



**HAL**  
open science

# ON COMPLEX LINE ARRANGEMENTS AND THEIR BOUNDARY MANIFOLDS

Vincent Florens, Benoît Guerville, Miguel Angel Marco Buzunariz

► **To cite this version:**

Vincent Florens, Benoît Guerville, Miguel Angel Marco Buzunariz. ON COMPLEX LINE ARRANGEMENTS AND THEIR BOUNDARY MANIFOLDS. 2013. hal-00825063v1

**HAL Id: hal-00825063**

**<https://hal.science/hal-00825063v1>**

Preprint submitted on 24 May 2013 (v1), last revised 29 Jun 2015 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON COMPLEX LINE ARRANGEMENTS AND THEIR BOUNDARY MANIFOLDS

V. FLORENS, B. GUERVILLE, AND M.A. MARCO BUZUNARIZ

ABSTRACT. Let  $\mathcal{A}$  be a line arrangement in the complex projective plane  $\mathbb{P}^2$ . We consider the boundary manifold, defined as the boundary of a close regular neighborhood of  $\mathcal{A}$  in  $\mathbb{P}^2$  and study the inclusion map on the complement. We give an explicit method to compute the map induced on the fundamental groups. This extends the work of E.Hironaka [6] on the homotopy type of the complement of (complexified) real arrangements to any complex arrangement.

## 1. INTRODUCTION

Line arrangements are finite collections of complex lines in the projective space  $\mathbb{P}^2$ . The fundamental group of their complement provides a rich invariant of their embedding, extensively studied. Rybnikov showed in particular that it is not in general of combinatorial nature, see [2, 9]. For a given arrangement  $\mathcal{A}$ , we consider its *boundary manifold*  $M(\mathcal{A})$ , defined as the boundary of a closed regular neighborhood in  $\mathbb{P}^2$ . Our aim is to describe explicitly the inclusion map on the complement  $E(\mathcal{A})$  and the map induced on their fundamental groups. This generalises the work of E. Hironaka [6] on the homotopy type of complements of (complexified) real arrangements to any complex arrangement.

The fundamental group of  $E(\mathcal{A})$  can be computed from the Zariski-Van Kampen method, adapted by W. Arvola, see [3]. The method involves the braid monodromy of the arrangement, determined by using *braided wiring diagrams* [4]. On the other hand, the boundary manifold  $M(\mathcal{A})$  is a compact graph manifold, in the sense of F. Waldhausen [10], modeled on the *incidence graph*  $\Gamma_{\mathcal{A}}$ . Its fundamental group can be computed from this description, see for example [5]. E. Hironaka [6] unifies these approaches and describes the map  $\pi_1(M(\mathcal{A})) \rightarrow \pi_1(E(\mathcal{A}))$  from an embedding of  $\Gamma_{\mathcal{A}}$  in  $M(\mathcal{A})$ , induced by a choice of a wiring diagram. We extend this construction to any complex arrangement, and give an explicit algorithm to compute this map in general. We focus our attention to a particular family of generators of  $\pi_1(M(\mathcal{A}))$ , coming from cycles of the graph  $\Gamma_{\mathcal{A}}$ . For real arrangements, they span the kernel of  $\pi_1(M(\mathcal{A})) \rightarrow \pi_1(E(\mathcal{A}))$ . We show that this is no longer the case for any complex arrangement and we describe how they retract to product of meridians of the lines.

Let us mention that the work of E.Artal [1] on the essential coordinate components of the characteristic varieties of an arrangement were one motivation for developing this method, required for the computation of special characters of the fundamental group (see in particular [1] Section 5.4).

In Section 2, we recall the basics on combinatorics of arrangements. We construct the boundary manifold  $M(\mathcal{A})$  from the incidence graph  $\Gamma_{\mathcal{A}}$  and give a presentation of its fundamental group. The Section 3 is devoted to the complement  $E(\mathcal{A})$  and the calculation of the fundamental group from the braided wiring diagram. In

---

2000 *Mathematics Subject Classification.* 32S22, 57M05.

Grupo Consolidado Geometría, ANR Project Interlow ANR-09-JCJC-0097-01.

Section 4, we present the method to compute the map induced by the inclusion on fundamental groups. We obtain a description of the homotopy type of the exterior where the boundary manifold appears explicitly. In the last Section 5, we illustrate the method with the example of MacLane arrangement.

We sometimes use both projective and affine point of views on arrangements. For a given arrangement  $\mathcal{A}$  in  $\mathbb{P}^2$  with  $n + 1$  lines, the line  $L_0$  will denote the arbitrary choice of the line at infinity. The arrangement  $\mathcal{A} - L_0$  in  $\mathbb{P}^2 - L_0 \simeq \mathbb{C}^2$  is an affine arrangement with  $n$  lines.

Along the different sections, the notions and computations are illustrated with the *didactic* example described by the following equations:

$$L_0 = \{z = 0\}, \quad L_1 = \{-(i+2)x + (2i+3)y = 0\}, \quad L_2 = \{-x + (i+2)y = 0\}, \\ L_3 = \{-x + 3y + iz = 0\}, \quad L_4 = \{-x + (2i+2)y = 0\}.$$

## 2. THE BOUNDARY MANIFOLD

The boundary manifold  $M(\mathcal{A})$  is the boundary of a closed regular neighborhood of  $\mathcal{A}$ , which can be constructed as a subcomplex of a triangulation of  $\mathbb{P}^2$  -the closed star of  $\mathcal{A}$  in the second barycentric subdivision. This is a compact connected, oriented graph 3-manifold, modeled on the incidence graph. In particular, it is combinatorially determined: any isomorphism of the incidence graph induces an isomorphism of the graph manifolds [6].

### 2.1. Incidence graph.

Let  $\mathcal{A}$  be an arrangement with set of singular points  $\mathcal{Q}$ . The incidence graph encodes the combinatorial information on  $\mathcal{A}$ , see [8] for details. For  $P \in \mathcal{Q}$ , let us denote  $\mathcal{A}_P = \{\ell \in \mathcal{A} \mid P \in \ell\}$ . The number  $m_P = \#\mathcal{A}_P \geq 2$  is called the *multiplicity* of  $P$ .

**Definition 2.1.** The *incidence graph*  $\Gamma_{\mathcal{A}}$  of  $\mathcal{A}$  is a non-oriented bipartite graph where the set of vertices  $V(\mathcal{A})$  decomposes as  $V_P(\mathcal{A}) \amalg V_L(\mathcal{A})$ , where

$$V_P(\mathcal{A}) = \{v_P \mid P \in \mathcal{Q}\}, \quad V_L(\mathcal{A}) = \{v_L \mid L \in \mathcal{A}\}.$$

The vertices of  $V_P(\mathcal{A})$  are called *point-vertices* and those of  $V_L(\mathcal{A})$  are called *line-vertices*. The edges of  $\Gamma_{\mathcal{A}}$  join  $v_L$  to  $v_P$  if and only if  $L \in \mathcal{A}_P$ . They are denoted  $e(L, P)$ .

A morphism between incidence graphs is a morphism of graphs preserving the vertex labelings, which send elements of  $V_P(\mathcal{A})$  (resp.  $V_L(\mathcal{A})$ ) to elements of  $V_P(\mathcal{A})$  (resp.  $V_L(\mathcal{A})$ ).

The incidence graph of the didactic example is pictured in Figure 1.

### 2.2. Fundamental group of $M(\mathcal{A})$ .

The fundamental group of  $M(\mathcal{A})$  is the group associated to the incidence graph, see [11], [5]. Two types of generators naturally appear: the *meridians* of the lines and the cycles related to the graph.

**Definition 2.2.** Let  $L$  be a line in  $\mathbb{P}^2$ , and  $b \in \mathbb{P}^2 - L$ . A homotopy class  $\alpha \in \pi_1(\mathbb{P}^2 \setminus L, b)$  is called a *meridian of  $L$*  if  $\alpha$  has a representative  $\delta$  satisfying the following properties:

- there is a smooth complex analytic disc  $\Delta \subset \mathbb{P}^2$  transverse to  $L$  such that  $\Delta \cap L = \{b'\} \subset L$ ;
- there is a path  $a$  in  $\mathbb{P}^2 - L$  from  $b$  to  $b' \in \partial\Delta$ ;
- $\delta = a.\beta.a^{-1}$ , where  $\beta$  is the closed path based in  $b'$  given by  $\partial\Delta$  (in the positive direction).

