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The complexity of deciding whether a graph admits an orientation with fixed weak diameter

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Abstract. An orientation of an undirected graph G has weak diameter k if, for every pair $\{u, v\}$ of vertices of G, there is a directed path with length at most k joining u and v in either direction. In this work, we show that deciding whether a graph admits an orientation with weak diameter k is NP-complete whenever $k \geq 2$.

1 Introduction

Let G be a simple undirected graph with vertex (resp. edge) set $V(G)$ (resp. $E(G)$). By *orienting* every edge uv of G, either from u to v or from v to u, one obtains an *orientation* $\overrightarrow{G}$ of G. This oriented graph $\overrightarrow{G}$ has the same vertex set as G, i.e. $V(\overrightarrow{G}) = V(G)$, and, for every edge $uv \in E(G)$, we have either $\overrightarrow{uv} \in E(\overrightarrow{G})$ or $\overrightarrow{vu} \in E(\overrightarrow{G})$ depending on the orientation assigned to uv.

The *distance* $dist(G, u, v)$ from u to v in G is the minimal length of a path joining u and v. We refer to the maximum distance between two vertices of G as its *diameter*, and denote it $diam(G)$. These definitions can be naturally adapted to the context of oriented graphs. A *k-dipath* $\overrightarrow{v_1 v_2 ... v_{k+1}}$ of $\overrightarrow{G}$ is a sequence of $k+1$ distinct vertices such that $\overrightarrow{v_i v_{i+1}} \in E(\overrightarrow{G})$ for every $i \in \{1, ..., k\}$. The *directed distance* $dist(\overrightarrow{G}, u, v)$ from u to v in $\overrightarrow{G}$ is the minimal length of a dipath starting from u and ending at v. Note that, contrary to the undirected case, we may have $dist(\overrightarrow{G}, u, v) \neq dist(\overrightarrow{G}, v, u)$. Therefore, two definitions of the oriented diameter can be adopted.

On the one hand, one can consider that two vertices u and v of $\overrightarrow{G}$ are close when one of $dist(\overrightarrow{G}, u, v)$ and $dist(\overrightarrow{G}, v, u)$ is small. In other words, we consider that two vertices are close if there is a dipath with small length joining them in $\overrightarrow{G}$ regardless of its direction. Let $dist_w(\overrightarrow{G}, u, v)$ be $\min\{dist(\overrightarrow{G}, u, v), dist(\overrightarrow{G}, v, u)\}$. The *weak diameter* of $\overrightarrow{G}$, denoted $diam_w(\overrightarrow{G})$, is the maximum $dist_w(\overrightarrow{G}, u, v)$ taken over all pairs $\{u, v\}$ of vertices of $\overrightarrow{G}$.

On the other hand, note that the existence of a dipath with small length from u to v in $\overrightarrow{G}$ does not guarantee that there is a short way from v to u. For this reason, the weak diameter of $\overrightarrow{G}$ may not be representative of its oriented distances. Let $dist_s(\overrightarrow{G}, u, v) = \max\{dist(\overrightarrow{G}, u, v), dist(\overrightarrow{G}, v, u)\}$. The *strong diameter* of $\overrightarrow{G}$, denoted $diam_s(\overrightarrow{G})$, is the maximum $dist_s(\overrightarrow{G}, u, v)$ regarding all possible pairs $\{u, v\}$ of vertices of $\overrightarrow{G}$.

An orientation of G is *k-weak* (resp. *k-strong*) if it has weak (resp. strong) diameter at most k. We only deal with k-weak orientations in this paper, mainly with the case $k = 2$ which is related to some graph colouring notions as explained below. A *k-colouring* of G is a partition of $V(G)$ into k parts such that two vertices of a same colour class are not adjacent. The least number of colours used by a colouring of G is referred to as the *chromatic number* of G, denoted $\chi(G)$. Clearly, we have $\chi(G) = |V(G)|$ if and only if G is a complete graph.

