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# The complexity of deciding whether a graph admits an orientation with fixed weak diameter

Julien Bensmail<sup>1</sup>, Romaric Duvignau<sup>1</sup>, and Sergey Kirgizov<sup>2</sup>

<sup>1</sup> CNRS and Univ. Bordeaux, LaBRI, UMR 5800, 33400 Talence, France

<sup>2</sup> CNRS and Univ. Pierre et Marie Curie, LIP6, UMR 7606, 75252 Paris, France

**Abstract.** An orientation of an undirected graph  $G$  has weak diameter  $k$  if, for every pair  $\{u, v\}$  of vertices of  $G$ , there is a directed path with length at most  $k$  joining  $u$  and  $v$  in either direction. In this work, we show that deciding whether a graph admits an orientation with weak diameter  $k$  is NP-complete whenever  $k \geq 2$ .

## 1 Introduction

Let  $G$  be a simple undirected graph with vertex (resp. edge) set  $V(G)$  (resp.  $E(G)$ ). By *orienting* every edge  $uv$  of  $G$ , either from  $u$  to  $v$  or from  $v$  to  $u$ , one obtains an *orientation*  $\vec{G}$  of  $G$ . This oriented graph  $\vec{G}$  has the same vertex set as  $G$ , i.e.  $V(\vec{G}) = V(G)$ , and, for every edge  $uv \in E(G)$ , we have either  $\vec{uv} \in E(\vec{G})$  or  $\vec{vu} \in E(\vec{G})$  depending on the orientation assigned to  $uv$ .

The *distance*  $dist(G, u, v)$  from  $u$  to  $v$  in  $G$  is the minimal length of a path joining  $u$  and  $v$ . We refer to the maximum distance between two vertices of  $G$  as its *diameter*, and denote it  $diam(G)$ . These definitions can be naturally adapted to the context of oriented graphs. A *k-dipath*  $\vec{v_1 v_2 \dots v_{k+1}}$  of  $\vec{G}$  is a sequence of  $k + 1$  distinct vertices such that  $\vec{v_i v_{i+1}} \in E(\vec{G})$  for every  $i \in \{1, \dots, k\}$ . The *directed distance*  $dist(\vec{G}, u, v)$  from  $u$  to  $v$  in  $\vec{G}$  is the minimal length of a dipath starting from  $u$  and ending at  $v$ . Note that, contrary to the undirected case, we may have  $dist(\vec{G}, u, v) \neq dist(\vec{G}, v, u)$ . Therefore, two definitions of the oriented diameter can be adopted.

On the one hand, one can consider that two vertices  $u$  and  $v$  of  $\vec{G}$  are close when one of  $dist(\vec{G}, u, v)$  and  $dist(\vec{G}, v, u)$  is small. In other words, we consider that two vertices are close if there is a dipath with small length joining them in  $\vec{G}$  regardless of its direction. Let  $dist_w(\vec{G}, u, v)$  be  $\min\{dist(\vec{G}, u, v), dist(\vec{G}, v, u)\}$ . The *weak diameter* of  $\vec{G}$ , denoted  $diam_w(\vec{G})$ , is the maximum  $dist_w(\vec{G}, u, v)$  taken over all pairs  $\{u, v\}$  of vertices of  $\vec{G}$ .

On the other hand, note that the existence of a dipath with small length from  $u$  to  $v$  in  $\vec{G}$  does not guarantee that there is a short way from  $v$  to  $u$ . For this reason, the weak diameter of  $\vec{G}$  may not be representative of its oriented distances. Let  $dist_s(\vec{G}, u, v) = \max\{dist(\vec{G}, u, v), dist(\vec{G}, v, u)\}$ . The *strong diameter* of  $\vec{G}$ , denoted  $diam_s(\vec{G})$ , is the maximum  $dist_s(\vec{G}, u, v)$  regarding all possible pairs  $\{u, v\}$  of vertices of  $\vec{G}$ .

