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THE STRONG GLOBAL DIMENSION OF PIECEWISE HEREDITARY ALGEBRAS

EDSON RIBEIRO ALVARES, PATRICK LE MEUR, AND EDUARDO N. MARCOS

In the memory of Dieter Happel

ABSTRACT. Let T be a tilting object in a triangulated category which is equivalent to the bounded derived category of a finite-dimensional hereditary algebra. The text investigates the strong global dimension, in the sense of Ringel, of the opposite algebra A of the endomorphism algebra of T . This invariant is expressed in terms of the lengths of the sequences T_0, \dots, T_ℓ of tilting objects such that $T_\ell = T$, each term arises from the preceding one by a tilting mutation, and the opposite of the endomorphism algebra of T_0 is a tilted algebra. It is also expressed in terms on the hereditary abelian subcategories of the triangulated category.

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INTRODUCTION

In homological algebra and representation theory of associative algebras, the global dimension is an important invariant, particularly to measure how difficult to understand the representation theory of a

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given algebra is. For instance: a noetherian local commutative algebra is regular if and only if its global dimension is finite. Also in the bounded derived category $\mathcal{D}^b(\text{mod } H)$ of finitely generated modules over a hereditary algebra (that is, with global dimension at most 1), any object is isomorphic to a direct sum of stalk complexes, and this is also true if one replaces $\text{mod } H$ by a hereditary abelian category. And the global dimension of a given finite-dimensional algebra A is finite if and only if $\mathcal{D}^b(\text{mod } A)$ is equivalent to the homotopy category of bounded complexes of finitely generated projective A -modules. This is also equivalent to the existence of a Serre duality (equivalently, an Auslander-Reiten structure) on $\mathcal{D}^b(\text{mod } A)$.

Recall that a finite-dimensional algebra A over an algebraically closed field k is called *piecewise hereditary* if $\mathcal{D}^b(\text{mod } A)$ is equivalent, as a triangulated category, to $\mathcal{D}^b(\mathcal{H})$ where \mathcal{H} is a hereditary abelian (k -)category with split idempotents and finite-dimensional Hom-spaces, and which has a tilting object. Happel and Reiten proved [22, 23] that such a hereditary abelian category is equivalent to $\text{mod } H$ for some finite-dimensional hereditary k -algebra, or to the category of coherent sheaves over a weighted projective line [19]. Among the piecewise hereditary algebras, the quasi-tilted algebras are those isomorphic to some $\text{End}_{\mathcal{H}}(T)^{\text{op}}$ where $T \in \mathcal{H}$ is a tilting object, and this algebra is called tilted when \mathcal{H} arises from a hereditary algebra.

The study of quasi-tilted algebras has had a strong impact in representation theory and geometry. Indeed, the trivial extensions of quasi-tilted algebras have been used intensively in the classification of self-injective algebras [8, 9, 13, 14, 15, 17, 29, 33, 34, 35, 39, 40, 41]. They also are used to describe and study cluster-tilted algebras [2]. The canonical algebras (which are fundamental examples of quasi-tilted algebras) have been useful to understand module varieties [10, 11, 12] and singularities (see [31]). The description made by Happel of $\mathcal{D}^b(\mathcal{H})$ [20] plays an essential role in the use of cluster categories to categorify cluster algebras [16]. This successful use of piecewise hereditary algebras is partly due to a good knowledge of their homological properties and Auslander-Reiten structure. This is illustrated by the homological characterisation of quasi-tilted algebras [24] or the Liu-Skowroński criterium for tilted algebras (see [3]). These characterisations confirm the intuitive idea that the quasi-tilted algebras are the closest piecewise hereditary algebras to hereditary ones, and it is the main objective of this text to give theoretical or numerical criteria to determine how far a piecewise hereditary algebra is from being hereditary.

One of the milestone results on piecewise hereditary algebras is the above-mentioned description of $\mathcal{D}^b(\mathcal{H})$: Happel proved that it is the additive closure

$$\mathcal{D}^b(\mathcal{H}) \simeq \bigvee_{i \in \mathbb{Z}} \mathcal{H}[i]$$

of all the possible suspensions of objects in \mathcal{H} , and that for given $X[i] \in \mathcal{H}[i]$ and $Y[j] \in \mathcal{H}[j]$ (where $[1]$ denotes the suspension functor), the space of morphisms $\text{Hom}(X[i], Y[j])$ in $\mathcal{D}^b(\mathcal{H})$ equals $\text{Hom}_{\mathcal{H}}(X, Y)$ if $j = i$, it equals $\text{Ext}_{\mathcal{H}}^1(X, Y)$ if $j = i + 1$, and it equals 0 otherwise. If $\mathcal{D}^b(\text{mod } A) \simeq \mathcal{D}^b(\mathcal{H})$ then there exists a tilting object $T \in \mathcal{D}^b(\mathcal{H})$ (that is, an object such that $\text{Hom}(T, T[i]) = 0$ for $i \in \mathbb{Z} \setminus \{0\}$, and such that $\mathcal{D}^b(\mathcal{H})$ is the smallest full triangulated subcategory of $\mathcal{D}^b(\mathcal{H})$ containing T and stable under taking direct summands) such that $A \simeq \text{End}(T)^{\text{op}}$ as k -algebras. Following the above description of $\mathcal{D}^b(\mathcal{H})$ there exists $s \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ such that $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[s+i]$. The intuition tells that when ℓ is large, then A should be more difficult to handle. However this might not be the case. The reader is referred to an example in [21] where $\text{End}(T)^{\text{op}}$ is a hereditary algebra, $T \in \mathcal{H} \vee \mathcal{H}[1]$, $T \notin \mathcal{H}$ and $T \notin \mathcal{H}[1]$.

This phenomenon is illustrated through another milestone result on piecewise hereditary algebras proved by Happel, Rickard and Schofield [25]. It asserts that if A and H are finite-dimensional k -algebras such that H is hereditary and $\mathcal{D}^b(\text{mod } A) \simeq \mathcal{D}^b(\text{mod } H)$ as triangulated categories, then there exists a sequence of algebras $A_0 = H, \dots, A_{\ell+1} = A$ where each A_i is the (opposite of the) endomorphisms algebra of a tilting A_{i-1} -module. In such case the global dimension of A does not exceed $\ell + 2$, and the

intuition tells that if ℓ is large then A should be more complex to understand. However, in many non trivial examples where ℓ is large, A appears to have global dimension quite small.

In the two previous situations the integer ℓ fails to give a precise measure of how far is a piecewise hereditary algebra from being quasi-tilted. The main reason for this is the non-uniqueness of the pair (\mathcal{H}, ℓ) such that $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[s+i]$ in the first case, and the non-uniqueness of the sequence $(A_0, \dots, A_{\ell+1})$ in the second case. Recently a new invariant for piecewise hereditary algebras has emerged and the present text aims at giving some evidence of its relevance to give such a measure. This invariant is the *strong global dimension*. Let k be an algebraically closed field and A be a finite-dimensional k -algebra. The strong global dimension, $\text{s.gl.dim. } A \in \mathbb{N} \cup \{+\infty\}$, was defined by Ringel as follows. Let X be an indecomposable object in the homotopy category of bounded complexes of finitely generated projective A -modules. Let

$$P: \dots \rightarrow 0 \rightarrow 0 \rightarrow P^r \rightarrow P^{r+1} \rightarrow \dots \rightarrow P^{s-1} \rightarrow P^s \rightarrow 0 \rightarrow 0 \rightarrow \dots,$$

be a minimal projective resolution of X , where $P^r \neq 0$ and $P^s \neq 0$. Then define the length of X as

$$\ell(X) = s - r.$$

The strong global dimension is

$$\text{s.gl.dim. } A = \sup_X \ell(X)$$

where X runs through all such indecomposable objects. It follows from the definition that $\text{s.gl.dim. } A = 1$ if and only if A is hereditary and not semi-simple. Ringel conjectured that A is piecewise hereditary if and only if the strong global dimension of A is finite. This has been studied by several authors. The case of radical square-zero algebras was treated in [30]. This work also proves an alternative characterisation for A to be piecewise hereditary when it is tame, that is, the push-down (or extension-of-scalars) functor $\text{mod } \hat{A} \rightarrow \text{mod } T(A)$ is dense. Here $T(A) = A \rtimes \text{Hom}_k(A, k)$ is the trivial extension and \hat{A} is the repetitive algebra. Note that a general study of $\mathcal{D}^b(\text{mod } A)$ is made in [5, 6] when A has a square-zero radical. The equivalence conjectured by Ringel was proved in the general case by Happel and Zacharia [27]. As a consequence of their techniques, they prove that $\text{s.gl.dim. } A = 2$ if and only if A is quasi-tilted and not hereditary.

Let \mathcal{T} be a triangulated category which is triangle equivalent to the bounded derived category of a hereditary algebra. Let $T \in \mathcal{T}$ be a tilting object and let A be the piecewise hereditary algebra $\text{End}(T)^{\text{op}}$. The purpose of this text is therefore to answer the following questions

- to what extent does $\text{s.gl.dim. } A$ measure how far A is from being quasi-tilted?
- Is it possible to compute the strong global dimension or to characterise it?

These questions are investigated from the point of view of the two milestone results recalled above. The first main result of this text gives an answer to these questions in terms of the first one of these milestone results. The first assertion of the theorem is just a reformulation of it.

Theorem 1. *Let \mathcal{T} be a triangulated category which is triangle equivalent to the bounded derived category of a finite-dimensional hereditary k -algebra. Let $T \in \mathcal{T}$ be a tilting object. Assume that $\text{End}(T)^{\text{op}}$ is not a hereditary algebra. There exists a full and additive subcategory $\mathcal{H} \subseteq \mathcal{T}$ which is hereditary and abelian, such that the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$, and such that*

$$T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i]$$

for some integer $\ell \geq 0$. Moreover

- (1) $\text{s.gl.dim. } \text{End}(T)^{\text{op}} \leq \ell + 2$ for any such pair (\mathcal{H}, ℓ) , and
- (2) there exists such a pair (\mathcal{H}, ℓ) verifying $\text{s.gl.dim. } \text{End}(T)^{\text{op}} = \ell + 2$.

The second main result of this text is related to the second above-mentioned milestone result. It is expressed in terms of *tilting mutations* in triangulated categories. This operation appeared with the reflection functors in the representation theory of quivers [7] and with APR (Auslander-Platzbeck-Reiten) tilting modules [4], and then formalised in the study of the combinatorial properties of tilting modules (see [26, 36]). Let $T \in \mathcal{T}$ be a tilting object; let $T = T_1 \oplus T_2$ be a direct sum decomposition such that $\text{Hom}(T_2, T_1) = 0$; then there exists a triangle

$$T'_2 \rightarrow M \rightarrow T_2 \rightarrow T'_2[1]$$

where $M \rightarrow T_2$ is a minimal right add T_1 -approximation; the object $T' = T_1 \oplus T'_2$ is then tilting (see 2.1.1 below). In this text, T' is called obtained from T by tilting mutation. The second main result of this text is the following.

Theorem 2. *Let \mathcal{T} be a triangulated category which is triangle equivalent to the bounded derived category of a finite-dimensional hereditary k -algebra. Let $T \in \mathcal{T}$ be a tilting object. Assume that $\text{End}(T)^{\text{op}}$ is not hereditary. Then there exists an integer $\ell \geq 0$ and a sequence $T^{(0)}, T^{(1)}, \dots, T^{(\ell)}$ of tilting objects in \mathcal{T} such that*

- $\text{End}(T^{(0)})^{\text{op}}$ is a tilted algebra, $T^{(\ell)} = T$, and
- for every i the object $T^{(i)}$ is obtained from $T^{(i-1)}$ by a tilting mutation.

For any such sequence, $\text{s.gl.dim. End}(T)^{\text{op}} \leq \ell + 2$. Moreover, there exists such a sequence which satisfies the following

- (1) $\text{s.gl.dim. End}(T^{(i)})^{\text{op}} = 2 + i$ for every i ,
- (2) $\text{s.gl.dim. End}(T)^{\text{op}} = 2 + \ell$.

