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# The complexity of deciding whether a graph admits an orientation with fixed weak diameter

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## Abstract

An orientation of an undirected graph  $G$  has weak diameter  $k$  if, for every pair  $\{u, v\}$  of vertices of  $G$ , there is a directed path with length at most  $k$  joining  $u$  and  $v$  in either direction. We show that deciding whether a graph admits an orientation with weak diameter  $k$  is NP-complete whenever  $k \geq 2$ .

*Keywords:* oriented graph, weak diameter, complexity

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## 1. Introduction

Let  $G$  be a simple undirected graph with vertex set  $V(G)$  and edge set  $E(G)$ . By *orienting* every edge  $uv$  of  $G$ , either from  $u$  to  $v$  or from  $v$  to  $u$ , one obtains an *orientation*  $\vec{G}$  of  $G$ . This oriented graph  $\vec{G}$  has the same vertex set as  $G$ , i.e.  $V(\vec{G}) = V(G)$ , and, for every edge  $uv \in E(G)$ , we have either  $\vec{uv} \in E(\vec{G})$  or  $\vec{vu} \in E(\vec{G})$  depending on the orientation assigned to  $uv$ .

The *distance*  $\text{dist}(G, u, v)$  from  $u$  to  $v$  in an undirected graph  $G$  is the minimal length of a path joining  $u$  and  $v$ . We refer to the maximum distance between two vertices of  $G$  as its *diameter*, and denote it  $\text{diam}(G)$ . These definitions can be naturally adapted to the context of oriented graphs. A  *$k$ -dipath*  $\vec{v}_1\vec{v}_2\dots\vec{v}_{k+1}$  of  $\vec{G}$  is a sequence of  $k+1$  distinct vertices such that  $\vec{v}_i\vec{v}_{i+1} \in E(\vec{G})$  for every  $i \in \{1, 2, \dots, k\}$ . The *directed distance*  $\text{dist}(\vec{G}, u, v)$  from  $u$  to  $v$  in  $\vec{G}$  is the minimal length of a dipath starting from  $u$  and ending at  $v$ . Note that, contrary to the undirected case, we may have  $\text{dist}(\vec{G}, u, v) \neq \text{dist}(\vec{G}, v, u)$ . Therefore, two definitions of the oriented diameter can be adopted.

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Let  $dist_w(\vec{G}, u, v)$  and  $dist_s(\vec{G}, u, v)$  be  $\min\{dist(\vec{G}, u, v), dist(\vec{G}, v, u)\}$  and  $\max\{dist(\vec{G}, u, v), dist(\vec{G}, v, u)\}$ , respectively. The *weak diameter* of  $\vec{G}$ , denoted  $diam_w(\vec{G})$ , is the maximum value of  $dist_w(\vec{G}, u, v)$  taken over all pairs  $\{u, v\}$  of vertices of  $\vec{G}$ . The *strong diameter* of  $\vec{G}$ , denoted  $diam_s(\vec{G})$ , is the maximum value of  $dist_s(\vec{G}, u, v)$  taken over all pairs  $\{u, v\}$  of vertices of  $\vec{G}$ .

An orientation of  $G$  is *k-weak* (resp. *k-strong*) if it has weak (resp. strong) diameter at most  $k$ . We only deal with *k-weak* orientations in this paper, our motivations being mainly supported by the case  $k = 2$  which is related to some graph colouring notions as explained below. A *proper k-colouring* of  $G$  is a partition of  $V(G)$  into  $k$  parts such that each colour class is a stable. The least number of colours used by a proper colouring of  $G$  is referred to as the *chromatic number* of  $G$ , denoted  $\chi(G)$ . Clearly, we have  $\chi(G) = |V(G)|$  if and only if  $G$  is a complete graph.

Now consider similar notions but for oriented graphs. A *proper oriented k-colouring* of  $\vec{G}$  is a partition of  $V(\vec{G})$  into  $k$  parts such that each colour class is a stable, and all arcs between two colour classes have the same direction. As usually, the least number of colours used by a proper oriented colouring of  $\vec{G}$  is defined as the *oriented chromatic number* of  $\vec{G}$ , denoted  $\chi_o(\vec{G})$ . Contrary to the undirected case, an oriented graph whose oriented chromatic number is exactly its order is not necessarily a tournament. As an illustration of that claim, remark that  $\chi_o(\vec{C}_5) = 5$ , where  $\vec{C}_5$  is the circuit on 5 vertices.

