Index)(A) := [\sigma(A)] \in K_1^{loc}(\Psi^0(M)/\Psi^{-1}(M)).$$

The operator $\mathbf{R}(A)$ given by formula (9) is a *local* operator and $[\mathbf{R}(A)] \in K_0^{loc}(\Psi^{-1}(M))$. This class, belonging to K_0^{loc} is by definition the *local analytical index class* of the elliptic operator A .

Definition 6.

$$(20) \quad (A^K.Index)(A) := [\mathbf{R}(A)] \in K_0^{loc}(\Psi^{-1}(M)).$$

Formula (9) justifies calling $[\mathbf{R}(A)]$ analytical index class of A . Indeed,

$$Index(A) = Tr S_0^2 - Tr S_1^2 = Tr \mathbf{R}(A).$$

Moreover, the Connes-Moscovici Theorem 3.9. [6] shows that $\mathbf{R}(A)$ contains the information about the analytical index of the operator A twisted with vector bundles (the Alexander-Spanier co-homology testing factor ϕ in the formulas (11), (13) and (14) should be seen as Chern character of vector bundles).

We expect the index theorem, at the *local* K -theory level, to become

Conjecture 7. *For any elliptic operator A*

$$(21) \quad \partial^{K,loc} (T^K.Index)(A) = (A^K.Index)(A),$$

where ∂^K is the connecting homomorphism (17), or

$$(22) \quad \partial^{K,loc}([\sigma(A)]) = [\mathbf{R}(A)],$$

6. INDEX THEOREM AT THE CYCLIC HOMOLOGY LEVEL.

6.1. Connes-Karoubi Chern character. To facilitate the reading of this paper we restrict ourselves to recalling the very basic elements of non-commutative geometry; to simplify the exposition we present these pre-requisites within the cyclic homology context rather than periodic cyclic homology.

Let \mathcal{A} be an algebra of functions or operators on the space M . We mean by this that for each element $f \in \mathcal{A}$ its support is well defined.

Let $C_r(\mathcal{A}) := (\mathcal{A} \otimes_{\mathbb{C}})^{r+1}$. An element $f_0 \otimes_{\mathbb{C}} f_1 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} f_r \otimes_{\mathbb{C}}$ is said to have *cyclic symmetry* provided

$$(23) \quad f_1 \otimes_{\mathbb{C}} f_2 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} f_r \otimes_{\mathbb{C}} f_0 \otimes_{\mathbb{C}} = (-1)^r f_0 \otimes_{\mathbb{C}} f_1 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} f_r \otimes_{\mathbb{C}}.$$

The *bar* boundary b' operator is by definition

$$b'(f_0 \otimes_{\mathbb{C}} f_1 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} f_r \otimes_{\mathbb{C}}) := \sum_{0 \leq s \leq r-1} (-1)^s f_0 \otimes_{\mathbb{C}} f_1 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} f_s \cdot f_{s+1} \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} f_r \otimes_{\mathbb{C}}$$

Let

$$C_r^\lambda(\mathcal{A}) := \{f \mid f \in C_r(\mathcal{A}), f \text{ is cyclic symmetric}\}.$$

Definition 8. (see Connes [5], [4], [12], [14]) -i) $\{C_*^\lambda(\mathcal{A}), b'\}$ is a homology complex, called the cyclic complex of the algebra \mathcal{A} ,

-ii) the homology of the cyclic complex $\{C_*^\lambda(\mathcal{A}), b'\}$ is called cyclic homology of the algebra \mathcal{A} , denoted $H_*^\lambda(\mathcal{A})$.

Theorem 9. Morita isomorphism.

The algebra \mathcal{A} and the matrix algebra $\mathbb{M}(\mathcal{A})$ have isomorphic cyclic homologies, see [14].

In this sub-section we use definitions and normalisation constants from Connes-Karoubi [7].

Definition 10. Let $[p] \in K_0(\mathcal{A})$ be represented by the idempotent $p = (p_i^j) \in \mathbb{M}(\mathcal{A})$.

