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# RECURRENCE OR TRANSIENCE OF RANDOM WALKS ON RANDOM GRAPHS GENERATED BY POINT PROCESSES IN $\mathbb{R}^d$

ARNAUD ROUSSELLE<sup>‡</sup>

ABSTRACT. We consider random walks associated with conductances on Delaunay triangulations, Gabriel graphs and skeletons of Voronoi tilings which are generated by point processes in  $\mathbb{R}^d$ . Under suitable assumptions on point processes and conductances, we show that, for almost any realization of the point process, these random walks are recurrent if  $d = 2$  and transient if  $d \geq 3$ . These results hold for a large variety of point processes including Poisson point processes, Matérn cluster and Matérn hardcore processes which have clustering or repulsive properties. In order to prove them, we state general criteria for recurrence or almost sure transience which apply to random graphs embedded in  $\mathbb{R}^d$ .

**Key words:** Random walk in random environment; recurrence; transience; Voronoi tessellation; Delaunay triangulation; Gabriel graph; point process; electrical network.

**AMS 2010 Subject Classification :** Primary: 60K37, 60D05; secondary: 60G55; 05C81.

## 1. INTRODUCTION AND MAIN RESULTS

We deal with the question of almost sure recurrence or transience for random walks on Delaunay triangulations, Gabriel graphs and skeletons of Voronoi tilings which are generated by point processes in  $\mathbb{R}^d$ . This fits into the broader context of random walks on random geometric graphs with conductances.

More precisely, consider a connected unoriented infinite locally finite random graph  $G = (V_G, E_G)$  embedded in  $\mathbb{R}^d$ . Let  $C$  be a nonnegative symmetric function on  $E_G$  satisfying  $w(u) := \sum_{v \sim u} C(u, v) > 0$  for any vertex  $u$ ;  $C$  is called a *conductance* on  $E_G$  and  $R = 1/C$  is the associated *resistance*. Then  $(G, C)$  is an infinite *random electrical network* and the random walk on  $G$  associated with  $C$  is the (time homogeneous) Markov chain  $(X_n)_n$  with transition probabilities given by:

$$\mathbb{P}[X_{n+1} = v | X_n = u] = \frac{C(u, v)}{w(u)}.$$

Of particular interest are the cases where  $C$  is either constant on the edge set (simple random walk on  $G$ ) or given by a decreasing function of edge length.

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We refer to [DS84, LP12] for introductions on electrical network and random walks associated with conductances. Random walks on random geometric networks are natural for describing flows, molecular diffusions, heat conduction or other problems from statistical mechanics in random and irregular media. In the last decades, most of the authors considered models where underlying graphs were the lattice  $\mathbb{Z}^d$  or a subgraph of  $\mathbb{Z}^d$ . Particular examples are random walks on percolation clusters (see [GKZ93] for the question of recurrence or transience and [BB07, Mat08, MP07] for invariance principles) and the so called random conductance model (see [Bis11] and references therein). To the best of our knowledge, very few articles deal with models where graphs are generated by general point processes in  $\mathbb{R}^d$ ; the exceptions are [CFG09, FSBS06, FM08, CF09]. In this papers, authors considered random walks on complete graphs generated by point processes with jump probability which is a decreasing function of the distance between points. They respectively give recurrence and transience results and state an annealed invariance principle.

Under suitable assumptions on the point process  $\mathcal{N}$  and conductance function  $C$ , we obtain in this paper almost sure recurrence or transience results, namely Theorem 1 and Theorem 2, when  $G$  is the Delaunay triangulation, the Gabriel graph or the skeleton of the Voronoi tiling of  $\mathcal{N}$ .

**1.1. Conditions on the point process.** In what follows the point process  $\mathcal{N}$  is supposed to be simple, stationary and almost surely in general position (see [Zes08]): there are no  $d$  points (resp.  $d + 1$  points) in a  $d - 1$ -dimensional affine subspace (resp. in a sphere).

We will need the following assumptions on the void probabilities **(V)** and on the deviation probabilities **(D<sub>2</sub>)** and **(D<sub>3+</sub>)**:

**Assumptions.**

**(V)** *There exists a constant  $c_1$  such that for  $L$  large enough:*

$$\mathbb{P}[\#([0, L]^d \cap \mathcal{N}) = 0] \leq e^{-c_1 L^d}.$$

**(D<sub>2</sub>)** *If  $d = 2$ , there are constants  $c_2, c_3$  such that for  $L, l$  large enough:*

$$\mathbb{P}[\#([0, L] \times [0, l] \cap \mathcal{N}) \geq c_2 L l] \leq e^{-c_3 L l}.$$

**(D<sub>3+</sub>)** *If  $d \geq 3$ , there exists  $c_4$  such that for  $L$  large enough and all  $m > 0$ :*

$$\mathbb{P}[\#([0, L]^d \cap \mathcal{N}) \geq m] \leq e^{c_4 L^d - m}.$$

For transience results, the following additional assumptions are needed:

**(FR<sub>k</sub>)**  *$\mathcal{N}$  has a finite range of dependence  $k$ , i.e., for any disjoint Borel sets  $A, B \subset \mathbb{R}^d$  with  $d(A, B) := \inf\{\|x - y\| : x \in A, y \in B\} \geq k$ ,  $\mathcal{N} \cap A$  and  $\mathcal{N} \cap B$  are independent.*

**(ND)** *Almost surely,  $\mathcal{N}$  does not have any descending chain, where a descending chain is a sequence  $(u_i)_{i \in \mathbb{N}} \subset \mathcal{N}$  such that:*

$$\forall i \in \mathbb{N}, \|u_{i+2} - u_{i+1}\| < \|u_{i+1} - u_i\|.$$

These assumptions are in particular satisfied if  $\mathcal{N}$  is:

- a homogeneous Poisson point process (PPP),
- a Matérn cluster process (MCP),

- a Matérn hardcore process I or II (MHP I/II).

Moreover, assumptions **(V)** and **(D<sub>2</sub>)** hold if  $\mathcal{N}$  is a stationary determinantal point process (DPP).

A brief overview on each of these point processes is given in Section 6. Note that these processes have different interaction properties: for PPPs there is no interaction between points, MCPs exhibit clustering effects whereas points in MHPs and DPPs repel each other.

**1.2. The graph structures.** Write  $\text{Vor}_{\mathcal{N}}(\mathbf{x}) := \{x \in \mathbb{R}^d : \|x - \mathbf{x}\| \leq \|x - \mathbf{y}\|, \forall \mathbf{y} \in \mathcal{N}\}$  for the Voronoi cell of  $\mathbf{x} \in \mathcal{N}$ ;  $\mathbf{x}$  is called the nucleus or the seed of the cell. The Voronoi diagram of  $\mathcal{N}$  is the collection of the Voronoi cells. It tessellates  $\mathbb{R}^d$  into convex polyhedra. See [Møl94, Cal10] for an overview on these tessellations.

The graphs considered in the sequel are:

**VS( $\mathcal{N}$ ):** the skeleton of the Voronoi tiling of  $\mathcal{N}$ . Its vertex (resp. edge) set consists of the collection of the vertices (resp. edges) on the boundaries of the Voronoi cells. Note that this is the only graph with bounded (actually constant) degree considered in the sequel.

**DT( $\mathcal{N}$ ):** the Delaunay triangulation of  $\mathcal{N}$ . It is the dual graph of its Voronoi tiling. It has  $\mathcal{N}$  as vertex set and there is an edge between  $\mathbf{x}$  and  $\mathbf{y}$  in  $\text{DT}(\mathcal{N})$  if  $\text{Vor}_{\mathcal{N}}(\mathbf{x})$  and  $\text{Vor}_{\mathcal{N}}(\mathbf{y})$  share a  $(d - 1)$ -dimensional face. Another useful characterization of  $\text{DT}(\mathcal{N})$  is the following: a simplex  $\Delta$  is a cell of  $\text{DT}(\mathcal{N})$  iff its circumscribed sphere has no point of  $\mathcal{N}$  in its interior. Note that this triangulation is well defined since  $\mathcal{N}$  is assumed to be in general position.

These two graphs are widely used in many fields such as astrophysics [RBFN01], cellular biology [Pou04], ecology [Roq97] and telecommunications [BB09].

**Gab( $\mathcal{N}$ ):** the Gabriel graph of  $\mathcal{N}$ . Its vertex set is  $\mathcal{N}$  and there is an edge between  $u, v \in \mathcal{N}$  if the ball of diameter  $[u, v]$  contains no point of  $\mathcal{N}$  in its interior. Note that  $\text{Gab}(\mathcal{N})$  is a subgraph of  $\text{DT}(\mathcal{N})$  and contains the Euclidean minimum spanning tree and the nearest neighbor graph. It has for example applications in geography, routing strategies, biology or tumor growth analysis (see [BBD02, GS69, MS80]).