*Oriented cliques* (or *o-clique* for short) were introduced as the analogues of cliques regarding proper oriented colouring. In other words, o-cliques are those oriented graphs whose oriented chromatic number is exactly their order. O-cliques have been mainly studied regarding the extremal theory point of view. The interested reader may refer to [1, 3, 5, 9] for interesting results regarding the size of o-cliques, and to [4, 7] for results on the maximum order of planar o-cliques.

The oriented chromatic number can also be defined for undirected graphs [8]. If  $G$  is an undirected graph, then  $\chi_o(G)$  is defined as

$$\chi_o(G) = \max\{\chi_o(\vec{G}), \vec{G} \text{ is an orientation of } G\}.$$

Hence, we have  $\chi_o(G) = |V(G)|$  if and only if  $G$  admits an orientation which is an o-clique. It follows from the definition that an oriented graph is an o-clique if and only if it has weak diameter at most 2 [4]. Therefore, we get that  $\chi_o(G) = |V(G)|$  if and only if  $G$  admits a 2-weak orientation, and finding 2-weak orientations of graphs becomes of interest.

From the algorithmic point of view, the complexity of deciding whether an undirected graph admits a 2-weak orientation was still unknown. Note that the analogous question for 2-strong orientations was settled by Chvátal and Thomassen, who showed this problem to be NP-complete [2]. We here settle this question by studying the following decision problem.

ORIENTATION WITH WEAK DIAMETER  $k$  -  $k$ -OWD

Instance: A graph  $G$ .

Question: Does  $G$  admit a  $k$ -weak orientation?

A graph admits a 1-weak orientation if and only if it is a complete graph. Therefore, 1-OWD is in P. In this work, we show the following.

**Theorem 1.**  $k$ -OWD is NP-complete for every  $k \geq 2$ .

Regarding our motivations above, we also deduce the following hardness result as an immediate corollary of the NP-completeness of 2-OWD.

**Corollary 2.** Deciding whether  $\chi_o(G) = |V(G)|$  for an undirected graph  $G$  is NP-complete.

## 2. Proof of Theorem 1

The problem  $k$ -OWD is in NP for every  $k$  since one can, given an orientation  $\vec{G}$  of  $G$ , check whether  $diam_w(\vec{G}) \leq k$ . For this purpose, one just has to check, for every possible pair  $\{u, v\}$  of distinct vertices of  $G$ , whether either  $u$  and  $v$  are adjacent, or there exist  $k' \leq k - 1$  vertices  $w_1, w_2, \dots, w_{k'}$  such that  $\overrightarrow{uw_1w_2\dots w_{k'}v}$  or  $\overrightarrow{vw_1w_2\dots w_{k'}u}$  is a dipath of  $\vec{G}$ . This naive witness algorithm runs in polynomial time with respect to the order of  $G$ .

Let  $k \geq 2$  be fixed. We show that  $k$ -OWD is NP-hard by reduction from the following problem, which is shown to be NP-complete in [6].

2-VERTEX-COLOURING OF 3-UNIFORM HYPERGRAPHS - 2COL

Instance: A 3-uniform hypergraph  $H$ .

Question: Is  $H$  2-colourable, i.e. can we colour each vertex of  $H$  either blue or red so that each edge of  $H$  has at least one blue and one red vertex?

Throughout this paper, we denote the vertices and edges of any hypergraph  $H$  with order  $n$  and size  $m$  by  $x_1, x_2, \dots, x_n$  and  $E_1, E_2, \dots, E_m$ , respectively. For every  $i \in \{1, 2, \dots, n\}$ , we further denote by  $n_i \geq 1$  the number of distinct edges of  $H$  which contain the vertex  $x_i$ . From a 3-uniform hypergraph  $H$ , we produce a graph  $G_H$  such that  $H$  is 2-colourable if and only if  $G_H$  admits a  $k$ -weak orientation  $\vec{G}_H$ . This reduction is achieved in polynomial time regarding the size of  $H$ .