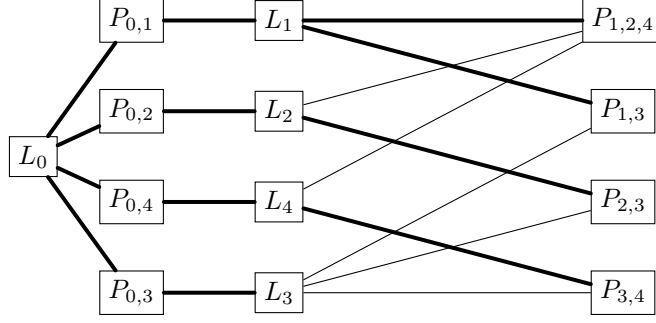


FIGURE 1. Incidence graph of the didactic example

Choose arbitrarily a line  $L_0$  of the arrangement. Note that the meridian of  $L_0$  is a product of meridians of the lines of  $\mathcal{A}$ , in  $\mathbb{P}^2 - \mathcal{A}$ . Let  $\mathcal{P}$  be the set of singular points of the affine arrangement  $\mathcal{A} - L_0$ .

**Definition 2.3.** A *cycle* of the incidence graph  $\Gamma_{\mathcal{A}}$  is an element of  $\pi_1(\Gamma_{\mathcal{A}}, v_{L_0})$ .

We construct a generating system  $\mathcal{E}$  of cycles of  $\Gamma_{\mathcal{A}}$  as follows. Let  $\mathcal{T}$  be the maximal tree of  $\Gamma_{\mathcal{A}}$  containing the following edges:

- $e(L, P)$  for all  $P \in L_0$ , and  $L \in \mathcal{A}$ ;
- $e(L_{\nu(P)}, P)$  for all  $P \in \mathcal{P}$  and  $\nu(P) = \min\{j \mid L_j \in \mathcal{A}_P\}$ .

The maximal tree  $\mathcal{T}$  of the didactic example is given by bold edges in Figure 1. An edge in  $\Gamma_{\mathcal{A}} - \mathcal{T}$  is of the form  $e(L_j, P)$ , with  $P \in \mathcal{P}$  and  $L_j \in \mathcal{A} - L_0$ . By definition of a maximal tree, there exists a unique path  $\lambda_{P,j}$  in  $\mathcal{T}$  joining  $v_P$  and  $v_{L_j}$ . The unique cycle of  $\Gamma_{\mathcal{A}}$  containing only the three vertices  $v_{L_0}$ ,  $v_{L_{\nu(P)}}$  and  $v_{L_j}$  is denoted:

$$\xi_{\nu(P),j} = \lambda_{P,j} \cup e(L_j, P).$$

Let  $\mathcal{E}$  be the set of cycles of  $\Gamma_{\mathcal{A}}$  of the form  $\xi_{s,t}$ . To each  $\xi_{s,t}$  in  $\mathcal{E}$  will correspond an element of  $\pi_1(M(\mathcal{A}))$ , that we denote  $\varepsilon_{s,t}$ .

**Notation 2.4.** We denote  $[a_1, \dots, a_m]$  the equality of all the cyclic commutators

$$\{a_1 \cdots a_m, a_2 \cdots a_m a_1, \dots, a_m a_1 \cdots a_{m-1}\}.$$

**Proposition 2.5.** For  $i = 1, \dots, n$ , let  $\alpha_i$  be the meridians of the lines  $L_i$ , and let  $\{\varepsilon_{s,t}\}$  be a set indexed by  $\mathcal{E}$ . For any singular point  $P = L_{i_1} \cap L_{i_2} \cap \dots \cap L_{i_m}$  with multiplicity  $m$  and  $i_1 = \nu(P)$ , let

$$\mathcal{R}_P = [\alpha_{i_m}^{c_{i_m}}, \dots, \alpha_{i_2}^{c_{i_2}}, \alpha_{i_1}], \text{ where } c_{i_j} = \varepsilon_{i_1, i_j} \text{ for all } j = 2, \dots, m.$$

The fundamental group of the boundary manifold  $M(\mathcal{A})$  admits the following presentation:

$$\pi_1(M(\mathcal{A})) = \langle \alpha_1, \dots, \alpha_n, \varepsilon_{s_1, t_1}, \dots, \varepsilon_{s_l, t_l} \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}_P \rangle.$$

It is worth noticing that the  $\varepsilon_{s,t}$  are not uniquely defined (see details in the proof).

*Proof.* The presentation follows from a plumbing description of a tubular neighborhood of  $\mathcal{A}$ . For every singular point  $P$  of  $\mathcal{A}$ , consider a 4-ball  $\mathbb{B}_P$  centered in  $P$ . Let  $F = \mathcal{A} - \left( \prod_{P \in \mathcal{Q}} (\mathcal{A} \cap \mathbb{B}_P) \right)$  be the surface

obtained by removing discs from the  $\mathbb{B}_P$ 's. One sees that  $F$  is a union  $\coprod_{k=0}^n F_k$  where each  $F_k$  corresponds to the line  $L_k$  of  $\mathcal{A}$ . Let  $\mathcal{N}_k = F_k \times S^1$  whose boundary is a union of disjoint tori. Then  $M(\mathcal{A})$  is obtained by identifications of the  $\mathcal{S}_P = \partial(\mathbb{B}_P) \setminus (\partial\mathbb{B}_P \cap \mathcal{A})$ , with  $P \in \mathcal{Q}$ , and the  $\mathcal{N}_k$ 's.

We have the following presentations of the fundamental groups of the  $\mathcal{N}_k$  and  $\mathcal{S}_P$ :

- If  $g_i$  are generators of  $\pi_1(F_k)$ , and  $\alpha_k$  is the meridian of  $L_k$ , then, for  $\mathcal{Q} \cap L_k = \{P_{k_1}, \dots, P_{k_l}\}$ :

$$\pi_1(\mathcal{N}_k) = \langle g_{k_1}, \dots, g_{k_l}, \alpha_k \mid \forall i \in \{k_1, \dots, k_l\}, \alpha_k^{-1} g_i \alpha_k = g_i \rangle$$

- For each singular point  $P = L_{i_1} \cap \dots \cap L_{i_m} \in \mathcal{Q}$  of multiplicity  $m$ , if  $y_{i_1}, \dots, y_{i_m}$  are the 'local' meridians of the line  $L_i$  in  $\mathbb{B}_P$ , then

$$\pi_1(\mathcal{S}_P) = \langle y_{k_1}, \dots, y_{k_m} \mid [y_{k_m}, \dots, y_{k_1}] \rangle$$

We apply Seifert-Van Kampen's theorem to the union of all  $\mathcal{S}_P$  and  $\mathcal{N}_k$  to obtain the result. Note that if  $P_i \in L_j$ , then the meridian  $\alpha_j$  is identified with  $y_j$  and  $g_i$  with the product of the generator of  $\pi_1(\mathcal{S}_P)$  different of  $y_i$ . The new generators  $\varepsilon_{s,t}$  appear with the gluing, in correspondence with the cycles  $\xi_{s,t}$  of the graph  $\Gamma_{\mathcal{A}}$ .  $\square$

**Example 2.6.** The fundamental group of the didactic example boundary manifold is :

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \varepsilon_{1,2}, \varepsilon_{1,3}, \varepsilon_{1,4}, \varepsilon_{2,3}, \varepsilon_{3,4} \mid [\alpha_4^{\varepsilon_{1,4}}, \alpha_2^{\varepsilon_{1,2}}, \alpha_1], [\alpha_3^{\varepsilon_{1,3}}, \alpha_1], [\alpha_3^{\varepsilon_{3,4}}, \alpha_4], [\alpha_3^{\varepsilon_{2,3}}, \alpha_2] \rangle .$$

### 3. THE COMPLEMENT

Let  $E(\mathcal{A})$  be the complement of a tubular neighborhood of  $\mathcal{A}$ . As before, we choice arbitrarily a line  $L_0$ , and let  $\mathbb{C}^2$  be  $\mathbb{P}^2 - L_0$ .

#### 3.1. Braided wiring diagrams.

Consider a linear projection  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ , *generic* in the sense that:

- For all  $i \in \{1, \dots, n\}$ , the restriction of  $\pi|_{L_i}$  is a homeomorphism.
- Each multiple point lie in a different fiber of  $\pi$ .

We suppose that the points  $x_i = \pi(P_i)$  have distinct real parts, and that we can order the points of  $\pi(\mathcal{P})$  by increasing real parts, so that  $Re(x_1) < Re(x_2) \dots < Re(x_k)$ . A smooth path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  emanating from  $x_0$  with  $Re(x_0) < Re(x_1)$ , passing through  $x_1, \dots, x_k$  in order, and horizontal in a neighborhood of each  $x_i$  is said to be *admissible*.