**1.3. Main results.** The obtained random graph is now equipped with conductance function  $C$  possibly depending on the graph structure. Our main result is:

**Theorem 1.** *Let  $\mathcal{N}$  be a homogeneous Poisson point process, a Matérn cluster process or a Matérn hardcore process of type I or II.*

- (1) *Let  $d = 2$ . If  $C$  is uniformly bounded from above, then for almost any realization of  $\mathcal{N}$  the random walks on  $\text{DT}(\mathcal{N})$ ,  $\text{Gab}(\mathcal{N})$  and  $\text{VS}(\mathcal{N})$  associated with  $C$  are recurrent.*
- (2) *Let  $d \geq 3$ . If  $C$  is uniformly bounded from below or a decreasing function of edge length, then for almost any realization of  $\mathcal{N}$  the random walks on  $\text{DT}(\mathcal{N})$ ,  $\text{Gab}(\mathcal{N})$  and  $\text{VS}(\mathcal{N})$  associated with  $C$  are transient.*

Moreover, (1) holds if  $\mathcal{N}$  is a stationary determinantal point process in  $\mathbb{R}^2$ .

Note that in the unpublished manuscript [ABS05], Addario-Berry & Sarkar announced similar results in the particular setting of simple random walks on the Delaunay triangulation generated by a PPP. In this paper, the proofs rely on a deviation result on the so-called *stabbing number* of  $\text{DT}(\mathcal{N})$  which was not established at that time. Actually, this deviation estimate has been proved since then in [PR12]. However, thanks to our method, such a strong estimate is no longer needed. This allows us to extend the results to more general settings.

In the sequel, we will actually prove Theorem 2 which implies Theorem 1 and deals with general point processes as described in Subsection 1.1:

**Theorem 2.** *Let  $\mathcal{N}$  be a stationary simple point process in  $\mathbb{R}^d$  almost surely in general position.*

- (1) *Let  $d = 2$ . Assume that  $\mathcal{N}$  satisfies **(V)** and **(D<sub>2</sub>)**. If  $C$  is uniformly bounded from above, then for almost any realization of  $\mathcal{N}$  the random walks on  $\text{DT}(\mathcal{N})$ ,  $\text{Gab}(\mathcal{N})$  and  $\text{VS}(\mathcal{N})$  associated with  $C$  are recurrent.*
- (2) *Let  $d \geq 3$ . Assume that  $\mathcal{N}$  has a finite range of dependence and satisfies **(V)** and **(D<sub>3+</sub>)**. If  $C$  is uniformly bounded from below or a decreasing function of edge length, then for almost any realization of  $\mathcal{N}$  the random walks on  $\text{DT}(\mathcal{N})$  and  $\text{VS}(\mathcal{N})$  associated with  $C$  are transient.*

*If in addition  $\mathcal{N}$  has almost surely no descending chain, the same conclusion holds on  $\text{Gab}(\mathcal{N})$ .*

In order to prove this result, we give recurrence and almost sure transience criteria, namely Criteria 3 and 6. These criteria have their own interest as they apply to the general framework of electrical geometric networks described earlier.

Sections 2 and 3 deal with recurrence and transience criteria. Sections 4 and 5 are devoted to the proof of Theorem 2. In Section 6, assumptions of Theorem 2 are proved to hold for the point processes considered in Theorem 1.

## 2. A RECURRENCE CRITERION

In this section, we give a recurrence criterion for random walks on a geometric graph  $G = (V_G, E_G)$  equipped with a conductance  $C$ . In the sequel  $G$  is assumed to be connected, infinite and locally finite. The criterion is established on a deterministic graph, but it gives a way to obtain almost sure results in the setting of random graphs (see section 4). Note that it is close in spirit to the proof of [ABS05, Theorem 4].

Let us define:

$$\begin{aligned} A_i &:= \{x \in \mathbb{R}^d : i - 1 \leq \|x\|_\infty < i\}, \\ B_i &:= [-i, i]^d, \\ \text{Ed}_G(i) &:= \{e = (u, v) \in E_G : u \in B_i, v \notin B_i\}. \end{aligned}$$

**Criterion 3.** *Assume that the conductance  $C$  is bounded from above and there exist functions  $L, N : \mathbb{N}^* \mapsto \mathbb{R}_+^*$  such that:*

- (1)  $\sum_i \frac{1}{L(i)N(i)} = +\infty$ ,
- (2) the length of any edge in  $Ed_G(i)$  is less than  $L(i)$ ,
- (3) the number of edges in  $Ed_G(i)$  is less than  $N(i)$ .

Then the random walk on  $G$  with conductance  $C$  is recurrent.

*Proof:* The basic idea is inherited from electrical network theory. One can define the *effective resistance to infinity* of the network which is known to be infinite if and only if the associated random walk is recurrent (see [LP12, §2.2]). We will reduce the network so that the resistance of any edge does not increase. Thanks to the Rayleigh monotonicity principle, the effective resistance to infinity does not increase either. It then suffices to show that the reduced network has infinite resistance to infinity.

First each edge  $e = (u, v)$  with  $u \in A_{i_1}, v \in A_{i_2}, i_1 < i_2$  is cut into  $j = i_2 - i_1$  resistors connected in series with endpoints in consecutive rings  $A_{i_1}, \dots, A_{i_2}$  each one having resistance  $(jC(e))^{-1}$ . Secondly, points of a ring  $A_i$  are merged together into a single point  $a_i$ . We hence obtain a new network with vertices  $(a_i)_{i \geq i_0}$  where  $i_0$  is the lowest index such that  $A_{i_0}$  contains a vertex of the original graph  $G$ . The resistance to infinity of the new network is lower than the original one and is equal to  $\sum_{i=i_0}^{+\infty} r_i$ , where  $r_i$  stands for the effective resistance between  $a_i$  and  $a_{i+1}$ .

It remains to show that:

$$\sum_{i=i_0}^{+\infty} r_i = +\infty.$$

Let us subdivide  $Ed_G(i)$  into the following subsets:

$$Ed_G(i, j) := \{e = (u, v) \in E_G : u \in A_{i_1}, v \in A_{i_2}, i_1 \leq i < i_2, i_2 - i_1 = j\}.$$

Note that an edge  $e \in Ed_G(i, j)$  provides a conductance of  $jC(e)$  between  $a_i$  and  $a_{i+1}$  in the new network. Thanks to the usual reduction rules and conditions (2) and (3), we get:

$$\begin{aligned} \frac{1}{r_i} &= \sum_{j=1}^{+\infty} \sum_{e \in Ed_G(i, j)} jC(e) \leq (\sup C) \sum_{j=1}^{+\infty} j \#Ed_G(i, j) \\ &\leq (\sup C) \sum_{j=1}^{L(i)} j \#Ed_G(i, j) \leq (\sup C)L(i) \sum_{j=1}^{L(i)} \#Ed_G(i, j) \\ &\leq (\sup C)L(i) \sum_{j=1}^{+\infty} \#Ed_G(i, j) \leq (\sup C)L(i)N(i). \end{aligned}$$

This concludes the proof since:

$$\sum_{i=i_0}^{+\infty} r_i \geq \sum_{i=i_0}^{+\infty} \frac{1}{(\sup C)L(i)N(i)}$$

and the r.h.s. is infinite by condition (1). □

## 3. A TRANSIENCE CRITERION

In this section, the graph  $G = (V_G, E_G)$  is obtained from a point process in  $\mathbb{R}^d$  and is equipped with a resistance  $R$  on  $E_G$ . The key idea is combining *discretization techniques* with a *rough embedding method*.

Let us define *rough embeddings* for unoriented networks in a similar way to [LP12, §2.3]:

**Definition 4.** *Let  $H$  and  $H'$  be two networks with resistances  $r$  and  $r'$ .*

*We say that a map  $\phi : V_H \rightarrow V_{H'}$  is a rough embedding from  $(H, r)$  to  $(H', r')$  if there exist  $\alpha, \beta < +\infty$  and a map  $\Phi$  from (unoriented) edges of  $H$  to (unoriented) paths in  $H'$  such that:*

- (1) *for every edge  $(u, v) \in E_H$ ,  $\Phi(u, v)$  is a non-empty simple path of edges of  $H'$  between  $\phi(u)$  and  $\phi(v)$  with*

$$\sum_{e' \in \Phi(u, v)} r'(e') \leq \alpha r(u, v);$$

- (2) *any edge  $e' \in E_{H'}$  is in the image under  $\Phi$  of at most  $\beta$  edges of  $H$ .*

A version of [LP12, Theorem 2.17] without orientation is needed to establish the criterion. Lyons and Peres attribute this result to Kanai [Kan86].