An orientation of  $G$  is  $k$ -weak (resp.  $k$ -strong) if it has weak (resp. strong) diameter at most  $k$ . We only deal with  $k$ -weak orientations in this paper, mainly with the case  $k = 2$  which is related to some graph colouring notions as explained below. A  $k$ -colouring of  $G$  is a partition of  $V(G)$  into  $k$  parts such that two vertices of a same colour class are not adjacent. The least number of colours used by a colouring of  $G$  is referred to as the *chromatic number* of  $G$ , denoted  $\chi(G)$ . Clearly, we have  $\chi(G) = |V(G)|$  if and only if  $G$  is a complete graph.

Now consider similar notions but for oriented graphs. An *oriented  $k$ -colouring* of  $\vec{G}$  is a partition of  $V(\vec{G})$  into  $k$  parts such that two arcs between two colour classes have the same direction. Note that this property implies that any two vertices from a same colour class cannot be adjacent. As usually, the least number of colours used by an oriented colouring of  $\vec{G}$  is defined as the *oriented chromatic number* of  $\vec{G}$ , denoted  $\chi_o(\vec{G})$ . Contrary to the undirected case, an oriented graph whose oriented chromatic number is exactly its order is not necessarily a tournament. As an illustration of that claim, remark that  $\chi_o(\vec{C}_5) = 5$ , where  $\vec{C}_5$  is the circuit on 5 vertices.

The observation above led to the introduction of *oriented cliques* (or *o-clique* for short), which are the analogues of cliques regarding oriented colouring. In other words, o-cliques are those oriented graphs whose oriented chromatic number is exactly their order. O-cliques have been mainly studied regarding the extremal theory point of view. The interested reader may refer to [1], [3], [4], [5] and [6] for interesting results regarding the order or the size of o-cliques.

By now extending the notion of oriented chromatic number to undirected graphs, finding 2-weak orientations of graphs becomes of some interest. If  $G$  is an undirected graph, then  $\chi_o(G)$  is defined as

$$\chi_o(G) = \max\{\chi_o(\vec{G}), \vec{G} \text{ is an orientation of } G\}.$$

Hence, we have  $\chi_o(G) = |V(G)|$  if and only if  $G$  admits an orientation which is an o-clique. An important result due to Klostermeyer and MacGillivray states that an oriented graph is an o-clique if and only if it has weak diameter at most 2 [4]. Therefore, we get that  $\chi_o(G) = |V(G)|$  if and only if  $G$  admits a 2-weak orientation.

From the algorithmic point of view, the complexity of deciding whether an undirected graph admits a 2-weak orientation was still unknown. Note that the analogous question for 2-strong orientations was settled down by Chvátal and Thomassen, who showed this problem to be NP-complete [2]. We here settle this question by studying the following decision problem.

ORIENTATION WITH WEAK DIAMETER  $k$  -  $k$ -OWD

Instance: A graph  $G$ .

Question: Does  $G$  admit a  $k$ -weak orientation?

Clearly, a graph admits a 1-weak orientation if and only if it is a complete graph. Therefore, 1-OWD is in P. In this work, we show that  $k$ -OWD is NP-complete for every  $k \geq 2$ . For this purpose, we first give the proof for the case  $k = 2$  in Section 2. We then explain how to generalize this result in Section 3.

## 2 Main result

We prove the following complexity result.

**Theorem 1.** *2-OWD is NP-complete.*

*Proof.*  $k$ -OWD is in NP for every  $k$  since one can, given an orientation  $\vec{G}$  of  $G$ , check whether  $\text{diam}_w(\vec{G}) \leq k$ . For this purpose, one just has to check, for every possible pair  $\{u, v\}$  of distinct vertices of  $G$ , whether either  $u$  and  $v$  are adjacent, or there exist  $k' \leq k - 1$  vertices  $w_1, \dots, w_{k'}$  such that  $\overrightarrow{uw_1 \dots w_{k'}v}$  or  $\overrightarrow{vw_1 \dots w_{k'}u}$  is a dipath of  $\vec{G}$ . This witness algorithm runs in polynomial time with respect to the order of  $G$ .

