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The complexity of deciding whether a graph admits an orientation with fixed weak diameter

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Abstract. An orientation of an undirected graph G has weak diameter k if, for every pair $\{u, v\}$ of vertices of G , there is a directed path with length at most k joining u and v in either direction. We show that deciding whether a graph admits an orientation with weak diameter k is NP-complete whenever $k \geq 2$.

1 Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. By *orienting* every edge uv of G , either from u to v or from v to u , one obtains an *orientation* \vec{G} of G . This oriented graph \vec{G} has the same vertex set as G , i.e. $V(\vec{G}) = V(G)$, and, for every edge $uv \in E(G)$, we have either $\vec{uv} \in E(\vec{G})$ or $\vec{vu} \in E(\vec{G})$ depending on the orientation assigned to uv .

The *distance* $dist(G, u, v)$ from u to v in an undirected graph G is the minimal length of a path joining u and v . We refer to the maximum distance between two vertices of G as its *diameter*, and denote it $diam(G)$. These definitions can be naturally adapted to the context of oriented graphs. A *k -dipath* $\vec{v}_1\vec{v}_2\dots\vec{v}_{k+1}$ of \vec{G} is a sequence of $k + 1$ distinct vertices such that $\vec{v}_i\vec{v}_{i+1} \in E(\vec{G})$ for every $i \in \{1, \dots, k\}$. The *directed distance* $dist(\vec{G}, u, v)$ from u to v in \vec{G} is the minimal length of a dipath starting from u and ending at v . Note that, contrary to the undirected case, we may have $dist(\vec{G}, u, v) \neq dist(\vec{G}, v, u)$. Therefore, two definitions of the oriented diameter can be adopted.

Let $dist_w(\vec{G}, u, v)$ and $dist_s(\vec{G}, u, v)$ be $\min\{dist(\vec{G}, u, v), dist(\vec{G}, v, u)\}$ and $\max\{dist(\vec{G}, u, v), dist(\vec{G}, v, u)\}$, respectively. The *weak diameter* of \vec{G} , denoted $diam_w(\vec{G})$, is the maximum value of $dist_w(\vec{G}, u, v)$ taken over all pairs $\{u, v\}$ of vertices of \vec{G} . The *strong diameter* of \vec{G} , denoted $diam_s(\vec{G})$, is the maximum value of $dist_s(\vec{G}, u, v)$ taken over all pairs $\{u, v\}$ of vertices of \vec{G} .

An orientation of G is *k -weak* (resp. *k -strong*) if it has weak (resp. strong) diameter at most k . We only deal with k -weak orientations in this paper, our motivations being mainly supported by the case $k = 2$ which is related to some graph colouring notions as explained below. A *k -colouring* of G is a partition of $V(G)$ into k parts such that each colour class is a stable. The least number of colours used by a colouring of G is referred to as the *chromatic number* of G , denoted $\chi(G)$. Clearly, we have $\chi(G) = |V(G)|$ if and only if G is a complete graph.

Now consider similar notions but for oriented graphs. An *oriented k -colouring* of \vec{G} is a partition of $V(\vec{G})$ into k parts such that each colour class is a stable, and all arcs between two colour classes have the same direction. As usually, the least number of colours used by an oriented colouring of \vec{G} is defined as the *oriented chromatic number* of \vec{G} , denoted $\chi_o(\vec{G})$. Contrary to the undirected case, an oriented graph whose oriented chromatic number is exactly its order is not necessarily a tournament. As an illustration of that claim, remark that $\chi_o(\vec{C}_5) = 5$, where \vec{C}_5 is the circuit on 5 vertices.

Oriented cliques (or *o-clique* for short) were introduced as the analogues of cliques regarding oriented colouring. In other words, o-cliques are those oriented graphs whose oriented chromatic number is exactly their order. O-cliques have been mainly studied regarding the extremal theory point of view. The interested reader may refer to [1], [3], [5] and [8] for interesting results regarding the size of o-cliques, and to [4] and [6] for results on the maximum order of planar o-cliques.