Now consider similar notions but for oriented graphs. An *oriented k-colouring* of $\overrightarrow{G}$ is a partition of $V(\overrightarrow{G})$ into k parts such that two arcs between two colour classes have the same direction. Note that this property implies that any two vertices from a same colour class cannot be adjacent. As usually, the least number of colours used by an oriented colouring of $\overrightarrow{G}$ is defined as the *oriented chromatic number* of $\overrightarrow{G}$, denoted $\chi_o(\overrightarrow{G})$. Contrary to the undirected case, an oriented graph whose oriented chromatic number is exactly its order is not necessarily a tournament. As an illustration of that claim, remark that $\chi_o(\overrightarrow{C_5}) = 5$, where $\overrightarrow{C_5}$ is the circuit on 5 vertices.

The observation above led to the introduction of *oriented cliques* (or *o-clique* for short), which are the analogues of cliques regarding oriented colouring. In other words, o-cliques are those oriented graphs whose oriented chromatic number is exactly their order. O-cliques have been mainly studied regarding the extremal theory point of view. The interested reader may refer to [1], [3], [4], [5] and [6] for interesting results regarding the order or the size of o-cliques.

By now extending the notion of oriented chromatic number to undirected graphs, finding 2-weak orientations of graphs becomes of some interest. If G is an undirected graph, then $\chi_o(G)$ is defined as

$$\chi_o(G) = \max\{\chi_o(\overrightarrow{G}), \overrightarrow{G} \text{ is an orientation of } G\}.$$

Hence, we have $\chi_o(G) = |V(G)|$ if and only if G admits an orientation which is an o-clique. An important result due to Klostermeyer and MacGillivray states that an oriented graph is an o-clique if and only if it has weak diameter at most 2 [4]. Therefore, we get that $\chi_o(G) = |V(G)|$ if and only if G admits a 2-weak orientation.

From the algorithmic point of view, the complexity of deciding whether an undirected graph admits a 2-weak orientation was still unknown. Note that the analogous question for 2-strong orientations was settled down by Chvátal and Thomassen, who showed this problem to be NP-complete [2]. We here settle this question by studying the following decision problem.

Orientation with Weak Diameter k - k-OWD
Instance: A graph G.
Question: Does G admit a k-weak orientation?

Clearly, a graph admits a 1-weak orientation if and only if it is a complete graph. Therefore, 1-OWD is in P. In this work, we show that k-OWD is NP-complete for every $k \geq 2$. For this purpose, we first give the proof for the case $k = 2$ in Section 2. We then explain how to generalize this result in Section 3.

2 Main result

We prove the following complexity result.

Theorem 1. 2-OWD *is NP-complete.*

Proof. k-OWD is in NP for every k since one can, given an orientation $\overrightarrow{G}$ of G, check whether $diam_w(\overrightarrow{G}) \leq k$. For this purpose, one just has to check, for every possible pair $\{u, v\}$ of distinct vertices of G, whether either u and v are adjacent, or there exist $k' \leq k-1$ vertices $w_1, ..., w_{k'}$ such that $\overrightarrow{uw_1...w_{k'}v}$ or $\overrightarrow{vw_1...w_{k'}u}$ is a dipath of $\overrightarrow{G}$. This witness algorithm runs in polynomial time with respect to the order of G.

We now show that 2-OWD is NP-hard by reduction from the following NP-complete problem.

Not-All-Equal 3-SAT - NAE-3SAT
Instance: A 3CNF formula F over variables $\{x_1, ..., x_n\}$ and clauses $\{C_1, ..., C_m\}$.
Question: Is F *nae-satisfiable*, i.e. is there a truth assignment of the variables of F such that no clause of F has its three literals having the same truth value?