**Theorem 5** (see [LP12]). *If there is a rough embedding from  $(H, r)$  to  $(H', r')$  and  $(H, r)$  is transient, then  $(H', r')$  is transient.*

Let us divide  $\mathbb{R}^d$  into boxes of side  $M \geq 1$ :

$$B_{\mathbf{z}} = B_{\mathbf{z}}^M := M\mathbf{z} + \left[ -\frac{M}{2}, \frac{M}{2} \right]^d, \mathbf{z} \in \mathbb{Z}^d.$$

We will prove the following criterion:

**Criterion 6.** *If  $d \geq 3$  and if one can find a subset of boxes, called good boxes, such that:*

- (1) *in each good box  $B_{\mathbf{z}}$ , one can choose a reference vertex  $v_{\mathbf{z}} \in B_{\mathbf{z}} \cap V_G$ ,*
- (2) *there exist  $K, L$  such that to each pair of neighboring good boxes  $B_{\mathbf{z}_1}$  and  $B_{\mathbf{z}_2}$ , one can associate a path  $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2})$  between the respective reference vertices  $v_{\mathbf{z}_1}$  and  $v_{\mathbf{z}_2}$  of these boxes satisfying:*
  - (a)  $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2}) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ ,
  - (b) *any edge of  $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2})$  has resistance at most  $K$ ,*
  - (c) *the length of  $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2})$  in the graph distance is bounded by  $L$ ,*
- (3) *the process  $X = \{X_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\} := \{\mathbb{1}_{B_{\mathbf{z}}} \text{ is good}, \mathbf{z} \in \mathbb{Z}^d\}$  stochastically dominates a supercritical independent Bernoulli site percolation process on  $\mathbb{Z}^d$ ,*

*then the random walk on  $G$  with resistance  $R$  is almost surely transient.*

**Remark 7.** *Thanks to [LSS97, Theorem 0.0], in order to show (3), it suffices to check that  $X = \{X_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\}$  is a  $k$ -dependent process so that  $\mathbb{P}[X_{\mathbf{z}} = 1] \geq p^* \in ]0, 1[$ , where  $p^*$  depends on  $k$  and  $d$  but not on  $M$ .*

*Proof:* Thanks to condition (3), one can define a random field  $(\sigma_1, \sigma_2) \in \{0, 1\}^{\mathbb{Z}^d} \times \{0, 1\}^{\mathbb{Z}^d}$ , where  $\sigma_1$  has the distribution of supercritical (independent) Bernoulli site percolation process,  $\sigma_2$  has the law of  $X$ , and the couple satisfies  $\sigma_1 \leq \sigma_2$  almost surely. Let  $\pi_\infty$  stand for the (a.s. unique) infinite percolation cluster. It is known that the simple random walk on  $\pi_\infty$  is a.s. transient when  $d \geq 3$  (see [GKZ93]). By Theorem 5, it is enough to exhibit an a.s. rough embedding from  $\pi_\infty$  (with resistance 1 on each edge) to  $(G, R)$ .

By stochastic domination, for any open site  $\mathbf{z} \in \pi_\infty$  the corresponding box  $B_{\mathbf{z}}$  is good. We set  $\phi(\mathbf{z}) := v_{\mathbf{z}}$  the reference vertex of  $B_{\mathbf{z}}$  given by (1). Fix  $\mathbf{z}_1 \sim \mathbf{z}_2 \in \pi_\infty$ . Then  $B_{\mathbf{z}_1}$  and  $B_{\mathbf{z}_2}$  are two neighboring good boxes and one can find a path  $\Phi(\mathbf{z}_1, \mathbf{z}_2)$  between  $v_{\mathbf{z}_1} = \phi(\mathbf{z}_1)$  and  $v_{\mathbf{z}_2} = \phi(\mathbf{z}_2)$  fully included in  $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ . Note that  $\Phi(\mathbf{z}_1, \mathbf{z}_2)$  can be assumed to be simple and that by (2)(b),(c), one has:

$$\sum_{e \in \Phi(\mathbf{z}_1, \mathbf{z}_2)} R(e) \leq \alpha := KL.$$

Moreover, an edge of  $E_G$  is in the image of at most  $\beta := 2d$  edges of  $\pi_\infty$  since a good box has at most  $2d$  neighboring good boxes and  $(v_{\mathbf{z}_1}, \dots, v_{\mathbf{z}_2}) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ .  $\square$

#### 4. RECURRENCE IN DIMENSION 2

Let us assume that  $\mathcal{N}$  is a stationary simple point process satisfying **(V)** and **(D<sub>2</sub>)**. The aim of this section is to prove the case  $d = 2$  of Theorem 2, i.e. almost sure recurrence of walks on  $\text{VS}(\mathcal{N})$ ,  $\text{DT}(\mathcal{N})$  and  $\text{Gab}(\mathcal{N})$ .

Since  $\text{Gab}(\mathcal{N})$  is a subgraph of  $\text{DT}(\mathcal{N})$ , recurrence on  $\text{DT}(\mathcal{N})$  implies recurrence on  $\text{Gab}(\mathcal{N})$  by Rayleigh monotonicity principle. Consequently, we only need to find functions  $L_{\text{DT}(\mathcal{N})}, L_{\text{VS}(\mathcal{N})}$  and  $N_{\text{DT}(\mathcal{N})}, N_{\text{VS}(\mathcal{N})}$  so that assumptions of Criterion 3 are satisfied for  $\text{DT}(\mathcal{N})$  and  $\text{VS}(\mathcal{N})$  respectively. In fact, for almost any realization of  $\mathcal{N}$ ,  $L.(i)$  can be chosen of order  $\sqrt{\log(i)}$  and  $N.(i)$  of order  $i\sqrt{\log(i)}$ .

**4.1. Delaunay triangulation case.** We first get an upper bound on the length of the edges in  $Ed_{\text{DT}(\mathcal{N})}(i)$  (i.e. the set of edges with only one endpoint in  $B_i$ ) via an extended version of [ABS05, Lemma 1].

We write  $\mathcal{A}_{\text{DT}}(i)$  for the event 'there exists an edge of  $Ed_{\text{DT}(\mathcal{N})}(i)$  with length greater than  $\sqrt{\log i}$ ' and  $\mathcal{B}_{\text{DT}}(i, j)$  for the event 'there exists an edge of  $Ed_{\text{DT}(\mathcal{N})}(i)$  with length between  $j\sqrt{\log i}$  and  $(j + 1)\sqrt{\log i}$ '.

**Lemma 8.** *Let  $\mathcal{N}$  be a stationary point process such that **(V)** holds. Then there exists a constant  $c_5 > 0$  such that for  $i$  large enough:*

$$\mathbb{P}[\mathcal{A}_{\text{DT}}(i)] \leq \frac{c_5}{i^2}.$$

*Proof:* If  $\mathcal{B}_{\text{DT}}(i, j)$  occurs, there is an edge  $e$  of length between  $j\sqrt{\log i}$  and  $(j + 1)\sqrt{\log i}$  having an endpoint in  $B_i$ . This edge is fully contained in  $B_{i'}$ , where  $i' := \lceil i + (j + 1)\sqrt{\log i} \rceil$ . Let  $m \leq i'$  be as large as possible such that  $s := \lfloor j\sqrt{\log i}/4 \rfloor$  divides  $m$  and  $l := m/s$ .



Now, divide  $B_m$  in  $l^2$  squares  $Q_1, \dots, Q_{l^2}$  of side  $s$ . Since  $e$  is an edge of  $\text{DT}(\mathcal{N})$ , one of the half-disks with diameter  $e$  does not intersect  $\mathcal{N}$ . Let us denote this half-disk by  $\mathfrak{D}$ . Note that  $\mathfrak{D}$  is included in  $B_{i'}$  because  $e$  has an endpoint in  $B_i$  and has length at most  $(j+1)\sqrt{\log i}$ . Thanks to the choice of  $s$ ,  $\mathfrak{D}$  contains one of the  $Q_k$ . So  $Q_k \cap \mathcal{N}$  is empty. By stationarity of  $\mathcal{N}$ , we obtain:

$$\mathbb{P}[\mathcal{B}_{\text{DT}}(i, j)] \leq l^2 \mathbb{P}[Q_1 \cap \mathcal{N} = \emptyset] \leq c_6 i^2 \mathbb{P}[Q_1 \cap \mathcal{N} = \emptyset]. \quad (1)$$

But, thanks to **(V)**, for  $i$  large enough:

$$\mathbb{P}[Q_1 \cap \mathcal{N} = \emptyset] \leq e^{-c_7 j^2 \log i} = \frac{1}{i^4} e^{-(c_7 j^2 - 4) \log i}.$$

Hence, with (1):

$$\mathbb{P}[\mathcal{B}_{\text{DT}}(i, j)] \leq \frac{c_6}{i^2} e^{-(c_7 j^2 - 4) \log i}.$$

Finally,

$$\mathbb{P}[\mathcal{A}_{\text{DT}}(i)] \leq \sum_{j=1}^{+\infty} \mathbb{P}[\mathcal{B}_{\text{DT}}(i, j)] \leq \frac{c_6}{i^2} \sum_{j=1}^{+\infty} e^{-(c_7 j^2 - 4) \log i} \leq \frac{c_5}{i^2}.$$

□

Thanks to Borel-Cantelli lemma, for almost any realization of  $\mathcal{N}$ ,  $\mathcal{A}_{\text{DT}}(i)$  fails for only finitely many  $i$ . Thus, for almost any realization of  $\mathcal{N}$ , one can choose  $L_{\text{DT}}(i)$  of order  $\sqrt{\log i}$  in Criterion 3.

We will see that we can choose  $N_{\text{DT}}(i)$  of order  $i\sqrt{\log i}$ . To do so, we show the following lemma and conclude with Borel-Cantelli lemma as before.

**Lemma 9.** *Let  $\mathcal{N}$  be a stationary point process such that **(V)** and **(D<sub>2</sub>)** hold.*

*Then, there exists  $c_8$  such that for  $i$  large enough:*

$$\mathbb{P}[\mathcal{C}_{\text{DT}}(i)] \leq \frac{c_8}{i^2},$$

where  $\mathcal{C}_{\text{DT}}(i)$  is the event ' $\#(Ed_{\text{DT}(\mathcal{N})}(i)) \geq 48c_2 i \sqrt{\log i}$ '.