The oriented chromatic number can also be defined for undirected graphs [7]. If G is an undirected graph, then $\chi_o(G)$ is defined as

$$\chi_o(G) = \max\{\chi_o(\vec{G}), \vec{G} \text{ is an orientation of } G\}.$$

Hence, we have $\chi_o(G) = |V(G)|$ if and only if G admits an orientation which is an o-clique. It follows from the definition that an oriented graph is an o-clique if and only if it has weak diameter at most 2 [4]. Therefore, we get that $\chi_o(G) = |V(G)|$ if and only if G admits a 2-weak orientation, and finding 2-weak orientations of graphs becomes of interest.

From the algorithmic point of view, the complexity of deciding whether an undirected graph admits a 2-weak orientation was still unknown. Note that the analogous question for 2-strong orientations was settled down by Chvátal and Thomassen, who showed this problem to be NP-complete [2]. We here settle this question by studying the following decision problem.

ORIENTATION WITH WEAK DIAMETER k - k -OWD

Instance: A graph G .

Question: Does G admit a k -weak orientation?

A graph admits a 1-weak orientation if and only if it is a complete graph. Therefore, 1-OWD is in P. In this work, we show the following.

Theorem 1. *k -OWD is NP-complete for every $k \geq 2$.*

Regarding our motivations above, we also deduce the following hardness result as an immediate corollary of the NP-completeness of 2-OWD.

Corollary 1. *Deciding whether $\chi_o(G) = |V(G)|$ for an undirected graph G is NP-complete.*

2 Proof of Theorem 1

The problem k -OWD is in NP for every k since one can, given an orientation \vec{G} of G , check whether $\text{diam}_w(\vec{G}) \leq k$. For this purpose, one just has to check, for every possible pair $\{u, v\}$ of distinct vertices of G , whether either u and v are adjacent, or there exist $k' \leq k - 1$ vertices $w_1, \dots, w_{k'}$ such that $\overrightarrow{uw_1 \dots w_{k'} v}$ or $\overrightarrow{vw_1 \dots w_{k'} u}$ is a dipath of \vec{G} . This naive witness algorithm runs in polynomial time with respect to the order of G .

Let $k \geq 2$ be fixed. We show that k -OWD is NP-hard by reduction from the following NP-complete problem.

NOT-ALL-EQUAL 3-SAT - NAE-3SAT

Instance: A 3CNF formula F over variables $\{x_1, \dots, x_n\}$ and clauses $\{C_1, \dots, C_m\}$.
 Question: Is F *nae-satisfiable*, i.e. is there a truth assignment of the variables of F such that no clause of F has its three literals having the same truth value?

The interesting property of NAE-3SAT is that it remains NP-complete even if none of its clauses contains a negated variable (MONOTONE NAE-3SAT). Note further that if a clause of F is of the form $(x_i \vee x_i \vee x_i)$, then F cannot be nae-satisfied. It is thus understood that F meets these two properties throughout this proof. From F , we produce a graph G_F such that F is nae-satisfiable if and only if G_F admits a k -weak orientation \vec{G}_F . This reduction is achieved in polynomial time regarding the size of F .

For every $i \in \{1, \dots, n\}$ we denote by n_i the number of distinct clauses that contain the variable x_i in F , and by m_j the number of distinct variables in the clause C_j for every $j \in \{1, \dots, m\}$. According to the remarks above, we have $n_i \geq 1$ and $m_j \in \{2, 3\}$ for every such integers.