We now show that 2-OWD is NP-hard by reduction from the following NP-complete problem.

NOT-ALL-EQUAL 3-SAT - NAE-3SAT

Instance: A 3CNF formula  $F$  over variables  $\{x_1, \dots, x_n\}$  and clauses  $\{C_1, \dots, C_m\}$ .

Question: Is  $F$  *nae-satisfiable*, i.e. is there a truth assignment of the variables of  $F$  such that no clause of  $F$  has its three literals having the same truth value?

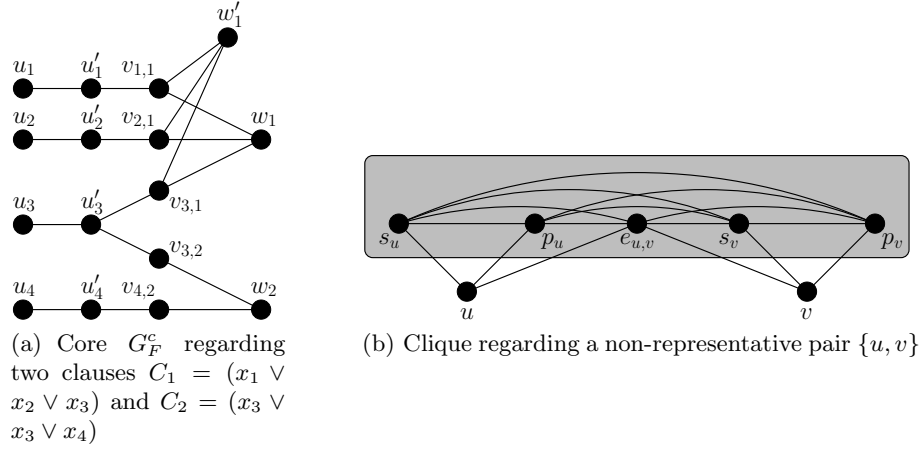
The interesting property of NAE-3SAT is that it remains NP-complete even if none of its clauses contains a negated variable (MONOTONE NAE-3SAT). Note further that if a clause of  $F$  is of the form  $(x_i \vee x_i \vee x_i)$ , then  $F$  cannot be nae-satisfied. It is thus understood that  $F$  meets these two properties throughout this proof. From  $F$ , we produce a graph  $G_F$  such that  $F$  is nae-satisfiable if and only if  $G_F$  admits a 2-weak orientation  $\vec{G}_F$ . This reduction is achieved in polynomial time regarding the size of  $F$ .

For every  $i \in \{1, \dots, n\}$  we denote by  $n_i$  the number of distinct clauses that contain the variable  $x_i$  in  $F$ , and by  $c_j$  the number of distinct variables in the clause  $C_j$  for every  $j \in \{1, \dots, m\}$ . Clearly, we have  $n_i \geq 1$  and  $c_j \in \{2, 3\}$  for every such integers.

We first describe the *core*  $G_F^c$  of  $G_F$ , i.e. the subgraph of  $G_F$  from which the equivalence with  $F$  follows. The subgraph  $G_F^c$  does not have diameter 2, but  $G_F$  will be augmented later so that it has diameter 2, and this without altering the equivalence. With each variable  $x_i$  of  $F$ , we associate  $n_i + 2$  vertices  $u_i, u'_i$ , and  $v_{i,j_1}, \dots, v_{i,j_{n_i}}$  in  $G_F^c$ , where  $j_1, \dots, j_{n_i}$  are the distinct indices of the clauses of  $F$  that contain  $x_i$ . These vertices are joined in  $G_F^c$  in such a way that they form a star whose central vertex is  $u'_i$ . Now, with every clause  $C_j$  in  $F$ , we associate a vertex  $w_j$  in  $G_F^c$ . In the case where  $c_j = 3$ , we also add a vertex  $w'_j$  to  $G_F^c$ . Finally, whenever a variable  $x_i$  belongs to a clause  $C_j$  in  $F$ , we add an edge linking  $v_{i,j}$  and  $w_j$  in  $G_F^c$ , and, if  $c_j = 3$ , an edge linking  $v_{i,j}$  and  $w'_j$ .