**Definition 3.1.** The *braided wiring diagram* associated to an admissible path  $\gamma$  is defined by :

$$W_{\mathcal{A}} = \{(x, y) \in \mathcal{A} \mid \exists t \in [0, 1], p(x, y) = \gamma(t)\} .$$

The trace  $\omega_i = W_{\mathcal{A}} \cap L_i$  is called the *wire* associated to the line  $L_i$ .

Note that if  $\mathcal{A}$  is a real complexified arrangement, then  $\gamma = [x_0 - \eta, x_k + \eta] \subset \mathbb{R}$ ; and  $W_{\mathcal{A}} \simeq \mathcal{A} \cap \mathbb{R}^2$ .

*Remark 3.2.*

- The braided wiring diagram depends on the path  $\gamma$ , and on the projection  $\pi$ .
- The set of singular points  $\mathcal{P}$  is contained in  $W_{\mathcal{A}}$ .

We re-index the lines  $L_1, \dots, L_n$  such that:

$$I_i < I_j \iff i < j,$$

where  $I_i = \text{Im}(L_i \cap \pi^{-1}(x_0))$ . On the representation described below of the braided wiring diagram, this re-indexation implies that the lines are ordered, at the left of the diagram, from the top to bottom.

Since the  $x$  coordinates of the points of  $W_{\mathcal{A}}$  are parametrized by  $\gamma$ , the wiring diagram can be seen as a one dimensional object inside  $\mathbb{R}^3 \simeq [0, 1] \times \mathbb{C}$ . Consider its image by a generic projection  $\gamma([0, 1]) \times \mathbb{C} \rightarrow \mathbb{R}^2$ . If we take a plane projection of this diagram (assume, for example, that it is in the direction of the vector  $(0, 0, 1)$  -that is, in the direction of the imaginary axis of the fibre-), we obtain a planar graph. Observe that there are nodes corresponding to the image of actual nodes in the wiring diagram in  $\mathbb{R}^3$  (that is, to a singular point of the arrangement). Other nodes appear from the projection of undergoing and overgoing branches of the wiring diagram in  $\mathbb{R}^3$ . The two types of nodes are called by Arvola *actual and virtual crossing*.

If we represent the virtual crossings in the same way that they are represented as in the case of braid diagrams, we obtain a schematic representation of the wiring diagram as in Figure 3. From now on, we will refer to this representation as the wiring diagram itself. By genericity, we assume that two crossings (actual or virtual) do not lie on the same vertical line.

It is worth noticing that from the braided wiring diagram, one may extract the braid monodromy of  $\mathcal{A}$ , related to the generic projection  $\pi$ . The local equation of a multiple point is of the form  $y^m - x^m$ , where  $m$  is the multiplicity, and the corresponding local monodromy is a full twist in the braid group with  $m$  strands.

### 3.2. Fundamental group of the complement.

We recall briefly the method due to Arvola [3] to obtain a presentation of the fundamental group of the complement from a braided wiring diagram  $W_{\mathcal{A}}$ . The algorithm goes as follows: start from the left of the diagram, assigning a generator  $\alpha_i$  to each strand. Then follow the diagram from the left to the right, assigning a new word to the strands each time going through a crossing. The rules for this new assignment are given in Figure 2, where the  $a_i$ 's are words in the  $\alpha_i$ 's.

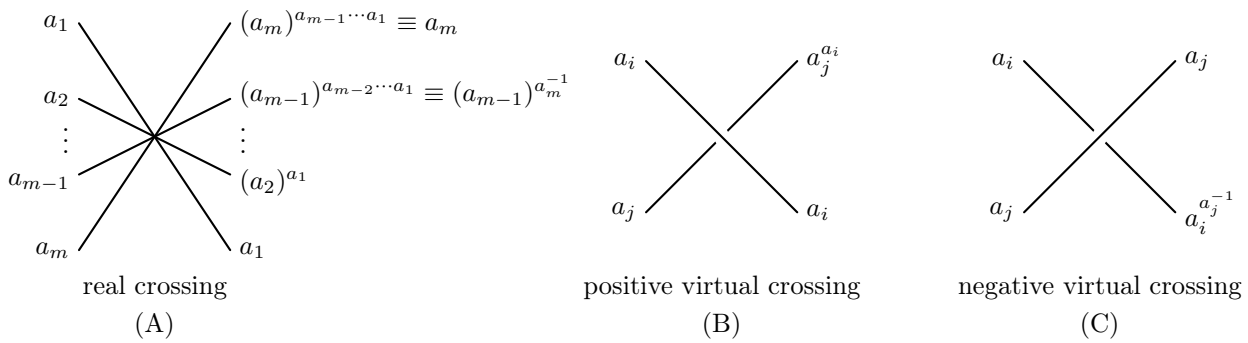


FIGURE 2. Computation of Arvola's words

The notation of Figure 2 is  $a_i^{a_j} = a_j^{-1} a_i a_j$ .

For each actual crossing  $P$  -that corresponds to a singular point of  $\mathcal{A}$ -, suppose that the strands are labelled with the words  $a_1, \dots, a_m$  with respect to their order in the diagram at this point  $P$ , from top to bottom,

where  $m = m_P$  is the multiplicity of  $P$ . Then the following relations are added to the presentation

$$R_P = [a_m, \dots, a_1] = \{a_m \cdots a_1 = a_1 a_m \cdots a_2 = \cdots = a_{m-1} \cdots a_1 a_m\}.$$

They correspond to the action of a half-twist on the free group, whereas the action of a virtual crossing is given by the corresponding braid.

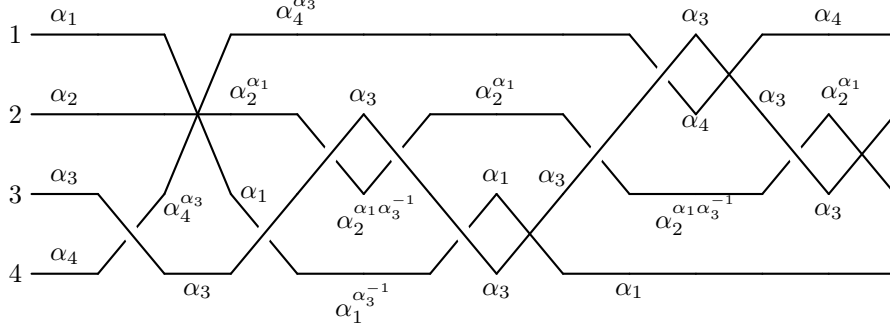


FIGURE 3. The braided wiring diagram of the affine part of the didactic example

**Theorem 3.3** (Arvola [3]). *For  $i = 0, \dots, n$ , let  $\alpha_i$  be the meridians of the lines  $L_i$ . The fundamental group of the exterior of  $\mathcal{A}$  admits the following presentation*

$$\pi_1(E(\mathcal{A})) = \langle \alpha_1, \dots, \alpha_n \mid \bigcup_P R_P \rangle,$$

where  $P$  ranges over all the actual crossings of the wiring diagram  $W_{\mathcal{A}}$ .

**Example 3.4.** The braided wiring diagram of the didactic example is pictured in Figure 3. Its fundamental group is :

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_4^{\alpha_3}, \alpha_2, \alpha_1], [\alpha_3, \alpha_1], [\alpha_4, \alpha_3], [\alpha_3, \alpha_2^{\alpha_1}] \rangle.$$

#### 4. THE INCLUSION MAP

The main result of the paper is a complete description of the map induced on the fundamental groups, by the inclusion of the boundary in the exterior of an arrangement  $\mathcal{A}$ . The computation is done in two main steps.

Let  $W_{\mathcal{A}}$  be the wiring diagram associated to the choice of a generic projection  $\pi$  and an admissible path  $\gamma$ . We start by choosing a generating system  $\mathcal{E} = \{\xi_{s,t}\}$  of cycles of the incidence graph  $\Gamma_{\mathcal{A}}$ . These cycles can be directly seen in  $W_{\mathcal{A}}$ , since it contains all the singular points and the vertices of  $\Gamma_{\mathcal{A}}$  can be identified with their corresponding wires between two singular points. Then, the first step is to "push" each cycle  $\xi_{s,t}$  from  $W_{\mathcal{A}}$  to the boundary manifold  $M(\mathcal{A})$ . The procedure is described in Section 4.1, and gives an explicit family  $\{\varepsilon_{s,t}\}$  of  $\pi_1(M(\mathcal{A}))$ , indexed by  $\mathcal{E}$ . This family, with the set of meridians of the lines, generates  $\pi_1(M(\mathcal{A}))$ . The second step is to compute the images of these generators  $\varepsilon_{s,t}$  by the inclusion map. We use an ad hoc Arvola's algorithm to make the computations directly from  $W_{\mathcal{A}}$ , see Section 4.2. Then the map is described in Theorem 4.4.