We first describe the *core* G_F^c of G_F , i.e. the subgraph of G_F from which the equivalence with F follows. The subgraph G_F^c does not have diameter k , but G_F will be augmented later so that it has diameter k , and this without altering the equivalence. The core G_F^c has the following vertices (see Figure 1). With each variable x_i of F , we associate $n_i + 2$ vertices u_i, u'_i , and $v_{i,j_1}, \dots, v_{i,j_{n_i}}$ in G_F^c , where j_1, \dots, j_{n_i} are the distinct indices of the clauses of F that contain x_i . We now associate some vertices (and potentially some edges) in G_F^c with each clause C_j in F , where $j \in \{1, \dots, m\}$. This association depends on m_j and on the parity of k . On the one hand, if k is even, then add a vertex a_j , and also another vertex a'_j if $m_j = 3$, to G_F^c . On the other hand, i.e. k is odd, proceed as follows. If $m_j = 2$, then just add an edge $a_j b_j$ to G_F^c . Otherwise, if $m_j = 3$, then add two cycles $a_j b_j c_j a_j$ and $a'_j b'_j c'_j a'_j$ with length 3 to G_F^c .

We now link the vertices of G_F^c by means of several vertex-disjoint paths. By joining a pair of vertices $\{u, v\}$ by a path", we mean that we identify the endvertices of a new path with u and v , respectively. Since this operation is used at most once for joining any pair $\{u, v\}$ of G_F^c , we use the notation uPv to denote the resulting path (if any). First, join every pair $\{u_i, u'_i\}$ by a path with length $\lfloor \frac{k}{2} \rfloor$. Then, join every pair $\{u'_i, v_{i,j}\}$ by a path with length $\lceil \frac{k}{2} \rceil$. Now, for every $j \in \{1, \dots, m\}$, add the following paths to G_F^c .

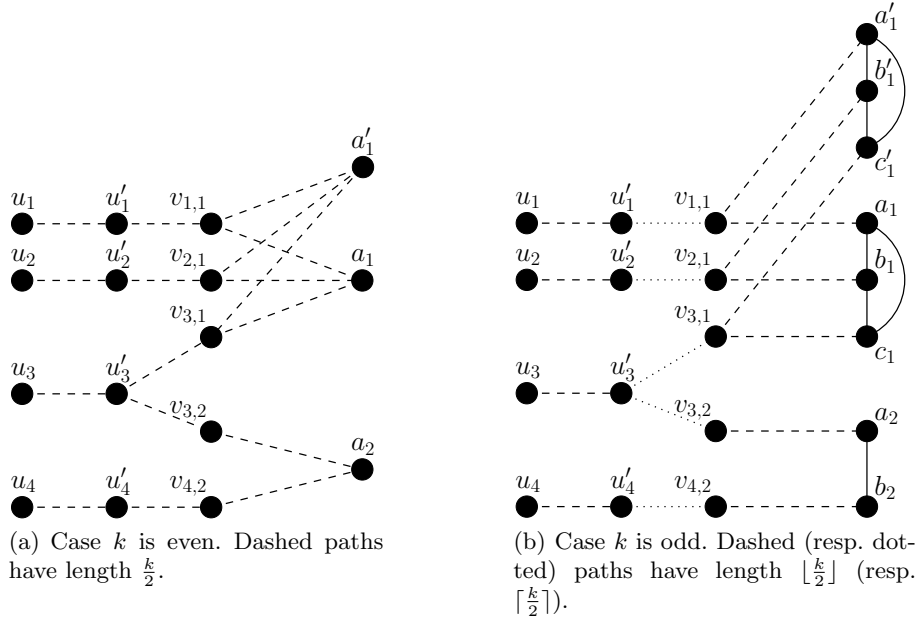


Fig. 1. The core subgraph G_F^c of G_F obtained assuming F contains two clauses $C_1 = (x_1 \vee x_2 \vee x_3)$ and $C_2 = (x_3 \vee x_3 \vee x_4)$.