This theorem is related to the second milestone result recalled above in the following way. Let A and H be algebras such that H is hereditary and $\mathcal{D}^b(\text{mod } A) \simeq \mathcal{D}^b(\text{mod } H)$. Assume that $A_0 = H, \dots, A_{\ell+1} = A$ is a sequence of algebras such that $A_i = \text{End}_{A_{i-1}}(M^{(i-1)})^{\text{op}}$ for a tilting A_{i-1} -module $M^{(i-1)}$ for every i . Then there exist tilting objects $T^{(0)}, \dots, T^{(\ell)}$ in $\mathcal{D}^b(\text{mod } H)$ such that $A_i \simeq \text{End}(T^{(i-1)})^{\text{op}}$ for every i , and which correspond to the tilting modules $M^{(0)}, \dots, M^{(\ell)}$ under suitable triangle equivalences $\mathcal{D}^b(\text{mod } H) \simeq \mathcal{D}^b(\text{mod } A_i)$. Then $\text{End}(T^{(0)})^{\text{op}}$ is tilted, and it follows from [28, Thm. 4.2] that $\text{s.gl.dim. End}(T^{(i-1)})^{\text{op}} \leq i + 2$ for every i . When H is of finite representation type the sequence $A_0, \dots, A_{\ell+1}$ may be chosen such that $M^{(i)}$ is an APR tilting module for every i . In such a situation $T^{(i)}$ is obtained from $T^{(i-1)}$ by a tilting mutation. From this point of view, Theorem 2 expresses the strong global dimension as the supremum of the number $\ell + 2$ of terms in all possible sequences $A_0, \dots, A_{\ell+1}$.

The proof of Theorem 1 and Theorem 2 is based on the description of $\text{s.gl.dim. End}(T)^{\text{op}}$ in terms of the connected components of the Auslander-Reiten quiver (or, Auslander-Reiten components) of \mathcal{T} in which some specific direct summands of T lie. Recall that the Auslander-Reiten structure of \mathcal{T} is described in [20].

The text is therefore organised as follows. Section 1 is devoted to preliminaries. There, an equivalent definition of $\text{s.gl.dim. End}(T)^{\text{op}}$ is given in terms of the triangulated structure of \mathcal{T} (and without using complexes or their lengths). This reformulation appeared first in [1]. It permits to give lower and upper bounds on the strong global dimension. Section 2 investigates the behaviour of strong global dimension under two classical operations. On the one hand the section compares the strong global dimensions of $\text{End}(T)^{\text{op}}$ and $\text{End}(T')^{\text{op}}$ when T' is obtained from T by tilting mutation. On the other hand it relates the strong global dimension arising from tilting objects in \mathcal{T} to that arising from tilting objects in \mathcal{T}' provided that \mathcal{T} and \mathcal{T}' are related by a pair of bi-adjoint functors. This is used to prove that the strong global dimension is unchanged when the algebra is replaced by a finite Galois covering with group whose order is non-zero in k . Section 3 is devoted to describing $\text{s.gl.dim. End}(T)^{\text{op}}$ when $T \in \mathcal{T}$ is a tilting

object, according to the Auslander-Reiten components in which T has indecomposable direct summands. Finally, Section 4 is devoted to the proof of Theorem 1 and Theorem 2.

Throughout the text k is an algebraically closed field, and \mathcal{T} is a triangulated k -category which is triangle equivalent to the bounded derived category of a finite-dimensional hereditary algebra. Given $X, Y \in \mathcal{T}$, the space $\text{Hom}_{\mathcal{T}}(X, Y)$ is denoted by $\text{Hom}(X, Y)$, and $\text{Ext}^i(X, Y)$ stands for $\text{Hom}(X, Y[i])$ for every $i \in \mathbb{Z}$. Given an additive category \mathcal{A} , the class of indecomposable objects in \mathcal{A} is denoted by $\text{ind } \mathcal{A}$. The standard duality functor $\text{Hom}(-, k)$ is denoted by D . By an *Auslander-Reiten component* of \mathcal{T} is meant a connected component of the Auslander-Reiten quiver of \mathcal{T} . By a *transjective component* is meant an Auslander-Reiten component which has only finitely many τ -orbits.

The reader is referred to [38, Chap. XIII] and [3, Chap. XVII] for a general account on the Auslander-Reiten structure (tubes, quasi-simples, components of shape $\mathbb{Z}A_{\infty}$) of hereditary algebras of tame and wild representation type, respectively.

1. PRELIMINARIES

This section gives a characterisation of the strong global dimension using only the triangulated structure of \mathcal{T} . This is applied to give, for a fixed tilting object $T \in \mathcal{T}$, upper and lower bounds on $\text{s.gl.dim. End}(T)^{\text{op}}$.

1.1. The definition of the strong global dimension revisited. The following was proved in [1, Lem. 5.6]. This is a description of the strong global dimension in terms of the morphisms in \mathcal{T} .

Let $T \in \mathcal{T}$. Let $X \in \mathcal{T}$. Define $\ell_T^+(X), \ell_T^-(X) \in \mathbb{Z} \cup \{-\infty, +\infty\}$ as follows

$$\begin{cases} \ell_T^+(X) = \sup \{n \in \mathbb{Z} \mid \text{Hom}(X, T[n]) \neq 0\}, \\ \ell_T^-(X) = \inf \{n \in \mathbb{Z} \mid \text{Hom}(T[n], X) \neq 0\}. \end{cases}$$

Proposition. *Let $T \in \mathcal{T}$ be a tilting object. Let $A = \text{End}(T)^{\text{op}}$. Then $-\infty < \ell_T^-(X) \leq \ell_T^+(X) < +\infty$ for every $X \in \mathcal{T}$ indecomposable, and*

$$\text{s.gl.dim. } A = \sup \{\ell_T^+(X) - \ell_T^-(X) \mid X \in \mathcal{T} \in \text{ind } \mathcal{T}\}.$$

In the sequel, if T is a tilting object in a hereditary triangulated category \mathcal{T} , and if $X \in \text{ind } \mathcal{T}$, then $\ell_T(X)$ denotes $\ell_T^+(X) - \ell_T^-(X)$. Also, $\text{s.gl.dim. } T$ denotes $\text{s.gl.dim. End}(T)^{\text{op}}$.

1.2. An upper bound on the strong global dimension.

Proposition. *Let $\mathcal{H} \subseteq \mathcal{T}$ be a full and additive subcategory which is hereditary and abelian, and such that the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$. Let $T \in \mathcal{T}$ be a tilting object. Assume that $\ell \geq 0$ is an integer such that*

$$T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i].$$

Then $\text{s.gl.dim. } T \leq \ell + 2$.

Proof. Let $X \in \mathcal{T}$ be indecomposable. Up to a shift, there is no loss of generality in assuming that $\ell_T^-(X) = 0$. Let $d = \ell_T^+(X) = \ell_T(X)$. Therefore there exist indecomposable direct summands T_1, T_2 of T such that

$$\text{Hom}(T_1, X) \neq 0 \quad \text{and} \quad \text{Hom}(X, T_2[d]) \neq 0.$$

Besides there exist integers $i \in \mathbb{Z}$ and $j, k \in \{0, \dots, \ell\}$ such that

$$X \in \mathcal{H}[i], \quad T_1 \in \mathcal{H}[j], \quad \text{and} \quad T_2 \in \mathcal{H}[k].$$

Therefore $0 \leq i - j \leq 1$ and $0 \leq (d + k) - i \leq 1$, and hence

$$d \leq 1 + i - k = 1 + \underbrace{(i - j)}_{\leq 1} + \underbrace{(j - k)}_{\leq \ell} \leq \ell + 2.$$

□

1.3. Lower bounds on the strong global dimension.

1.3.1. Let $\mathcal{H} \subseteq \mathcal{T}$ be an additive and full subcategory which is hereditary and abelian, and such that the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$. Let $T \in \mathcal{T}$ be a tilting object, and let $\ell \geq 0$ be an integer such that there exists indecomposable direct summands M_0, M_1 of T satisfying the following

- $M_0 \in \mathcal{H}$, and
- $M_1 \in \mathcal{H}[\ell]$ lies in a non-transjective Auslander-Reiten component of $\mathcal{H}[\ell]$.

Since \mathcal{T} is triangle equivalent to the derived category of a hereditary abelian category with tilting object, the non-transjective Auslander-Reiten components are either tubes or of the shape $\mathbb{Z}A_\infty$.

Let $X = \tau^{-1}M_0[\ell + 1]$ and $Y = \tau M_1[1]$. Let $Z \rightarrow M_1$ be a minimal right almost split morphism in \mathcal{T} . In particular, Z is indecomposable if and only if M_1 is quasi-simple.

Lemma. *Under the setting described previously, the following holds true.*

- (1) If $\text{Hom}(X, Y) \neq 0$, then $\ell_T(\tau^{-1}M_0) \geq \ell + 2$. In particular $\text{s.gl.dim. } T \geq \ell + 2$.
- (2) If $\text{Ext}^1(Y, X) \neq 0$, then $\ell_T(\tau Z) \geq \ell + 2$ or $\ell_T(\tau^2 M_1) \geq \ell + 2$, according to whether M_1 is quasi-simple or is not. In particular, $\text{s.gl.dim. } T \geq \ell + 2$.
- (3) If $\text{Hom}(X, Y) = 0$ and $\text{Hom}(Y, X) \neq 0$, then $\ell_T(\tau Z) \geq \ell + 1$ or $\ell_T(\tau^2 M_1) \geq \ell + 1$, according to whether M_1 is quasi-simple or is not. In particular $\text{s.gl.dim. } T \geq \ell + 1$.
- (4) If $\text{Ext}^1(X, Y) \neq 0$, then $\ell_T(\tau M_1) \geq \ell + 1$. In particular $\text{s.gl.dim. } T \geq \ell + 1$.

Proof. (1) Serre duality gives

$$\begin{cases} 0 \neq \text{Hom}(X, Y) = \text{Hom}(\tau^{-1}M_0[\ell + 1], \tau M_1[1]) = D\text{Hom}(M_1, \tau^{-1}M_0[\ell + 1]), \\ 0 \neq \text{Hom}(\tau^{-1}M_0[\ell + 1], \tau^{-1}M_0[\ell + 1]) = D\text{Hom}(\tau^{-1}M_0[\ell + 1], M_0[\ell + 2]). \end{cases}$$

Thus $\ell_T(\tau^{-1}M_0[\ell + 1]) \geq \ell + 2$.

(2) The hypothesis imply that $0 \neq \text{Ext}^1(Y, X) = \text{Hom}(\tau^2 M_1, M_0[\ell + 1])$, and hence $\ell_T^+(\tau^2 M_1) \geq \ell + 1$.

Assume first that M_1 is not quasi-simple. Therefore $\text{Hom}(\tau M_1, M_1) \neq 0$. Serre duality then implies $\text{Hom}(M_1[-1], \tau^2 M_1)$. Hence $\ell_T^-(\tau^2 M_1) \leq -1$, and thus $\ell_T(\tau^2 M_1) \geq \ell + 2$.

Assume now that M_1 is quasi-simple. Since M_1 lies in an Auslander-Reiten component of $\mathcal{H}[\ell]$ which is a tube or of the shape $\mathbb{Z}A_\infty$, there exists an almost split triangle $\tau^2 M_1 \rightarrow \tau Z \rightarrow \tau M_1 \rightarrow \tau^2 M_1[1]$. Since $M_0[\ell + 1] \in \mathcal{H}[\ell + 1]$ and $\tau^2 M_1 \in \mathcal{H}[\ell]$, the hypothesis $0 \neq \text{Ext}^1(Y, X)$ entails (see above):

$$0 \neq \text{Ext}^1(Y, X) = \text{Hom}(\tau^2 M_1, M_0[\ell + 1]) \subseteq \text{rad}(\tau^2 M_1, M_0[\ell + 1]).$$

In particular there exists a non-zero morphism $\tau^2 M_1 \rightarrow M_0[\ell + 1]$ which factors through $\tau^2 M_1 \rightarrow \tau Z$. Hence $\text{Hom}(\tau Z, M_0[\ell + 1]) \neq 0$, and therefore

$$\ell_T^+(\tau Z) \geq \ell + 1.$$

Moreover, using Serre duality yields

$$0 \neq \text{Hom}(\tau Z, \tau M_1) = D\text{Hom}(M_1[-1], \tau Z).$$

Hence $\ell_T^-(\tau Z) \leq -1$, and thus $\ell_T(\tau Z) \geq \ell + 2$.