In Section 4.3, we examine the kernel of the map; this provides an exact sequence involving  $\pi_1(M(\mathcal{A}))$  and  $\pi_1(E(\mathcal{A}))$ , see Theorem 4.5. We deduce in Section 4.4 a presentation of  $\pi_1(E(\mathcal{A}))$  where the generators of  $\pi_1(M(\mathcal{A}))$  appear explicitly. This presentation provides a complex whose homotopy type is the same that  $E(\mathcal{A})$ , see Proposition 4.8.

#### 4.1. Cycles of the boundary manifold.

We suppose that the admissible path  $\gamma$  emanates from  $x_0$  and goes through  $x_1, \dots, x_k$ , the images of the singular points of  $\mathcal{A}$  by  $\pi$ , ordered by their real parts. Let  $\mathcal{E} = \{\xi_{s,t}\}$  be the generating set defined in Section 2 of cycles of  $\Gamma_{\mathcal{A}}$ .

Each cycle  $\xi_{s,t} \in \mathcal{E}$  is sent to  $M(\mathcal{A})$  via  $W_{\mathcal{A}}$ , as follows. Let  $X_0 = (x_0, y_0)$  be a point of  $\mathcal{N}_0$  such that  $x_0 = \gamma(0)$ . The vertices of  $\xi_{s,t}$  of the form  $v_{L_0}$  or  $v_P$  with  $P \in L_0$  and the edges of the form  $e(L_0, P)$ , with  $P \in L_0$ , are all sent to the point  $X_0$ . The edges  $e(L_i, P)$ , with  $i \neq 0$  and  $P \in L_0$ , are sent on segments from  $X_0$  to the points  $L_i \cap \pi^{-1}(x_0)$ . Then the remaining vertices of the form  $v_L \in V_L(\mathcal{A})$  are sent to  $L \cap \pi^{-1}(x_0)$ . Let  $\xi_{s,t}$  denotes now the cycle of  $W_{\mathcal{A}}$ , relative to the left endpoints, where the vertices  $v_P \in V_P(\mathcal{A})$  are identified with the singular points  $P$ , and the edges with their corresponding wire of  $W_{\mathcal{A}}$ .

A *framed cycle* in  $M(\mathcal{A})$  is obtained as a perturbation of a cycle  $\xi_{s,t}$ . This cycle  $\xi_{s,t}$  consists of two arcs, the parts in  $L_s$  and  $L_t$ . Each arc goes through several actual crossings of  $W_{\mathcal{A}}$  and can be viewed as a union of small arcs. Each of them is projected to  $M(\mathcal{A})$  in the direction  $(0, i)$  and their images are glued together as follows. For each actual crossing  $P$ , modify  $\gamma$  slightly so that it makes a half circle of (small) radius  $\eta_P$  around  $x = \pi(P)$  in the positive sense. Choose  $\eta_P$  so that the preimage of this half circle lies in  $M(\mathcal{A})$ . Note that we avoid the singular point  $P = L_s \cap L_t$  as indicated in Figure 6. The class of the obtained cycle in  $\pi_1(M(\mathcal{A}), X_0)$  is  $\varepsilon_{s,t}$  (see Proposition 2.5). The following map  $\sigma$  is a morphism:

$$\begin{aligned} \sigma : \pi_1(\Gamma_{\mathcal{A}}, \mathcal{T}) &\longrightarrow \pi_1(M(\mathcal{A}), X_0), \\ \xi_{s,t} &\longmapsto \varepsilon_{s,t}. \end{aligned}$$

In order to compute the image of the framed cycles in the exterior  $E(\mathcal{A})$ , we introduce *geometric cycles*, defined as parallel copies of the  $\xi_{s,t}$ 's. Indeed, let  $\tilde{\xi}_{s,t}$  in  $\pi_1(M(\mathcal{A}))$  be the image of  $\xi_{s,t}$  by the projection in the direction  $(0, i)$ . We define an *unknotting map*:

$$\begin{aligned} \delta : \pi_1(M(\mathcal{A}), X_0) &\longrightarrow \pi_1(M(\mathcal{A}), X_0), \\ \varepsilon_{s,t} &\longmapsto \delta_{s,t}^l \varepsilon_{s,t} \delta_{s,t}^r, \end{aligned}$$

where  $\delta_{s,t}^l$  (resp.  $\delta_{s,t}^r$ ) corresponds to the arc in  $L_s$  (resp.  $L_t$ ) of  $\varepsilon_{s,t}$ . They are defined as the products over all actual crossing  $P$  of the arc  $L_s$  (resp.  $L_t$ ) of  $\xi_{s,t}$ , different from  $L_s \cap L_t$ , of the following words. Suppose that  $P = L_{i_1} \cap \dots \cap L_{i_m}$  where the order of the lines corresponds to Figure 4.

- If  $P \in L_s$ , let  $h \in \{1, \dots, m\}$  be such that  $i_h = s$ , then  $P$  contributes to  $\delta_{s,t}^l$  by:

$$\varepsilon_{i_1, i_h}^{-1} \left( \alpha_{i_1}^{-1} (\varepsilon_{i_1, i_2} \alpha_{i_2}^{-1} \varepsilon_{i_1, i_2}^{-1}) \cdots (\varepsilon_{i_1, i_{h-1}} \alpha_{i_{h-1}}^{-1} \varepsilon_{i_1, i_{h-1}}^{-1}) \right) \varepsilon_{i_1, i_h},$$

- If  $P \in L_t$ , let  $h \in \{1, \dots, m\}$  be such that  $i_h = t$ , then  $P$  contributes to  $\delta_{s,t}^r$  by:

$$\varepsilon_{i_1, i_h}^{-1} \left( (\varepsilon_{i_1, i_{h-1}} \alpha_{i_{h-1}}^{-1} \varepsilon_{i_1, i_{h-1}}^{-1}) \cdots (\varepsilon_{i_1, i_2} \alpha_{i_2}^{-1} \varepsilon_{i_1, i_2}^{-1}) \alpha_{i_1} \right) \varepsilon_{i_1, i_h}.$$

**Proposition 4.1.** *The image of a framed cycle by the unknotting map  $\delta$  is a geometric cycle. In particular, we have*

$$\delta(\varepsilon_{s,t}) = \delta_{s,t}^l \varepsilon_{s,t} \delta_{s,t}^r = \tilde{\xi}_{s,t}.$$



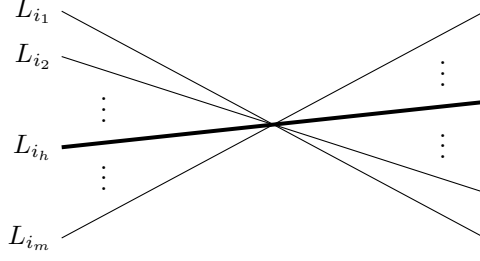


FIGURE 4

*Proof.* The contribution of  $P$  is induced by the action of a half-twist, given by the pre-image by  $\gamma$  of the half circle around each  $x = \pi(P)$ , in the positive sense. We obtain the description of  $\delta_{s,t}^l$  and  $\delta_{s,t}^r$  above, and then  $\varepsilon_{s,t} = (\delta_{s,t}^l)^{-1} \tilde{\xi}_{s,t} (\delta_{s,t}^r)^{-1}$ .  $\square$

**Example 4.2.** The images of the  $\varepsilon_{s,t}$  of the didactic example by the unknotting map are:

$$\begin{aligned} \delta(\varepsilon_{1,2}) &= \varepsilon_{1,2}, & \delta(\varepsilon_{1,4}) &= \varepsilon_{1,4}, \\ \delta(\varepsilon_{1,3}) &= \varepsilon_{1,3}, & \delta(\varepsilon_{2,3}) &= (\varepsilon_{1,2}^{-1} \alpha_1^{-1} \varepsilon_{1,2}) \varepsilon_{2,3} (\varepsilon_{1,3}^{-1} \alpha_1 \varepsilon_{1,3}), \\ \delta(\varepsilon_{3,4}) &= (\varepsilon_{1,3}^{-1} \alpha_1^{-1} \varepsilon_{1,3}) \varepsilon_{3,4} (\varepsilon_{1,4}^{-1} (\alpha_1 \varepsilon_{1,2} \alpha_2 \varepsilon_{1,2}^{-1}) \varepsilon_{1,4}). \end{aligned}$$