- If $m_j = 2$, and x_{i_1} and x_{i_2} are the distinct variables of C_j , then join every pair of $\{v_{i_1,j}, v_{i_2,j}\} \times \{a_j\}$ by a path with length $\frac{k}{2}$ if k is even, or join each of $\{v_{i_1,j}, a_j\}$ and $\{v_{i_2,j}, b_j\}$ by a path with length $\lfloor \frac{k}{2} \rfloor$ otherwise.
- If $m_j = 3$, and x_{i_1}, x_{i_2} and x_{i_3} are the variables of C_j , then, if k is even, join every pair of $\{v_{i_1,j}, v_{i_2,j}, v_{i_3,j}\} \times \{a_j, a'_j\}$ by means of a path with length $\frac{k}{2}$. Otherwise, if k is odd, then join every pair of $\{v_{i_1,j}\} \times \{a_j, a'_j\}$, $\{v_{i_2,j}\} \times \{b_j, b'_j\}$, and $\{v_{i_3,j}\} \times \{c_j, c'_j\}$ by a path with length $\lfloor \frac{k}{2} \rfloor$.

Note that, by construction, exactly one pair $\{v_{i,j}, s(v_{i,j})\}$ (resp. $\{v_{i,j}, s'(v_{i,j})\}$) was joined by a path with length $\lfloor \frac{k}{2} \rfloor$, where $s(v_{i,j})$ (resp. $s'(v_{i,j})$) is a vertex of the form a_j, b_j or c_j (resp. a'_j, b'_j or c'_j). The notation $s(v_{i,j})$ and $s'(v_{i,j})$ are used throughout this section. In particular, observe that if k is even, then we have $s(v_{i_1,j}) = s(v_{i_2,j}) = a_j$ or $s(v_{i_1,j}) = s(v_{i_2,j}) = s(v_{i_3,j}) = a_j$ whenever a clause C_j of F has two or three distinct variables whose indexes are i_1, i_2 , and possibly i_3 . Besides, if $m_j = 3$, then $s'(v_{i_1,j}) = s'(v_{i_2,j}) = s'(v_{i_3,j}) = a'_j$.

A pair $\{u, v\}$ of distinct vertices of G_F^c is said *representative* whenever it matches one of the following forms.

1. $\{u_i, v_{i,j}\}$ where $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$.
2. $\{u'_i, s(v_{i,j})\}$ where $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, and $x_i \in C_j$.
3. $\{v_{i_1,j}, v_{i_2,j}\}$ where $\{i_1, i_2\} \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, and $x_{i_1}, x_{i_2} \in C_j$.

An orientation of G_F^c is *good* if two vertices forming a representative pair are linked by a k -dipath in either direction. Note that, in this definition, there is no requirement on the oriented distance between two vertices which are at distance at least $k + 1$. A representative pair is a pair of vertices which are not adjacent in G_F , and for which there are at most two paths with length at most k joining it. All of these paths belong to G_F^c so that the existence of a k -weak orientation of G_F depends on the existence of a good orientation of G_F^c .

We prove that we have an equivalence between nae-satisfying F and finding a good orientation of G_F^c . This relies on the following claims.

Claim 1. *Suppose x_i belongs to the clauses $C_{j_1}, \dots, C_{j_{n_i}}$ of F . Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c , $\overrightarrow{u_i Pu'_i P v_{i,j} Ps(v_{i,j})}$ is a dipath for every $j \in \{j_1, \dots, j_{n_i}\}$, or $\overrightarrow{s(v_{i,j}) P v_{i,j} P u'_i P u_i}$ is a dipath for every $j \in \{j_1, \dots, j_{n_i}\}$.*

Proof. Note that because $u_i P u'_i P v_{i,j_1}$ is the only path with length at most k joining u_i and v_{i,j_1} in G_F^c , either $\overrightarrow{u_i P u'_i P v_{i,j_1}}$ or $\overrightarrow{v_{i,j_1} P u'_i P u_i}$ must be a dipath of $\overrightarrow{G_F^c}$. Assume $\overrightarrow{u_i P u'_i P v_{i,j_1}}$ is a dipath of $\overrightarrow{G_F^c}$ without loss of generality. Since $\overrightarrow{u_i P u'_i}$ is now a dipath of $\overrightarrow{G_F^c}$, $\overrightarrow{u'_i P v_{i,j}}$ must also be a dipath for every $j \in \{j_1, \dots, j_{n_i}\}$ since $u_i P u'_i P v_{i,j}$ is the only path with length at most k joining u_i and $v_{i,j}$ in G_F^c .