We first describe the *core*  $G_H^c$  of  $G_H$ , i.e. the subgraph of  $G_H$  from which the equivalence with  $H$  follows. The subgraph  $G_H^c$  does not have diameter  $k$ , but  $G_H$  will be augmented later so that it has diameter  $k$ , and this without altering the equivalence. The core  $G_H^c$  has the following vertices (see Figure 1). With each vertex  $x_i$  of  $H$ , we associate  $n_i + 2$  vertices  $u_i, u'_i$ , and  $v_{i,j_1}, v_{i,j_2}, \dots, v_{i,j_{n_i}}$  in  $G_H^c$ , where  $j_1, j_2, \dots, j_{n_i}$  are the distinct indices of the edges of  $H$  which contain  $x_i$ . We now associate additional vertices in  $G_H^c$  with each edge  $E_j$  of  $H$ , where  $j \in \{1, 2, \dots, m\}$ . This association depends on the parity of  $k$ .

- If  $k$  is even, then add two vertices  $a_j$  and  $a'_j$  to  $G_H^c$ .

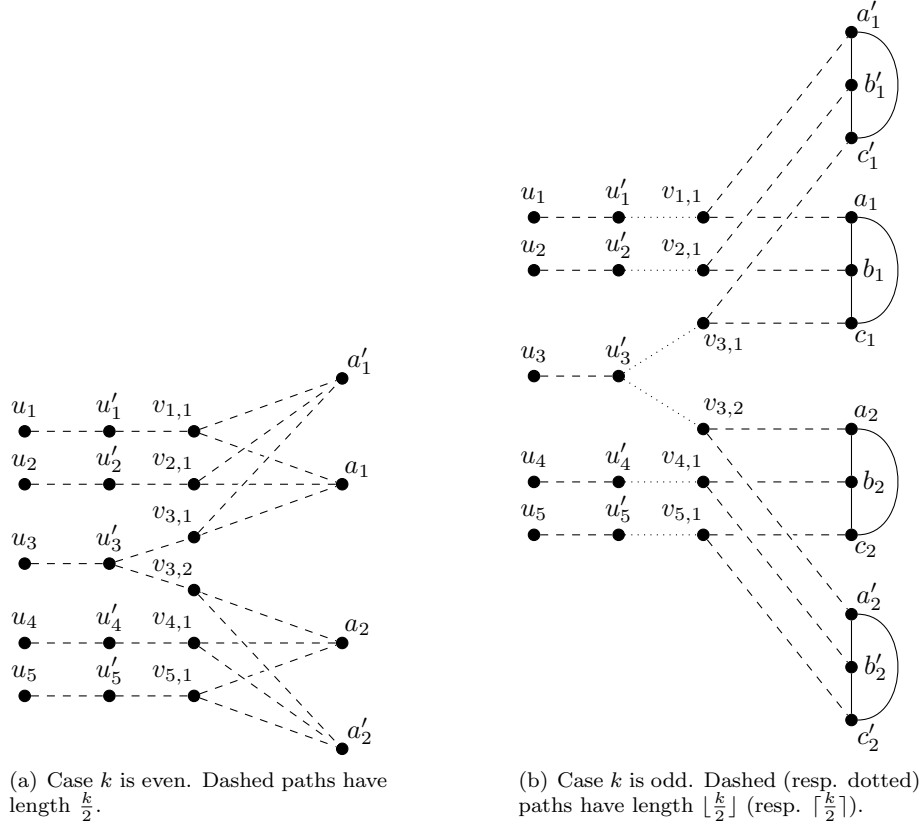


Figure 1: The core subgraph  $G_H^c$  of  $G_H$  obtained assuming  $H$  has two edges  $E_1 = \{x_1, x_2, x_3\}$  and  $E_2 = \{x_3, x_4, x_5\}$ .

- Otherwise, if  $k$  is odd, then add two cycles  $a_j b_j c_j a_j$  and  $a'_j b'_j c'_j a'_j$  with length 3 to  $G_H^c$ .