(3) The hypotheses imply that $0 \neq \text{Hom}(Y, X) = \text{Hom}(\tau^2 M_1, M_0[\ell])$. Hence $\ell_T^+(\tau^2 M_1) \geq \ell$.

Assume first that M_1 is not quasi-simple. The argument used in (2) also applies here and shows that $\ell_T^-(\tau^2 M_1) \leq -1$. Thus $\ell_T(\tau^2 M_1) \geq \ell + 1$.

Assume now that M_1 is quasi-simple. It follows from the hypotheses that

$$0 = \text{Hom}(X, Y) = \text{Hom}(\tau^{-1} M_0[\ell + 1], \tau M_1[1]) = \text{Hom}(M_0[\ell], \tau^2 M_1).$$

In particular $M_0[\ell] \not\cong \tau^2 M_1$, and therefore (see above)

$$0 \neq \text{Hom}(Y, X) = \text{Hom}(\tau^2 M_1, M_0[\ell]) \subseteq \text{rad}(\tau^2 M_1, M_0[\ell]).$$

Hence there exists a non-zero morphism $\tau^2 M_1 \rightarrow M_0[\ell]$ which factors through $\tau^2 M_1 \rightarrow \tau Z$. Therefore $\text{Hom}(\tau Z, M_0[\ell]) \neq 0$, and thus

$$\ell_T^+(\tau Z) \geq \ell.$$

The arguments used in (2) to prove that $\ell_T^-(\tau Z) \leq -1$ also apply here. Thus $\ell_T(\tau Z) \geq \ell + 1$.

(4) Serre duality gives

$$\begin{cases} 0 \neq \text{Ext}^1(X, Y) = \text{Hom}(\tau^{-1} M_0[\ell + 1], \tau M_1[2]) = D\text{Hom}(\tau M_1, M_0[\ell]), \\ 0 \neq \text{Hom}(M_1, M_1) = D\text{Hom}(M_1[-1], \tau M_1). \end{cases}$$

Thus $\ell_T(\tau M_1) \geq \ell + 1$. □

1.3.2. The following lemma is fundamental in the description of $\text{s.gl.dim. } T$ using the Auslander-Reiten structure of \mathcal{T} .

Lemma. *Under the setting described in 1.3.1, assume that both M_0 and M_1 lie in non-transjective Auslander-Reiten components of \mathcal{T} .*

- (1) *Then $\text{s.gl.dim. } T \geq \ell + 1$.*
- (2) *If M_0 lies in a tube of the Auslander-Reiten quiver of \mathcal{T} , and if M_1 lies in the ℓ -th suspension of this tube, then $\text{s.gl.dim. } T \geq \ell + 2$.*
- (3) *If both M_0 and M_1 lie in Auslander-Reiten components of the form $\mathbb{Z}A_\infty$, then $\text{s.gl.dim. } T \geq \ell + 2$.*

Proof. (1) Let \mathcal{C} be the Auslander-Reiten component of \mathcal{T} such that $M_0[\ell + 1] \in \mathcal{C}$. Let Γ be the unique tranjective Auslander-Reiten component of \mathcal{T} such that

$$(\forall V \in \mathcal{C}) (\exists U \in \Gamma) \quad \text{Hom}(U, V) \neq 0.$$

Let \mathcal{R} be the disjoint union of the non-transjective Auslander-Reiten components of \mathcal{T} such that

$$(\forall V \in \mathcal{R}) (\exists U \in \Gamma) \quad \text{Hom}(U, V) \neq 0.$$

Therefore

- $\mathcal{C} \subseteq \mathcal{R}$,
- \mathcal{R} is the family of regular Auslander-Reiten components of $\mathcal{H}[\ell + 1]$, and
- $\mathcal{R}[-1]$ is the family of regular Auslander-Reiten components of $\mathcal{H}[\ell]$; in particular $M_1 \in \mathcal{R}[-1]$.

In view of proving that $\text{s.gl.dim. } T \geq \ell + 1$, it is useful to prove the existence of $S_0 \in \Gamma$ such that

$$\text{Hom}(M_1, S_0) \neq 0 \quad \text{and} \quad \text{Hom}(S_0, M_0[\ell + 1]) \neq 0.$$

First there exists a slice Σ un Γ such that

$$(\forall S \in \Sigma) \quad \text{Hom}(S, M_0[\ell + 1]) \neq 0.$$

Next define the full subcategory $\mathcal{H}' \subseteq \mathcal{T}$ as $\mathcal{H}' = \{V \in \mathcal{T} \mid \text{Ext}^i(V, S) = 0, (\forall i \neq 0) (\forall S \in \Sigma)\}$. Then

- \mathcal{H}' is hereditary and abelian,
- the indecomposable injectives of \mathcal{H}' are the objects in Σ , up to isomorphism, and
- $\mathcal{R}[-1]$ is the family of regular Auslander-Reiten components of \mathcal{H}' ; in particular $M_1 \in \mathcal{H}'$.

Thus there exists $S_0 \in \Sigma$ such that $\text{Hom}(M_1, S_0) \neq 0$, and, by hypothesis, $\text{Hom}(S_0, M_0[\ell + 1]) \neq 0$.

(2) It follows from the hypothesis that there exists a tube \mathcal{C} of the Auslander-Reiten quiver of \mathcal{T} such that $Y = \tau M_1[1] \in \mathcal{C}$ and $X = \tau^{-1} M_0[\ell + 1] \in \mathcal{C}$. Moreover there exist infinite sectional paths in \mathcal{C}

$$X \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \quad \text{and} \quad \cdots \rightarrow \bullet \rightarrow \bullet \rightarrow Y.$$

Since \mathcal{C} is a tube, the two paths intersect. Hence there exist $S \in \mathcal{C}$ and sectional paths in \mathcal{C}

$$\tau^{-1} M_0[\ell + 1] \rightarrow \cdots \rightarrow S \quad \text{and} \quad S \rightarrow \cdots \rightarrow \tau M_1[1].$$

Since the composition of morphisms along a sectional path does not vanish, there exist non-zero morphisms $\tau^{-1} M_0[\ell + 1] \rightarrow S$ and $S \rightarrow \tau M_1[1]$. Using Serre duality, this implies that $\text{Hom}(S, M_0[\ell + 2]) \neq 0$ and $\text{Hom}(M_1, S) \neq 0$. Thus $\ell_T(S) \geq \ell + 2$.

(3) By hypothesis, $X = \tau^{-1} M_0[\ell + 1]$ and $Y = \tau M_1[1]$ both lie in $\mathcal{H}[\ell + 1]$. Let \mathcal{C}_X and \mathcal{C}_Y be the Auslander-Reiten components of $\mathcal{H}[\ell + 1]$ such that $X \in \mathcal{C}_X$ and $Y \in \mathcal{C}_Y$. Then \mathcal{C}_X and \mathcal{C}_Y are of the shape $\mathbb{Z}A_\infty$. In particular, \mathcal{H} is equivalent to the module category of a finite-dimensional hereditary algebra of wild type. In the rest of the proof, \mathcal{H} is considered as a full subcategory of \mathcal{T} . However, it is useful to mention when a morphism between objects in \mathcal{H} is epimorphic (or monomorphic) as a morphism in the abelian category \mathcal{H} .

There exist quasi-simples $S_X \in \mathcal{C}_X$ and $S_Y \in \mathcal{C}_Y$, together with sectional paths in \mathcal{C}_X and \mathcal{C}_Y , respectively:

$$\begin{cases} X \rightarrow \cdots \rightarrow S_X & \text{made of irreducible epimorphisms in } \mathcal{H}, \\ S_Y \rightarrow \cdots \rightarrow Y & \text{made of irreducible monomorphisms in } \mathcal{H}. \end{cases}$$

Taking the composite morphisms yields non-zero morphisms $X \rightarrow S_X$ and $S_Y \rightarrow Y$ which are epimorphic and monomorphic, respectively, as morphisms in \mathcal{H} . Let S be any quasi-simple regular object in \mathcal{H} (e.g. $S = S_Y$). Since S , S_Y and S_X lie in Auslander-Reiten components of \mathcal{H} of the form $\mathbb{Z}A_\infty$, there exist integers $n_1, n_2 \geq 0$ such that

$$\begin{cases} (\forall n \geq n_1) & \text{Hom}(S_X, \tau^n S) \neq 0 \quad \text{and} \\ (\forall n \geq n_2) & \text{Hom}(S, \tau^n S_Y) \neq 0. \end{cases}$$

Let $n = \max(n_1, n_2)$. There are infinite sectional paths

$$\tau^n S \rightarrow X_2 \rightarrow X_3 \rightarrow \cdots \quad \text{and} \quad \cdots \rightarrow Y_3 \rightarrow Y_2 \rightarrow \tau^{-n} S$$

such that $X_n = Y_n$, and such that the former (or, the latter) is made of irreducible monomorphisms (or, epimorphisms, respectively) in \mathcal{H} . Taking the composite morphisms yields non-zero morphisms

$\tau^n S \rightarrow X_n$ and $X_n \rightarrow \tau^{-n} S$ which are monomorphic and epimorphic, respectively, as morphisms in \mathcal{H} . Since $\text{Hom}(S_X, \tau^n S) \neq 0$ and $\text{Hom}(\tau^{-n} S, S_Y) \neq 0$, there are non-zero composite morphisms in \mathcal{H}

$$X \rightarrow S_X \xrightarrow{\neq 0} \tau^n S \hookrightarrow X_n \quad \text{and} \quad X_n \rightarrow \tau^{-n} S \xrightarrow{\neq 0} S_Y \hookrightarrow Y.$$

Hence $\text{Hom}(X, X_n) \neq 0$ and $\text{Hom}(X_n, Y) \neq 0$, and thus (using Serre duality) $\text{Hom}(M_1, X_n) \neq 0$ and $\text{Hom}(X_n, M_0[\ell + 2]) \neq 0$. This proves that $\ell_T(X_n) \geq \ell + 2$. \square

1.3.3. Let Γ be a transjective Auslander-Reiten component of \mathcal{T} . Let Σ be a slice of Γ . Let $\mathcal{H} \subseteq \mathcal{T}$ be the full subcategory

$$\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma) (\forall i \neq 0) \text{Hom}(S, X) = 0\}.$$

Let $T \in \mathcal{T}$ be a tilting object and let $\ell \geq 1$ be an integer such that

- the sources S_1, \dots, S_n of the full subquiver Σ of Γ are all indecomposable summands of T ,
- there exists an indecomposable summand L of T lying in $\mathcal{H}[\ell]$.

Lemma. *Under the previous setting, there exists $M \in \tau^{-1}\Sigma[\ell + 1]$, and there exist non-zero morphisms $L \rightarrow M$ and $M \rightarrow \bigoplus_{i=1}^n S_i[\ell + 2]$. Hence $\ell_T(M) \geq \ell + 2$. In particular, $\text{s.gl.dim. } T \geq \ell + 2$.*

Proof. It is useful to prove first that $L \in \tau^{-2}\mathcal{H}[\ell]$. For this purpose note that

$$(\text{ind } \mathcal{H}[\ell]) \setminus (\text{ind } \tau^{-2}\mathcal{H}[\ell]) = \Sigma[\ell] \cup \tau^{-1}\Sigma[\ell],$$

as sets of indecomposable objects. Since $L \in \mathcal{H}[\ell]$, the claim therefore deals with proving that $L \notin \Sigma[\ell]$ and $L \notin \tau^{-1}\Sigma[\ell]$. Using $\text{Hom}(T[\ell], T) = 0$, $\text{Hom}(T, T[\ell + 1]) = 0$ and Serre duality implies that

$$\text{Hom}\left(\bigoplus_{i=1}^n S_i[\ell], L\right) = 0 \quad \text{and} \quad \text{Hom}\left(\bigoplus_{i=1}^n \tau^{-1}S_i[\ell], L\right) = 0.$$

Since $S_1[\ell], \dots, S_n[\ell]$ (or $\tau^{-1}S_1[\ell], \dots, \tau^{-1}S_n[\ell]$) are the sources of the slice $\Sigma[\ell]$ (or $\tau^{-1}\Sigma[\ell]$, respectively), this entails that $L \notin \Sigma[\ell]$ and $L \notin \tau^{-1}\Sigma[\ell]$. This proves the claim: $L \in \tau^{-2}\mathcal{H}[\ell]$.