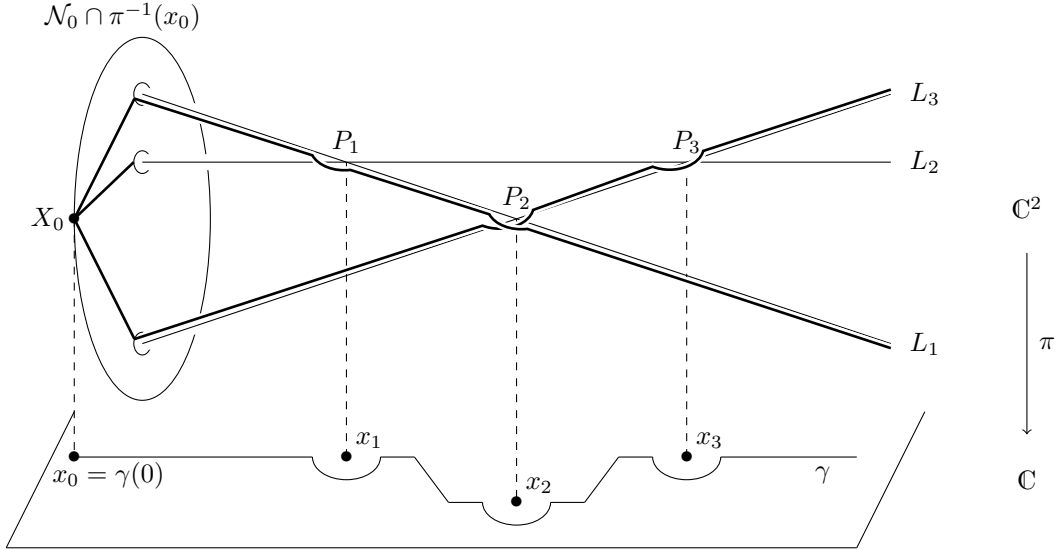


FIGURE 5

#### 4.2. Inclusion map.

Geometric cycles were constructed by taking parallel copies of cycles of  $\Gamma_{\mathcal{A}}$ , via  $W_{\mathcal{A}}$ , to the boundary manifold  $M(\mathcal{A})$ . Their image in  $E(\mathcal{A})$  can then be computed directly from  $W_{\mathcal{A}}$ .

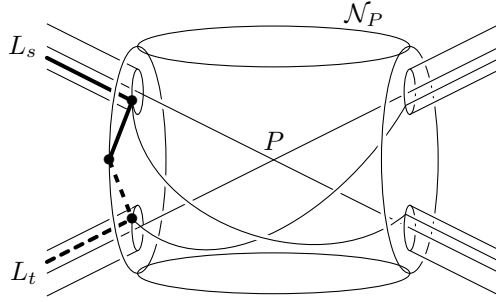


FIGURE 6

Let  $\xi_{s,t}$  be a cycle of  $W_{\mathcal{A}}$ , relative to the left endpoints. An *over arc*  $\varsigma$  is an arc of  $W_{\mathcal{A}}$  that goes over  $\xi_{s,t}$  through a virtual crossing. Denote  $\text{sgn}(\varsigma) \in \{\pm 1\}$  the sign of the crossing. It is positive if the orientations of  $\varsigma$  and  $\xi_{s,t}$  (in this order) at the crossing form a positive base, and is negative otherwise.

Let  $S_{\xi_{s,t}}$  be the set of over arcs of  $\xi_{s,t}$ , oriented from left to right. The element  $\mu_{s,t}$  is defined by :

$$\mu_{s,t} = \prod_{\varsigma \in S_{\xi_{s,t}}} a_{\varsigma}^{\text{sgn}(\varsigma)},$$

where  $a_{\varsigma}$  is the word associate to the arc  $\varsigma$  by the Arvola's algorithm, and the order in the product respect the order of the crossings in the cycle  $\xi_{s,t}$ . Note that  $\mu_{s,t}$  is a product of conjugated of meridians.

**Example 4.3.** All the  $\mu_{s,t}$  of the didactic example are equal to 1, except  $\mu_{1,4} = \alpha_3^{-1}$ .

**Theorem 4.4.** For  $i = 1, \dots, n$ , let  $\alpha_i$  be the meridians of the lines and  $\{\varepsilon_{s,t}\}$  be indexed by a generating system  $\mathcal{E}$  of cycles of the incidence graph  $\Gamma_{\mathcal{A}}$ . Then  $\{\alpha_1, \dots, \alpha_n, \varepsilon_{s_1, t_1}, \dots, \varepsilon_{s_i, t_i}\}$  generates  $\pi_1(M(\mathcal{A}))$  and the map  $i_* : \pi_1(M(\mathcal{A})) \rightarrow \pi_1(E(\mathcal{A}))$  induced by the inclusion is described as follows :

$$i_* : \begin{cases} \alpha_i & \mapsto \alpha_i, \\ \varepsilon_{s,t} & \mapsto (\delta_{s,t}^l)^{-1} \mu_{s,t} (\delta_{s,t}^r)^{-1}, \end{cases}$$

It is worth noticing that by a recursive argument on the set of  $\varepsilon_{s,t}$ , the words  $(\delta_{s,t}^l)^{-1} \mu_{s,t} (\delta_{s,t}^r)^{-1}$  are products of conjugated of the meridians  $\alpha_1, \dots, \alpha_n$ .

*Proof.* Since  $M(\mathcal{A}) \subset E(\mathcal{A})$ , then a class in  $\pi_1(M(\mathcal{A}))$  can be view as a class in  $\pi_1(E(\mathcal{A}))$ , and the both are denoted in the same way.

By Proposition 4.1, each class  $\delta_{s,t}^l \varepsilon_{s,t} \delta_{s,t}^r$  in  $\pi_1(M(\mathcal{A}))$  can be represented by a geometric cycle  $\tilde{\xi}_{s,t}$ , obtained as a parallel copy of  $\xi_{s,t}$  from  $W_{\mathcal{A}}$  to  $M(\mathcal{A})$ . Consider a 2-cell homotopic to a disc with  $\text{card}(S_{\xi_{s,t}})$  holes. Then glue the boundary of the disc to  $\tilde{\xi}_{s,t}$  and the other boundary components to the meridians of over arcs  $\varsigma \in S_{\xi_{s,t}}$ . As the 2-cell is in  $E(\mathcal{A})$ , then, in the exterior,  $\tilde{\xi}_{s,t}$  can be retracted to the product  $\mu_{s,t}$  of the  $a_{\varsigma}$ , with  $\varsigma \in S_{\xi_{s,t}}$ . It follows that in  $\pi_1(E(\mathcal{A}))$ ,  $\delta_{s,t}^l \varepsilon_{s,t} \delta_{s,t}^r = \mu_{s,t}$ .  $\square$

### 4.3. Exact sequence.

**Theorem 4.5.** The following sequence is exact

$$0 \longrightarrow K \xrightarrow{\phi} \pi_1(M(\mathcal{A})) \xrightarrow{i_*} \pi_1(E(\mathcal{A})) \longrightarrow 0,$$

where  $K$  is the normal subgroup of  $\pi_1(M(\mathcal{A}))$  generated by the elements of the form  $\delta_{s,t}^l \varepsilon_{s,t} \delta_{s,t}^r \mu_{s,t}^{-1}$ .

*Proof.* By theorem 4.4, the map  $i_*$  is onto and  $K$  is included in  $\ker(i_*)$ . It remains to show that the relations induced by the images  $i_*(\varepsilon_{s,t})$  are enough to determine a presentation of  $\pi_1(E(\mathcal{A}))$ . We compare these relations to those coming from braid monodromy and Zariski-Van Kampen's method, see [7] for example.

Let  $P = L_{i_1} \cap \cdots \cap L_{i_m}$  (as in Figure 4), be a singular point of  $\mathcal{A}$ , with  $i_1 = \nu(P)$ . Consider a small ball in  $\mathbb{P}^2$  with center  $P$  and a local base point  $b$  in its boundary sphere. Let  $\lambda$  be a path from  $X_0$  to  $b$ , and let  $y_j$  be the (local) meridian of  $L_j$  with base  $b$ , for  $j = 1, \dots, m$ . The path  $\lambda$  can be chosen in such a way that the Zariski-Van Kampen relation associated to  $P$  are :

$$[y_{i_m}^\lambda, \dots, y_{i_1}^\lambda].$$

We can assume that  $b$  is a point of  $\varepsilon_{i_1, j}$ , for all  $j = i_2, \dots, i_m$ . Then we write  $\varepsilon_{i_1, j} = \beta_j^{-1} \beta_{i_1}$  where  $\beta_{i_1}$  goes from  $X_0$  to  $b$ , and  $\beta_j^{-1}$  from  $b$  to  $X_0$ . We get

$$\begin{aligned} [\alpha_{i_m}^{\varepsilon_{i_1, i_m}}, \dots, \alpha_{i_2}^{\varepsilon_{i_1, i_2}}, \alpha_{i_1}] &\Leftrightarrow [\alpha_{i_m}^{\beta_{i_1}^{-1} \beta_{i_1}}, \dots, \alpha_{i_2}^{\beta_{i_1}^{-1} \beta_{i_1}}, \alpha_{i_1}^{\beta_{i_1}^{-1} \beta_{i_1}}], \\ &\Leftrightarrow [\alpha_{i_m}^{\beta_{i_1}^{-1}}, \dots, \alpha_{i_2}^{\beta_{i_1}^{-1}}, \alpha_{i_1}^{\beta_{i_1}^{-1}}] \beta_{i_1}, \\ &\Leftrightarrow [\alpha_{i_m}^{\beta_{i_1}^{-1}}, \dots, \alpha_{i_2}^{\beta_{i_1}^{-1}}, \alpha_{i_1}^{\beta_{i_1}^{-1}}], \\ &\Leftrightarrow [\alpha_{i_m}^{\beta_{i_1}^{-1}}, \dots, \alpha_{i_2}^{\beta_{i_1}^{-1}}, \alpha_{i_1}^{\beta_{i_1}^{-1}}]^\lambda. \end{aligned}$$