We now link the vertices of  $G_H^c$  by means of several vertex-disjoint paths. By “joining a pair of vertices  $\{u, v\}$  by a path”, we mean that we identify the endvertices of a new path with  $u$  and  $v$ , respectively. Since this operation is used at most once for joining any pair  $\{u, v\}$  of  $G_H^c$ , we use the notation  $uPv$  to denote the resulting path (if any). First, join every pair  $\{u_i, u'_i\}$  by a path with length  $\lfloor \frac{k}{2} \rfloor$ . Then also join every pair  $\{u'_i, v_{i,j}\}$  by a path with length  $\lceil \frac{k}{2} \rceil$ . Now consider each edge  $E_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$  of  $H$ , and add the following paths to  $G_H^c$ .

- If  $k$  is even, join every pair of  $\{v_{i_1,j}, v_{i_2,j}, v_{i_3,j}\} \times \{a_j, a'_j\}$  by means of a path with length  $\frac{k}{2}$ .
- Otherwise, if  $k$  is odd, then join every pair of  $\{v_{i_1,j}\} \times \{a_j, a'_j\}$ ,  $\{v_{i_2,j}\} \times \{b_j, b'_j\}$ , and  $\{v_{i_3,j}\} \times \{c_j, c'_j\}$  by a path with length  $\lfloor \frac{k}{2} \rfloor$ .

Note that, by construction, exactly one pair  $\{v_{i,j}, s(v_{i,j})\}$  (resp.  $\{v_{i,j}, s'(v_{i,j})\}$ ) was joined by a path with length  $\lfloor \frac{k}{2} \rfloor$ , where  $s(v_{i,j})$  (resp.  $s'(v_{i,j})$ ) is a vertex of the form  $a_j, b_j$  or  $c_j$  (resp.  $a'_j, b'_j$  or  $c'_j$ ). The notation  $s(v_{i,j})$  and  $s'(v_{i,j})$  are used throughout this section. In particular, observe that if  $k$  is even, then we have  $s(v_{i_1,j}) = s(v_{i_2,j}) = s(v_{i_3,j}) = a_j$  for every edge  $E_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$  of  $H$ . We analogously have  $s'(v_{i_1,j}) = s'(v_{i_2,j}) = s'(v_{i_3,j}) = a'_j$ .

A pair  $\{u, v\}$  of distinct vertices of  $G_H^c$  is said *representative* whenever it matches one of the following forms.

1.  $\{u_i, v_{i,j}\}$  where  $i \in \{1, 2, \dots, n\}$ ,  $j \in \{1, 2, \dots, m\}$ , and  $x_i \in E_j$ .
2.  $\{u'_i, s(v_{i,j})\}$  where  $i \in \{1, 2, \dots, n\}$ ,  $j \in \{1, 2, \dots, m\}$ , and  $x_i \in E_j$ .
3.  $\{v_{i_1,j}, v_{i_2,j}\}$  where  $i_1, i_2 \in \{1, 2, \dots, n\}$ ,  $j \in \{1, 2, \dots, m\}$ , and  $x_{i_1}, x_{i_2} \in E_j$ .

An orientation of  $G_H^c$  is *good* if two vertices forming a representative pair are linked by a  $k$ -dipath in either direction. Note that, in this definition, there is no requirement on the oriented distance between two vertices which are at distance at least  $k + 1$ . A representative pair is a pair of vertices which are not adjacent in  $G_H$ , and for which there are at most two paths with length at most  $k$  joining it. All of these paths belong to  $G_H^c$  so that the existence of a  $k$ -weak orientation of  $G_H$  depends on the existence of a good orientation of  $G_H^c$ .

We prove below that we have an equivalence between finding a proper 2-vertex-colouring of  $H$  and a good orientation of  $G_H^c$ . The proof relies on the following claims.

**Claim 1.** Suppose the vertex  $x_i$  belongs to the edges  $E_{j_1}, E_{j_2}, \dots, E_{j_{n_i}}$  of  $H$ . Then, in any good orientation  $\overrightarrow{G_H^c}$  of  $G_H^c$ ,  $\overrightarrow{u_i Pu'_i P v_{i,j} P s(v_{i,j})}$  is a dipath for every  $j \in \{j_1, j_2, \dots, j_{n_i}\}$ , or  $\overrightarrow{s(v_{i,j}) P v_{i,j} P u'_i P u_i}$  is a dipath for every  $j \in \{j_1, j_2, \dots, j_{n_i}\}$ .