The category $\tau^{-2}\mathcal{H}[\ell]$ is abelian and its indecomposable injectives are the objects in $\tau^{-1}\Sigma[\ell + 1]$, up to isomorphism. Hence

$$(\exists M \in \tau^{-1}\Sigma[\ell + 1]) \quad \text{Hom}(L, M) \neq 0.$$

Besides $\text{Hom}(\bigoplus_{i=1}^n \tau^{-1}S_i[\ell + 1], M) \neq 0$ for $\tau^{-1}S_i[\ell + 1], \dots, \tau^{-1}S_n[\ell + 1]$ are the sources of the slice $\tau^{-1}\Sigma[\ell + 1]$ of $\Gamma[\ell + 1]$. Using Serre duality this implies that $\text{Hom}\left(M, \bigoplus_{i=1}^n S_i[\ell + 2]\right) \neq 0$. \square

2. THE STRONG GLOBAL DIMENSION AND SOME CLASSICAL OPERATIONS OF REPRESENTATION THEORY

2.1. Behaviour under tilting mutation.

2.1.1. *Setting.* The following setting is used throughout the subsection. Let $T \in \mathcal{T}$ be a tilting object. Suppose that there is a decomposition

$$T = T_1 \oplus T_2$$

such that $\text{Hom}(T_2, T_1) = 0$. Let $M \rightarrow T_2$ be a minimal right add T_1 -approximation. It fits into a triangle

$$(\Delta) \quad T'_2 \rightarrow M \rightarrow T_2 \rightarrow T'_2[1].$$

Let $T' = T_1 \oplus T'_2$.

Proposition. *Under the previous setting, T' is a tilting object in \mathcal{T} .*

Proof. Because of (Δ) , the smallest triangulated subcategory of \mathcal{T} containing T' and stable under direct summands is \mathcal{T} . Hence it suffices to prove that $\text{Hom}(T', T'[i]) = 0$ for every $i \neq 0$.

First, $\text{Hom}(T_1, T_1[i]) = 0$ for every $i \neq 0$ because T is tilting. Next there is an exact sequence obtained by applying $\text{Hom}(T_1, -)$ to (Δ) , for every i

$$\text{Hom}(T_1, M[i - 1]) \rightarrow \text{Hom}(T_1, T_2[i - 1]) \rightarrow \text{Hom}(T_1, T'_2[i]) \rightarrow \text{Hom}(T_1, M[i]).$$

Since T is tilting and $M \rightarrow T_2$ is an add T_1 -approximation it follows that

$$(\forall i \neq 0) \quad \text{Hom}(T_1, T'_2[i]) = 0.$$

Next there is an exact sequence obtained by applying $\text{Hom}(-, T_1[i])$ to (Δ) for every i

$$\text{Hom}(M, T_1[i]) \rightarrow \text{Hom}(T'_2, T_1[i]) \rightarrow \text{Hom}(T_2, T_1[i + 1]).$$

Since T is tilting and $\text{Hom}(T_2, T_1) = 0$ it follows that

$$(\forall i \neq 0) \quad \text{Hom}(T'_2, T_1[i]) = 0.$$

Next there is an exact sequence obtained by applying $\text{Hom}(-, T_2[i])$ to (Δ) for every i

$$\text{Hom}(M, T_2[i]) \rightarrow \text{Hom}(T'_2, T_2[i]) \rightarrow \text{Hom}(T_2, T_2[i+1]).$$

Hence $\text{Hom}(T'_2, T_2[i]) = 0$ for every $i \neq 0, -1$ because T is tilting. Finally there is an exact sequence obtained by applying $\text{Hom}(T'_2, -)$ to (Δ) for every i

$$\text{Hom}(T'_2, M[i-1]) \rightarrow \text{Hom}(T'_2, T_2[i-1]) \rightarrow \text{Hom}(T'_2, T_2[i]) \rightarrow \text{Hom}(T'_2, M[i]).$$

In this exact sequence,

- (i) $\text{Hom}(T'_2, M[i]) = 0$ for every $i \neq 0$ as observed earlier and because $M \in \text{add}(T_1)$;
- (ii) $\text{Hom}(T'_2, T_2[i]) = 0$ for every $i \neq 0, -1$ as explained earlier;
- (iii) The leftmost arrow is an epimorphism if $i = 1$. Indeed let $u \in \text{Hom}(T'_2, T_2)$. Consider the following diagram where the top horizontal row is the suspension of (Δ)

$$\begin{array}{ccccccc} T_2[-1] & \xrightarrow{a} & T'_2 & \xrightarrow{b} & M & \xrightarrow{c} & T_2 \\ & & \downarrow u & \swarrow v & \downarrow w & & \\ & & T_2 & \xleftarrow{c} & M & & \end{array}$$

Then $ua = 0$ because T is tilting. Hence there exists $v \in \text{Hom}(M, T_2)$ such that $u = vb$. Besides there exists $w \in \text{Hom}(M, M)$ such that $v = cw$ because $M \rightarrow T_2$ is an $\text{add } T_1$ -approximation. In other words $u = cwb$ lies in the image of the mapping $\text{Hom}(c, -): \text{Hom}(T'_2, M) \rightarrow \text{Hom}(T'_2, T_2)$.

It follows from (i), (ii) and (iii) that

$$(\forall i \neq 0) \quad \text{Hom}(T'_2, T_2[i]) = 0.$$

These considerations prove that T' is tilting. □

The following paragraphs deal with the comparison of $\text{s.gl.dim } T$ and $\text{s.gl.dim } T'$ under the above setting.

2.1.2. Let Z be an indecomposable direct summand of T_2 . Let $N \rightarrow Z$ be a minimal right $\text{add } T_1$ -approximation. Then there exists a triangle

$$X \rightarrow N \rightarrow Z \rightarrow X[1].$$

The following result relates the two triangles $T'_2 \rightarrow M \rightarrow T_2 \rightarrow T'_2[1]$ and $X \rightarrow N \rightarrow Z \rightarrow X[1]$.

Lemma. *Under the previous setting the following assertions hold true*

- (1) X is indecomposable;
- (2) X is a direct summand of T'_2 ;
- (3) if $\text{Hom}(T_1, Z) = 0$ then $X = Z[-1]$;
- (4) if $\text{Hom}(T_1, Z) \neq 0$ then $\text{Hom}(X, N) \neq 0$ and $\text{Hom}(Z[-1], X) \neq 0$.

Proof. (1) Let $e \in \text{End}(X)$ be an idempotent. Since T is tilting and $N, Z \in \text{add } T$ it follows that $\text{Hom}(Z[-1], N) = 0$. Hence there exists $f \in \text{End}(N)$ such that the following diagram (whose rows are triangles) is commutative

$$\begin{array}{ccccccc} Z[-1] & \longrightarrow & X & \longrightarrow & N & \longrightarrow & Z \\ & & \downarrow e & & \downarrow f & & \\ Z[-1] & \longrightarrow & X & \longrightarrow & N & \longrightarrow & Z. \end{array}$$

Therefore there exists $g \in \text{End}(Z)$ such that the following diagram commutes

$$\begin{array}{ccccccc} Z[-1] & \longrightarrow & X & \longrightarrow & N & \longrightarrow & Z \\ g[-1] \downarrow & & \downarrow e & & \downarrow f & & \downarrow g \\ Z[-1] & \longrightarrow & X & \longrightarrow & N & \longrightarrow & Z. \end{array}$$

Since Z is indecomposable then g is either invertible or nilpotent. If g is invertible then so is f , for $N \rightarrow Z$ is a minimal right add T_1 -approximation, and thus so is e ; since $e^2 = e$ this implies that $e = 1 \in \text{End}(X)$. If g is nilpotent then there exists an integer $n \geq 0$ such that $g^n = 0$, and therefore the following diagram commutes

$$\begin{array}{ccccccc} Z[-1] & \longrightarrow & X & \longrightarrow & N & \longrightarrow & Z \\ \parallel & & \downarrow 1-e^n & & \downarrow 1-f^n & & \parallel \\ Z[-1] & \longrightarrow & X & \longrightarrow & N & \longrightarrow & Z. \end{array}$$

Note that $1 - e^n = 1 - e$. The above argument where g is invertible still applies here, and entails that $1 - e = 1$, that is $e = 0$. This proves that X is indecomposable.

(2) Let $Z \rightarrow T_2$ be a section and $T_2 \rightarrow Z$ be its retraction. Since $N \rightarrow Z$ and $M \rightarrow T_2$ are add T_1 -approximations there is a commutative diagram whose arrows are triangles

$$\begin{array}{ccccccc} X & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T'_2 & \longrightarrow & M & \longrightarrow & T_2 & \longrightarrow & T'_2[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

Since $N \rightarrow Z$ is right minimal and the composite morphism $Z \rightarrow T_2 \rightarrow Z$ is identity, it follows that the composite morphism $N \rightarrow M \rightarrow N$ is an isomorphism, and hence so is the composite morphism $X \rightarrow T'_2 \rightarrow X$. Therefore $T'_2 \rightarrow X$ is a retraction, and thus X is a direct summand of T'_2 .

(3) If $\text{Hom}(T_1, Z) = 0$ then $N = 0$, and thus $X = Z[-1]$.

(4) Assume that $\text{Hom}(T_1, Z) \neq 0$. Therefore $N \neq 0$. Since X and $Z[-1]$ are indecomposable then $Z[-1] \not\cong X \oplus N[-1]$; moreover $N[-1] \not\cong Z[-1] \oplus X[-1]$, for $\text{Hom}(T_2, T_1) = 0$, and $Z \in \text{add } T_2$ and $N \in \text{add } T_1$. Therefore, in the following triangles

$$\begin{array}{l} N[-1] \rightarrow Z[-1] \rightarrow X \rightarrow N, \text{ and} \\ X[-1] \rightarrow N[-1] \rightarrow Z[-1] \rightarrow X, \end{array}$$

the morphisms $X \rightarrow N$ and $Z[-1] \rightarrow X$ are non-zero. □

2.1.3. The respective positions of T'_2 and M in \mathcal{T} .

Lemma. (1) If X is an indecomposable direct summand of T'_2 then $X \in \text{add } M$ if $\text{Hom}(T_2[-1], X) = 0$; if $\text{Hom}(X, M) = 0$ then $X \in \text{add}(T_2[-1])$.

(2) Let $\mathcal{H} \subseteq \mathcal{T}$ be an additive and full subcategory which is hereditary abelian and such that the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$. Let $\ell \geq 1$ be an integer such that

$$T_1 \in \bigvee_{i=0}^{\ell-1} \mathcal{H}[i] \quad \text{and} \quad T_2 \in \mathcal{H}[\ell].$$

Then $M \in \mathcal{H}[\ell - 1]$ and $T'_2 \in \mathcal{H}[\ell - 1]$.

Proof. (1) Let $X \hookrightarrow T'_2$ be a section and $T'_2 \twoheadrightarrow X$ be a retraction. In the following diagram the row is a triangle

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow & & \\ T_2[-1] & \longrightarrow & T'_2 & \longrightarrow & M \longrightarrow T_2 \\ & & \downarrow & & \\ & & X & & \end{array}$$

Therefore if $\text{Hom}(X, M) = 0$ then the section factors through $T_2[-1] \rightarrow T'_2$; this gives a section $X \rightarrow T_2[-1]$; thus $X \in \text{add}(T_2[-1])$. If $\text{Hom}(T_2[-1], X) = 0$ then the retraction factors through $T'_2 \rightarrow M$; this gives a retraction $M \rightarrow X$; thus $X \in \text{add } M$.