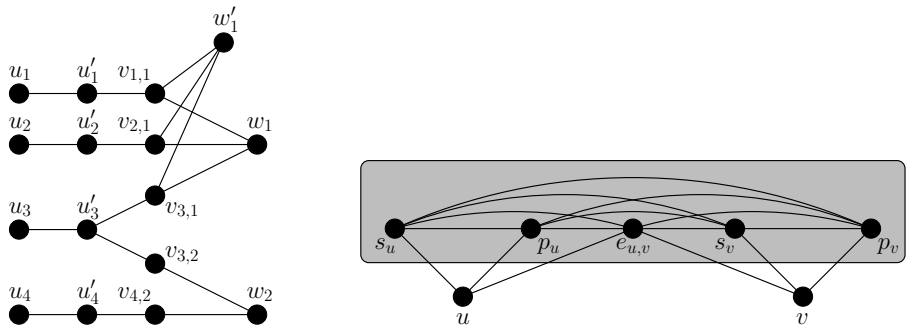
The interesting property of NAE-3SAT is that it remains NP-complete even if none of its clauses contains a negated variable (MONOTONE NAE-3SAT). Note further that if a clause of F is of the form $(x_i \vee x_i \vee x_i)$, then F cannot be nae-satisfied. It is thus understood that F meets these two properties throughout this proof. From F, we produce a graph G_F such that F is nae-satisfiable if and only if G_F admits a 2-weak orientation $\overrightarrow{G_F}$. This reduction is achieved in polynomial time regarding the size of F.

For every $i \in \{1, ..., n\}$ we denote by n_i the number of distinct clauses that contain the variable x_i in F, and by c_j the number of distinct variables in the clause C_j for every $j \in \{1, ..., m\}$. Clearly, we have $n_i \geq 1$ and $c_j \in \{2, 3\}$ for every such integers.

We first describe the *core* G_F^c of G_F, i.e. the subgraph of G_F from which the equivalence with F follows. The subgraph G_F^c does not have diameter 2, but G_F will be augmented later so that it has diameter 2, and this without altering the equivalence. With each variable x_i of F, we associate $n_i + 2$ vertices u_i, u'_i, and $v_{i,j_1}, ..., v_{i,j_{n_i}}$ in G_F^c, where $j_1, ..., j_{n_i}$ are the distinct indices of the clauses of F that contain x_i. These vertices are joined in G_F^c in such a way that they form a star whose central vertex is u'_i. Now, with every clause C_j in F, we associate a vertex w_j in G_F^c. In the case where $c_j = 3$, we also add a vertex w'_j to G_F^c. Finally, whenever a variable x_i belongs to a clause C_j in F, we add an edge linking $v_{i,j}$ and w_j in G_F^c, and, if $c_j = 3$, an edge linking $v_{i,j}$ and w'_j.

This construction is depicted in Figure 1 (a). A pair $\{u, v\}$ of distinct vertices of G_F^c is *representative* if it matches one of the following forms.

1. $\{u, v\} = \{u_i, v_{i,j}\}$ where $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$.
2. $\{u, v\} = \{u'_i, w_j\}$ where $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$, and $x_i \in C_j$.
3. $\{u, v\} = \{v_{i_1,j}, v_{i_2,j}\}$ where $i_1 \neq i_2$, $j \in \{1, ..., m\}$, and $x_{i_1}, x_{i_2} \in C_j$.



(a) Core G_F^c regarding two clauses $C_1 = (x_1 \vee x_2 \vee x_3)$ and $C_2 = (x_3 \vee x_3 \vee x_4)$

(b) Clique regarding a non-representative pair $\{u, v\}$

Fig. 1. Subgraphs obtained while constructing the reduced graph G_F

An orientation of G_F^c is *good* if two vertices forming a representative pair are linked by a 2-dipath in either direction. Note that, in this definition, there is no requirement on the oriented distance between two vertices which are at distance at least 3. A representative pair is a pair of vertices which are not adjacent in G_F, and for which there are only a few paths with length at most 2 between them. All of these paths belong to G_F^c so that the existence of a 2-weak orientation of G_F depends on the existence of a good orientation of G_F^c.