Similarly, since, for every $j \in \{j_1, \dots, j_{n_i}\}$, the only path with length at most k joining u'_i and $s(v_{i,j})$ in G_F^c is $u'_i P v_{i,j} P s(v_{i,j})$, and $\overrightarrow{u'_i P v_{i,j}}$ is a dipath of $\overrightarrow{G_F^c}$, then $\overrightarrow{v_{i,j} P s(v_{i,j})}$ has to be a dipath of $\overrightarrow{G_F^c}$. Thus, $\overrightarrow{u_i P u'_i P v_{i,j} Ps(v_{i,j})}$ belongs to $\overrightarrow{G_F^c}$ for every $j \in \{j_1, \dots, j_{n_i}\}$ assuming that $\overrightarrow{u_i P u'_i P v_{i,j_1}}$ belongs to the orientation. The claim follows analogously from the assumption that $\overrightarrow{v_{i,j_1} P u'_i P u_i}$ is a dipath of $\overrightarrow{G_F^c}$.

Claim 2. *Suppose k is even, and C_j has only two distinct variables x_{i_1} and x_{i_2} in F . Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c , either $\overrightarrow{v_{i_1,j} Ps(v_{i_1,j}) P v_{i_2,j}}$ or $\overrightarrow{v_{i_2,j} Ps(v_{i_2,j}) P v_{i_1,j}}$ is a dipath.*

Proof. Recall that when k is even and $m_j = 2$, we have $s(v_{i_1,j}) = s(v_{i_2,j}) = a_j$. Then the claim follows from the fact that $v_{i_1,j} P a_j P v_{i_2,j}$ is the only path with length at most k joining $v_{i_1,j}$ and $v_{i_2,j}$ in G_F^c .

Claim 3. *Suppose k is odd, and C_j has only two distinct variables x_{i_1} and x_{i_2} in F . Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c , either $\overrightarrow{v_{i_1,j} Ps(v_{i_1,j}) s(v_{i_2,j}) P v_{i_2,j}}$ or $\overrightarrow{v_{i_2,j} Ps(v_{i_2,j}) s(v_{i_1,j}) P v_{i_1,j}}$ is a dipath.*

Proof. The claim follows similarly from the fact that $v_{i_1,j} P s(v_{i_1,j}) P s(v_{i_2,j}) P v_{i_2,j}$ is the only path with length at most k joining $v_{i_1,j}$ and $v_{i_2,j}$ in G_F^c .

Claim 4. *Suppose k is even, and C_j has three distinct variables x_{i_1} , x_{i_2} and x_{i_3} in F . Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c , either $\overrightarrow{v_{i,j} Ps(v_{i,j})}$ or $\overrightarrow{s(v_{i,j}) P v_{i,j}}$ is a dipath for every $i \in \{i_1, i_2, i_3\}$. Besides these three dipaths cannot be all directed from or towards the $s(v_{i,j})$'s.*

Proof. Recall that $s(v_{i_1,j}) = s(v_{i_2,j}) = s(v_{i_3,j}) = a_j$ and $s'(v_{i_1,j}) = s'(v_{i_2,j}) = s'(v_{i_3,j}) = a'_j$ when k is even and $m_j = 3$. Note further that there are only two paths with length at most k joining any two of $v_{i_1,j}$, $v_{i_2,j}$, and $v_{i_3,j}$. These pass by a_j and a'_j , respectively. If the statement of the claim is not fulfilled, then there is no k -dipath of $\overrightarrow{G_F^c}$ joining any two of $v_{i_1,j}$, $v_{i_2,j}$, and $v_{i_3,j}$ passing by a_j . So there must be three k -dipaths joining these vertices passing by a'_j , but this is impossible.