(2) Let Y be an indecomposable direct summand of M . Then $\text{Hom}(Y, T_2) \neq 0$ for $M \rightarrow T_2$ is right minimal. Since $T_2 \in \mathcal{H}[\ell]$ and $M \in \bigvee_{i=0}^{\ell-1} \mathcal{H}[i]$ it follows that $Y \in \mathcal{H}[\ell - 1]$. Thus $M \in \mathcal{H}[\ell - 1]$.

Let $X \in \text{add } T'_2$ be indecomposable. Note that $M, T_2[-1] \in \mathcal{H}[\ell - 1]$. Hence, if $X \in \text{add}(M \oplus T_2[-1])$, then $X \in \mathcal{H}[\ell - 1]$. Otherwise, it follows from (1) that $\text{Hom}(X, M) \neq 0$ and $\text{Hom}(T_2[-1], X) \neq 0$. This implies that $X \in \mathcal{H}[\ell - 1]$. Thus $T'_2 \in \mathcal{H}[\ell - 1]$. \square

2.1.4. *The length with respect to T' expressed using the length with respect to T .*

Lemma. *Let $X \in \mathcal{T}$. Assume (up to a suspension) that $\ell_{T'}^-(X) = 0$. Let $\ell_{T'}(X) = \ell$. Then $\ell_T(X)$ is given by the table below.*

	$\text{Ext}^1(T_2, X) \neq 0$	$\text{Ext}^1(T_2, X) = 0$
$\text{Hom}(X, T_1[\ell]) \neq 0$	$\ell_T^-(X) = -1, \ell_T^+(X) = \ell - 1, \text{ and } \ell_T(X) = \ell$	$\ell_T^-(X) = 0, \ell_T^+(X) = \ell - 1, \text{ and } \ell_T(X) = \ell - 1$
$\text{Hom}(X, T_1[\ell]) = 0$	$\ell_T^-(X) = -1, \ell_T^+(X) = \ell, \text{ and } \ell_T(X) = \ell + 1$	$\ell_T^-(X) = 0, \ell_T^+(X) = \ell, \text{ and } \ell_T(X) = \ell$

Proof. All the exact sequences in this proof are obtained applying either $\text{Hom}(X, -)$ or $\text{Hom}(-, X)$ to (Δ) . First note that $\text{Hom}(T_1[i], X) = \text{Hom}(M[i], X) = 0$ if $i < 0$, for $T_1, M \in \text{add } T'$ and $\ell_{T'}^-(X) = 0$. From the exact sequence

$$\underbrace{\text{Hom}(T'_2[i+1], X)}_{= 0 \text{ if } i < -1} \rightarrow \text{Hom}(T_2[i], X) \rightarrow \underbrace{\text{Hom}(M[i], X)}_{= 0 \text{ if } i < 0}$$

it follows that $\text{Hom}(T_2[i], X) = 0$ if $i < -1$, and therefore $\ell_T^-(X) \geq -1$. In particular

$$\text{Hom}(T_2[-1], X) \neq 0 \Rightarrow \ell_T^-(X) = -1.$$

Assume that $\text{Hom}(T_2[-1], X) = 0$. Then $\text{Hom}(T_1, X) \neq 0$. Indeed, by absurd, if $\text{Hom}(T_1, X) = 0$ then $\text{Hom}(T'_2, X) \neq 0$ because $\ell_{T'}^-(X) = 0$; this contradicts the exactness of the sequence

$$\underbrace{\text{Hom}(M, X)}_{=0} \rightarrow \text{Hom}(T'_2, X) \rightarrow \underbrace{\text{Hom}(T_2[-1], X)}_{=0}.$$

Therefore $\text{Hom}(T_1, X) \neq 0$, and thus

$$\text{Hom}(T_2[-1], X) = 0 \Rightarrow \ell_{T'}^-(X) = 0.$$

Next note that $\text{Hom}(X, T_1[i]) = \text{Hom}(X, M[i]) = \text{Hom}(X, T'_2[i]) = 0$ if $i > \ell$, for $T_1, M, T'_2 \in \text{add } T'$ and $\ell_{T'}^+(X) = \ell$. From the exact sequence

$$\underbrace{\text{Hom}(X, M[i])}_{=0 \text{ if } i > \ell} \rightarrow \text{Hom}(X, T_2[i]) \rightarrow \underbrace{\text{Hom}(X, T'_2[i+1])}_{=0 \text{ if } i > \ell - 1}$$

it follows that $\text{Hom}(X, T_2[i]) = 0$ if $i > \ell$, and therefore $\ell_T^+(X) \leq \ell$. In particular

$$\text{Hom}(X, T_1[\ell]) \neq 0 \Rightarrow \ell_T^+(X) = \ell.$$

Assume that $\text{Hom}(X, T_1[\ell]) = 0$. Then $\text{Hom}(X, T'_2[\ell]) \neq 0$ because $\ell_{T'}^+(X) = \ell$. Therefore the exact sequence

$$\text{Hom}(X, T_2[\ell - 1]) \rightarrow \underbrace{\text{Hom}(X, T'_2[\ell])}_{\neq 0} \rightarrow \underbrace{\text{Hom}(X, M[\ell])}_{=0 \text{ (} M \in \text{add } T_1)}$$

entails that $\text{Hom}(X, T_2[\ell - 1]) \neq 0$. Thus

$$\text{Hom}(X, T_1[\ell]) = 0 \Rightarrow \ell_T^+(X) = \ell - 1.$$

□

2.1.5. *Comparison of s.gl.dim. T and s.gl.dim. T' .* Using the previous lemma, the following is immediate.

Proposition. *Under the setting presented at the beginning of 2.1.1:*

$$|\text{s.gl.dim. } T - \text{s.gl.dim. } T'| \leq 1.$$

Besides, $\text{s.gl.dim. } T = 1 + \text{s.gl.dim. } T'$ if and only if the following conditions are satisfied:

- (a) for every indecomposable $X \in \mathcal{T}$ such that $\ell_T^-(X) = 0$ and $\ell_T(X) = \text{s.gl.dim. } T$, the mapping $\alpha_{X, \ell_T(X)} : \text{Hom}(X, M[\ell_T(X)]) \rightarrow \text{Hom}(X, T_2[\ell_T(X)])$ is an epimorphism and $\text{Hom}(T_1, X) = 0$,
- (b) for every indecomposable $X \in \mathcal{T}$ such that $\ell_T^-(X) = 0$ and $\ell_T(X) = -1 \text{s.gl.dim. } T$, the mapping $\alpha_{X, \ell_T(X)} : \text{Hom}(X, M[\ell_T(X)]) \rightarrow \text{Hom}(X, T_2[\ell_T(X)])$ is an epimorphism or else $\text{Hom}(T_1, X) = 0$.

2.2. Behaviour under split bi-adjunction and under taking finite Galois coverings.

2.2.1. *Behaviour under split bi-adjunction.* Let \mathcal{T} and \mathcal{T}' be Krull-Schmidt Hom-finite triangulated categories. Assume that there exists a bi-adjoint pair (F, G) of triangle functors

$$\begin{array}{ccc} & \mathcal{T}' & \\ F \uparrow & & \downarrow G \\ & \mathcal{T} & \end{array}$$

such that

- X is a direct summand of $GF X$, for every $X \in \mathcal{T}'$, and
- M is a direct summand of $FG M$, for every $M \in \mathcal{T}$.

Proposition. *Let $T \in \mathcal{T}$ be an object such that $\text{add } T = \text{add } FGT$. The following conditions are equivalent for every indecomposable objects $M \in \mathcal{T}$ and $X \in \mathcal{T}'$*

- (i) $-\infty < \ell_T^-(M) \leq \ell_T^+(M) < +\infty$;
- (ii) $-\infty < \ell_{GT}^-(X) \leq \ell_{GT}^+(X) < +\infty$.

When these two conditions are satisfied then $\text{s.gl.dim. } T = \text{s.gl.dim. } GT$.

Proof. If $X \in \mathcal{T}'$ (or $M \in \mathcal{T}$) is indecomposable, then X (or M) is a direct summand of GFX (or FGM); hence there exists an indecomposable direct summand $M_X \in \mathcal{T}$ of FX (or $X_M \in \mathcal{T}'$ of GM) such that X (or M) is a direct summand of FM_X (or GX_M , respectively). In order to prove the result it is useful to first prove the following facts

- (1) $\ell_{GT}^-(X) = \ell_T^-(FX)$ and $\ell_{GT}^+(X) = \ell_T^+(FX)$;
- (2) $\ell_T^-(M) = \ell_{GT}^-(GM)$ and $\ell_T^+(M) = \ell_{GT}^+(GM)$;
- (3) $\ell_T^-(M_X) = \ell_{GT}^-(X)$ and $\ell_T^+(M_X) = \ell_{GT}^+(X)$;
- (4) $\ell_T^-(M) = \ell_{GT}^-(X_M)$ and $\ell_T^+(M) = \ell_{GT}^+(X_M)$.

Indeed,

- (1) if $i \in \mathbb{Z}$ then $\text{Hom}(GT[i], X) = \text{Hom}(T[i], FX)$, and hence $\ell_{GT}^-(X) = \ell_T^-(FX)$ (the equality $\ell_{GT}^+(X) = \ell_T^+(FX)$ follows from a dual argument);
- (2) If $i \in \mathbb{Z}$ then, using the hypothesis asserting that $\text{add } T = \text{add } FGT$ gives

$$\text{Hom}(T[i], M) = 0 \Leftrightarrow \text{Hom}(FGT[i], M) = 0 \Leftrightarrow \text{Hom}(GT[i], GM) = 0,$$

and hence $\ell_T^-(M) = \ell_{GT}^-(GM)$ (the equality $\ell_T^+(M) = \ell_{GT}^+(GM)$ follows from a dual argument);

- (3) It follows from (1) and (2) that

$$\ell_T^-(FX) \underset{(M_X \in \text{add } FX)}{\leq} \ell_T^-(M_X) \underset{(2)}{=} \ell_{GT}^-(GM_X) \underset{(X \in \text{add } GM_X)}{\leq} \ell_{GT}^-(X) \underset{(1)}{=} \ell_T^-(FX),$$

and hence $\ell_{GT}^-(X) = \ell_T^-(M_X)$ (the equality $\ell_{GT}^+(X) = \ell_T^+(M_X)$ follows from a dual argument);

- (4) Using a similar argument gives

$$\ell_T^-(FX_X) \underset{(M \in \text{add } FX_X)}{\leq} \ell_T^-(M) \underset{(2)}{=} \ell_{GT}^-(GM) \underset{(X_M \in \text{add } GM)}{\leq} \ell_{GT}^-(X_M) \underset{(1)}{=} \ell_T^-(FX_X),$$

and hence $\ell_{GT}^-(X_M) = \ell_T^-(M)$ (the equality $\ell_{GT}^+(X_M) = \ell_T^+(M)$ follows from a dual argument);

The equivalence (i) \Leftrightarrow (ii), and the equality $\text{s.gl.dim. } T = \text{s.gl.dim. } GT$ follows from (3) and (4). \square

2.2.2. Application to Galois coverings. The previous proposition applies to compare the strong global dimensions of two algebras the first of which is a Galois covering of second. In [32, Thm. C] it was proved that if the latter is derived equivalent to a hereditary algebra then so is the former. When the group of the Galois covering is finite the following corollary generalises this result and makes it more precise in terms of the strong global dimension.