Note that during this computation, base point may have changed, but the first and the last relations are based in  $X_0$ . Since  $\alpha_j^{\beta_j^{-1}} = y_{i,j}$ , for all  $j = i_1, \dots, i_m$ , then:

$$\begin{aligned} [\alpha_{i_m}^{\varepsilon_{i_1, i_m}}, \dots, \alpha_{i_2}^{\varepsilon_{i_1, i_2}}, \alpha_{i_1}] &\Leftrightarrow [y_{i_m}, \dots, y_{i_2}, y_{i_1}]^\lambda, \\ &\Leftrightarrow [y_{i_m}^\lambda, \dots, y_{i_2}^\lambda, y_{i_1}^\lambda]. \end{aligned}$$

□

#### 4.4. Homotopy type of the complement.

From Theorem 4.3, we obtain a presentation of the fundamental group of  $\pi_1(E(\mathcal{A}))$ .

**Corollary 4.6.** *For  $i = 1, \dots, n$ , let  $\alpha_i$  be the meridians of the lines  $L_i$ . For any singular point  $P = L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_m}$  with  $i_1 = \nu(P)$ , let*

$$\mathcal{R}_P = [\alpha_{i_m}^{c_{i_m}}, \dots, \alpha_{i_2}^{c_{i_2}}, \alpha_{i_1}], \text{ where } c_{i_j} = \left( \delta_{i_1, i_j}^l \right)^{-1} \mu_{i_1, i_j} \left( \delta_{i_1, i_j}^r \right)^{-1} \text{ for all } j = 2, \dots, m.$$

The fundamental group of  $E(\mathcal{A})$  admits the following presentation:

$$\pi_1(M(\mathcal{A})) = \langle \alpha_1, \dots, \alpha_n \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}_P \rangle.$$

*Proof.* For each  $\varepsilon_{s,t}$ , let  $r_{s,t}$  be the relation  $\varepsilon_{s,t} = (\delta_{s,t}^l)^{-1} \mu_{s,t} (\delta_{s,t}^r)^{-1}$ , and for each point  $P \in \mathcal{P}$  (with  $P = L_{i_1} \cap \cdots \cap L_{i_m}$  and  $i_1 = \nu(P)$ ), we define the relation  $\mathcal{R}'_P : [\alpha_{i_m}^{\varepsilon_{i_1, i_m}}, \dots, \alpha_{i_2}^{\varepsilon_{i_1, i_2}}, \alpha_{i_1}]$ . Then, Theorem 4.5

imply that we have the following presentation:

$$\pi_1(E(\mathcal{A})) = \langle \alpha_1, \dots, \alpha_n, \varepsilon_{s_1, t_1}, \dots, \varepsilon_{s_l, t_l} \mid \bigcup_{P \in \mathcal{P}} \mathcal{R}'_P, \bigcup_{i=1}^l r_{s_i, t_i} \rangle.$$

Consider the total order on the set  $\{\varepsilon_{s,t}\}$ :  $(\varepsilon_{s,t} < \varepsilon_{s',t'}) \Leftrightarrow (s \leq s' \text{ and } t < t')$ . By construction,  $\delta_{s,t}^l$  and  $\delta_{s,t}^r$  depend on  $\varepsilon_{s',t'}$  if and only if  $\varepsilon_{s',t'} < \varepsilon_{s,t}$ . Since  $\mu_{s,t}$  is a product of meridians, then the smallest  $\varepsilon_{s,t}$  is a product of meridians. And by induction, the relation  $r_{s,t}$  express any  $\varepsilon_{s,t}$  as a product of  $\alpha_i$ .  $\square$

**Example 4.7.** The presentation of the fundamental group of the didactic example is:

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_4^{\alpha_3^{-1}}, \alpha_2, \alpha_1], [\alpha_3, \alpha_1], [\alpha_3^{\alpha_1 \alpha_2^{-1} \alpha_1^{-1}}, \alpha_4], [\alpha_3^{\alpha_1 \alpha_4^{-1}}, \alpha_2] \rangle$$

**Proposition 4.8.** *The 2-complex modeled on the minimal presentation given in Corollary 4.6 is homotopy equivalent to  $E(\mathcal{A})$ .*

*Proof.* The proof of Theorem 4.5 shows in particular that the relations of the presentation in Corollary 4.6 are equivalent to the Zariski-Van Kampen relations, based on the braid monodromy. It is shown in [7] that the 2-complex modeled on a minimal presentation equivalent to the Zariski-Van Kampen presentation is homotopy equivalent to  $E(\mathcal{A})$ .  $\square$

## 5. THE EXAMPLE OF POSITIVE MACLANE LINE ARRANGEMENT

In this section, we illustrate Theorem 4.4 with an arrangement  $Q^+$  introduced by Mac Lane, given by the following equations

$$\begin{aligned} L_0 &= \{z = 0\}; & L_1 &= \{z - x = 0\}; & L_2 &= \{x = 0\}; & L_3 &= \{y = 0\}; \\ L_4 &= \{z + \omega^2 x + \omega y = 0\}; & L_5 &= \{y - x = 0\}; & L_6 &= \{z - x - \omega^2 y = 0\}; & L_7 &= \{z + \omega y = 0\}, \end{aligned}$$

where  $\omega = \exp(\frac{2i\pi}{3})$  is a primitive root of unity of order 3.

The incidence graph  $\Gamma$  of  $Q^+$  is given in Figure 7.

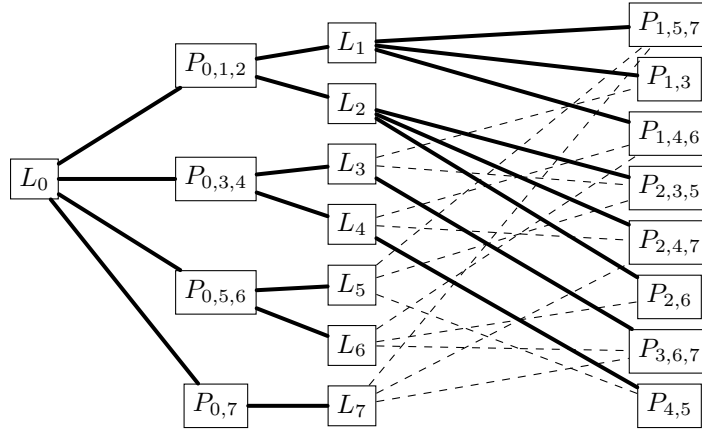


FIGURE 7. Incidence graph of the MacLane arrangement  $Q^+$

It is worth mentioning that  $Q^+$  is one of the only two topological realisations of this combinatorial data by an arrangement in  $\mathbb{P}^2$ . The other realisation  $Q^-$  corresponds to  $\omega = \exp(\frac{-2i\pi}{3})$ . These two arrangements do not admit real equations.