*Proof.* Note that because  $u_i P v_{i,j_1}$  is the only path with length at most  $k$  joining  $u_i$  and  $v_{i,j_1}$  in  $G_H^c$ , either  $\overrightarrow{u_i P u'_i P v_{i,j_1}}$  or  $\overrightarrow{v_{i,j_1} P u'_i P u_i}$  must be a dipath of  $\overrightarrow{G_H^c}$ . Assume  $\overrightarrow{u_i P u'_i P v_{i,j_1}}$  is a dipath of  $\overrightarrow{G_H^c}$ . Since  $\overrightarrow{u_i P u'_i}$  is now a dipath of  $\overrightarrow{G_H^c}$ ,  $\overrightarrow{u'_i P v_{i,j}}$  must also be a dipath for every  $j \in \{j_1, j_2, \dots, j_{n_i}\}$  since  $u_i P u'_i P v_{i,j}$  is the only path with length at most  $k$  joining  $u_i$  and  $v_{i,j}$  in  $G_H^c$ .

Similarly, since, for every  $j \in \{j_1, j_2, \dots, j_{n_i}\}$ , the only path with length at most  $k$  joining  $u'_i$  and  $s(v_{i,j})$  in  $G_H^c$  is  $\overrightarrow{u'_i P v_{i,j} P s(v_{i,j})}$ , and  $\overrightarrow{u'_i P v_{i,j}}$  is a dipath of  $\overrightarrow{G_H^c}$ , then  $\overrightarrow{v_{i,j} P s(v_{i,j})}$  has to be a dipath of  $\overrightarrow{G_H^c}$ . Thus,  $\overrightarrow{u_i P u'_i P v_{i,j} P s(v_{i,j})}$  belongs to  $\overrightarrow{G_H^c}$  for every  $j \in \{j_1, j_2, \dots, j_{n_i}\}$  assuming that  $\overrightarrow{u_i P u'_i P v_{i,j_1}}$  belongs to the orientation. The claim follows analogously from the assumption that  $\overrightarrow{v_{i,j_1} P u'_i P u_i}$  is a dipath of  $\overrightarrow{G_H^c}$ .  $\square$

**Claim 2.** Suppose  $k$  is even, and  $E_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$  is an edge of  $H$ . Then, in any good orientation  $\overrightarrow{G_H^c}$  of  $G_H^c$ , either  $\overrightarrow{v_{i,j} P s(v_{i,j})}$  or  $\overrightarrow{s(v_{i,j}) P v_{i,j}}$  is a dipath for every  $i \in \{i_1, i_2, i_3\}$ . Furthermore, these three dipaths cannot be all directed from or towards the  $s(v_{i,j})$ 's.

*Proof.* Recall that  $s(v_{i_1,j}) = s(v_{i_2,j}) = s(v_{i_3,j}) = a_j$  and  $s'(v_{i_1,j}) = s'(v_{i_2,j}) = s'(v_{i_3,j}) = a'_j$  when  $k$  is even. Note further that there are only two paths with length at most  $k$  joining any two of  $v_{i_1,j}$ ,  $v_{i_2,j}$ , and  $v_{i_3,j}$ . These include  $a_j$  and  $a'_j$ , respectively. If the statement of the claim is not fulfilled, then there is no  $k$ -dipath of  $\overrightarrow{G_H^c}$  joining any two of  $v_{i_1,j}$ ,  $v_{i_2,j}$ , and  $v_{i_3,j}$  including  $a_j$ . So there must be three  $k$ -dipaths joining these vertices including  $a'_j$ , but this is impossible.  $\square$

**Claim 3.** Suppose  $k$  is odd, and  $E_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$  is an edge of  $H$ . Then, in any good orientation  $\overrightarrow{G_H^c}$  of  $G_H^c$ , either  $\overrightarrow{v_{i,j}Ps(v_{i,j})}$  or  $\overrightarrow{s(v_{i,j})Pv_{i,j}}$  is a dipath for every  $i \in \{i_1, i_2, i_3\}$ . Besides these three dipaths cannot be all directed from or towards the  $s(v_{i,j})$ 's.