We claim that we have an equivalence between nae-satisfying F and finding a good orientation of G_F^c. This relies on the following three observations.

Claim 1. *Suppose x_i belongs to the clauses $C_{j_1}, ..., C_{j_{n_i}}$ of F. Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c, $\overrightarrow{u_i u'_i v_{i,j} w_j}$ is a 3-dipath for every $j \in \{j_1, ..., j_{n_i}\}$, or $\overrightarrow{w_j v_{i,j} u'_i u_i}$ is a 3-dipath for every $j \in \{j_1, ..., j_{n_i}\}$.*

Proof. Suppose $\overrightarrow{u_i u'_i} \in E(\overrightarrow{G_F^c})$. Because u_i is at distance 2 from the $v_{i,j}$'s in G_F^c and u'_i is the only neighbour of u_i, we have to add $\overrightarrow{u'_i v_{i,j_1}}, ..., \overrightarrow{u'_i v_{i,j_{n_i}}}$ to $E(\overrightarrow{G_F^c})$ so that there are 2-dipaths joining u_i and the $v_{i,j}$'s in $\overrightarrow{G_F^c}$. Similarly, because $v_{i,j}$ is the only vertex joining u'_i and w_j in G_F^c, and $\overrightarrow{u'_i v_{i,j}}$ belongs to $E(\overrightarrow{G_F^c})$ for every $j \in \{j_1, ..., j_{n_i}\}$, we have to add $\overrightarrow{v_{i,j_1} w_{j_1}}, ..., \overrightarrow{v_{i,j_{n_i}} w_{j_{n_i}}}$ to $E(\overrightarrow{G_F^c})$ so that there are 2-dipaths joining u'_i and the w_j's. The claim follows similarly from the assumption $\overrightarrow{u'_i u_i} \in E(\overrightarrow{G_F^c})$.

Claim 2. *Suppose C_j has only two distinct variables x_{i_1} and x_{i_2} in F. Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c, either $\overrightarrow{v_{i_1,j} w_j v_{i_2,j}}$ or $\overrightarrow{v_{i_2,j} w_j v_{i_1,j}}$ is a 2-dipath.*

Proof. The claim follows from the fact that $v_{i_1,j} w_j v_{i_2,j}$ is the only path with length at most 2 joining $v_{i_1,j}$ and $v_{i_2,j}$ in G_F^c.

Claim 3. *Suppose C_j has three distinct variables x_{i_1}, x_{i_2} and x_{i_3} in F. Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c, the three edges $v_{i_1,j}w_j$, $v_{i_2,j}w_j$ and $v_{i_3,j}w_j$ cannot be all directed from or towards w_j.*

Proof. Note that there are only two paths with length at most 2 joining any two of $v_{i_1,j}$, $v_{i_2,j}$, and $v_{i_3,j}$. These pass through w_j and w'_j, respectively. Note thus that if one of the two situations of the claim occurs, then there is no 2-dipath of $\overrightarrow{G_F^c}$ joining two of these vertices which passes by w_j. So there must be three 2-dipaths joining these vertices passing by w'_j, but this is impossible.

Remark that if two of the edges $v_{i_1,j}w_j$, $v_{i_2,j}w_j$ and $v_{i_3,j}w_j$ have the same direction while the third one is oriented in the opposite direction, e.g. suppose that $\overrightarrow{v_{i_1,j}w_j}$, $\overrightarrow{v_{i_2,j}w_j}$ and $\overrightarrow{w_jv_{i_3,j}}$ are edges of $\overrightarrow{G_F^c}$, then we can obtain three 2-dipaths joining any two of $v_{i_1,j}$, $v_{i_2,j}$, and $v_{i_3,j}$. So far, note that there are two 2-dipaths starting from $v_{i_1,j}$ and $v_{i_2,j}$, respectively, and ending at $v_{i_3,j}$. The last 2-dipath starting from $v_{i_1,j}$ and ending at $v_{i_2,j}$ can be obtained by orienting the edges incident with w'_j, e.g. by adding $\overrightarrow{v_{i_1,j}w'_j}$ and $\overrightarrow{w'_jv_{i_2,j}}$ to $E(\overrightarrow{G_F^c})$.