The $2q$ -degree component of the Connes-Karoubi Chern character of $[p]$ is the cyclic homology class of the cycle

$$(24) \quad \begin{aligned} Ch_{2q}(p) &= \\ &= 1/q! \cdot p_{i_0}^{i_{2q+1}} \otimes_{\mathbb{C}} p_{i_{2q+1}}^{i_{2q}} \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} p_{i_2}^{i_1} \otimes_{\mathbb{C}} p_{i_1}^{i_0} \quad (2q + 1 \text{ factors}). \end{aligned}$$

Definition 11. Let $[u] \in K_1(\mathcal{A})$ be represented by the invertible matrix $u = (u_i^j) \in \mathbb{M}(\mathcal{A})$.

The $2q - 1$ -degree component of the Connes-Karoubi Chern character of $[u]$ is the cyclic homology class of the cycle

$$(25) \quad \begin{aligned} Ch_{2q-1}(u) &:= \\ &= (-1)^{q-1} \frac{(q-1)!}{(2q-1)!} (u^{-1} - 1)_{i_{2q}}^{i_0} \otimes_{\mathbb{C}} (u - 1)_{i_{2q-1}}^{i_{2q}} \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} (u^{-1} - 1)_{i_1}^{i_2} \otimes_{\mathbb{C}} (u - 1)_{i_0}^{i_1} \quad (2q \text{ factors}). \end{aligned}$$

6.2. Local cyclic homology. Local Chern character.

Definition 12. Let \mathcal{A} be an associative algebra; suppose the support of any element of the algebra \mathcal{A} is defined.

Let $\{C_*^{\lambda,U}(\mathcal{A}), b'\}_*$ be the sub-complex consisting of cyclic elements whose supports lay in the neighbourhood U .

The local cyclic complex of the algebra \mathcal{A} is

$$(26) \quad \{C_*^{\lambda,loc}(\mathcal{A}), b'\}_* = \text{proj lim}_U \{C_*^{\lambda,U}(\mathcal{A}), b'\}_*.$$

The homology of the local cyclic complex is denoted $H_*^{\lambda,loc}(\mathcal{A})$.

An element $m \in \mathbb{M}(\mathcal{A})$ is called *local* provided its support is small.

Definition 13. Let $[p] \in K_0(\mathcal{A})$ be represented by the local idempotent $p = (p_i^j) \in \mathbb{M}(\mathcal{A})$.

The $2q$ -degree component of the local Chern character of $[p]$ is the local cyclic homology class of the cycle

$$(27) \quad \begin{aligned} Ch_{2q}(p) &= \\ &= 1/q! \cdot p_{i_0}^{i_{2q+1}} \otimes_C p_{i_{2q+1}}^{i_{2q}} \otimes_C \dots \otimes_C p_{i_2}^{i_1} \otimes_C p_{i_1}^{i_0} \quad (2q + 1 \text{ factors}). \end{aligned}$$

The summation with respect to the indices i_r used in this formula is by definition the trace, denoted tr .

Definition 14. Let $[u] \in K_1(\mathcal{A})$ be represented by the local invertible matrix $u = (u_i^j) \in \mathbb{M}(\mathcal{A})$.

The $2q - 1$ -degree component of the local Chern character of $[u]$ is the local cyclic homology class of the cycle

$$(28) \quad \begin{aligned} Ch_{2q-1}(u) &:= \\ &= (-1)^{q-1} \frac{(q-1)!}{(2q-1)!} (u^{-1}-1)_{i_{2q}}^{i_0} \otimes_C (u-1)_{i_{2q-1}}^{i_{2q}} \otimes_C \dots \otimes_C (u^{-1}-1)_{i_1}^{i_2} \otimes_C (u-1)_{i_0}^{i_1} \quad (2q \text{ factors}). \end{aligned}$$

6.3. Connes, Moscovici [6] local index theorem vs. local cyclic homology. We come back to the Connes, Moscovici [6] local index theorem. Formula (13) defines $Ind_{(-)}A$; this is a current on M . This current is identified by formulas (11) - (14). The Connes-Moscovici procedure obtains this form by duality, i.e. by pairing it with the Alexander-Spanier co-homology, given by formula (11). The pairing procedure bypasses certain difficulties which appear in the current construction:

-i) the element $\mathbf{R}(\mathbf{A}) = \mathbf{P} - \mathbf{e}$ is not an idempotent and it does not have a trace. The Connes-Karoubi Chern character construction may not be applied onto the element $\mathbf{R}(\mathbf{A})$.