Claim 5. *Suppose k is odd, and C_j has three distinct variables x_{i_1} , x_{i_2} and x_{i_3} in F . Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c , either $\overrightarrow{v_{i,j}Ps(v_{i,j})}$ or $\overrightarrow{s(v_{i,j})Pv_{i,j}}$ is a dipath for every $i \in \{i_1, i_2, i_3\}$. Besides these three dipaths cannot be all directed from or towards the $s(v_{i,j})$'s.*

Proof. Similarly as for the previous claim, if the statement of the claim is not fulfilled by $\overrightarrow{G_F^c}$, then there is no dipath with length at most k joining any two of $v_{i_1,j}$, $v_{i_2,j}$, and $v_{i_3,j}$ passing by the $s(v_{i,j})$'s. Then there cannot be three k -dipaths joining every pair of these vertices passing by the $s'(v_{i,j})$'s, and this no matter how the paths $v_{i_1,j}s'(v_{i_1,j})$, $v_{i_2,j}s'(v_{i_2,j})$ and $v_{i_3,j}s'(v_{i_3,j})$ are oriented, and how the edges of the cycles $a_jb_jc_ja_j$ and $a'_jb'_jc'_ja'_j$ are oriented.

Remark that if two of the dipaths obtained by orienting the paths $\overrightarrow{v_{i_1,j}Ps(v_{i_1,j})}$, $\overrightarrow{v_{i_2,j}Ps(v_{i_2,j})}$ and $\overrightarrow{v_{i_3,j}Ps(v_{i_3,j})}$ have the same direction, i.e. from or towards the $s(v_{i,j})$'s while the third one is oriented in the opposite direction, then we can obtain three k -dipaths joining any two of $v_{i_1,j}$, $v_{i_2,j}$, and $v_{i_3,j}$. Suppose e.g. that $\overrightarrow{v_{i_1,j}Ps(v_{i_1,j})}$, $\overrightarrow{v_{i_2,j}Ps(v_{i_2,j})}$ and $\overrightarrow{s(v_{i_3,j})Pv_{i_3,j}}$ are dipaths of $\overrightarrow{G_F^c}$. So far, note that there are two k -dipaths starting from $v_{i_1,j}$ and $v_{i_2,j}$, respectively, and ending at $v_{i_3,j}$ (when k is odd, these are obtained by adding $\overrightarrow{s(v_{i_1,j})s(v_{i_3,j})}$ and $\overrightarrow{s(v_{i_2,j})s(v_{i_3,j})}$ to $E(\overrightarrow{G_F^c})$). The last k -dipath starting from $v_{i_1,j}$ and ending at $v_{i_2,j}$ can be obtained e.g. by orienting the edges of G_F^c in such a way that $\overrightarrow{v_{i_1,j}Ps'(v_{i_1,j})}$ and $\overrightarrow{s'(v_{i_2,j})Pv_{i_2,j}}$ are dipaths, and $\overrightarrow{s'(v_{i_1,j})s'(v_{i_2,j})}$ is an arc when k is odd.

According to Claims 1 to 5, we have an equivalence between nae-satisfying F and finding a good orientation of G_F^c . Indeed, assume that having the dipath $\overrightarrow{u_iPu_i}$ (resp. $\overrightarrow{u'_iPu_i}$) in an orientation of G_F^c simulates an assignment of the variable x_i to true (resp. false), and that having the dipath $\overrightarrow{v_{i,j}Ps(v_{i,j})}$ (resp. $\overrightarrow{s(v_{i,j})Pv_{i,j}}$) simulates the fact that the variable x_i is evaluated true (resp. false) in C_j . Claim 1 reflects the fact that, in a truth assignment of the variables of F , the variable x_i has the same truth value in every clause it appears in. Claims 2 to 5 depict the fact that all variables of a clause cannot have the same truth value. Thus, from a satisfying truth assignment of the variables of F we can deduce a good orientation of G_F^c , and vice-versa.