*Proof.* Similarly as for previous Claim 2, if the statement of the claim is not fulfilled by  $\overrightarrow{G_H^c}$ , then there is no dipath with length at most  $k$  joining any two of  $v_{i_1,j}$ ,  $v_{i_2,j}$ , and  $v_{i_3,j}$  including the  $s(v_{i,j})$ 's. Then there cannot be three  $k$ -dipaths, including the  $s'(v_{i,j})$ 's, joining every pair of these vertices, and this no matter how the paths  $v_{i_1,j}s'(v_{i_1,j})$ ,  $v_{i_2,j}s'(v_{i_2,j})$  and  $v_{i_3,j}s'(v_{i_3,j})$  are oriented, and how the edges of the cycles  $a_jb_jc_ja_j$  and  $a'_jb'_jc'_ja'_j$  are oriented.  $\square$

Regarding previous Claims 2 and 3, remark that if two of the dipaths obtained by orienting the paths  $\overrightarrow{v_{i_1,j}Ps(v_{i_1,j})}$ ,  $\overrightarrow{v_{i_2,j}Ps(v_{i_2,j})}$  and  $\overrightarrow{v_{i_3,j}Ps(v_{i_3,j})}$  have the same direction, i.e. from or towards the  $s(v_{i,j})$ 's while the third one is oriented in the opposite direction, then we can obtain three  $k$ -dipaths joining any two of  $v_{i_1,j}$ ,  $v_{i_2,j}$ , and  $v_{i_3,j}$ . Suppose e.g. that  $\overrightarrow{v_{i_1,j}Ps(v_{i_1,j})}$ ,  $\overrightarrow{v_{i_2,j}Ps(v_{i_2,j})}$  and  $\overrightarrow{s(v_{i_3,j})Pv_{i_3,j}}$  are dipaths of  $\overrightarrow{G_H^c}$ . So far, note that there are two  $k$ -dipaths starting from  $v_{i_1,j}$  and  $v_{i_2,j}$ , respectively, and ending at  $v_{i_3,j}$  (when  $k$  is odd, these are obtained by adding  $\overrightarrow{s(v_{i_1,j})s(v_{i_3,j})}$  and  $\overrightarrow{s(v_{i_2,j})s(v_{i_3,j})}$  to  $E(\overrightarrow{G_H^c})$ ). The last  $k$ -dipath starting from  $v_{i_1,j}$  and ending at  $v_{i_2,j}$  can be obtained e.g. by orienting the edges of  $G_H^c$  in such a way that  $\overrightarrow{v_{i_1,j}Ps'(v_{i_1,j})}$  and  $\overrightarrow{s'(v_{i_2,j})Pv_{i_2,j}}$  are dipaths, and  $\overrightarrow{s'(v_{i_1,j})s'(v_{i_2,j})}$  is an arc when  $k$  is odd.

According to Claims 1, 2 and 3, we have an equivalence between finding a proper 2-vertex-colouring of  $H$  and a good orientation of  $G_H^c$ . Indeed, assume that having the dipath  $\overrightarrow{u_iPu'_i}$  (resp.  $\overrightarrow{u'_iPu_i}$ ) in an orientation of  $G_H^c$  simulates that the vertex  $x_i$  of  $H$  is coloured blue (resp. red), and that having the dipath  $\overrightarrow{v_{i,j}Ps(v_{i,j})}$  (resp.  $\overrightarrow{s(v_{i,j})Pv_{i,j}}$ ) simulates the fact that the vertex  $x_i$  is counted as a blue (resp. red) vertex in  $E_j$ . Claim 1 reflects the fact that if  $x_i$  is coloured, say, blue by a proper 2-vertex-colouring of  $H$ , then  $x_i$  counts as a blue vertex in every edge which contains it. Claims 2 and 3 depict the fact that all vertices from a single edge of  $H$  cannot have the same colour. Thus, from a proper 2-vertex-colouring of  $H$  we can deduce a good orientation of  $G_H^c$ , and vice-versa.