Corollary. *Let A and A' be finite-dimensional algebras. Let G be a finite group such that $\text{Card } G \in k^\times$. Assume that A and A' are basic and that there exists a Galois covering $A' \rightarrow A$ with group G (in the sense of [18, 3]). Then A is piecewise hereditary if and only if so is A' . When this is the case then $\text{s.gl.dim. } A = \text{s.gl.dim. } A'$.*

Proof. Let $\mathcal{T} = \mathcal{D}^b(\text{mod } A)$ and $\mathcal{T}' = \mathcal{D}^b(\text{mod } A')$. Let $T = A \in \mathcal{T}$. Following [18, 3.2], the restriction-of-scalars and the extension-of-scalars functors form a bi-adjoint pair of exact functors

$$A \otimes_{A'} - \left(\begin{array}{c} \text{mod } A' \\ \text{res} \\ \text{mod } A \end{array} \right)$$

such that there exists a functorial isomorphism $\text{res} \left(A \otimes_{A'} X \right) \simeq \bigoplus_{\gamma \in \Gamma} {}^\gamma X$ for every $X \in \text{mod } A'$, and such that $\text{res}(A) = \bigoplus_{\gamma \in \Gamma} {}^\gamma A'$ and $A \otimes_{A'} \text{res}(A) \simeq A^{\text{Card}(\Gamma)}$. Here $(\gamma, X) \mapsto {}^\gamma X$ denotes the action of Γ on $\text{mod } A'$. Since $\text{Card}(\Gamma) \in k^\times$ there also exists a functorial section $M \rightarrow A \otimes_{A'} \text{res}(M)$ for every $M \in \text{mod } A$ ([18, 3.4]).

The algebras A and A' have the same global dimension hence there is no loss of generality in assuming that these are finite. Then the exact functors res and $A \otimes_{A'} -$ induce triangle functors $F: \mathcal{T}' \rightarrow \mathcal{T}$ and $G: \mathcal{T} \rightarrow \mathcal{T}'$ forming a bi-adjoint pair ([32, Prop. 2.1])

$$F \left(\begin{array}{c} \mathcal{T}' \\ G \\ \mathcal{T} \end{array} \right)$$

Besides the above properties of $(A \otimes_{A'} -, \text{res})$ entail that (F, G) fulfills the hypothesis in 2.2.1. The conclusion thus follows from 2.2.1 and 1.1 □

3. THE STRONG GLOBAL DIMENSION THROUGH AUSLANDER-REITEN THEORY

Let Γ be a transjective component. If Σ is a slice in Γ then the full and additive subcategory $\mathcal{H} \subseteq \mathcal{T}$ defined by

$$\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma) (\forall i \neq 0) \text{Hom}(S, X[i]) \neq 0\}$$

is equivalent to the module category of the finite-dimensional hereditary algebra $\text{End}(\bigoplus_{X \in \Sigma} X)^{\text{op}}$. The indecomposable projective (or injective) objects of \mathcal{H} are the objects in Σ (or in $\tau\Sigma[1]$, respectively) up to isomorphism. Also, the embedding $\mathcal{H} \subseteq \mathcal{T}$ extends to triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$.

3.1. Setting: where does T start and end in the triangulated category? Let Γ be a transjective Auslander-Reiten component of \mathcal{T} . Then,

- either Γ is the whole Auslander-Reiten quiver of \mathcal{T} , in which case \mathcal{T} is equivalent to the bounded derived category of a finite-dimensional hereditary algebra of finite representation type,
- or else the Auslander-Reiten quiver of \mathcal{T} is the disjoint union

$$\dots \cup \Gamma[-1] \cup \mathcal{R}[-1] \cup \Gamma \cup \mathcal{R} \cup \Gamma[1] \cup \mathcal{R}[1] \cup \dots$$

where \mathcal{R} is a disjoint union of non-transjective Auslander-Reiten components (that are either all stable tubes, or all of the shape $\mathbb{Z}A_\infty$), and characterised as follows, for every indecomposable $X \in \mathcal{T}$ lying in none of $\Gamma[i]$, $i \in \mathbb{Z}$, then

$$X \in \mathcal{R} \Leftrightarrow (\exists Y \in \Gamma) \text{Hom}(Y, X) \neq 0.$$

Let $T \in \mathcal{T}$ be a tilting object. In view of describing $\text{s.gl.dim. } T$ in terms of the Auslander-Reiten quiver of \mathcal{T} , it may be assumed that one of the two following conditions is satisfied, up to a shift and using the duality over k :

- (1) $T \in \text{add}(\Gamma \cup \mathcal{R} \cup \Gamma[1] \cup \mathcal{R}[1] \cup \dots)$ and T has an indecomposable direct summand in Γ (it is here implicitly assumed that when Γ is the whole Auslander-Reiten quiver of \mathcal{T} , then $\mathcal{R} = \emptyset$ and $T \in \text{add}\Gamma$),
- (2) there exists an integer $\ell \geq 0$ such that

$$T \in \text{add}(\mathcal{R} \cup \Gamma[1] \cup \mathcal{R}[1] \cup \dots \cup \Gamma[\ell] \cup \mathcal{R}[\ell])$$

and T has an indecomposable direct summand in \mathcal{R} and in $\mathcal{R}[\ell]$.

For simplicity, in the first (or second) case, T is said to *start* in Γ (or, to start in \mathcal{R} , and to end in $\mathcal{R}[\ell]$ of non-transjective Auslander-Reiten components, respectively). The aim of this section is to describe $\text{s.gl.dim. } T$ in these two cases.

3.2. The strong global dimension when T starts in a transjective component.

Proposition. *Let $T \in \mathcal{T}$ be a tilting object. Assume that T starts in the transjective Auslander-Reiten component Γ . Then*

- (1) *there exists a slice Σ in Γ such that every source of Σ is an indecomposable direct summand of T , and for every indecomposable direct summand Y of T lying in Γ there exists a path in Γ with source in Σ and target Y ;*
- (2) *Let $\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma) (\forall i \neq 0) \text{Hom}(S, X[i]) = 0\}$. Then \mathcal{H} is a hereditary abelian category such that the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$. Moreover there exists an integer $\ell \geq 0$ such that*

$$T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i],$$

and such that T has an indecomposable summand in \mathcal{H} and in $\mathcal{H}[\ell]$;

- (3) *If $\text{End}(T)^{\text{op}}$ is not a hereditary algebra then $\text{s.gl.dim. } T = \ell + 2$.*

Proof. (1) Let S_1, \dots, S_n be the indecomposable direct summands of T such that $\text{Hom}(\bigoplus_{i=1}^n S_i, X) \neq 0$ for every indecomposable direct summand X of T lying in Γ , and such that $\text{Hom}(S_i, S_j) = 0$ if $i \neq j$. Let Σ be the full subquiver of Γ the vertices of which are those $X \in \Gamma$ such that X is the successor in Γ of at least one of S_1, \dots, S_n , and such that any path in Γ from any of S_1, \dots, S_n to X is sectional.

It follows from its definition that Σ is a convex subquiver of Γ such that there is no $X \in \Gamma$ verifying $X, \tau X \in \Sigma$. Let $X \in \Gamma$. Since Γ is the repetitive quiver of some quiver without oriented cycles there exists $n \in \mathbb{Z}$ such that $\tau^n X$ is a successor in Γ of one of the vertices in Σ , and $\tau^{n+1} X$ is the successor in Γ of none of the vertices in Σ . Let

$$(\gamma) \quad S_i \rightarrow L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_r = \tau^n X$$

be any path in Γ from one of S_1, \dots, S_n to $\tau^n X$. If (γ) were not sectional there would exist some hook

$$L_{t-1} \rightarrow L_t \rightarrow L_{t+1} = \tau^{-1} L_{t-1},$$

and hence a path in Γ

$$S_i \rightarrow L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_{t-1} = \tau L_{t+1} \rightarrow \tau L_{t+2} \rightarrow \dots \rightarrow \tau L_r = \tau^{n+1} X$$

which would contradict the definition of n . The path (γ) is therefore sectional. This proves that Σ is a slice in Γ fitting the requirements of (1).

(2) The first assertion follows from the fact that Σ is a section of Γ . In particular $\mathcal{T} = \bigvee_{i \in \mathbb{Z}} \mathcal{H}[i]$, and the indecomposable objects in \mathcal{H} are, up to isomorphism, the objects $X \in \Gamma$ which are successors in Γ of some $S_i \in \Sigma$; the objects $X \in \mathcal{R}$; and the objects $X \in \Gamma[1]$ which are predecessors in $\Gamma[1]$ of some

$\tau S_i[1] \in \tau \Sigma[1]$. The second assertion therefore follows from these considerations and from the fact that T starts in Γ and from the definition of Σ .

(3) If $\ell = 0$ then $T \in \mathcal{H}$. Therefore $\text{End}(T)^{\text{op}}$ is a tilted algebra which is not hereditary, and thus $\text{s.gl.dim. } T = 2$ ([27, Prop. 3.3]). If $\ell \geq 1$ then the conclusion follows from 1.2 and 1.3.3. \square

3.3. The strong global dimension when T starts and ends in family of tubes.

3.3.1. *Setting.* Throughout this subsection, it is assumed that \mathcal{T} is triangle equivalent to the bounded derived category of a finite-dimensional hereditary algebra of tame representation type. Hence \mathcal{R} is a disjoint union of tubes which are pairwise orthogonal in \mathcal{T} .

Let $T \in \mathcal{T}$ be a tilting object. Let $\ell \geq 0$ be an integer. In this subsection it is assumed that T starts in \mathcal{R} and ends in $\mathcal{R}[\ell]$; the objective is to determine $\text{s.gl.dim. } T$. For this purpose a major issue is to determine the tubes in \mathcal{R} and $\mathcal{R}[\ell]$ containing an indecomposable direct summand of T . Therefore it is convenient to use the following notation: let \mathcal{S}_T (or \mathcal{E}_T) be the set of tubes in \mathcal{R} (or in $\mathcal{R}[\ell]$, respectively) containing an indecomposable direct summand of T .

Following [37] there is no regular tilting module over a finite-dimensional hereditary algebra of tame representation type. Therefore $\ell \geq 1$.

3.3.2. The strong global dimension when T starts in a tube and ends in a shift of this tube.

Lemma. *Let \mathcal{C} be a tube in \mathcal{R} . Assume that $\mathcal{S}_T = \{\mathcal{C}\}$ and $\mathcal{E}_T = \{\mathcal{C}[\ell]\}$. Then $\text{s.gl.dim. } T = \ell + 2$.*

Proof. It follows from 1.3.2 (part (2)) that $\text{s.gl.dim. } T \geq \ell + 2$. Let Σ be any slice in Γ . Let

$$\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma) (\forall i \neq 0) \text{Hom}(S, X[i]) \neq 0\}.$$

Therefore \mathcal{R} is the family of regular Auslander-Reiten components of the hereditary abelian category \mathcal{H} . Since T starts in \mathcal{R} and ends in $\mathcal{R}[\ell]$ it follows that $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[\ell]$. Hence $\text{s.gl.dim. } T \leq \ell + 2$ (1.2). Thus $\text{s.gl.dim. } T = \ell + 2$. \square

The rest of the subsection proves that the hypotheses of the previous lemma actually hold true, that is, there exists a tube $\mathcal{C} \subseteq \mathcal{R}$ such that $\mathcal{S}_T = \{\mathcal{C}\}$ and $\mathcal{E}_T = \{\mathcal{C}[\ell]\}$. This is done in several steps for some of them are used later.

3.3.3. The particular case where $\ell = 1$.

Lemma. *Under the setting described in 3.3.1, assume that $\ell = 1$. Then*

- (1) $\text{s.gl.dim. } T = 3$ and there exists a tube $\mathcal{C} \subseteq \mathcal{R}$ such that $\mathcal{C} \in \mathcal{S}_T$ and $\mathcal{C}[1] \in \mathcal{E}_T$;
- (2) $\mathcal{S}_T = \{\mathcal{C}\}$ and $\mathcal{E}_T = \{\mathcal{C}[1]\}$.

Proof. (1) Note that if $\mathcal{H}' \subseteq \mathcal{T}$ is a full and additive subcategory which is hereditary and abelian, and such that the embedding $\mathcal{H}' \subseteq \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}') \simeq \mathcal{T}$, then there exists a unique $i \in \mathbb{Z}$ such that $\mathcal{R}[i]$ is the regular part of the Auslander-Reiten quiver of \mathcal{H}' . In particular $\mathcal{H}' \cap \mathcal{R}[j] = \emptyset$ if $i \neq j$. Therefore T lies in no such \mathcal{H}' , and thus $\text{End}(T)^{\text{op}}$ is not quasi-tilted. This proves that $\text{s.gl.dim. } T \geq 3$ [27, Prop. 3.3].