This construction is depicted in Figure 1 (a). A pair  $\{u, v\}$  of distinct vertices of  $G_F^c$  is *representative* if it matches one of the following forms.

1.  $\{u, v\} = \{u_i, v_{i,j}\}$  where  $i \in \{1, \dots, n\}$  and  $j \in \{1, \dots, m\}$ .
2.  $\{u, v\} = \{u'_i, w_j\}$  where  $i \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, m\}$ , and  $x_i \in C_j$ .
3.  $\{u, v\} = \{v_{i_1,j}, v_{i_2,j}\}$  where  $i_1 \neq i_2$ ,  $j \in \{1, \dots, m\}$ , and  $x_{i_1}, x_{i_2} \in C_j$ .



**Fig. 1.** Subgraphs obtained while constructing the reduced graph  $G_F$

An orientation of  $G_F^c$  is *good* if two vertices forming a representative pair are linked by a 2-dipath in either direction. Note that, in this definition, there is no requirement on the oriented distance between two vertices which are at distance at least 3. A representative pair is a pair of vertices which are not adjacent in  $G_F$ , and for which there are only a few paths with length at most 2 between them. All of these paths belong to  $G_F^c$  so that the existence of a 2-weak orientation of  $G_F$  depends on the existence of a good orientation of  $G_F^c$ .

We claim that we have an equivalence between nae-satisfying  $F$  and finding a good orientation of  $G_F^c$ . This relies on the following three observations.

**Claim 1.** *Suppose  $x_i$  belongs to the clauses  $C_{j_1}, \dots, C_{j_{n_i}}$  of  $F$ . Then, in any good orientation  $\overrightarrow{G_F^c}$  of  $G_F^c$ ,  $\overrightarrow{u_i u'_i v_{i,j} w_j}$  is a 3-dipath for every  $j \in \{j_1, \dots, j_{n_i}\}$ , or  $\overrightarrow{w_j v_{i,j} u'_i u_i}$  is a 3-dipath for every  $j \in \{j_1, \dots, j_{n_i}\}$ .*

*Proof.* Suppose  $\overrightarrow{u_i u'_i} \in E(\overrightarrow{G_F^c})$ . Because  $u_i$  is at distance 2 from the  $v_{i,j}$ 's in  $G_F^c$  and  $u'_i$  is the only neighbour of  $u_i$ , we have to add  $\overrightarrow{u'_i v_{i,j_1}}, \dots, \overrightarrow{u'_i v_{i,j_{n_i}}}$  to  $E(\overrightarrow{G_F^c})$  so that there are 2-dipaths joining  $u_i$  and the  $v_{i,j}$ 's in  $\overrightarrow{G_F^c}$ . Similarly, because  $v_{i,j}$  is the only vertex joining  $u'_i$  and  $w_j$  in  $G_F^c$ , and  $\overrightarrow{u'_i v_{i,j}}$  belongs to  $E(\overrightarrow{G_F^c})$  for every  $j \in \{j_1, \dots, j_{n_i}\}$ , we have to add  $\overrightarrow{v_{i,j_1} w_{j_1}}, \dots, \overrightarrow{v_{i,j_{n_i}} w_{j_{n_i}}}$  to  $E(\overrightarrow{G_F^c})$  so that there are 2-dipaths joining  $u'_i$  and the  $w_j$ 's. The claim follows similarly from the assumption  $\overrightarrow{u'_i u_i} \in E(\overrightarrow{G_F^c})$ .

**Claim 2.** *Suppose  $C_j$  has only two distinct variables  $x_{i_1}$  and  $x_{i_2}$  in  $F$ . Then, in any good orientation  $\overrightarrow{G_F^c}$  of  $G_F^c$ , either  $\overrightarrow{v_{i_1,j} w_j v_{i_2,j}}$  or  $\overrightarrow{v_{i_2,j} w_j v_{i_1,j}}$  is a 2-dipath.*

*Proof.* The claim follows from the fact that  $v_{i_1,j} w_j v_{i_2,j}$  is the only path with length at most 2 joining  $v_{i_1,j}$  and  $v_{i_2,j}$  in  $G_F^c$ .