According to Claims 1, 2 and 3, we have an equivalence between nae-satisfying F and finding a good orientation of G_F^c. Indeed, assume that adding $\overrightarrow{u_iu'_i}$ (resp. $\overrightarrow{u'_iu_i}$) to $E(\overrightarrow{G_F^c})$ simulates an assignment of the variable x_i to true (resp. false). Claim 1 reflects the fact that, in a truth assignment of the variables of F, the variable x_i has the same truth value in every clause it appears in. Claims 2 and 3 depict the fact that all variables of a clause cannot have the same truth value. Thus, from a satisfying truth assignment of the variables of F we can deduce a good orientation of G_F^c, and vice-versa.

We now augment G_F with additional vertices so that there is a path with length 2 joining every two non-adjacent vertices of G_F^c that do not form a representative pair. This is done in such a way that there is an orientation of the edges of $E(G_F) - E(G_F^c)$ so that every two vertices of G_F that do not form a representative pair are joined by a 2-dipath. In this way, the existence of a 2-weak orientation of G_F only relies on the existence of a good orientation of G_F^c.

First, for every vertex v of G_F^c, add two new vertices s_v and p_v, and the edges vs_v and vp_v. Then, for every non-representative pair $\{u, v\}$ of G_F^c, add a new vertex $e_{u,v}$, and the edges $ue_{u,v}$ and $ve_{u,v}$. Finally, turn the subgraph of G_F induced by $\bigcup_{v\in V(G_F^c)}\{s_v, p_v\} \cup \bigcup_{\{u,v\}\in NR}\{e_{u,v}\}$ into a clique, where NR denotes the set of non-representative pairs of G_F^c. This construction is depicted in Figure 1 (b).

Note that, thanks to our modifications, there is an orientation of the edges of G_F such that every two vertices u and v of G_F that do not form a representative pair of G_F^c are joined by a 2-dipath. This may be obtained for instance as follows. Define an arbitrary ordering $\sigma = (v_1, ..., v_{|V(G_F^c)|})$ of the vertices of G_F^c. Now for every vertex v_i of G_F^c, let $\overrightarrow{v_is_{v_i}}$ and $\overrightarrow{p_{v_i}v_i}$ be arcs. Moreover, for every vertex v_j such that $i < j$ (if any), let $\overrightarrow{s_{v_i}s_{v_j}}$, $\overrightarrow{s_{v_i}p_{v_j}}$, $\overrightarrow{p_{v_i}s_{v_j}}$ and $\overrightarrow{p_{v_i}p_{v_j}}$ be arcs. If the pair

$\{v_i, v_j\}$ is non-representative, then let also $\overrightarrow{v_i e_{v_i,v_j}}$ and $\overrightarrow{e_{v_i,v_j} v_j}$ be arcs of the partial orientation. Finally, for every other vertex v_k such that $i < j, k$ (resp. $j, k < i$) and $\{v_j, v_k\}$ is a non-representative pair, let $\overrightarrow{s_{v_i} e_{v_j,v_k}}$ and $\overrightarrow{p_{v_i} e_{v_j,v_k}}$ (resp. $\overrightarrow{e_{v_j,v_k} s_{v_i}}$ and $\overrightarrow{e_{v_j,v_k} p_{v_i}}$) be arcs. Any other edge can be oriented arbitrarily.