We now augment  $G_H$  with additional vertices so that there is a path with length at most  $k$  joining every two non-adjacent vertices of  $G_H^c$  that do not form

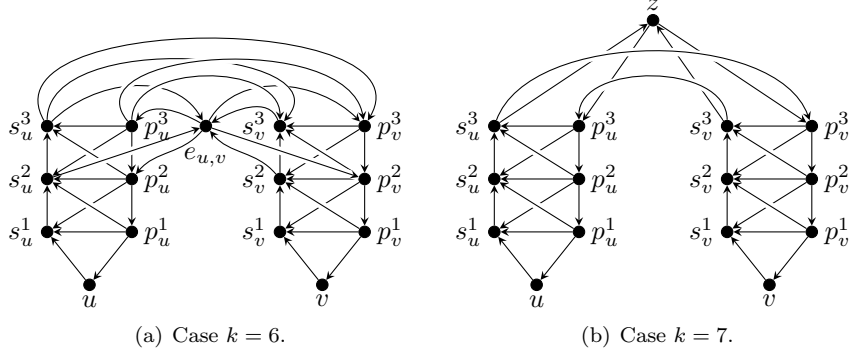


Figure 2: The gadgets  $G_u$  and  $G_v$  obtained for a pair  $\{u, v\}$  which is not representative.

a representative pair. This is done in such a way that there is an orientation of the edges of  $E(G_H) - E(G_H^c)$  so that every two vertices of  $G_H$  that do not form a representative pair are joined by a dipath with length at most  $k$ . In this way, the existence of a  $k$ -weak orientation of  $G_H$  only relies on the existence of a good orientation of  $G_H^c$ .

The augmentation consists in associating a gadget  $G_v$  with each vertex  $v$  of  $G_H^c$ , and then connecting all the resulting gadgets in such a way there is a path with length at most  $k$  between any two vertices from different gadgets  $G_u$  and  $G_v$ . In the case where  $\{u, v\}$  is not a representative pair, we add a *shortcut* between  $G_u$  and  $G_v$ , i.e. an alternative shorter path for joining two vertices of  $G_u$  and  $G_v$ . This is done in such a way that every vertex  $u'$  of  $G_u$  is at distance at most  $k$  from any vertex  $v'$  of  $G_v$ , unless  $u' = u$ ,  $v' = v$  and  $\{u, v\}$  is a representative pair. However, in the situation where  $\{u, v\}$  is not representative, there is a path with length  $k$  joining  $u$  and  $v$  that uses the shortcut between  $G_u$  and  $G_v$ .

Set  $x = \lfloor \frac{k}{2} \rfloor$ . For every  $i \in \{1, 2, \dots, x\}$ , add two new vertices  $s_v^i$  and  $p_v^i$  to  $G_v$ . These two vertices form the  $i^{\text{th}}$  level of  $G_v$ , and are said to be  $i$ -vertices. Next, for every  $i \in \{1, 2, \dots, x-1\}$ , add all possible edges between the  $i$ - and  $(i+1)$ -vertices of  $G_v$  so that two consecutive levels of  $G_v$  form a clique on 4 vertices. Finally, add an edge between  $v$  and every 1-vertex of  $G_v$ .

We finish the construction of  $G_H$  by adding some connection between the gadgets. We distinguish two cases depending on the parity of  $k$ .

- If  $k$  is even, then we turn the subgraph induced by all  $x$ -vertices of  $G_H$  into a clique. Next, for every pair  $\{u, v\}$  of vertices of  $G_H^c$  which is not representative, add a shortcut vertex  $e_{u,v}$  to the clique constructed just before. Finally, add every edge between  $e_{u,v}$  and the vertices from the  $(x-1)^{\text{th}}$  levels of  $G_u$  and  $G_v$  if  $k \geq 4$ , or the edges  $ue_{u,v}$  and  $e_{u,v}v$  when  $k = 2$ .
- Otherwise, if  $k$  is odd, then add a new vertex  $z$  to  $G_H$ , and add all possible edges between  $z$  and  $x$ -vertices. For every pair  $\{u, v\}$  of  $G_H^c$  that is not



representative, also add the shortcut edges  $s_u^x p_v^x$  and  $p_u^x s_v^x$  to  $G_H$ .

This construction is illustrated in Figure 2 for  $k = 6$  and  $k = 7$ . Note that no new path with length at most  $k$  between two vertices composing a representative pair of  $G_H$  arose from the modifications. Therefore, the equivalence between finding a proper 2-vertex-colouring of  $H$  and a good orientation of  $G_H^c$  is preserved. We finally show that there is an orientation of the edges we just added so that every pair of vertices of  $G_H$  which is not representative is joined by a  $k$ -dipath in either direction.