**Claim 3.** Suppose  $C_j$  has three distinct variables  $x_{i_1}$ ,  $x_{i_2}$  and  $x_{i_3}$  in  $F$ . Then, in any good orientation  $\overrightarrow{G_F^c}$  of  $G_F^c$ , the three edges  $v_{i_1,j}w_j$ ,  $v_{i_2,j}w_j$  and  $v_{i_3,j}w_j$  cannot be all directed from or towards  $w_j$ .

*Proof.* Note that there are only two paths with length at most 2 joining any two of  $v_{i_1,j}$ ,  $v_{i_2,j}$ , and  $v_{i_3,j}$ . These pass through  $w_j$  and  $w'_j$ , respectively. Note thus that if one of the two situations of the claim occurs, then there is no 2-dipath of  $\overrightarrow{G_F^c}$  joining two of these vertices which passes by  $w_j$ . So there must be three 2-dipaths joining these vertices passing by  $w'_j$ , but this is impossible.

Remark that if two of the edges  $v_{i_1,j}w_j$ ,  $v_{i_2,j}w_j$  and  $v_{i_3,j}w_j$  have the same direction while the third one is oriented in the opposite direction, e.g. suppose that  $\overrightarrow{v_{i_1,j}w_j}$ ,  $\overrightarrow{v_{i_2,j}w_j}$  and  $\overrightarrow{w_jv_{i_3,j}}$  are edges of  $\overrightarrow{G_F^c}$ , then we can obtain three 2-dipaths joining any two of  $v_{i_1,j}$ ,  $v_{i_2,j}$ , and  $v_{i_3,j}$ . So far, note that there are two 2-dipaths starting from  $v_{i_1,j}$  and  $v_{i_2,j}$ , respectively, and ending at  $v_{i_3,j}$ . The last 2-dipath starting from  $v_{i_1,j}$  and ending at  $v_{i_2,j}$  can be obtained by orienting the edges incident with  $w'_j$ , e.g. by adding  $\overrightarrow{v_{i_1,j}w'_j}$  and  $\overrightarrow{w'_jv_{i_2,j}}$  to  $E(\overrightarrow{G_F^c})$ .

According to Claims 1, 2 and 3, we have an equivalence between nae-satisfying  $F$  and finding a good orientation of  $G_F^c$ . Indeed, assume that adding  $\overrightarrow{u_iu'_i}$  (resp.  $\overrightarrow{u'_iu_i}$ ) to  $E(\overrightarrow{G_F^c})$  simulates an assignment of the variable  $x_i$  to true (resp. false). Claim 1 reflects the fact that, in a truth assignment of the variables of  $F$ , the variable  $x_i$  has the same truth value in every clause it appears in. Claims 2 and 3 depict the fact that all variables of a clause cannot have the same truth value. Thus, from a satisfying truth assignment of the variables of  $F$  we can deduce a good orientation of  $G_F^c$ , and vice-versa.

We now augment  $G_F$  with additional vertices so that there is a path with length 2 joining every two non-adjacent vertices of  $G_F^c$  that do not form a representative pair. This is done in such a way that there is an orientation of the edges of  $E(G_F) - E(G_F^c)$  so that every two vertices of  $G_F$  that do not form a representative pair are joined by a 2-dipath. In this way, the existence of a 2-weak orientation of  $G_F$  only relies on the existence of a good orientation of  $G_F^c$ .

First, for every vertex  $v$  of  $G_F^c$ , add two new vertices  $s_v$  and  $p_v$ , and the edges  $vs_v$  and  $vp_v$ . Then, for every non-representative pair  $\{u, v\}$  of  $G_F^c$ , add a new vertex  $e_{u,v}$ , and the edges  $ue_{u,v}$  and  $ve_{u,v}$ . Finally, turn the subgraph of  $G_F$  induced by  $\bigcup_{v \in V(G_F^c)} \{s_v, p_v\} \cup \bigcup_{\{u,v\} \in NR} \{e_{u,v}\}$  into a clique, where  $NR$  denotes the set of non-representative pairs of  $G_F^c$ . This construction is depicted in Figure 1 (b).