Let Σ be a slice in Γ . Let $\mathcal{H} \subseteq \mathcal{T}$ be the full subcategory such that

$$\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma) (\forall i \neq 0) \text{Hom}(S, X[i]) = 0\}.$$

Hence \mathcal{H} is hereditary abelian, and the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$. Since T starts in \mathcal{R} and ends in $\mathcal{R}[1]$, it follows that $T \in \mathcal{H} \vee \mathcal{H}[1]$. Therefore $\text{s.gl.dim. } T \leq 3$ (1.2), and thus $\text{s.gl.dim. } T = 3$.

Let $X \in \mathcal{T}$ be indecomposable such that $\ell_T^-(X) = 0$ and $\ell_T^+(X) = 3$. Let

$$T = T_0 \oplus T_t \oplus T_1$$

be a decomposition such that $T_0 \in \text{add } \mathcal{R}$, $T_t \in \text{add}(\Gamma[1])$ and $T_1 \in \text{add}(\mathcal{R}[1])$. Since $\text{Hom}(T, X) \neq 0$ and $\text{Hom}(X, T[3]) \neq 0$ then

$$X \in \mathcal{R} \cup \Gamma[1] \cup \mathcal{R}[1] \cup \Gamma[2] \cup \mathcal{R}[2] \quad \text{and} \quad X \in \mathcal{R}[2] \cup \Gamma[3] \cup \mathcal{R}[3] \cup \Gamma[4] \cup \mathcal{R}[4].$$

Hence $X \in \mathcal{R}[2]$, and therefore $\text{Hom}(T_0 \oplus T_t, X) = \text{Hom}(X, T_t[3] \oplus T_1[3]) = 0$. In other words

$$\text{Hom}(T_1, X) \neq 0 \quad \text{and} \quad \text{Hom}(X, T_0[3]) \neq 0.$$

Let M_0, M_1 be indecomposable direct summands of T_0 and T_1 , respectively, such that $\text{Hom}(M_1, X) \neq 0$ and $\text{Hom}(X, M_0[3]) \neq 0$. Therefore

- $\tau M_1[1], X, \tau^{-1} M_0[2] \in \mathcal{R}[2]$,
- $\text{Hom}(X, \tau M_1[1]) \neq 0$, and
- $\text{Hom}(\tau^{-1} M_0[2], X) \neq 0$.

Since \mathcal{R} is a disjoint union of tubes which are pairwise orthogonal in \mathcal{T} , this entails that $\tau M_1[1], X$ and $\tau^{-1} M_0[2]$ all lie in a same tube from $\mathcal{R}[2]$. Therefore there exists a tube $\mathcal{C} \subseteq \mathcal{R}$ such that $M_0 \in \mathcal{C}$ and $M_1 \in \mathcal{C}[1]$. Thus $\mathcal{C} \in \mathcal{S}(T)$ and $\mathcal{C}[1] \in \mathcal{E}(T)$.

(2) The proof proceeds by contradiction. Assume first that T has at least one indecomposable direct summand lying in $\mathcal{R}[1] \setminus \mathcal{C}[1]$. Let

$$T = T_1 \oplus T_2$$

be another decomposition such that $T_2 \in \text{add}(\mathcal{C}[1])$ and T_1 has no indecomposable direct summand lying in $\mathcal{C}[1]$. The tubes in $\mathcal{R}[1]$ are pairwise orthogonal in \mathcal{T} , and $\text{Hom}(\mathcal{R}[1], \Gamma) = \text{Hom}(\mathcal{R}[1], \mathcal{R}) = 0$. Therefore

$$\text{Hom}(T_2, T_1) = 0.$$

Let $M \rightarrow T_2$ be a minimal right $\text{add } T_1$ -approximation. It fits into a triangle

$$T'_2 \rightarrow M \rightarrow T_2 \rightarrow T'_2[1].$$

Thus $T' := T_1 \oplus T'_2$ is a tilting object in \mathcal{T} (2.1.1).

Let X be an indecomposable direct summand of T'_2 . If X is a direct summand of $T_2[-1]$ then $X \in \mathcal{C}$. If X is a direct summand of M then so is it one of T_1 , and hence $X \in \mathcal{R} \cup \Gamma[1] \cup \mathcal{R}[1]$ and $X \notin \mathcal{C}[1]$. Otherwise $\text{Hom}(T_2[-1], X) \neq 0$ and $\text{Hom}(X, M) \neq 0$ (2.1.3); Therefore $\text{Hom}(X, \tau T_2) \neq 0$ and $\text{Hom}(X, M) \neq 0$, which prevents X from lying in $\mathcal{R}[1]$ (indeed an object $Y \in \mathcal{R}[1]$ such that $\text{Hom}(Y, \tau T_2) \neq 0$ must lie in $\mathcal{C}[1]$, for $\tau T_2 \in \text{add}(\mathcal{C}[1])$ and the tubes in $\mathcal{R}[1]$ are pairwise orthogonal, on the other hand there is no non-zero morphism from an object in $\mathcal{C}[1]$ to M , for M lies in $\text{add}(\mathcal{R} \cup \Gamma[1] \cup \mathcal{R}[1] \setminus \mathcal{C}[1])$; in particular if $X \in \mathcal{R}$ then the property $\text{Hom}(T_2[-1], X) \neq 0$ forces X to lie in \mathcal{C} ; thus $X \in \mathcal{C} \cup \Gamma[1]$.

By hypothesis, $T_1 \in \text{add}(\mathcal{R} \cup \Gamma[1] \cup \mathcal{R}[1])$, also T_1 has an indecomposable summand lying in $\mathcal{R}[1]$ and none of them lie in $\mathcal{C}[1]$. Thus T' starts in \mathcal{R} ; it ends in $\mathcal{R}[1]$; and it is such that $\mathcal{S}(T') = \mathcal{S}(T)$ and $\mathcal{E}(T') = \mathcal{E}(T) \setminus \{\mathcal{C}[1]\}$.

All this procedure (or its dual version) starting with T and ending with T' may be repeated as long as there exists a tube \mathcal{C}' in \mathcal{R} such that $\mathcal{C}' \in \mathcal{S}(T')$ and $\mathcal{C}'[1] \in \mathcal{E}(T')$. Eventually the procedure ends up with a tilting object T'' which starts in \mathcal{R} , which ends in $\mathcal{R}[1]$, and such that there is no tube $\mathcal{C}' \subseteq \mathcal{R}$ such that $\mathcal{C}' \in \mathcal{S}(T'')$ and $\mathcal{C}'[1] \in \mathcal{E}(T'')$. This contradicts (1). Thus every indecomposable direct summand of T lying in $\mathcal{R}[1]$ does lie in $\mathcal{C}[1]$. A dual arguments shows that every indecomposable direct summand of T lying in \mathcal{R} does lie in \mathcal{C} . \square

3.3.4. *The behaviour of \mathcal{S}_T and \mathcal{E}_T under the modification of T using approximations.* Note that, by definition, if T' is a tilting object which starts in \mathcal{R} and ends in $\mathcal{R}[\ell - 1]$ then $\mathcal{E}_{T'}$ is the set of tubes $\mathcal{C} \subseteq \mathcal{R}[\ell - 1]$ containing an indecomposable direct summand of T' .

Lemma. *Assume that $\ell \geq 2$. Let $T = T_1 \oplus T_2$ be the decomposition such that $T_2 \in \text{add}(\Gamma[\ell] \cup \mathcal{R}[\ell])$ and T_1 has no indecomposable direct summand in $\Gamma[\ell] \cup \mathcal{R}[\ell]$. Let*

$$T'_2 \rightarrow M \rightarrow T_2 \rightarrow T'_2[1]$$

be a triangle where $M \rightarrow T_2$ is a minimal right $\text{add } T_1$ -approximation. Then $T' := T_1 \oplus T'_2$ is a tilting object in \mathcal{T} such that: T' starts in \mathcal{R} and ends in $\mathcal{R}[\ell - 1]$; and $\mathcal{S}_{T'} = \mathcal{S}_T$ and $\{\mathcal{C}[-1] \mid \mathcal{C} \in \mathcal{E}_T\} \subseteq \mathcal{E}_{T'}$.

Proof. There is non non-zero morphism from an object in $\Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1]$ to an object in $\Gamma[i] \cup \mathcal{R}[i]$ if $i < \ell - 1$. Therefore $\text{Hom}(T_2, T_1) = 0$. Using 2.1.1 it follows that T' is tilting.

Note that $T_1 \in \text{add}(\mathcal{R} \cup \Gamma[1] \cup \dots \cup \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1])$, and T_1 has an indecomposable direct summand in \mathcal{R} . This and the right minimality of $M \rightarrow T_2$ imply that $M \in \text{add}(\Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1])$. Let $X \in \text{add } T'_2$ be indecomposable. If $X \in \text{add}(T_2[-1] \oplus M)$ then $X \in \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1]$. Otherwise $\text{Hom}(T_2[-1], X) \neq 0$ and $\text{Hom}(X, M) \neq 0$ (2.1.3, part (2)); this entails $X \in \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1]$ because $M, T_2[-1] \in \text{add}(\Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1])$. Thus $X \in \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1]$ in all cases.

Therefore there only remains to prove that if $\mathcal{C} \subseteq \mathcal{R}[\ell]$ is a tube containing an indecomposable direct summand of T then $\mathcal{C}[-1]$ contains an indecomposable direct summand of T' . Let $Z \in \text{add } T$ be indecomposable such that $Z \in \mathcal{C}$. Hence $Z \in \text{add } T_2$. If $\text{Hom}(T_1 x s, Z) = 0$ then $Z[-1] \in \text{add } T'_2$ (2.1.2, part (3)), and thus $\mathcal{C}[-1] \in \mathcal{E}_{T'}$. Otherwise $\text{Hom}(T_1, Z) \neq 0$, and hence $\text{Hom}(M, Z) \neq 0$ because $M \rightarrow T_2$ is an $\text{add } T_1$ -approximation; moreover $\text{Hom}(Z, M) = 0$ because $\text{Hom}(T_2, T_1) = 0$; therefore, in the diagram below where the row is a triangle and $Z[-1] \hookrightarrow T_2[-1]$ is a section,

$$\begin{array}{ccccccc} & & Z[-1] & & & & \\ & & \downarrow & & & & \\ M[-1] & \longrightarrow & T_2[-1] & \longrightarrow & T'_2 & \longrightarrow & M \end{array}$$

the section does not factor through $M[-1] \rightarrow T_2[-1]$; this entails that the composite morphism $Z[-1] \rightarrow T_2[-1] \rightarrow T'_2$ is non-zero; hence $\text{Hom}(M, Z) \neq 0$ and $\text{Hom}(Z[-1], T'_2) \neq 0$ with $Z[-1] \in \mathcal{C}[-1] \subseteq \mathcal{R}[\ell - 1]$ and $M, T'_2 \in \text{add}(\Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1])$; thus there exists $Z' x s \in \text{add } T'_2$ indecomposable such that $\text{Hom}(Z[-1], Z') \neq 0$ and $Z' \in \mathcal{R}[\ell - 1]$; in particular $Z' \in \mathcal{C}[-1]$ because $\mathcal{R}[\ell - 1]$ is a disjoint union of tubes which are orthogonal in \mathcal{T} . This proves that $\mathcal{C}[-1] \in \mathcal{E}_{T'}$. Therefore T' ends in $\mathcal{R}[\ell - 1]$ and $\{\mathcal{C}[-1] \mid \mathcal{C} \in \mathcal{E}_T\} \subseteq \mathcal{E}_{T'}$. \square

3.3.5. *The uniqueness of the tube in which T starts and ends (up to a shift).* Summarising 3.3.3, 3.3.4 and 3.3.5 gives the following structure result on tilting objects which start and end in non-transjective Auslander-Reiten components.