*Proof:* Note that:

$$\mathbb{P}[\mathcal{C}_{\text{DT}}(i)] = \mathbb{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)] + \mathbb{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)^c] \leq \mathbb{P}[\mathcal{A}_{\text{DT}}(i)] + \mathbb{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)^c],$$

with  $\mathcal{A}_{\text{DT}}(i)$  as in Lemma 8. So, it remains to show:

$$\mathbb{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)^c] \leq \frac{c_9}{i^2}.$$

On the event  $\mathcal{A}_{\text{DT}}(i)^c$ , edges in  $Ed_{\text{DT}(\mathcal{N})}(i)$  have lengths at most  $\sqrt{\log i}$ , thus these edges are fully included in the ring  $R(i) := [-i - \sqrt{\log i}, i + \sqrt{\log i}]^2 \setminus [-i + \sqrt{\log i}, i - \sqrt{\log i}]^2$ . The restriction of  $\text{DT}(\mathcal{N})$  to  $R(i)$  is a planar graph with  $\#(\mathcal{N} \cap R(i))$  vertices. Thanks to a corollary of Euler's formula (see [Bol98, Theorem 16, p.22]), it has at most  $3\#(\mathcal{N} \cap R(i)) - 6$  edges. Thus  $\#(Ed_{\text{DT}(\mathcal{N})}(i))$  is bounded from above by  $3\#(\mathcal{N} \cap R(i))$ .

So, with **(D<sub>2</sub>)**:

$$\begin{aligned} \mathbb{P}[\mathcal{C}_{\text{DT}}(i) \cap \mathcal{A}_{\text{DT}}(i)^c] &\leq \mathbb{P}\left[\#(\mathcal{N} \cap R(i)) \geq 16c_2i\sqrt{\log i}\right] \\ &\leq 4\mathbb{P}\left[\#(\mathcal{N} \cap ([0, 2i] \times [0, 2\sqrt{\log i}])) \geq 4c_2i\sqrt{\log i}\right] \\ &\leq 4e^{-4c_3i\sqrt{\log i}} \leq \frac{c_9}{i^2}. \end{aligned}$$

□

**4.2. Skeleton of the Voronoi tiling case.** In order to estimate the lengths of edges in  $Ed_{\text{VS}(\mathcal{N})}(i)$ , the following analogue of Lemma 8 is needed:

**Lemma 10.** *Let  $\mathcal{N}$  be a stationary point process such that **(V)** holds.*

*Then, there exists  $c_{10}$  such that for  $i$  large enough:*

$$\mathbb{P}[\mathcal{A}_{\text{VS}}(i)] \leq \frac{c_{10}}{i^2}$$

where  $\mathcal{A}_{\text{VS}}(i)$  is the event 'there exists an edge of  $Ed_{\text{VS}(\mathcal{N})}(i)$  with length greater than  $4\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$ '.

*Proof:* Fix  $\varepsilon_0 > 0$  and set  $\overline{B}_i := B_i + B(0, 4\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} + \varepsilon_0)$  where  $B(x, r)$  stands for the Euclidean ball centered at  $x$  and of radius  $r$ . We say that  $\mathcal{B}_{\text{VS}}(i)$  holds if, when  $\overline{B}_i$  is covered with  $O(i^2/\log i)$  disjoint squares of side  $2c_1^{-\frac{1}{2}}\sqrt{\log i}$ , any of these squares contains a point of  $\mathcal{N}$ .

We will show that, on  $\mathcal{B}_{\text{VS}}(i)$ , any Voronoi cell intersecting  $\partial B_i$  is contained in a ball centered at its nucleus and of radius  $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$ . This will imply that  $\mathcal{B}_{\text{VS}}(i) \subset \mathcal{A}_{\text{VS}}(i)^c$ .

Let us assume that  $\mathcal{B}_{\text{VS}}(i)$  holds. Then any point in  $\partial B_i$  and the nucleus  $\mathbf{x}$  of its Voronoi cell are distant by at most  $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$ . Note that, for  $0 < \varepsilon < \varepsilon_0$ ,  $B(\mathbf{x}, 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} + \varepsilon) \subset \overline{B}_i$ . Hence, points in  $\partial B(\mathbf{x}, 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} + \varepsilon)$  are within a distance of at most  $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$  from nuclei of their respective Voronoi cells which cannot be  $\mathbf{x}$ . Thus  $\text{Vor}_{\mathcal{N}}(\mathbf{x}) \subset B(\mathbf{x}, 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i})$ . So, if  $\mathcal{A}_{\text{VS}}(i)$  holds,  $\mathcal{B}_{\text{VS}}(i)$  fails.

Finally, with **(V)**, for  $i$  large enough:

$$\begin{aligned} \mathbb{P}[\mathcal{A}_{\text{VS}}(i)] &\leq \mathbb{P}[\mathcal{B}_{\text{VS}}(i)^c] \leq c_{10} \frac{i^2}{\log i} \mathbb{P}[[0, 2c_1^{-\frac{1}{2}}\sqrt{\log i}]^2 \cap \mathcal{N} = \emptyset] \\ &\leq c_{10}i^2e^{-4\log i} = \frac{c_{10}}{i^2}. \end{aligned}$$

□

By Borel-Cantelli lemma, for almost any realization of  $\mathcal{N}$ ,  $\mathcal{A}_{\text{VS}}(i)$  fails only finitely many times. Thus one can choose  $L_{\text{VS}(\mathcal{N})}$  of order  $\sqrt{\log i}$ .

It remains to show that one can choose  $N_{\text{VS}}(i)$  of order  $i\sqrt{\log i}$ . To do this, we state the following lemma and conclude as usual with Borel-Cantelli lemma.

**Lemma 11.** *Let  $\mathcal{N}$  be a stationary point process such that **(V)** and **(D<sub>2</sub>)** hold.*

*Then there exists  $c_{11}$  such that for  $i$  large enough:*

$$\mathbb{P}[\mathcal{C}_{\text{VS}}(i)] \leq \frac{c_{11}}{i^2},$$

where  $\mathcal{C}_{\text{VS}}(i)$  is the event ' $\#(Ed_{\text{VS}(\mathcal{N})}(i)) \geq 32\sqrt{2}c_2c_1^{-\frac{1}{2}}i\sqrt{\log i}$ '.

*Proof:* One has:

$$\mathbb{P}[\mathcal{C}_{\text{VS}}(i)] = \mathbb{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)^c] + \mathbb{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)] \leq \mathbb{P}[\mathcal{B}_{\text{VS}}(i)^c] + \mathbb{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)],$$

with  $\mathcal{B}_{\text{VS}}(i)$  as in the proof of Lemma 10. It remains to show that:

$$\mathbb{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)] \leq \frac{c_{12}}{i^2}.$$

One can see that edges in  $Ed_{\text{VS}(\mathcal{N})}(i)$  belong to boundaries of Voronoi cells intersecting  $\partial B_i$  and that  $\partial B_i$  intersects at most two sides of a given Voronoi cell. The number of edges in  $Ed_{\text{VS}(\mathcal{N})}(i)$  is then bounded by the number of Voronoi cells crossing  $\partial B_i$ . But, as noticed during the proof of Lemma 10, on the event  $\mathcal{B}_{\text{VS}}(i)$ , points of  $\partial B_i$  are within a distance of at most  $2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}$  from nuclei of their respective Voronoi cells. Thus, nuclei of cells intersecting  $\partial B_i$  are in the ring:

$$R(i) := \left[ -i - 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}, i + 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} \right]^2 \setminus \left[ -i + 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}, i - 2\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i} \right]^2.$$

So, on  $\mathcal{B}_{\text{VS}}(i)$ ,  $\#(Ed_{\text{VS}(\mathcal{N})}(i))$  is bounded from above by the number of points in  $\mathcal{N} \cap R(i)$ .

Thus, with **(D<sub>2</sub>)**, for  $i$  large enough:

$$\begin{aligned} \mathbb{P}[\mathcal{C}_{\text{VS}}(i) \cap \mathcal{B}_{\text{VS}}(i)] &\leq \mathbb{P}[\#(\mathcal{N} \cap R(i)) \geq 32\sqrt{2}c_2c_1^{-\frac{1}{2}}i\sqrt{\log i}] \\ &\leq 4\mathbb{P}[\#(\mathcal{N} \cap ([0, 2i] \times [0, 4\sqrt{2}c_1^{-\frac{1}{2}}\sqrt{\log i}]]) \geq 8\sqrt{2}c_2c_1^{-\frac{1}{2}}i\sqrt{\log i}] \\ &\leq 4e^{-8\sqrt{2}c_3c_1^{-\frac{1}{2}}i\sqrt{\log i}} \leq \frac{c_{11}}{i^2}. \end{aligned}$$

□

## 5. TRANSCIENCE IN HIGHER DIMENSIONS

This section is devoted to the proof of the second part of Theorem 2, i.e. almost sure transience of walks on  $\text{VS}(\mathcal{N})$ ,  $\text{DT}(\mathcal{N})$  and  $\text{Gab}(\mathcal{N})$  under the assumptions on  $\mathcal{N}$ .