Note that, thanks to our modifications, there is an orientation of the edges of  $G_F$  such that every two vertices  $u$  and  $v$  of  $G_F$  that do not form a representative pair of  $G_F^c$  are joined by a 2-dipath. This may be obtained for instance as follows. Define an arbitrary ordering  $\sigma = (v_1, \dots, v_{|V(G_F^c)|})$  of the vertices of  $G_F^c$ . Now for every vertex  $v_i$  of  $G_F^c$ , let  $\overrightarrow{v_i s_{v_i}}$  and  $\overrightarrow{p_{v_i} v_i}$  be arcs. Moreover, for every vertex  $v_j$  such that  $i < j$  (if any), let  $\overrightarrow{s_{v_i} s_{v_j}}$ ,  $\overrightarrow{s_{v_i} p_{v_j}}$ ,  $\overrightarrow{p_{v_i} s_{v_j}}$  and  $\overrightarrow{p_{v_i} p_{v_j}}$  be arcs. If the pair

$\{v_i, v_j\}$  is non-representative, then let also  $\overrightarrow{v_i e_{v_i, v_j}}$  and  $\overrightarrow{e_{v_i, v_j} v_j}$  be arcs of the partial orientation. Finally, for every other vertex  $v_k$  such that  $i < j, k$  (resp.  $j, k < i$ ) and  $\{v_j, v_k\}$  is a non-representative pair, let  $\overrightarrow{s_{v_i} e_{v_j, v_k}}$  and  $\overrightarrow{p_{v_i} e_{v_j, v_k}}$  (resp.  $\overrightarrow{e_{v_j, v_k} s_{v_i}}$  and  $\overrightarrow{e_{v_j, v_k} p_{v_i}}$ ) be arcs. Any other edge can be oriented arbitrarily.

It is easy to check that, under the orientation given above, there is a dipath with length at most 2 joining every two non-adjacent vertices  $u$  and  $v$  that do not form a representative pair of  $G_F$ . First suppose that  $u = v_i$  and  $v = v_j$  are such vertices. Then, note that  $\overrightarrow{u e_{u, v}}$  belongs to the orientation. Now, if  $u = v_i \in V(G_F^c)$  and  $v \notin V(G_F^c)$ , then  $v$  is a vertex from the clique and is adjacent to some vertex  $v_j \neq v_i$  of  $G_F^c$ . Then, note that either  $\overrightarrow{u s_u v}$  or  $\overrightarrow{v p_u u}$  is a 2-dipath of  $G_F$  assuming that  $i < j$  or  $i > j$ , respectively.

Finally remark that, for every representative pair  $\{u, v\}$  of  $G_F$ , there is no new path with length 2 joining  $u$  and  $v$  passing through new vertices resulting from the modification. Indeed, such a path would have to start from  $u$ , enter the clique, cross the clique, and end at  $v$ . Thus, such a path has length at least 3. It follows that the distances between vertices from representative pairs of  $G_F$  are not altered by the modifications, and thus that the equivalence between nae-satisfying  $F$  and finding a good orientation of  $G_F^c$  is preserved. Therefore,  $F$  is nae-satisfiable if and only if  $G_F$  admits a 2-weak orientation.

### 3 Generalisation

Although our motivations for restricting our intention on 2-OWD are supported by a colouring problem, one could naturally ask about the complexity of any problem  $k$ -OWD with  $k \geq 3$ .