### Generating set of cycles of $\Gamma_{Q^+}$ .

Consider the maximal tree  $\mathcal{T}$  in  $\Gamma_{Q^+}$  indicated with thick lines in Figure 7. Let  $\mathcal{E}$  be the generating system of cycles induced by  $\mathcal{T}$  (it is in one-to-one correspondance with the dotted lines in Figure 7):

$$\mathcal{E} = \{\xi_{2,3}, \xi_{2,5}, \xi_{2,4}, \xi_{2,7}, \xi_{2,6}, \xi_{4,5}, \xi_{3,6}, \xi_{3,7}, \xi_{1,5}, \xi_{1,7}, \xi_{1,3}, \xi_{1,4}, \xi_{1,6}\}.$$

### Group of the boundary manifold.

By Section 4.1, the images  $\varepsilon$  of the cycles  $\xi$  by the application  $\sigma$  provide a family of cycles in  $\pi_1(M(Q^+))$ . Proposition 2.5 applies to this explicit family, and  $\pi_1(M(Q^+))$  admits a presentation with generators:

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\} \cup \{\varepsilon_{2,3}, \varepsilon_{2,5}, \varepsilon_{2,4}, \varepsilon_{2,7}, \varepsilon_{2,6}, \varepsilon_{4,5}, \varepsilon_{3,6}, \varepsilon_{3,7}, \varepsilon_{1,5}, \varepsilon_{1,7}, \varepsilon_{1,3}, \varepsilon_{1,4}, \varepsilon_{1,6}\},$$

and relations:

$$[\alpha_7^{\varepsilon_{1,7}}, \alpha_5^{\varepsilon_{1,5}}, \alpha_1], [\alpha_3^{\varepsilon_{1,3}}, \alpha_1], [\alpha_6^{\varepsilon_{1,6}}, \alpha_4^{\varepsilon_{1,4}}, \alpha_1], [\alpha_5^{\varepsilon_{2,5}}, \alpha_3^{\varepsilon_{2,3}}, \alpha_2], [\alpha_7^{\varepsilon_{2,7}}, \alpha_4^{\varepsilon_{2,4}}, \alpha_2], [\alpha_6^{\varepsilon_{2,6}}, \alpha_2], [\alpha_7^{\varepsilon_{3,7}}, \alpha_6^{\varepsilon_{3,6}}, \alpha_3], [\alpha_5^{\varepsilon_{4,5}}, \alpha_4].$$

### Geometric cycles and unknotting map.

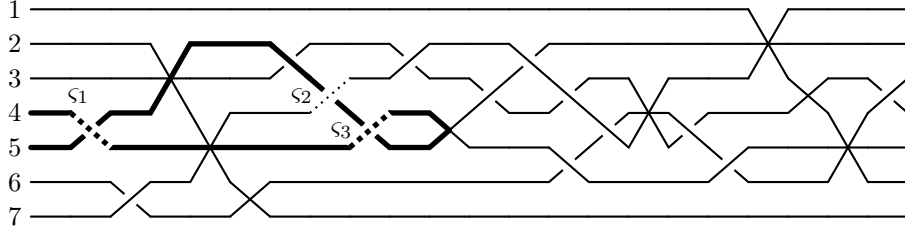


FIGURE 8. Wiring diagram of the positive MacLane arrangement

Let  $W_{Q^+}$  be the braided wiring diagram of  $Q^+$  given in Figure 8.

Note that  $W_{Q^+}$  differs from the wiring diagram considered in [5] by an axial symmetry and a local move on the wires corresponding to  $L_3, L_5, L_7$ .

The diagram  $W_{Q^+}$  is used to compute the unknotting map  $\delta$ , and the images of the cycles  $\varepsilon$  in terms of geometric cycles, see Proposition 4.1. The thick lines in Figure 8 represent the cycle  $\xi_{4,5}$ , divided into two arcs of  $L_4$  and  $L_5$ . The first arc  $L_4$  meets the triple point  $v_{P_{2,4,7}}$ . This gives  $\delta_{4,5}^l = \varepsilon_{2,4}^{-1} \alpha_2^{-1} \varepsilon_{2,4}$ . The second arc  $L_5$  meets  $v_{P_{2,3,5}}$ , and  $\delta_{4,5}^r = \varepsilon_{2,5}^{-1} \varepsilon_{2,3} \alpha_3 \varepsilon_{2,3}^{-1} \alpha_2 \varepsilon_{2,5}$ . This implies that

$$(1) \quad \delta(\varepsilon_{4,5}) = (\varepsilon_{2,4}^{-1} \alpha_2^{-1} \varepsilon_{2,4}) \varepsilon_{4,5} (\varepsilon_{2,5}^{-1} \varepsilon_{2,3} \alpha_3 \varepsilon_{2,3}^{-1} \alpha_2 \varepsilon_{2,5}).$$

Similarly, one computes:

$$\begin{aligned}
\delta(\varepsilon_{2,3}) &= \varepsilon_{2,3}, & \delta(\varepsilon_{2,5}) &= \varepsilon_{2,5}, \\
\delta(\varepsilon_{2,4}) &= \varepsilon_{2,4}, & \delta(\varepsilon_{2,7}) &= \varepsilon_{2,7}, \\
\delta(\varepsilon_{2,6}) &= \varepsilon_{2,6}, & \delta(\varepsilon_{4,5}) &= (\varepsilon_{2,4}^{-1}\alpha_2^{-1}\varepsilon_{2,4})\varepsilon_{4,5}(\varepsilon_{2,5}^{-1}\varepsilon_{2,3}\alpha_3\varepsilon_{2,3}^{-1}\alpha_2\varepsilon_{2,5}), \\
\delta(\varepsilon_{3,6}) &= (\varepsilon_{2,3}^{-1}\alpha_2^{-1}\varepsilon_{2,3})\varepsilon_{3,6}(\varepsilon_{2,6}^{-1}\alpha_2^{-1}\varepsilon_{2,6}), & \delta(\varepsilon_{3,7}) &= (\varepsilon_{2,3}^{-1}\alpha_2\varepsilon_{2,3})\varepsilon_{3,7}(\varepsilon_{2,7}^{-1}\varepsilon_{2,4}\alpha_4\varepsilon_{2,4}^{-1}\alpha_2\varepsilon_{2,7}), \\
\delta(\varepsilon_{1,5}) &= \varepsilon_{1,5}(\varepsilon_{4,5}^{-1}\alpha_4\varepsilon_{4,5}\varepsilon_{2,5}^{-1}\varepsilon_{2,3}\alpha_3\varepsilon_{2,3}^{-1}\alpha_2\varepsilon_{2,5}), & \delta(\varepsilon_{1,7}) &= \varepsilon_{1,7}(\varepsilon_{3,7}^{-1}\varepsilon_{3,6}\alpha_6\varepsilon_{3,6}^{-1}\alpha_3\varepsilon_{3,7}\varepsilon_{2,7}^{-1}\varepsilon_{2,4}\alpha_4\varepsilon_{2,4}^{-1}\alpha_2\varepsilon_{2,7}), \\
\delta(\varepsilon_{1,3}) &= \varepsilon_{1,3}(\varepsilon_{2,3}^{-1}\alpha_2\varepsilon_{2,3}), & \delta(\varepsilon_{1,4}) &= \varepsilon_{1,4}(\varepsilon_{2,4}^{-1}\alpha_2\varepsilon_{2,4}), \\
\delta(\varepsilon_{1,6}) &= \varepsilon_{1,6}(\varepsilon_{3,6}^{-1}\alpha_3\varepsilon_{3,6}\varepsilon_{2,6}^{-1}\alpha_2\varepsilon_{2,6}).
\end{aligned}$$

### Retractions of geometric cycles.

We now compute the family of  $\mu_{s,t}$ , required to obtain the inclusion map, see Section 4.2. The arcs of the wiring diagram  $W_{Q^+}$  are labelled by the algorithm of Arvola, see Section 3.2.

The case of  $\mu_{4,5}$  is drawn in thick in Figure 8. The over arcs  $\varsigma_1$ ,  $\varsigma_2$  and  $\varsigma_3$  are dotted in Figure 8. Arvola's labellings of these arcs are respectively :  $a_{\varsigma_1} = \alpha_4$ ,  $a_{\varsigma_2} = \alpha_7$  and  $a_{\varsigma_3} = \alpha_7^{-1}\alpha_4\alpha_7$ . Furthermore,  $\text{sgn}(\varsigma_1) = -1$ ,  $\text{sgn}(\varsigma_2) = 1$  and  $\text{sgn}(\varsigma_3) = 1$ . We obtain  $\mu_{4,5} = (\alpha_7^{-1}\alpha_4\alpha_7)\alpha_7\alpha_4^{-1}$ , which gives

$$\mu_{4,5} = \alpha_7^{-1}\alpha_4\alpha_7^2\alpha_4^{-1}.$$

Similarly:

$$\begin{aligned}
\mu_{2,3} &= 1, & \mu_{2,5} &= 1, \\
\mu_{2,4} &= 1, & \mu_{2,7} &= 1, \\
\mu_{2,6} &= 1, & \mu_{4,5} &= (\alpha_7^{-1}\alpha_4\alpha_7)\alpha_7\alpha_4^{-1}, \\
\mu_{3,6} &= \alpha_4^{-1}\alpha_5\alpha_4\alpha_7^{-2}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7^2\alpha_6^{-1}, & \mu_{3,7} &= \alpha_4^{-1}\alpha_5\alpha_4\alpha_7^{-2}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7^2, \\
\mu_{1,5} &= \alpha_4^{-1}\alpha_5^{-1}, & \mu_{1,7} &= 1, \\
\mu_{1,3} &= \alpha_7^{-1}\alpha_4^{-1}\alpha_7^2\alpha_6^{-1}\alpha_7^{-2}\alpha_4\alpha_7^{-1}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_7^{-2}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_5^{-1}\alpha_4, & \mu_{1,4} &= 1, \\
\mu_{1,6} &= \alpha_7.
\end{aligned}$$

### Images in the group of the complement.

Following Theorem 4.4, we can compute  $i_* : \pi_1(M(Q^+)) \rightarrow \pi_1(E(Q^+))$ . The computations above describe the relations induced by the images of the cycles  $\varepsilon$  in  $\pi_1(E(Q^+))$ . By the previous computations,  $\varepsilon_{2,3}, \varepsilon_{2,5}, \varepsilon_{2,4}, \varepsilon_{2,7}, \varepsilon_{2,6}$  are equal to 1 (i.e. they are contractible in  $E(\mathcal{A})$ ). They are the relations  $r_{2,3}, r_{2,5}, r_{2,4}, r_{2,7}$  and  $r_{2,6}$ .