We now augment G_F with additional vertices so that there is a path with length at most k joining every two non-adjacent vertices of G_F^c that do not form

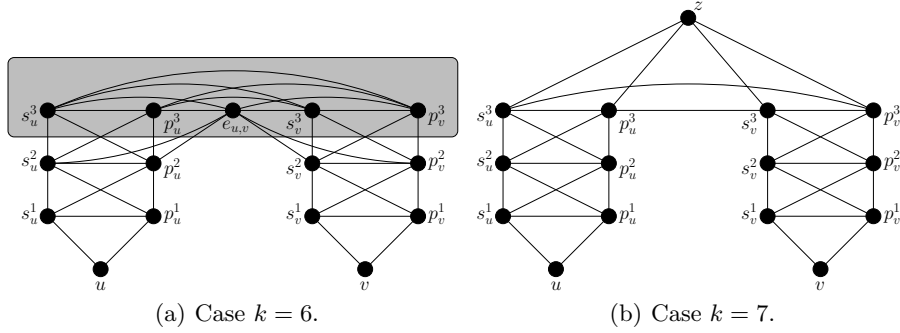


Fig. 2. The gadgets G_u and G_v obtained for a pair $\{u, v\}$ which is not representative.

a representative pair. This is done in such a way that there is an orientation of the edges of $E(G_F) - E(G_F^c)$ so that every two vertices of G_F that do not form a representative pair are joined by a dipath with length at most k . In this way, the existence of a k -weak orientation of G_F only relies on the existence of a good orientation of G_F^c .

The augmentation consists in associating a gadget G_v with each vertex v of G_F^c , and then connecting all the resulting gadgets in such a way there is a path with length at most k between any two vertices from different gadgets G_u and G_v . In the case where $\{u, v\}$ is not a representative pair, we add a *shortcut* between G_u and G_v , i.e. an alternative shorter path for joining two vertices of G_u and G_v . Typically, every vertex u' of G_u is at distance at most k from any vertex v' of G_v , unless $u' = u$, $v' = v$ and $\{u, v\}$ is a representative pair. However, in the situation where $\{u, v\}$ is not representative, there is a path with length k joining u and v that uses the shortcut between G_u and G_v .

Set $x = \lfloor \frac{k}{2} \rfloor$. For every $i \in \{1, \dots, x\}$, add two new vertices s_v^i and p_v^i to G_v . These two vertices form the i^{th} level of G_v , and are said to be i -vertices. Next, for every $i \in \{1, \dots, x-1\}$, add all possible edges between the i - and $(i+1)$ -vertices of G_v so that two consecutive levels of G_v form a clique on 4 vertices. Finally, add an edge between v and every 1-vertex of G_v .

We finish the construction of G_F by adding some connection between the gadgets. We distinguish two cases depending on the parity of k .

- If k is even, then we turn the subgraph induced by all x -vertices of G_F into a clique. Next, for every pair $\{u, v\}$ of vertices of G_F^c which is not representative, add a shortcut vertex $e_{u,v}$ to the clique constructed just before. Finally, add every edge between $e_{u,v}$ and the vertices from the $(x-1)^{\text{th}}$ levels of G_u and G_v .
- Otherwise, if k is odd, then add a new vertex z to G_F , and add all possible edges between z and x -vertices. For every pair $\{u, v\}$ of G_F^c that is not representative, also add the shortcut edges $s_u^x p_v^x$ and $p_u^x s_v^x$ to G_F .