In what follows, we explain how to generalize the reduction we gave for the case  $k = 2$  to any fixed  $k \geq 3$ . From a 3CNF formula  $F$ , we first produce the core  $G_F^c$  of  $G_F$  and define an orientation of  $G_F^c$  to be good if and only if some (representative) pairs of vertices of  $G_F^c$  are linked by a dipath with length exactly  $k$  in either direction. The structure of  $G_F^c$  is representative of the structure of  $F$  so that a good orientation of  $G_F^c$  implies a nae-assignment of  $F$ , and vice-versa. We then augment  $G_F$  with additional gadgets so that there is an orientation of the new edges under which every two vertices from a non-representative pair of  $G_F$  are joined by a dipath with length at most  $k$  in either direction. This is done in such a way that the existence of a good orientation of  $G_F^c$  does not depend of the edges resulting from the modifications.

This augmentation consists mainly in *rooting* a gadget  $G_v$  in each vertex  $v$  of  $G_F^c$ . All gadgets of  $G_F$  are connected in a specific way. In the case where  $\{u, v\}$  is not a representative pair, we add a *shortcut* between  $G_u$  and  $G_v$ , i.e. an alternative shorter path for joining two vertices of  $G_u$  and  $G_v$ . Typically, every vertex  $u'$  of  $G_u$  is at distance at most  $k$  from any vertex  $v'$  of  $G_v$ , unless  $u' = u$ ,  $v' = v$  and  $\{u, v\}$  is a representative pair. In the proof of Theorem 1, i.e. for the case  $k = 2$ , the gadget  $G_v$  is the triangle induced by  $v$ ,  $s_v$ , and  $p_v$ , and, assuming  $\{u, v\}$  is not representative, the shortcut between  $G_u$  and  $G_v$  is the vertex  $e_{u, v}$ .

Typically, the regular path between  $u$  and  $v$ , i.e. the one passing through the clique, has length 3, while the path passing by the shortcut  $e_{u,v}$  has length 2.

Let  $k \geq 3$  be fixed. The core  $G_F^c$  has the following vertices. Similarly as in the case  $k = 2$ , let  $u_i, u'_i, v_{i,j}$  be vertices for every  $i \in \{1, \dots, n\}$  and  $j \in \{i_1, \dots, i_{n_i}\}$ , where  $i_1, \dots, i_{n_i}$  are the distinct indices of the clauses of  $F$  that contain the variable  $x_i$ . The remaining vertices of  $G_F^c$  depend on the parity of  $k$ . If  $k$  is even, then, for every  $j \in \{1, \dots, m\}$ , add the vertex  $w_j$ , and additionally  $w'_j$  when  $c_j = 3$ , as in the case  $k = 2$ . Otherwise, if  $k$  is odd, then, for every  $j \in \{1, \dots, m\}$ , proceed as follows. First, if  $c_j = 2$ , then just add an edge  $a_j b_j$  to  $G_F^c$ . Otherwise, if  $c_j = 3$ , then add two cycles  $a_j w_j b_j a_j$  and  $a'_j w'_j b'_j a'_j$  with length 3 to  $G_F^c$ .

By joining two vertices  $u$  and  $v$  by a path, we mean that we identify the endvertices of a new path with  $u$  and  $v$ , respectively. We now link all the vertices of  $G_F^c$  by means of several vertex-disjoint paths. First, join every pair  $\{u_i, u'_i\}$  by a path with length  $\lfloor \frac{k}{2} \rfloor$ . Then, join every pair  $\{u'_i, v_{i,j}\}$  by a path with length  $\lceil \frac{k}{2} \rceil$ . Now, for every  $j \in \{1, \dots, m\}$ , add the following paths to  $G_F^c$ .

- If  $c_j = 2$ , and  $x_{i_1}$  and  $x_{i_2}$  are the distinct variables of  $C_j$ , then join each of  $\{v_{i_1,j}, w_j\}$  and  $\{v_{i_2,j}, w_j\}$  by a path with length  $\frac{k}{2}$  if  $k$  is even, or join each of  $\{v_{i_1,j}, a_j\}$  and  $\{v_{i_2,j}, b_j\}$  by a path with length  $\lfloor \frac{k}{2} \rfloor$  otherwise.
- If  $c_j = 3$ , and  $x_{i_1}, x_{i_2}$  and  $x_{i_3}$  are the variables of  $C_j$ , then, if  $k$  is even, join every vertex among  $\{v_{i_1,j}, v_{i_2,j}, v_{i_3,j}\}$  and each vertex in  $\{w_j, w'_j\}$  by means of a path with length  $\frac{k}{2}$ . Otherwise, if  $k$  is odd, then join  $v_{i_1,j}$  and both  $a_j$  and  $a'_j$ ,  $v_{i_2,j}$  and both  $w_j$  and  $w'_j$ , and  $v_{i_3,j}$  and both  $b_j$  and  $b'_j$  by a path with length  $\lfloor \frac{k}{2} \rfloor$  otherwise.