The assumption that  $\mathcal{N}$  has almost surely no descending chain is only used in the proof of transience on  $\text{Gab}(\mathcal{N})$ . Since  $\text{Gab}(\mathcal{N})$  is a subgraph of  $\text{DT}(\mathcal{N})$ , transience on  $\text{Gab}(\mathcal{N})$  implies transience on  $\text{DT}(\mathcal{N})$ . Actually, one can directly prove transience on  $\text{DT}(\mathcal{N})$ .

without this additional condition. This proof is very similar to the  $\text{VS}(\mathcal{N})$  case and is omitted here.

We will define 'good boxes' and use Criterion 6 for  $\text{Gab}(\mathcal{N})$  and  $\text{VS}(\mathcal{N})$ : we construct paths needed in Criterion 6 and check that  $\mathbb{P}[X_{\mathbf{z}} = 1]$  is large enough if parameters are well chosen. Thanks to the finite range of dependence assumptions (say of range  $k$ ) and to the definitions of 'good boxes' it will be clear that  $\{X_{\mathbf{z}}\}$  is a  $k$ -dependent Bernoulli process on  $\mathbb{Z}^d$ .

### 5.1. Skeleton of the Voronoi tiling case.

5.1.1. *Good boxes.* For  $M \geq 1$  to be determined later, consider a partition of  $\mathbb{R}^d$  into boxes of side  $M$ :

$$B_{\mathbf{z}} = B_{\mathbf{z}}^M := M\mathbf{z} + \left[ -\frac{M}{2}, \frac{M}{2} \right]^d, \mathbf{z} \in \mathbb{Z}^d.$$

We say that a box  $B_{\mathbf{z}}$  is  $M$ -good for VS if the following conditions are satisfied:

- i-  $\#(B_{\mathbf{z}} \cap \mathcal{N}) \leq 2c_4 M^d$ ,
- ii- when  $B_{\mathbf{z}}$  is (regularly) cut into  $\alpha_d^d := (6\lceil\sqrt{d}\rceil)^d$  sub-boxes  $b_i^z$  of side  $M/\alpha_d$ , each of these sub-boxes contains at least one point of  $\mathcal{N}$ .

5.1.2. *Construction of paths.* First of all, for each good box  $B_{\mathbf{z}_i}$ , one can fix a reference vertex  $v_{\mathbf{z}_i}$  on the boundary of the Voronoi cell of  $M\mathbf{z}_i$  as required by condition (1) in Criterion 6. Note that, thanks to -ii- in the definition of good boxes, the cell containing  $M\mathbf{z}_i$  is included in  $B_{\mathbf{z}_i}$  so that  $v_{\mathbf{z}_i} \in B_{\mathbf{z}_i}$ .

Consider two neighboring good boxes  $B_{\mathbf{z}_1}$  and  $B_{\mathbf{z}_2}$ . One must show that there exists a path in  $\text{VS}(\mathcal{N})$  between  $v_{\mathbf{z}_1}$  and  $v_{\mathbf{z}_2}$  so that condition (2) of Criterion 6 holds. To do so, note that one can find a self-avoiding path between  $v_{\mathbf{z}_1}$  and  $v_{\mathbf{z}_2}$  with edges belonging to boundaries of the Voronoi cells crossing the line segment  $[M\mathbf{z}_1, M\mathbf{z}_2]$ . Clearly, such a path is contained in  $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$  as soon as Voronoi cells intersecting  $[M\mathbf{z}_1, M\mathbf{z}_2]$  are included in  $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ . Since sub-boxes  $b_1^{\mathbf{z}_1}, \dots, b_{\alpha_d^d}^{\mathbf{z}_1}, b_1^{\mathbf{z}_2}, \dots, b_{\alpha_d^d}^{\mathbf{z}_2}$  are non-empty (of points of  $\mathcal{N}$ ), points in  $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$  are distant by at most  $\sqrt{d}M/\alpha_d$  from nuclei of their respective Voronoi cell. As in the proof of Lemma 10, one can see that the Voronoi cells intersecting  $[M\mathbf{z}_1, M\mathbf{z}_2]$  are included in  $[M\mathbf{z}_1, M\mathbf{z}_2] + B(0, 2\sqrt{d}M/\alpha_d) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ . Moreover, nuclei of Voronoi cells which are neighbors of cells intersecting  $[M\mathbf{z}_1, M\mathbf{z}_2]$  are in  $[M\mathbf{z}_1, M\mathbf{z}_2] + B(0, 3\sqrt{d}M/\alpha_d) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ . Note that the path between  $v_{\mathbf{z}_1}$  and  $v_{\mathbf{z}_2}$  has chemical length (i.e. for the graph distance) bounded by the number of vertices on the boundaries of Voronoi cells crossing  $[M\mathbf{z}_1, M\mathbf{z}_2]$ . This is less than the number of Voronoi cells crossing  $[M\mathbf{z}_1, M\mathbf{z}_2]$  times the maximal number of vertices on such a cell. Since these cells are included in  $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$  and  $B_{\mathbf{z}_1}, B_{\mathbf{z}_2}$  are good, there are no more than  $4c_4 M^d$  cells intersecting  $[M\mathbf{z}_1, M\mathbf{z}_2]$ . Each of these cells has at most  $4c_4 M^d$  neighboring cells which nuclei are in  $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ . The total number of vertices on the boundary of such a cell is generously bounded by  $\binom{4c_4 M^d}{d}$ . Indeed, any of these vertices is obtained as the intersection of  $d$  bisecting hyperplanes separating the cell from one of its neighbors (which are at most  $4c_4 M^d$ ). So, one can choose  $L := 4c_4 M^d \binom{4c_4 M^d}{d}$  in Criterion 6 (2)(c).

Finally, if  $C$  is uniformly bounded from below set  $K := \max 1/C$  in Criterion 6 (2)(b). If  $C$  is given by a decreasing function  $\varphi$  of edge length, set  $K := 1/\varphi(\sqrt{d} + 3M)$ . Indeed  $\sqrt{d} + 3M$  is the maximal (Euclidean) length of an edge in the path.

5.1.3.  $\mathbb{P}[X_{\mathbf{z}} = 1]$  is large enough. It remains to show that if  $M$  is appropriately chosen (3) in Criterion 6 is satisfied. Since  $\mathcal{N}$  has a finite range of dependence, the process  $\{X_{\mathbf{z}}\}$  is  $k$ -dependent, so, as noticed in Remark 7, it suffices to show that for  $M$  large enough:

$$\mathbb{P}[X_{\mathbf{z}} = 1] \geq p^*$$

where  $p^* = p^*(d, k) < 1$  is large enough to ensure that  $\{X_{\mathbf{z}}\}$  dominates site percolation on  $\mathbb{Z}^d$ .

Indeed, with **(V)** and **(D<sub>3+</sub>)**, one has:

$$\begin{aligned} \mathbb{P}[X_{\mathbf{z}} = 0] &\leq \mathbb{P}\left[\#(B_{\mathbf{z}} \cap \mathcal{N}) > 2c_4M^d \text{ or one of the } b_i^{\mathbf{z}}\text{'s is empty}\right] \\ &\leq \mathbb{P}\left[\#([0, M]^d \cap \mathcal{N}) > 2c_4M^d\right] + \alpha_d^d \mathbb{P}\left[\#\left(\left[0, \frac{M}{\alpha_d}\right]^d \cap \mathcal{N}\right) = 0\right] \\ &\leq \exp(-c_4M^d) + (6\lceil\sqrt{d}\rceil)^d \exp\left(-c_1\frac{M^d}{6^d\lceil\sqrt{d}\rceil^d}\right) \end{aligned}$$

which is as small as we wish for large  $M$ .

## 5.2. Gabriel graph case.

5.2.1. *A geometric lemma.* We shall state a generalization of [BBD02, Lemma 1] which allows us to control the behavior of paths as needed in Criterion 6.

**Lemma 12.** *Let  $\mathcal{N}$  be a locally finite subset of  $\mathbb{R}^d$  without descending chain.*

*Then, for any  $x, y \in \mathcal{N}$ , there exists a Gabriel path  $(x_1 = x, \dots, x_n = y)$  from  $x$  to  $y$  such that:*

$$\sum_{i=1}^{n-1} \|x_{i+1} - x_i\|^2 \leq \|y - x\|^2. \quad (2)$$

*Proof:* We use the following recursive algorithm:

```

γ=path(x, y)
if B([x, y]) ∩ N \ {x, y} = ∅ then
  set γ:=(x, y)
else
  pick z in B([x, y]) ∩ N \ {x, y}
  set γ:=path(x, z)*path(z, y)
end

```

where  $B([x, y])$  stands for the ball of diameter  $[x, y]$  and  $\star$  for concatenation of paths. Let us show that this algorithm terminates and gives the desired result.