-ii) Even if such a *homological* Chern character would be defined, it would take values in the cyclic homology of the algebra of compact operators. It is known that the Hochschild and the non-localised cyclic homology of the algebra of compact operators is trivial, see e. g. Cuntz [9].

Firstly, we are going to discuss problem -i). A solution to this problem is proposed in [23]. We summarise it here. Consider $\mathbf{R} = \mathbf{P} - \mathbf{e}$ given by formula (9); it is a difference of idempotents P and e , which belong to the algebra $\mathbb{M}(A)$ of matrices with entries in A .

In view of Definition 13, if \mathbf{R} were an idempotent for any $q \in \mathbb{N}$

$$(29) \quad 1/q! \cdot \text{tr}(\mathbf{R} \otimes_{\mathbb{C}} \dots, \otimes_{\mathbb{C}} \mathbf{R} \otimes_{\mathbb{C}}), \quad (2q+1) \text{ factors}$$

would be a cycle in the cyclic complex of the algebra $\mathbb{M}(A)$ and its homology class would be the Connes-Karoubi Chern character of the idempotent.

Let $S = \mathbb{C} + \mathbb{C}e$. This is a separable sub-ring of the algebra of $\mathbb{M}(A)$. Consider the S -localised cyclic complex

$$\{(\mathbb{M}(A) \otimes_S)^r, b^r\}_{r \in \mathbb{N}},$$

consisting of cyclic elements (i.e. elements satisfying equation (20), with $\otimes_{\mathbb{C}}$ replaced by \otimes_S); let $\{C_{*,S}^\lambda(\mathbb{M}(A))\}$ denote this complex and let $H_{S,*}^\lambda(\mathbb{M}(A))$ denote its homology. As the sub-ring S is separable, see Loday [14] §1.2.12-13, the localisation at S does not modify the cyclic homology. Using Theorem 9, Morita isomorphism, we have

$$(30) \quad H_{S,*}^\lambda(\mathbb{M}(A)) = H_*^\lambda(\mathbb{M}(A)) = H_*^\lambda(\mathbb{M}(A)) = H_*^\lambda(A).$$

As said before, this homology is not reach enough to give interesting information.

Although \mathbf{R} is not an idempotent, it satisfies the identity

$$(31) \quad \mathbf{R}^2 = \mathbf{R} - (e\mathbf{R} + \mathbf{R}e).$$

We show in [23] that the identity (31) implies

$$(32) \quad \tilde{Ch}_{2q}(\mathbf{R}) := 1/q! \cdot [\text{tr}(\mathbf{R} \otimes_S)^{2q+1}] \in \mathbf{H}_{S,2q}^\lambda(\mathbb{M}(\Psi^{-1}(\mathbf{M})))$$

is a well defined homology class in the S -local cyclic complex. This provides a solution to the problem -i).

Now we address the problem -ii). Although the S -localisation provides the correct homological setting, formula (30) states that the homology of the complex $\{C_{S,*}^\lambda(\mathbb{M}(A))\}$ is still not adequate to provide the correct information.

In order to get correct results, we have to *further localise* the complex $\{C_{S,*}^\lambda(\mathbb{M}(A))\}$ along the Alexander-Spanier co-homology construction, i.e. to consider the *projective limit* of sub-complexes defined by those chains which have supports in smaller and smaller neighbourhoods U of the diagonal.

Definition 15. *Define*

$$(33) \quad \{C_{S,*}^{\lambda,loc}(\mathbb{M}(A))\} := \text{proj} \lim_U \{C_{S,*}^{\lambda,U}(\mathbb{M}(A))\}$$

and denote its homology by $H_{S,*}^{\lambda,loc}(\mathbb{M}(A))$.

[24] provides a positive result in this direction.

Theorem 16. (Teleman [24], Theorem 18).

The local Hochschild homology of the algebra of Hilbert-Schmidt operators on the homogeneous simplicial space X is isomorphic to the singular homology of X .

6.4. Local Chern Character. In this sub-section we define the *local* Chern character of elements in K_i^{loc} , $i = 0, 1$.