It is easy to check that, under the orientation given above, there is a dipath with length at most 2 joining every two non-adjacent vertices u and v that do not form a representative pair of G_F. First suppose that $u = v_i$ and $v = v_j$ are such vertices. Then, note that $\overrightarrow{u e_{u,v} v}$ belongs to the orientation. Now, if $u = v_i \in V(G_F^c)$ and $v \notin V(G_F^c)$, then v is a vertex from the clique and is adjacent to some vertex $v_j \neq v_i$ of G_F^c. Then, note that either $\overrightarrow{u s_u v}$ or $\overrightarrow{v p_u u}$ is a 2-dipath of G_F assuming that $i < j$ or $i > j$, respectively.

Finally remark that, for every representative pair $\{u, v\}$ of G_F, there is no new path with length 2 joining u and v passing through new vertices resulting from the modification. Indeed, such a path would have to start from u, enter the clique, cross the clique, and end at v. Thus, such a path has length at least 3. It follows that the distances between vertices from representative pairs of G_F are not altered by the modifications, and thus that the equivalence between nae-satisfying F and finding a good orientation of G_F^c is preserved. Therefore, F is nae-satisfiable if and only if G_F admits a 2-weak orientation.

3 Generalisation

Although our motivations for restricting our intention on 2-OWD are supported by a colouring problem, one could naturally ask about the complexity of any problem k-OWD with $k \geq 3$.

In what follows, we explain how to generalize the reduction we gave for the case $k = 2$ to any fixed $k \geq 3$. From a 3CNF formula F, we first produce the core G_F^c of G_F and define an orientation of G_F^c to be good if and only if some (representative) pairs of vertices of G_F^c are linked by a dipath with length exactly k in either direction. The structure of G_F^c is representative of the structure of F so that a good orientation of G_F^c implies a nae-assignment of F, and vice-versa. We then augment G_F with additional gadgets so that there is an orientation of the new edges under which every two vertices from a non-representative pair of G_F are joined by a dipath with length at most k in either direction. This is done in such a way that the existence of a good orientation of G_F^c does not depend of the edges resulting from the modifications.

This augmentation consists mainly in *rooting* a gadget G_v in each vertex v of G_F^c. All gadgets of G_F are connected in a specific way. In the case where $\{u, v\}$ is not a representative pair, we add a *shortcut* between G_u and G_v, i.e. an alternative shorter path for joining two vertices of G_u and G_v. Typically, every vertex u' of G_u is at distance at most k from any vertex v' of G_v, unless $u' = u$, $v' = v$ and $\{u, v\}$ is a representative pair. In the proof of Theorem 1, i.e. for the case $k = 2$, the gadget G_v is the triangle induced by v, s_v, and p_v, and, assuming $\{u, v\}$ is not representative, the shortcut between G_u and G_v is the vertex $e_{u,v}$.

Typically, the regular path between u and v, i.e. the one passing though the clique, has length 3, while the path passing by the shortcut $e_{u,v}$ has length 2.

Let $k \geq 3$ be fixed. The core G_F^c has the following vertices. Similarly as in the case $k = 2$, let u_i, u'_i, $v_{i,j}$ be vertices for every $i \in \{1, ..., n\}$ and $j \in \{i_1, ..., i_{n_i}\}$, where $i_1, ..., i_{n_i}$ are the distinct indices of the clauses of F that contain the variable x_i. The remaining vertices of G_F^c depend on the parity of k. If k is even, then, for every $j \in \{1, ..., m\}$, add the vertex w_j, and additionally w'_j when $c_j = 3$, as in the case $k = 2$. Otherwise, if k is odd, then, for every $j \in \{1, ..., m\}$, proceed as follows. First, if $c_j = 2$, then just add an edge $a_j b_j$ to G_F^c. Otherwise, if $c_j = 3$, then add two cycles $a_j w_j b_j a_j$ and $a'_j w'_j b'_j a'_j$ with length 3 to G_F^c.