Proposition. *Assume that \mathcal{T} is a triangulated category triangle equivalent to the bounded derived category of a finite-dimensional k -algebra of tame representation type. Let $T \in \mathcal{T}$ be a tilting object. Assume that $\ell \geq 0$ is an integer and \mathcal{R} is a family of non-transjective Auslander-Reiten components such that T starts in \mathcal{R} and ends in $\mathcal{R}[\ell]$. Then $\ell \geq 1$ and there exists a tube $\mathcal{C} \subseteq \mathcal{R}$ such that the indecomposable direct summands of T lying in \mathcal{R} (or in $\mathcal{R}[\ell]$) all lie in \mathcal{C} (or in $\mathcal{C}[\ell]$, respectively).*

Proof. As explained in 3.3.1, it is necessary that $\ell \geq 1$. The proof of the lemma is an induction on $\ell \geq 1$. If $\ell = 1$ the result follows from 3.3.3.

Assume that $\ell \geq 2$ and that the conclusion of the lemma holds true for every tilting object in \mathcal{T} which starts in \mathcal{R} and ends in $\mathcal{R}[\ell - 1]$. Applying 3.3.4 to T gives rise to the tilting object T' . The

induction hypothesis therefore applies to T' : Hence there exists a tube $\mathcal{C} \subseteq \mathcal{R}$ such that $\mathcal{S}_{T'} = \{\mathcal{C}\}$ and $\mathcal{E}_{T'} = \{\mathcal{C}[\ell - 1]\}$. The conclusion of 3.3.4 then implies that $\mathcal{S}_T = \{\mathcal{C}\}$ and $\mathcal{E}_T = \{\mathcal{C}[\ell]\}$. \square

3.3.6. The following proposition follows directly from 3.3.2 and 3.3.5

Proposition. *Assume that \mathcal{T} is triangle equivalent to the bounded derived category of a finite-dimensional hereditary algebra of tame representation type. Let $T \in \mathcal{T}$ be a tilting object. Assume that there exists a family \mathcal{R} of non-transjective Auslander-Reiten components, and an integer $\ell \geq 0$ such that T starts in \mathcal{R} and ends in $\mathcal{R}[\ell]$. Then $\text{s.gl.dim. } T = \ell + 2$.*

3.4. **The strong global dimension when T starts and ends in a family of $\mathbb{Z}A_\infty$ components.**

Proposition. *Assume that \mathcal{T} is triangle equivalent to the bounded derived category of a finite-dimensional hereditary algebra of wild representation type. Let $T \in \mathcal{T}$ be a tilting object. Assume that there exists a family \mathcal{R} of non-transjective Auslander-Reiten components, and an integer $\ell \geq 0$ such that T starts in \mathcal{R} and ends in $\mathcal{R}[\ell]$. Then $\text{s.gl.dim. } T = \ell + 2$.*

Proof. Let Σ be a slice in Γ . Let $\mathcal{H} \subseteq \mathcal{T}$ be the additive and full subcategory such that

$$\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma) (\forall i \neq 0) \text{Hom}(S, X[i]) = 0\}.$$

Therefore \mathcal{R} is the family of regular Auslander-Reiten components of the hereditary abelian category \mathcal{H} . Therefore $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i]$, and thus $\text{s.gl.dim. } T \leq 2 + \ell$ (1.2). On the other hand 1.3.2, part (3), implies that $\text{s.gl.dim. } T \geq \ell + 2$. Thus $\text{s.gl.dim. } T = \ell + 2$. \square

4. PROOF OF THE MAIN THEOREMS

4.1. Proof of Theorem 1.

Proof. The assertion (1) follows from 1.2.

The proof of the existence of a pair (\mathcal{H}, ℓ) satisfying (2) distinguishes two cases according to whether T starts (or ends) in a transjective Auslander-Reiten component, or T starts in a family of non-transjective Auslander-Reiten components and ends in a family of non-transjective Auslander-Reiten components.

Assume first that T starts in a transjective Auslander-Reiten component Γ . Let (\mathcal{H}, ℓ) be like in 3.2. Then (\mathcal{H}, ℓ) satisfies (2). If T ends in a transjective Auslander-Reiten component then the same conclusion follows from dual arguments.

Assume next that there exists an integer $\ell \geq 0$ and a family \mathcal{R} of non-transjective Auslander-Reiten components, such that T starts in \mathcal{R} and ends in $\mathcal{R}[\ell]$. Let Γ be the transjective Auslander-Reiten component such that

$$(\forall X \in \mathcal{R}) (\exists Y \in \Gamma) \text{Hom}(Y, X) \neq 0.$$

Let Σ be any slice in Γ . Let $\mathcal{H} \subseteq \mathcal{T}$ be the full and additive subcategory

$$\mathcal{H} = \{X \in \mathcal{T} \mid (\forall S \in \Sigma) (\forall i \neq 0) \text{Hom}(S, X[i]) = 0\}.$$

Then \mathcal{H} is hereditary and abelian, and the embedding $\mathcal{H} \hookrightarrow \mathcal{T}$ extends to a triangle equivalence $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{T}$. Moreover, $\mathcal{R}[i]$ is the family of regular Auslander-Reiten components of the abelian category $\mathcal{H}[i]$. Therefore $T \in \bigvee_{i=0}^{\ell} \mathcal{H}[i]$ since T starts in \mathcal{R} and ends in $\mathcal{R}[\ell]$. Finally it follows from 3.3.6 and 3.4 that $\text{s.gl.dim. } T = \ell + 2$. \square

4.2. Proof of Theorem 2.

Proof. Let $T^{(0)}, \dots, T^{(\ell)}$ be such a sequence. Then $|\text{s.gl.dim. } T^{(i)} - \text{s.gl.dim. } T^{(i+1)}| \leq 1$ for every i (2.1.5). Thus $\text{s.gl.dim. } T \leq \ell + 2$. This proves (1).

Let $d = \text{s.gl.dim. } T$. Then $d \geq 2$ for $\text{End}(T)^{\text{op}}$ is not hereditary. The existence of a sequence $T^{(0)}, \dots, T^{(\ell)}$ verifying (2) is proved by induction on d . If $d = 2$ then $\text{End}(T)^{\text{op}}$ is a tilted algebra, and thus taking $\ell = 0$ and $T^{(0)} = T$ does the trick.

Assume now that $d > 2$ and that part (2) of the theorem holds true for any tilting object T' such that $\text{s.gl.dim. } T' = d - 1$. In order to prove the same for T it suffices to prove the following claim: There exists a decomposition

$$T = T_1 \oplus T_2$$

such that $\text{Hom}(T_2, T_1) = 0$ and the tilting object $T' = T_1 \oplus T_2'$ (2.1.1) has strong global dimension equal to $d - 1$, where T_2' is the first term of a triangle

$$T_2' \rightarrow M \rightarrow T_2 \rightarrow T_2'[1]$$

where $M \rightarrow T_2$ is a minimal right $\text{add } T_1$ -approximation. This claim is proved distinguishing two cases according to whether T starts (or ends) in a transjective Auslander-Reiten component, or T starts in a family of non-transjective Auslander-Reiten components and ends in a family of non-transjective Auslander-Reiten components.

Assume first that T starts in a transjective Auslander-Reiten component Γ (the case where T ends in Γ is dealt with using dual arguments). Let Σ, \mathcal{H} and ℓ be like in 3.2 applied to T . In particular $\ell = d - 2 > 0$. Let

$$T = T_1 \oplus T_2$$

be a decomposition such that $T_1 \in \bigvee_{i=0}^{\ell-1} \mathcal{H}[i]$ and $T_2 \in \mathcal{H}[\ell]$. In particular $\text{Hom}(T_2, T_1) = 0$. Let $M \rightarrow T_2$ be a minimal right $\text{add } T_1$ -approximation. Let

$$T_2' \rightarrow M \rightarrow T_2 \rightarrow T_2'[1]$$

be a triangle. Let $T' = T_1 \oplus T_2'$. Therefore T' is tilting, and $T_2' \in \mathcal{H}[\ell - 1]$ (2.1.3). Since $\ell \geq 1$ it follows that any indecomposable direct summand of T in Γ is a direct summand of T' . Hence if Σ', \mathcal{H}' and ℓ' are like in 3.2 applied to T' then

$$\Sigma' = \Sigma, \mathcal{H}' = \mathcal{H}, \ell' = \ell - 1, \text{ and } \text{s.gl.dim. } T' = \ell' + 2 = \ell + 1 = d - 1.$$

This proves the claim when T starts in Γ .

Assume now that there exists an integer $\ell \geq 0$ and a family \mathcal{R} of non-transjective Auslander-Reiten components such that T starts in \mathcal{R} and ends in $\mathcal{R}[\ell]$. Then $\text{s.gl.dim. } T = \ell + 2$ (3.3.6, 3.4). Let Γ and \mathcal{H} be like in the proof in 4.1 in the corresponding situation. Then

$$T \in \text{add}(\mathcal{R} \cup \Gamma[1] \cup \dots \cup \Gamma[\ell] \cup \mathcal{R}[\ell]).$$

Let

$$T = T_1 \oplus T_2$$

be a decomposition such that $T_1 \in \text{add}(\mathcal{R} \cup \Gamma[1] \cup \dots \cup \mathcal{R}[\ell - 1])$ and $T_2 \in \text{add}(\Gamma[\ell] \cup \mathcal{R}[\ell])$. In particular $\text{Hom}(T_2, T_1) = 0$. Let $M \rightarrow T_2$ be a minimal right $\text{add } T_1$ -approximation. Let

$$T_2' \rightarrow M \rightarrow T_2 \rightarrow T_2'[1]$$

be a triangle. Let $T' = T_1 \oplus T_2'$. Then T' is tilting. Note that the choice made for the decomposition $T = T_1 \oplus T_2$ and the fact that $M \rightarrow T_2$ is right minimal entail that $M \in \text{add}(\Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1])$. It appears that T' starts in \mathcal{R} and ends in $\mathcal{R}[\ell - 1]$. Indeed, first

$$T_1 \in \text{add}(\mathcal{R} \cup \Gamma[1] \cup \dots \cup \mathcal{R}[\ell - 1]) \text{ and } (\text{add } T_1) \cap \mathcal{R} \neq \emptyset.$$

Next, if X is an indecomposable direct summand of T'_2 then (2.1.3) at least one of the following assertions is true

- $X \in \text{add}(T_2[-1])$, in which case $X \in \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1]$,
- $X \in \text{add } M$, in which case $X \in \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1]$,
- $\text{Hom}(X, M) \neq 0$ and $\text{Hom}(T_2[-1], X) \neq 0$, which imply that

$$X \in \Gamma[\ell - 2] \cup \mathcal{R}[\ell - 2] \cup \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1], \text{ and } X \in \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1] \cup \Gamma[\ell] \cup \mathcal{R}[\ell],$$

that is, $X \in \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1]$.

Therefore

$$T'_2 \in \text{add}(\Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1]).$$

Hence there only remains to prove that T' has an indecomposable direct summand lying in $\mathcal{R}[\ell - 1]$. Let $Z \in \mathcal{R}[\ell]$ be an indecomposable direct summand of T . Let $X \rightarrow N \rightarrow Z \rightarrow X[1]$ be a triangle like in 2.1.2. Therefore X is an indecomposable direct summand of T' . Note that $N \in \text{add}(\mathcal{R}[\ell - 1])$ because $N \rightarrow Z$ is a minimal right $\text{add } T_1$ -approximation and $Z \in \mathcal{R}[\ell]$. If $\text{Hom}(T_1, Z) = 0$ then $X = Z[-1] \in \mathcal{R}[\ell - 1]$. Otherwise $\text{Hom}(X, N) \neq 0$ and $\text{Hom}(Z[-1], X) \neq 0$. The former implies that $X \in \mathcal{R}[\ell - 2] \cup \Gamma[\ell - 1] \cup \mathcal{R}[\ell - 1]$. The latter implies that $X \in \mathcal{R}[\ell - 1] \cup \Gamma[\ell] \cup \mathcal{R}[\ell]$ because $Z \in \mathcal{R}[\ell]$. Thus $X \in \mathcal{R}[\ell - 1]$. This proves that T' starts in \mathcal{R} and ends in $\mathcal{R}[\ell - 1]$. The claim is therefore proved, and so is the theorem. \square

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