If no recursive call is made during the execution of  $\text{path}(x, y)$ , then  $B([x, y]) \cap \mathcal{N} = \emptyset$  and  $(x, y)$  is a Gabriel edge. The returned path trivially satisfies (2). Otherwise, we write

$z := \text{IP}(x, y)$  the intermediary point of  $\mathcal{N}$  appearing in the first recursive calls to  $\text{path}$ . The section of  $B([x, y])$  by the plane generated by  $x, y, z$  is a circle with diameter  $[x, y]$ . Thus we have:

$$\|z - x\|^2 + \|y - z\|^2 \leq \|y - x\|^2.$$

Assume that  $\text{path}(x, z) = (x_1 = x, \dots, x_k = z)$  and  $\text{path}(z, y) = (x_k = z, \dots, x_n = y)$  satisfy (2). Then, so does their concatenation  $\text{path}(x, y)$ :

$$\begin{aligned} \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|^2 &= \sum_{i=1}^{k-1} \|x_{i+1} - x_i\|^2 + \sum_{i=k}^{n-1} \|x_{i+1} - x_i\|^2 \\ &\leq \|z - x\|^2 + \|y - z\|^2 \leq \|y - x\|^2. \end{aligned}$$

Note also that  $\|z - x\|, \|y - z\| < \|y - x\|$ .

It remains to prove that this algorithm terminates for any input. If not, one can obtain a contradiction by constructing a descending chain  $(u_i)_{i \in \mathbb{N}}$  as follows.

We assume that  $\text{path}(x, y)$  does not terminate and we prove by induction that for any  $i \geq 1$  there exist  $u_0, \dots, u_i, z_1^{i+1} \in \mathcal{N}$  such that:

- $\|u_{j+1} - u_j\| < \|u_j - u_{j-1}\|$  for  $j = 1, \dots, i - 1$ ,
- $\|z_1^{i+1} - u_i\| < \|u_i - u_{i-1}\|$ ,
- $\text{path}(u_i, z_1^{i+1})$  does not terminate.

For  $i = 1$ , one takes  $z_2^1 := \text{IP}(x, y)$  the intermediary point of  $\mathcal{N}$  appearing in the first recursive calls in  $\text{path}(x, y)$ . Either  $\text{path}(x, z_2^1)$  or  $\text{path}(z_2^1, y)$  does not terminate and we may assume, without loss of generality, that it is  $\text{path}(x, z_2^1)$ . We then set  $u_0 := y, u_1 := x$  and  $z_1^2 := z_2^1$ .

Once  $u_0, \dots, u_i, z_1^{i+1} \in \mathcal{N}$  are constructed, we write  $z_2^{i+1} := \text{IP}(u_i, z_1^{i+1})$ . There are then two possibilities:

- 1)  **$\text{path}(z_2^{i+1}, z_1^{i+1})$  does not terminate:** We set  $u_{i+1} := z_1^{i+1}$  and  $z_1^{i+2} := z_2^{i+1}$ . Since  $u_{i+1}$  is inside the ball  $B([u_{i-1}, u_i])$  and  $z_1^{i+2}$  is inside the ball  $B([u_i, u_{i+1}])$ , we have:

$$\|z_1^{i+2} - u_{i+1}\| < \|u_{i+1} - u_i\| < \|u_i - u_{i-1}\| < \dots < \|u_1 - u_0\|.$$

Moreover,  $\text{path}(u_{i+1}, z_1^{i+2})$  does not terminate.

- 2)  **$\text{path}(z_2^{i+1}, z_1^{i+1})$  terminates:** Then  $\text{path}(u_i, z_2^{i+1})$  does not end. We set  $z_3^{i+1} := \text{IP}(u_i, z_2^{i+1})$ . If  $\text{path}(z_3^{i+1}, z_2^{i+1})$  does not end, we proceed as in 1). Otherwise, we repeat the procedure until  $\text{path}(z_{n_{i+1}+1}^{i+1}, z_{n_{i+1}}^{i+1})$  does not terminate. One can see that:

$$\|z_{n_{i+1}+1}^{i+1} - z_{n_{i+1}}^{i+1}\| < \|z_{n_{i+1}}^{i+1} - u_i\| < \dots < \|z_1^{i+1} - u_i\| < \|u_i - u_{i-1}\|.$$

In particular, such  $n_{i+1}$  does exist. Indeed, if not, there would be infinitely many points of  $\mathcal{N}$  in the ball  $B(u_i, \|u_i - u_{i-1}\|)$  and  $\mathcal{N}$  would not be locally finite. We set  $u_{i+1} := z_{n_{i+1}+1}^{i+1}$  and  $z_1^{i+2} := z_{n_{i+1}+1}^{i+1}$ . Then:

$$\|z_1^{i+2} - u_{i+1}\| < \|u_{i+1} - u_i\| < \|u_i - u_{i-1}\| < \dots < \|u_1 - u_0\|,$$

and  $\text{path}(u_{i+1}, z_1^{i+2})$  does not terminate.

Finally,  $\mathcal{N}$  has a descending chain. This contradicts the original assumption.  $\square$

5.2.2. *Good boxes.* For  $M \geq 1$ , consider as before a partition of  $\mathbb{R}^d$  into boxes of side  $M$ ,  $\{B_{\mathbf{z}}, \mathbf{z} \in \mathbb{Z}^d\}$ . For  $m \in \mathbb{N}^*$ , write  $\alpha_{d,m}$  for the odd integer such that:

$$\beta_d m^2 + \sqrt{d} + 1 \leq \alpha_{d,m} < \beta_d m^2 + \sqrt{d} + 3,$$

where  $\beta_d := 2^{2d+2}(d+3+2(d+3)^{3/2})$ . We say that a box  $B_{\mathbf{z}}$  is  $(M, m)$ -good for the Gabriel graph if when  $B_{\mathbf{z}}$  is cut into  $\alpha_{d,m}^d$  sub-boxes  $b_i^{\mathbf{z}}$  of side  $M/\alpha_{d,m}$ , each of these sub-boxes contains at least one and at most  $m$  points of  $\mathcal{N}$ .

5.2.3. *Construction of paths.* Let  $B_{\mathbf{z}_1}$  and  $B_{\mathbf{z}_2}$  be two neighboring good boxes of side  $M$  and  $b_1^{\mathbf{z}_1, \mathbf{z}_2}, \dots, b_{\alpha_{d,m}^d}^{\mathbf{z}_1, \mathbf{z}_2}$  be the sub-boxes of side  $M/\alpha_{d,m}$  intersecting the line segment  $[M\mathbf{z}_1, M\mathbf{z}_2]$ . Write  $c_i$  (resp.  $v_i$ ) for the center of  $b_i^{\mathbf{z}_1, \mathbf{z}_2}$  (resp. the point of  $b_i^{\mathbf{z}_1, \mathbf{z}_2} \cap \mathcal{N}$  which is the closest to  $c_i$ ). Vertices  $v_1$  and  $v_{\alpha_{d,m}^d}$  are reference vertices of  $B_{\mathbf{z}_1}$  and  $B_{\mathbf{z}_2}$  respectively. One must prove that there exists a Gabriel path from  $v_1$  to  $v_{\alpha_{d,m}^d}$  which is included in  $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ . To this end, it suffices to check that, for all  $i \in \{1, \dots, \alpha_{d,m}^d\}$ , there exists a Gabriel path from  $v_i$  to  $v_{i+1}$  which is included in  $B(c_i, M/2)$ . Proceeding along the same lines as in the proof of [BBD02, Lemma 3], we show that the path  $\gamma_i := (x_1 = v_i, \dots, x_n = v_{i+1})$  given by Lemma 12 for  $v_i$  and  $v_{i+1}$  satisfies this property. Observe that this path can be assumed to be simple and satisfies:

$$\sum_{j=1}^{n-1} \|x_{j+1} - x_j\|^2 \leq \|v_{i+1} - v_i\|^2 \leq (d+3) \frac{M^2}{\alpha_{d,m}^2}. \quad (3)$$

In particular,  $\gamma_i$  does not contain any edge with length greater than  $\sqrt{d+3}M/\alpha_{d,m}$  and contains at most  $2^{2d+2}(d+3)m^2$  edges with length between  $M/(2^{d+1}\alpha_{d,m}m)$  and  $\sqrt{d+3}M/\alpha_{d,m}$ , called *long* edges in the following. Indeed, with (3),

$$\frac{M^2}{2^{2d+2}\alpha_{d,m}^2 m^2} \#\{e : \text{long edge of } \gamma_i\} \leq \sum_{e: \text{long edge}} \|e\|^2 \leq (d+3) \frac{M^2}{\alpha_{d,m}^2}.$$

Hence,  $\gamma_i$  consists of at most  $2^{2d+2}(d+3)m^2$  long edges and at most  $2^{2d+2}(d+3)m^2 + 1$  groups of consecutive *short* edges with length lower than  $M/(2^{d+1}\alpha_{d,m}m)$ . Let us consider a simple path starting at  $v_i$  and having  $N_l \in \{0, \dots, 2^{2d+2}(d+3)m^2\}$  long edges and  $N_s \in \{0, \dots, 2^{2d+2}(d+3)m^2 + 1\}$  groups of consecutive short edges. Each group of short edges has total Euclidean length bounded by  $M/(2\alpha_{d,m})$ . Indeed, one can prove by induction that the vertex at the beginning of each group of consecutive short edges satisfies the assumption of Lemma 13, stated at the end of this subsection for sake of readability. Consequently,

the distance from  $c_i$  to the farthest point of the path is less than:

$$\begin{aligned}
 \|v_i - c_i\| + N_l \frac{\sqrt{d+3}M}{\alpha_{d,m}} + N_s \frac{M}{2\alpha_{d,m}} \\
 \leq \sqrt{d} \frac{M}{2\alpha_{d,m}} + N_l \frac{\sqrt{d+3}M}{\alpha_{d,m}} + N_s \frac{M}{2\alpha_{d,m}} \\
 \leq \sqrt{d} \frac{M}{2\alpha_{d,m}} + 2^{2d+2}(d+3)^{\frac{3}{2}}m^2 \frac{M}{\alpha_{d,m}} + (2^{2d+2}(d+3)m^2 + 1) \frac{M}{2\alpha_{d,m}} \\
 = \frac{M}{2} \frac{\beta_d m^2 + \sqrt{d} + 1}{\alpha_{d,m}} \leq \frac{M}{2}.
 \end{aligned}$$

Thus  $\gamma_i$  is included in  $B(c_i, M/2)$  and there exists a Gabriel path  $\gamma := (v_1, \dots, v_{\alpha_{d,m}+1})$  from  $v_1$  to  $v_{\alpha_{d,m}+1}$  contained in  $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ .

It remains to choose  $L$  and  $K$  in Criterion 6 (2)(b) and (2)(c). Since  $\gamma \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$  and can be supposed simple, it has chemical length at most  $\#((B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}) \cap \mathcal{N}) - 1 \leq 2\alpha_{d,m}^d m - 1$ . Thus, one can set  $L := 2\alpha_{d,m}^d m - 1$ . If  $C(\cdot) = \varphi(\|\cdot\|)$  is a decreasing function of edges lengths, we set  $K := 1/\varphi(\sqrt{d+3}M)$ ; if  $C$  is uniformly bounded from below, we set  $K := \max 1/C$ .

**Lemma 13.** *Let  $B_{\mathbf{z}_1}, B_{\mathbf{z}_2}$  be two neighboring good boxes and  $\gamma$  be a simple Gabriel path with edges of length bounded by  $M/(2^{d+1}\alpha_{d,m})$  with a vertex  $u$  such that  $B(u, M/(2\alpha_{d,m})) \subset B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$ .*

*Then,  $\gamma$  consists in at most  $2^d m - 1$  edges. In particular, it has total Euclidean length bounded by  $M/(2\alpha_{d,m})$ .*

*Proof:* Note that  $B(u, M/(2\alpha_{d,m}))$  intersects at most  $2^d$  sub-boxes of side  $M/\alpha_{d,m}$ . Since it is furthermore included in  $B_{\mathbf{z}_1} \cup B_{\mathbf{z}_2}$  which are good boxes, it contains at most  $2^d m$  points of  $\mathcal{N}$ . Assume that  $\gamma$  contains more than  $2^d m$  edges. Hence,  $\gamma$  has a (sub-)path  $\gamma'$  of  $2^d m$  edges (and  $2^d m + 1$  vertices) such that  $u \in \gamma'$ . But  $\gamma'$  is included in  $B(u, M/(2\alpha_{d,m}))$  which provides a contradiction.  $\square$

5.2.4.  $\mathbb{P}[X_{\mathbf{z}} = 1]$  is large enough. The process  $\{X_{\mathbf{z}}\}$  is  $k$ -dependent because of the definition of good boxes and the fact that  $\mathcal{N}$  has a finite range of dependence. It remains to show that if  $M, m$  are suitably chosen:

$$\mathbb{P}[X_{\mathbf{z}} = 1] \geq p^*$$

where  $p^* = p^*(d, k) < 1$  is large enough to ensure that  $\{X_{\mathbf{z}}\}$  dominates supercritical site percolation on  $\mathbb{Z}^d$ .

Thanks to the choice of  $\alpha_{d,m} \sim \beta_d m^2$ , for  $m \in \mathbb{N}^*$  large enough, we can choose  $M$  so that:

$$\frac{\alpha_{d,m}^d}{c_1} \log \left( \frac{2\alpha_{d,m}^d}{1-p^*} \right) \leq M^d \leq \frac{\alpha_{d,m}^d}{c_4} \left( m - \log \left( \frac{2\alpha_{d,m}^d}{1-p^*} \right) \right).$$



With **(V)** and **(D<sub>3+</sub>)**:

$$\mathbb{P}\left[\#\left(\left[0, \frac{M}{\alpha_{d,m}}\right]^d \cap \mathcal{N}\right) = 0\right] \leq \exp\left(-c_1 \frac{M^d}{\alpha_{d,m}^d}\right) \leq \frac{1-p^*}{2\alpha_{d,m}^d},$$

and

$$\mathbb{P}\left[\#\left(\left[0, \frac{M}{\alpha_{d,m}}\right]^d \cap \mathcal{N}\right) > m\right] \leq \exp\left(c_4 \frac{M^d}{\alpha_{d,m}^d} - m\right) \leq \frac{1-p^*}{2\alpha_{d,m}^d}.$$

Finally, using stationarity of  $\mathcal{N}$ :

$$\begin{aligned} \mathbb{P}[X_{\mathbf{z}} = 0] &\leq \sum_{i=1}^{\alpha_{d,m}^d} \left\{ \mathbb{P}[\#(b_i^{\mathbf{z}} \cap \mathcal{N}) = 0] + \mathbb{P}[\#(b_i^{\mathbf{z}} \cap \mathcal{N}) > m] \right\} \\ &= \alpha_{d,m}^d \left\{ \mathbb{P}\left[\#\left(\left[0, \frac{M}{\alpha_{d,m}}\right]^d \cap \mathcal{N}\right) = 0\right] + \mathbb{P}\left[\#\left(\left[0, \frac{M}{\alpha_{d,m}}\right]^d \cap \mathcal{N}\right) > m\right] \right\} \\ &\leq 1 - p^*. \end{aligned}$$

## 6. EXAMPLES OF POINT PROCESSES

In this section, an overview on point processes appearing in Theorem 1 is given. Assumptions of Theorem 2 are checked for these processes. Note that the probability estimates given here are rough but good enough to verify **(V)**-**(D<sub>3+</sub>)**.

**6.1. Poisson point processes.** For homogeneous Poisson point processes (PPPs), stationarity and the finite range of dependence condition are clear. Note that stationary Poisson point processes are almost surely in general position. We refer to [HM96] or [DL05] for the almost sure absence of descending chain. Moreover, we check by standard computations that, for any Borel set  $A$ , a PPP  $\mathcal{N}$  of intensity  $\lambda$  satisfies:

$$\mathbb{P}[\#(A \cap \mathcal{N}) = 0] = e^{-\lambda \text{Vol}_{\mathbb{R}^d}(A)},$$

and

$$\mathbb{P}[\#(A \cap \mathcal{N}) > m] \leq e^{\lambda(e-1) \text{Vol}_{\mathbb{R}^d}(A) - m}.$$

This implies **(V)**-**(D<sub>3+</sub>)**.

**6.2. Matérn cluster processes.** Matérn cluster processes (MCPs) are particular cases of Neyman-Scott Poisson processes (see [SKM87, p. 142]). Cluster processes are used as models for spatial phenomena, e.g. galaxy locations in space [KPBS<sup>+</sup>99] or epicenters of micro-earthquake locations [VJ70].

MCPs are constructed as follows. One first chooses a PPP  $Y$  of intensity  $\lambda$  called the *parent process*. For any  $y \in Y$ , a centered *daughter process*  $\mathcal{N}_y$  is then chosen such that, given  $Y$ , one has:

- (1)  $\{\#\mathcal{N}_y\}_{y \in Y}$  are i.i.d. with Poisson distribution of mean  $\mu \text{Vol}_{\mathbb{R}^d}(B(0, R))$ ,
- (2)  $\{\mathcal{N}_y\}_{y \in Y}$  are independent,
- (3) points of  $\mathcal{N}_y$  are uniformly distributed in  $B(0, R)$ .

Then,  $\mathcal{N} := \bigcup_{y \in Y} (y + \mathcal{N}_y)$  is a MCP with parameters  $\lambda, \mu, R$ . It is clear that such processes are stationary. Since its parent process has a finite range of dependence and daughter processes have bounded supports, any MCP has a finite range of dependence. Thanks to [HNS, Proposition 2.3], MCPs have almost surely no descending chains. MCPs can be seen as doubly stochastic processes or Cox processes (see [SKM87, p. 132] for a definition). Their (diffusive) random intensity measures  $\mu_Y$  have densities  $\sum_{y \in Y} \mathbb{1}_{B(y, R)}(x)$  w.r.t. Lebesgue measure. In particular,  $(d - 1)$  dimensional hyperplanes and spheres are  $\mu_Y$ -null sets and MCPs are almost surely in general position.