This construction is illustrated in Figure 2 for $k = 6$ and $k = 7$. Note that no new path with length at most k between two vertices composing a representative

pair of G_F arose from the modifications. Therefore, the equivalence between nae-satisfying F and finding a good orientation of G_F^c is preserved. We finally show that there is an orientation of the edges we just added so that every pair of vertices of G_F which is not representative is joined by a k -dipath in either direction.

Define an arbitrary ordering $\sigma = (v_1, \dots, v_{|V(G_F^c)|})$ over all vertices of G_F^c , and consider the following partial orientation. First, for every vertex v of G_F^c , let $\overrightarrow{vs_v^1}$ and $\overrightarrow{p_v^1v}$ be arcs. Then, for every level $i \in \{1, \dots, x\}$ of G_v , let $\overrightarrow{p_v^i s_v^i}$ be an arc. Next, for every $i \in \{1, \dots, x-1\}$, add the arcs $\overrightarrow{s_v^i s_v^{i+1}}$, $\overrightarrow{p_v^{i+1} p_v^i}$, $\overrightarrow{p_v^i s_v^{i+1}}$ and $\overrightarrow{p_v^{i+1} s_v^i}$ to the partial orientation. Now, on the one hand, if k is even, then, for every shortcut vertex e of G_F , add the arcs $\overrightarrow{s_v^x e}$ and $\overrightarrow{e p_v^x}$. Next, for every $i < j$ consider $\overrightarrow{s_{v_i}^x s_{v_j}^x}$, $\overrightarrow{s_{v_i}^x p_{v_j}^x}$, $\overrightarrow{p_{v_i}^x s_{v_j}^x}$ and $\overrightarrow{p_{v_i}^x p_{v_j}^x}$ as arcs of the partial orientation. Additionally, if $\{v_i, v_j\}$ is not a representative pair, then let $\overrightarrow{s_{v_j}^{x-1} e_{v_i, v_j}}$, $\overrightarrow{e_{v_i, v_j} p_{v_i}^{x-1}}$, $\overrightarrow{s_{v_i}^{x-1} e_{v_i, v_j}}$, and $\overrightarrow{e_{v_i, v_j} p_{v_j}^{x-1}}$ be arcs. On the other hand, if k is odd, then let $\overrightarrow{s_{v_i}^x z}$ and $\overrightarrow{z p_{v_i}^x}$ be arcs. Finally, if $\{v_i, v_j\}$ is not representative, then let $\overrightarrow{s_{v_i}^x p_{v_j}^x}$ and $\overrightarrow{s_{v_j}^x p_{v_i}^x}$ be arcs.

Note that, under the partial orientation given above, any vertex u' from a gadget G_u can directly access⁷ the upper or lower level of G_u . Besides, there is a dipath with length at most k joining u' and any vertex v' from another gadget G_v , unless $u' = u$, $v' = v$, and $\{u, v\}$ is a representative pair. Such a path typically goes up across G_u , then exits G_u to enter G_v (either directly from the x^{th} levels or via z), and finally goes down across G_v . Because the gadgets have $x = \lfloor \frac{k}{2} \rfloor$ levels, the length of such a path does not exceed k . Finally observe that if $\{u, v\}$ is representative, then there is no path with length at most k joining u and v passing across the gadgets. On the contrary, if $\{u, v\}$ is not a representative pair, then there is a path with length exactly k joining u and v . This path necessarily passes by the shortcut between G_u and G_v , i.e. the vertex $e_{u,v}$ if k is even or an edge linking the x^{th} levels of G_u and G_v otherwise.

Hence, G_F admits a k -weak orientation if and only if G_F^c admits a good orientation. Besides, G_F^c admits a good orientation if and only if F is nae-satisfiable. By transitivity, we get that G_F admits a k -weak orientation if and only if F is nae-satisfiable, and thus that k -OWD is NP-complete. \square

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