By joining two vertices u and v by a path, we mean that we identify the endvertices of a new path with u and v, respectively. We now link all the vertices of G_F^c by means of several vertex-disjoint paths. First, join every pair $\{u_i, u'_i\}$ by a path with length $\lfloor \frac{k}{2} \rfloor$. Then, join every pair $\{u'_i, v_{i,j}\}$ by a path with length $\lceil \frac{k}{2} \rceil$. Now, for every $j \in \{1, ..., m\}$, add the following paths to G_F^c.

- If $c_j = 2$, and x_{i_1} and x_{i_2} are the distinct variables of C_j, then join each of $\{v_{i_1,j}, w_j\}$ and $\{v_{i_2,j}, w_j\}$ by a path with length $\frac{k}{2}$ if k is even, or join each of $\{v_{i_1,j}, a_j\}$ and $\{v_{i_2,j}, b_j\}$ by a path with length $\lfloor \frac{k}{2} \rfloor$ otherwise.
- If $c_j = 3$, and x_{i_1}, x_{i_2} and x_{i_3} are the variables of C_j, then, if k is even, join every vertex among $\{v_{i_1,j}, v_{i_2,j}, v_{i_3,j}\}$ and each vertex in $\{w_j, w'_j\}$ by means of a path with length $\frac{k}{2}$. Otherwise, if k is odd, then join $v_{i_1,j}$ and both a_j and a'_j, $v_{i_2,j}$ and both w_j and w'_j, and $v_{i_3,j}$ and both b_j and b'_j by a path with length $\lfloor \frac{k}{2} \rfloor$ otherwise.

A pair of vertices of G_F^c is representative whenever it is of the form $\{u_i, v_{i,j}\}$, $\{u'_i, s\}$, where s is of the form a_j, b_j, or w_j such that there is a path with length $\lfloor \frac{k}{2} \rfloor$ between a vertex $v_{i,j}$ and s, or $\{v_{i_1,j}, v_{i_2,j}\}$. An orientation of G_F^c is good if every two vertices of G_F^c composing a representative pair are joined by a k-dipath in either direction. Note that, by construction, every two such vertices are at distance exactly k in G_F^c, and that the possibilities for joining them by a k-dipath are quite limited. For these reasons, Claims 1, 2 and 3 introduced in the proof of Theorem 1 can be derived for the general case. We get that nae-satisfying F is equivalent to finding a good orientation of G_F^c.

We finally augment G_F so that every two of its vertices are at distance at most k. This is done by adding a gadget G_v to G_F for every vertex v of G_F^c as explained above. Set $x = \lfloor \frac{k}{2} \rfloor$. For every $i \in \{1, ..., x\}$, add two new vertices s_v^i and p_v^i to G_v. These two vertices form the ith *level* of G_v, and are said to be *i-vertices*. Next, for every $i \in \{1, ..., x-1\}$, add all possible edges between the i- and $(i+1)$-vertices of G_v so that two consecutive levels of G_v form a clique on 4 vertices. Finally, add an edge between v and every 1-vertex of G_v.

We finish the construction of G_F by adding some connection between the gadgets. We distinguish two cases depending on the parity of k.

- If k is even, then we turn the subgraph induced by all x-vertices of G_F into a clique. Next, for every non-representative pair $\{u, v\}$ of G_F^c, add a shortcut

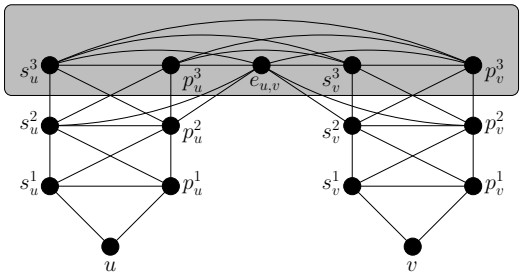


Fig. 2. The gadgets G_u and G_v obtained for a non-representative pair $\{u, v\}$ and $k = 6$

vertex $e_{u,v}$ to the clique constructed just before. Finally, add every edge between $e_{u,v}$ and the vertices from the $(x-1)^{th}$ levels of G_u and G_v.