It remains to check assumptions **(V)**-**(D<sub>3+</sub>)**. Let  $L > 2R$ , then,

$$\begin{aligned} \mathbb{P}[\#(\mathcal{N} \cap [0, L]^d) = 0] &= \mathbb{E}[\mathbb{P}[\#(\mathcal{N} \cap [0, L]^d) = 0 | Y]] \\ &= \mathbb{E}\left[\mathbb{P}\left[\bigcap_{y \in Y} \{\#(\mathcal{N}_y \cap [0, L]^d) = 0\} \middle| Y\right]\right] \\ &= \mathbb{E}\left[\prod_{y \in Y} \mathbb{P}\left[\{\#(\mathcal{N}_y \cap [0, L]^d) = 0\} \middle| Y\right]\right] \\ &= \mathbb{E}\left[\prod_{y \in Y} \exp(-\mu \text{Vol}_{\mathbb{R}^d}(B(y, R) \cap [0, L]^d))\right] \end{aligned}$$

so, with [SW08, Theorem 3.2.4],

$$\begin{aligned} \mathbb{P}[\#(\mathcal{N} \cap [0, L]^d) = 0] &= \exp\left(-\lambda \int_{[0, L]^d + B(0, R)} \{1 - \exp(-\mu \text{Vol}_{\mathbb{R}^d}([0, L]^d \cap B(y, R)))\} dy\right) \\ &\leq \exp\left(-\lambda \int_{[0, L]^d - B(0, R)} \{1 - \exp(-\mu \text{Vol}_{\mathbb{R}^d}(B(0, R)))\} dy\right) \end{aligned}$$

where  $A - B := \{x \in \mathbb{R}^d : \forall y \in B, x + y \in A\}$ . Thus,

$$\mathbb{P}[\#(\mathcal{N} \cap [0, L]^d) = 0] \leq \exp\left(-\lambda \{1 - \exp(-\mu \text{Vol}_{\mathbb{R}^d}(B(0, R)))\} (L - 2R)^d\right)$$

and **(V)** holds.

In order to show  $(\mathbf{D}_2)$  and  $(\mathbf{D}_{3+})$  for MCPs one can use exponential Markov inequality and the following estimate. For any bounded Borel set  $A$ :

$$\begin{aligned}
\mathbb{E}[e^{\#(\mathcal{N} \cap A)}] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\sum_{y \in Y} \#(\mathcal{N}_y \cap A)\right) \middle| Y\right]\right] \\
&= \mathbb{E}\left[\prod_{y \in Y \cap (A+B(0,R))} \mathbb{E}\left[e^{\#(\mathcal{N}_y \cap A)} \middle| Y\right]\right] \\
&\leq \mathbb{E}\left[\prod_{y \in Y \cap (A+B(0,R))} \mathbb{E}\left[e^{\#\mathcal{N}_y} \middle| Y\right]\right] \\
&= \mathbb{E}\left[\exp\left(\mu \text{Vol}_{\mathbb{R}^d}(B(0,R))(e-1)\#\{y \in Y \cap (A+B(0,R))\}\right)\right] \\
&= e^{c \text{Vol}_{\mathbb{R}^d}(A+B(0,R))}
\end{aligned}$$

where  $c = c(\lambda, \mu, R)$ .

**6.3. Matérn hardcore processes.** In 1960, Matérn introduced several hardcore models for point processes. These processes are dependent thinnings of PPPs and spread more regularly in space than PPPs. Such models are useful when competition for resources exists (e.g. tree or city locations, see [Mat86] and references therein).

Let  $\mathcal{N}$  be a marked PPP of intensity  $\lambda$ , with independent marks  $\{T_x\}_{x \in \mathcal{N}}$  uniformly distributed in  $[0, 1]$ . Then, Matérn I/II hardcore processes (MHP I/II)  $\mathcal{N}_{\text{I}}$  and  $\mathcal{N}_{\text{II}}$  are defined, for a given  $R > 0$  by:

$$\begin{aligned}
\mathcal{N}_{\text{I}} &:= \{x \in \mathcal{N} : \|x - y\| > R, \forall y \in \mathcal{N} \setminus \{x\}\}, \\
\mathcal{N}_{\text{II}} &:= \{x \in \mathcal{N} : T_x < T_y, \forall y \in \mathcal{N} \cap B(x, R)\}.
\end{aligned}$$

Clearly, MHPs are stationary and have finite range of dependence. Note that  $\mathcal{N}_{\text{I}} \subset \mathcal{N}_{\text{II}} \subset \mathcal{N}$ . Hence, the facts that  $\mathcal{N}_{\text{I}}$ ,  $\mathcal{N}_{\text{II}}$  are almost surely in general position and that they have almost surely no descending chains are inherited from these properties for PPPs. Moreover, inequalities  $(\mathbf{D}_2)$  and  $(\mathbf{D}_{3+})$  are immediate from those for PPPs, and it suffices to show  $(\mathbf{V})$  for  $\mathcal{N}_{\text{I}}$ . To this end, first assume there is an integer  $n$  such that  $L = 3nR$  and cut  $[0, L]^d$  into  $3^d n^d$  disjoint sub-boxes  $b_j$  of side  $R$ . Note that if there is a sub-box  $b_j$  with  $\#(b_j \cap \mathcal{N}) = 1$  having all neighboring sub-boxes empty, then  $[0, L]^d \cap \mathcal{N}_{\text{I}} \neq \emptyset$ . Thus,

$$\mathbb{P}[\#([0, L]^d \cap \mathcal{N}_{\text{I}}) = 0] \leq \mathbb{P}\left[\bigcap_j A_j\right],$$

where  $A_j$  stands for the event: ' $\#(b_j \cap \mathcal{N}) \neq 1$  or a neighbor of  $b_j$  contains at least a point of  $\mathcal{N}$ '. One can choose a collection of  $n^d$  sub-boxes  $b_j$  so that, if  $i \neq j$ , events  $A_i$  and  $A_j$  are independent. So,

$$\mathbb{P}[\#([0, L]^d \cap \mathcal{N}_{\text{I}}) = 0] \leq \mathbb{P}[A_{j_0}]^{n^d} = e^{-c_{12}n^d} = e^{-\frac{c_{12}}{3^d}L^d}.$$

For general  $L \geq 3R$ , it suffices to fix  $n$  so that  $3nR \leq L \leq 3(n+1)R$  to obtain:

$$\mathbb{P}[\#([0, L]^d \cap \mathcal{N}_{\text{I}}) = 0] \leq \mathbb{P}[\#([0, 3nR]^d \cap \mathcal{N}) = 0] \leq e^{-\frac{c_{12}}{6^d}L^d}.$$

**6.4. Determinantal processes.** In 1975, Macchi [Mac75] introduced determinantal point processes (DPPs) in order to model fermions in quantum mechanics. These processes also arise in various other settings such as eigenvalues of random matrices, random spanning trees, carries processes when adding a list of random numbers; see for example [BP93, Dia03, Gin65, ST03a, ST03b].

Let  $K$  be a self-adjoint non-negative locally trace class operator acting on  $L^2(\mathbb{R}^d)$  whose integral kernel  $k$  is properly chosen such that it satisfies the so called *local trace formula*:

$$\mathrm{tr} K|_A = \int_A k(x, x) dx$$

for all bounded Borel set  $A \subset \mathbb{R}^d$ . A DPP with kernel  $K$  is a simple point process  $\mathcal{N}$  whose correlation functions  $\rho_k$  satisfy:

$$\rho_k(x_1, \dots, x_k) = \det (K(x_i, x_j))_{1 \leq i, j \leq k}$$

for all  $k \geq 1$  and all  $x_1, \dots, x_k \in \mathbb{R}^d$ . If  $\mathcal{N}$  is stationary, one can fix  $K_0 := k(0, 0) = k(x, x) > 0$  for almost all  $x$ . Thus, previous formula reduces in this case to:

$$\mathrm{tr} K|_A = K_0 \mathrm{Vol}_{\mathbb{R}^d}(A). \quad (4)$$

where  $\mathrm{Vol}_{\mathbb{R}^d}(A)$  denotes the volume of  $A$  for the Lebesgue measure on  $\mathbb{R}^d$ . We refer to [BHKPV06], [Sos00] and the appendix in [GY05] for more details.

Theorem 7 in [BHKPV06] provides the following useful fact for DPPs: the number of points of a DPP  $\mathcal{N}$  with kernel  $K$  falling in a bounded set  $A$  has the law of the sum of independent Bernoulli random variables with parameters the eigenvalues of the restriction  $K|_A$  of  $K$  to  $A$ . Thus writing  $\lambda_k$  for the eigenvalues of  $K|_A$ , using that for  $s \geq 0$ ,  $1 - s \leq e^{-s}$  and formula (4), we have:

$$\begin{aligned} \mathbb{P}[\#(A \cap \mathcal{N}) = 0] &= \prod_k (1 - \lambda_k) \\ &\leq \prod_k e^{-\lambda_k} = e^{-\mathrm{tr}(K|_A)} \\ &= e^{-K_0 \mathrm{Vol}_{\mathbb{R}^d}(A)} \end{aligned}$$

which implies **(V)**. Similarly, using that  $1 + s \leq e^s$  and formula (4):

$$\begin{aligned} \mathbb{E}[e^{\log(\frac{3}{2})\#(\mathcal{N} \cap A)}] &= \prod_k (1 + \frac{\lambda_k}{2}) \\ &\leq \prod_k e^{\frac{\lambda_k}{2}} = e^{\frac{1}{2} \mathrm{tr}(K|_A)} \\ &= e^{\frac{1}{2} K_0 \mathrm{Vol}_{\mathbb{R}^d}(A)}. \end{aligned}$$

This implies **(D<sub>2</sub>)** using exponential Markov inequality. Actually, one can obtain **(D<sub>3+</sub>)** in a similar way, but we are unable to check assumption (3) in Criterion 6 because of the lack of independence in this case.

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