Definition 17. - 1) For any local invertible element $u \in \mathbb{G}L(\mathcal{A})$, the images of the cycles (28) in the local cyclic homology $H_{odd}^{\lambda,loc}$ constitute the local Chern character of u , denoted $Ch_{odd}^{\lambda,loc}(u) \in H_{odd}^{\lambda,loc}(\mathcal{A})$.

- 2) For any local idempotent $p \in \mathbb{M}(\mathcal{A})$, the images of the cycles (27) in the local complex $\{C_{S,even}^{\lambda,loc}(\mathbb{M}(\mathcal{A}))\}$ constitute the Chern character of $[p] \in K_0(\mathcal{A})$, denoted $Ch_{S,even}^{\lambda,loc}(A) \in H_{even}^{\lambda,loc}(\mathbb{M}(\mathcal{A}))$ (here $S = \mathbb{C}$).

- 3) For any local operator $\mathbf{R} = P - e \in \mathbb{M}(\mathcal{A})$, where P and e are idempotents, the images of the cycles (see formula (32))

$$(34) \quad Ch_{2q}(\mathbf{R}) := 1/q! \cdot [tr(\mathbf{R} \otimes_S)^{2q+1}] \in H_{S,2q}^{\lambda}(\mathbb{M}(\Psi^{-1}(M)))$$

in the local complex $\{C_{S,even}^{\lambda,loc}(\mathbb{M}(\mathcal{A}))\}$ constitute the local Chern character of $[\mathbf{R}] \in K_0(\mathcal{A})$, denoted $Ch_{S,even}^{\lambda,loc}(\mathcal{A}) \in H_{even}^{\lambda,loc}(\mathbb{M}(\mathcal{A}))$.

Definition 18. For any elliptic operator (15) we define

- the local cyclic, topological index class

$$(35) \quad (T_S^{\lambda,loc}.Index)(A) := Ch_{odd}^{\lambda,loc}[\sigma(A)] \in H_{odd}^{\lambda,loc}(\mathbb{M}(\Psi^0/\Psi^{-1})(M)),$$

and

- the local cyclic, S -localised analytic index class

$$(36) \quad (A^{\lambda,loc}.Index)(A) := Ch_{S,even}^{\lambda,loc}[\mathbf{R}(A)] \in H_{even}^{\lambda,loc}(\mathbb{M}(\Psi^{-1})(M))$$

6.5. The Index Theorem at the Local Cyclic Level.

Conjecture 19. For any local elliptic operator A (15)

$$(37) \quad \partial^{\lambda,loc}(T_S^{\lambda,loc}.Index)(A) = (A^{\lambda,loc}.Index)(A),$$

where $\partial^{\lambda,loc}$ is the connecting homomorphism in the local cyclic complex composed with the isomorphism given by the S -localisation (see formula (30)).

7. THE INDEX THEOREM AT THE DIFFERENTIAL GEOMETRY LEVEL

Definition 20. Let ∇ denote the restriction of local cyclic classes to the diagonal, see -3) from §2.

Definition 21. For any elliptic operator (15) define

- the differential geometry topological index class

$$(38) \quad (T^{\nabla}.Index)(A) := \nabla_*(T_{odd}^{\lambda,loc}.Index(A)) = \nabla_*(Ch_{odd}^{\lambda,loc}[\sigma(A)]) \in H_*^{AS}(M),$$

and

- the differential geometry analytic index class

$$(39) \quad (A^{\nabla}.Index)(A) := \nabla_*(A_{S,even}^{\lambda,loc}(A)) = \nabla_*(Ch_{S,even}^{\lambda,loc}[\mathbf{R}(A)]) \in H_*^{AS}(M).$$

Once Conjecture 19 is stated, the next Conjecture 22 is a direct consequence.

Conjecture 22. *For any local elliptic operator A (15), one has*

$$(40) \quad \partial^{AS}(T^\nabla \text{.Index})(A) = (A^\nabla \text{.Index})(A).$$

8. CLASSICAL ATIYAH-SINGER INDEX FORMULA [1], [2].

For a discussion of this point, the reader is invited to read *Classical Atiyah-Singer index formula* §2, -4).

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