– Otherwise, if k is odd, then add a new vertex z to G_F, and add all possible edges between z and x-vertices. For every non-representative pair $\{u, v\}$ of G_F^c, also add the shortcut edges $p_u^x s_v^x$ and $s_u^x p_v^x$ to G_F.

This construction is depicted in Figure 2. Note that no new path with length at most k between two vertices composing a representative pair of G_F arose from the modifications. Therefore, the equivalence between nae-satisfying F and finding a good orientation of G_F^c is still correct. We claim, as in the case $k = 2$, that there is an orientation of the edges we just added so that every non-representative pair of vertices of G_F is joined by a k-dipath.

To see that claim, define an arbitrary ordering $\sigma = (v_1, ..., v_{|V(G_F^c)|})$ over all vertices of G_F^c, and consider the following partial orientation. First, for every vertex v of G_F^c, let $\overrightarrow{vs_v^1}$ and $\overrightarrow{p_v^1 v}$ be arcs. Then, for every level $i \in \{1, ..., x\}$ of G_v, let $\overrightarrow{s_v^i p_v^i}$ be an arc. Next, for every $i \in \{1, ..., x-1\}$, add the arcs $\overrightarrow{s_v^i s_v^{i+1}}$, $\overrightarrow{p_v^{i+1} p_v^i}$, $\overrightarrow{p_v^i s_v^{i+1}}$ and $\overrightarrow{p_v^{i+1} s_v^i}$ to the partial orientation. Now, on the one hand, if k is even, then, for every shortcut vertex e of G_F, add the arcs $\overrightarrow{s_v^x e}$ and $\overrightarrow{ep_v^x}$. Next, for every $i < j$ consider $\overrightarrow{s_{v_i}^x s_{v_j}^x}$, $\overrightarrow{s_{v_i}^x p_{v_j}^x}$, $\overrightarrow{p_{v_i}^x s_{v_j}^x}$ and $\overrightarrow{p_{v_i}^x p_{v_j}^x}$ as arcs of the partial orientation. Additionally, if $\{v_i, v_j\}$ is a representative pair, then let $\overrightarrow{s_{v_j}^{x-1} e_{v_i,v_j}}$, $\overrightarrow{e_{v_i,v_j} p_{v_i}^{x-1}}$, $\overrightarrow{s_{v_i}^{x-1} e_{v_i,v_j}}$, and $\overrightarrow{e_{v_i,v_j} p_{v_j}^{x-1}}$ be arcs. On the other hand, if k is odd, then let $\overrightarrow{s_{v_i}^x z}$ and $\overrightarrow{zp_{v_i}^x}$ be arcs. Finally, if $\{v_i, v_j\}$ is representative, then let $\overrightarrow{s_{v_i}^x p_{v_j}^x}$ and $\overrightarrow{s_{v_j}^x p_{v_i}^x}$ be arcs. Orient all the remaining edges arbitrarily.

Note that, under the partial orientation given above, any vertex from a gadget can directly access the upper or lower level of the same gadget. Besides, any such non-root vertex can also reach or be reached by any other vertex from another gadget by means of a dipath with length at most k. As in the case $k = 2$, such a path typically goes up across a first gadget, then exits the first gadget to enter the second one (either directly from the xth level or via z), and finally goes down across the second gadget. Because the gadgets have $x = \lfloor \frac{k}{2} \rfloor$ levels, the length of such a path does not exceed k. Finally observe that if $\{u, v\}$ is representative, then there is no path with length at most k joining u and v passing across the

gadgets. On the contrary, if $\{u, v\}$ is non-representative, then there is a path with length exactly k joining u and v. This path necessarily passes through the shortcut between G_u and G_v.

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