Define an arbitrary ordering  $\sigma = (v_1, v_2, \dots, v_{|V(G_H^c)|})$  over all vertices of  $G_H^c$ , and consider the following partial orientation. First, for every vertex  $v$  of  $G_H^c$ , let  $\overrightarrow{vs_v^1}$  and  $\overrightarrow{p_v^1 v}$  be arcs. Then, for every level  $i \in \{1, 2, \dots, x\}$  of  $G_v$ , let  $\overrightarrow{p_v^i s_v^i}$  be an arc. Next, for every  $i \in \{1, 2, \dots, x-1\}$ , add the arcs  $\overrightarrow{s_v^i s_v^{i+1}}$ ,  $\overrightarrow{p_v^{i+1} p_v^i}$ ,  $\overrightarrow{p_v^i s_v^{i+1}}$  and  $\overrightarrow{p_v^{i+1} s_v^i}$  to the partial orientation. The partial orientation is completed depending on the parity of  $k$ .

- If  $k$  is even, then, for every shortcut vertex  $e$  of  $G_H$ , add the arcs  $\overrightarrow{s_v^x e}$  and  $\overrightarrow{ep_v^x}$ . Next, for every  $i < j$  consider  $\overrightarrow{s_{v_i}^x s_{v_j}^x}$ ,  $\overrightarrow{s_{v_i}^x p_{v_j}^x}$ ,  $\overrightarrow{p_{v_i}^x s_{v_j}^x}$  and  $\overrightarrow{p_{v_i}^x p_{v_j}^x}$  as arcs of the partial orientation. Additionally, if  $\{v_i, v_j\}$  is not a representative pair, then let  $\overrightarrow{s_{v_j}^{x-1} e_{v_i, v_j}}$ ,  $\overrightarrow{e_{v_i, v_j} p_{v_i}^{x-1}}$ ,  $\overrightarrow{s_{v_i}^{x-1} e_{v_i, v_j}}$ , and  $\overrightarrow{e_{v_i, v_j} p_{v_j}^{x-1}}$  be arcs if  $k \geq 4$ , or  $\overrightarrow{v_i e_{v_i, v_j}}$  and  $\overrightarrow{e_{v_i, v_j} v_j}$  be arcs when  $k = 2$ .
- If  $k$  is odd, then let  $\overrightarrow{s_{v_i}^x z}$  and  $\overrightarrow{z p_{v_i}^x}$  be arcs. Finally, if  $\{v_i, v_j\}$  is not representative, then let  $\overrightarrow{s_{v_i}^x p_{v_j}^x}$  and  $\overrightarrow{s_{v_j}^x p_{v_i}^x}$  be arcs.

Note that, under the partial orientation given above, any vertex  $u'$  from a gadget  $G_u$  can directly “access” the upper or lower level of  $G_u$ . Besides, there is a dipath with length at most  $k$  joining  $u'$  and any vertex  $v'$  from another gadget  $G_v$ , unless  $u' = u$ ,  $v' = v$ , and  $\{u, v\}$  is a representative pair. Such a path typically goes up across  $G_u$ , then exits  $G_u$  to enter  $G_v$  (either directly from the  $x^{\text{th}}$  levels or via  $z$ ), and finally goes down across  $G_v$ . Because the gadgets have  $x = \lfloor \frac{k}{2} \rfloor$  levels, the length of such a path does not exceed  $k$ . Finally observe that if  $\{u, v\}$  is representative, then there is no path with length at most  $k$  joining  $u$  and  $v$  going across the gadgets. On the contrary, if  $\{u, v\}$  is not a representative pair, then there is a path with length exactly  $k$  joining  $u$  and  $v$ . This path necessarily includes the shortcut between  $G_u$  and  $G_v$ , i.e. the vertex  $e_{u, v}$  if  $k$  is even or an edge linking the  $x^{\text{th}}$  levels of  $G_u$  and  $G_v$  otherwise.

Hence,  $G_H$  admits a  $k$ -weak orientation if and only if  $G_H^c$  admits a good orientation. Besides,  $G_H^c$  admits a good orientation if and only if  $H$  is 2-colourable. By transitivity, we get that  $G_H$  admits a  $k$ -weak orientation if and only if  $H$  is 2-colourable, and thus that  $k$ -OWD is NP-complete.  $\square$

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