The case of  $r_{4,5}$ :

$$r_{4,5} : \varepsilon_{2,4}^{-1}\alpha_2^{-1}\varepsilon_{2,4}\varepsilon_{4,5}\varepsilon_{2,5}^{-1}\varepsilon_{2,3}\alpha_3\varepsilon_{2,3}^{-1}\alpha_2\varepsilon_{2,5} = \alpha_7^{-1}\alpha_4\alpha_7^2\alpha_4^{-1}.$$

Then using  $r_{2,4}, r_{2,5}$  and  $r_{2,3}$  we obtain that:

$$r_{4,5} : \varepsilon_{4,5} = \alpha_2\alpha_7^{-1}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_2^{-1}\alpha_3^{-1}.$$

The others relations are:

- $r_{3,6} : \varepsilon_{3,6} = \alpha_2\alpha_4^{-1}\alpha_5\alpha_4\alpha_7^{-2}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7^2\alpha_6^{-1}\alpha_7^{-1}\alpha_2^{-1}\alpha_7$ ,
- $r_{3,7} : \varepsilon_{3,7} = \alpha_2\alpha_4^{-1}\alpha_5\alpha_4\alpha_7^{-2}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7^2\alpha_2^{-1}\alpha_4^{-1}$ ,

- $r_{1,5}$  :  $\varepsilon_{1,5} = \alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7\alpha_2^{-1}\alpha_4^{-1}\alpha_2\alpha_7^{-1}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_2^{-1}\alpha_3^{-1}$ ,
- $r_{1,7}$  :  $\varepsilon_{1,7} = (\alpha_7^{-2}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_5\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_7^2\alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_2^{-1})(\alpha_3^{-1})$   
 $(\alpha_2\alpha_4^{-1}\alpha_5\alpha_4\alpha_7^{-2}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7^2\alpha_6^{-1}\alpha_7^{-1}\alpha_2^{-1}\alpha_7)(\alpha_6)(\alpha_7^{-1}\alpha_2\alpha_7\alpha_6\alpha_2^{-1}\alpha_4^{-1})$ ,
- $r_{1,3}$  :  $\varepsilon_{1,3} = \alpha_7^{-1}\alpha_4^{-1}\alpha_7^2\alpha_6^{-1}\alpha_7^{-2}\alpha_4\alpha_7^{-1}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_7^{-2}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_2^{-1}$ ,
- $r_{1,4}$  :  $\varepsilon_{1,4} = \alpha_2^{-1}$ ,
- $r_{1,6}$  :  $\varepsilon_{1,6} = \alpha_7\alpha_2^{-1}(\alpha_7^{-1}\alpha_2\alpha_7\alpha_6\alpha_7^{-2}\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_5\alpha_4\alpha_7^2\alpha_4^{-1}\alpha_7^2\alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_2^{-1})\alpha_3^{-1}$   
 $(\alpha_2\alpha_4^{-1}\alpha_5\alpha_4\alpha_7^{-2}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_5^{-1}\alpha_4\alpha_7^{-2}\alpha_4^{-1}\alpha_7^2\alpha_6^{-1}\alpha_7^{-1}\alpha_2^{-1}\alpha_7)$ .

From the proof of Corollary 4.6, we obtain the presentation of  $\pi_1(E(Q^+))$ .

$$\begin{aligned} \pi_1(E(\mathcal{A}^+)) = < \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \varepsilon_{2,3}, \varepsilon_{2,5}, \varepsilon_{2,4}, \varepsilon_{2,7}, \varepsilon_{2,6}, \varepsilon_{4,5}, \varepsilon_{3,6}, \varepsilon_{3,7}, \varepsilon_{1,5}, \varepsilon_{1,7}, \varepsilon_{1,3}, \varepsilon_{1,4}, \varepsilon_{1,6} \mid \\ r_{2,3}, r_{2,5}, r_{2,4}, r_{2,7}, r_{2,6}, r_{4,5}, r_{3,6}, r_{3,7}, r_{1,5}, r_{1,7}, r_{1,3}, r_{1,4}, r_{1,6}, \\ [\alpha_7^{\varepsilon_{1,7}}, \alpha_5^{\varepsilon_{1,5}}, \alpha_1], [\alpha_3^{\varepsilon_{1,3}}, \alpha_1], [\alpha_6^{\varepsilon_{1,6}}, \alpha_4^{\varepsilon_{1,4}}, \alpha_1], [\alpha_5^{\varepsilon_{2,5}}, \alpha_3^{\varepsilon_{2,3}}, \alpha_2], \\ [\alpha_7^{\varepsilon_{2,7}}, \alpha_4^{\varepsilon_{2,4}}, \alpha_2], [\alpha_6^{\varepsilon_{2,6}}, \alpha_2], [\alpha_7^{\varepsilon_{3,7}}, \alpha_6^{\varepsilon_{3,6}}, \alpha_3], [\alpha_5^{\varepsilon_{4,5}}, \alpha_4] > . \end{aligned}$$

#### ACKNOWLEDGMENTS

The authors thank E. Artal Bartolo for all several helpfull comments and suggestions.

#### REFERENCES

1. E. Artal, *Topology of arrangements and position of singularities*, Annales de la fac. des sciences de Toulouse, to appear.
2. E. Artal, J. Carmona, J.I. Cogolludo-Agustín, and M. Marco, *Topology and combinatorics of real line arrangements*, Compos. Math. **141** (2005), no. 6, 1578–1588.
3. W.A. Arvola, *The fundamental group of the complement of an arrangement of complex hyperplanes*, Topology **31** (1992), no. 4, 757–765.
4. D.C. Cohen and A.I. Suciú, *The braid monodromy of plane algebraic curves and hyperplane arrangements*, Comment. Math. Helv. **72** (1997), no. 2, 285–315.
5. ———, *The boundary manifold of a complex line arrangement*, Groups, homotopy and configuration spaces, Geom. Topol. Monogr., vol. 13, Geom. Topol. Publ., Coventry, 2008, pp. 105–146.
6. E. Hironaka, *Boundary manifolds of line arrangements*, Math. Ann. **319** (2001), no. 1, 17–32.
7. A. Libgober, *On the homotopy type of the complement to plane algebraic curves*, J. Reine Angew. Math. **367** (1986), 103–114.
8. P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften, vol. 300, Springer-Verlag, Berlin, 1992.
9. G. Rybnikov, *On the fundamental group of the complement of a complex hyperplane arrangement*, Funct. Anal. Appl. **45** (2011), 137–148.
10. F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, II*, Invent. Math. **3** (1967), 308–333; *ibid.* **4** (1967), 87–117.
11. E. Westlund, *The boundary manifold of an arrangement*, Ph.D. thesis, university of Wisconsin - Madison, 1997.

LMA, UMR CNRS 5142 UNIVERSITE DE PAU ET DES PAYS DE L'ADOUR 64000 PAU FRANCE

*E-mail address:* `vincent.florens@univ-pau.fr`

LMA, UMR CNRS 5142 UNIVERSITE DE PAU ET DES PAYS DE L'ADOUR 64000 PAU FRANCE

*E-mail address:* `benoit.guerville@univ-pau.fr`

ICMAT: CSIC-COMPLUTENSE-AUTONOMA-CARLOS III, DEPARTAMENTO DE ALGEBRA - FACULTAD DE CC. MATEMATICAS  
- PLAZA DE LAS CIENCIAS, 3, 28040 MADRID, SPAIN

*E-mail address:* `mmarco@unizar.es`