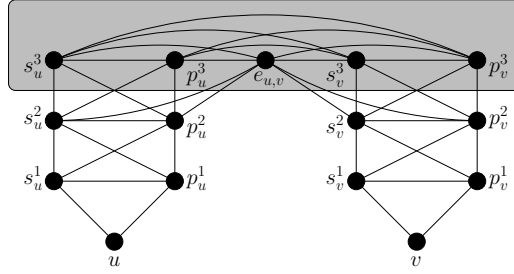
A pair of vertices of  $G_F^c$  is representative whenever it is of the form  $\{u_i, v_{i,j}\}$ ,  $\{u'_i, s\}$ , where  $s$  is of the form  $a_j, b_j$ , or  $w_j$  such that there is a path with length  $\lfloor \frac{k}{2} \rfloor$  between a vertex  $v_{i,j}$  and  $s$ , or  $\{v_{i_1,j}, v_{i_2,j}\}$ . An orientation of  $G_F^c$  is good if every two vertices of  $G_F^c$  composing a representative pair are joined by a  $k$ -dipath in either direction. Note that, by construction, every two such vertices are at distance exactly  $k$  in  $G_F^c$ , and that the possibilities for joining them by a  $k$ -dipath are quite limited. For these reasons, Claims 1, 2 and 3 introduced in the proof of Theorem 1 can be derived for the general case. We get that nae-satisfying  $F$  is equivalent to finding a good orientation of  $G_F^c$ .

We finally augment  $G_F$  so that every two of its vertices are at distance at most  $k$ . This is done by adding a gadget  $G_v$  to  $G_F$  for every vertex  $v$  of  $G_F^c$  as explained above. Set  $x = \lfloor \frac{k}{2} \rfloor$ . For every  $i \in \{1, \dots, x\}$ , add two new vertices  $s_v^i$  and  $p_v^i$  to  $G_v$ . These two vertices form the  $i^{\text{th}}$  level of  $G_v$ , and are said to be  $i$ -vertices. Next, for every  $i \in \{1, \dots, x-1\}$ , add all possible edges between the  $i$ - and  $(i+1)$ -vertices of  $G_v$  so that two consecutive levels of  $G_v$  form a clique on 4 vertices. Finally, add an edge between  $v$  and every 1-vertex of  $G_v$ .

We finish the construction of  $G_F$  by adding some connection between the gadgets. We distinguish two cases depending on the parity of  $k$ .

- If  $k$  is even, then we turn the subgraph induced by all  $x$ -vertices of  $G_F$  into a clique. Next, for every non-representative pair  $\{u, v\}$  of  $G_F^c$ , add a shortcut





**Fig. 2.** The gadgets  $G_u$  and  $G_v$  obtained for a non-representative pair  $\{u, v\}$  and  $k = 6$

- vertex  $e_{u,v}$  to the clique constructed just before. Finally, add every edge between  $e_{u,v}$  and the vertices from the  $(x-1)^{th}$  levels of  $G_u$  and  $G_v$ .
- Otherwise, if  $k$  is odd, then add a new vertex  $z$  to  $G_F$ , and add all possible edges between  $z$  and  $x$ -vertices. For every non-representative pair  $\{u, v\}$  of  $G_F^c$ , also add the shortcut edges  $p_u^x s_v^x$  and  $s_u^x p_v^x$  to  $G_F$ .

This construction is depicted in Figure 2. Note that no new path with length at most  $k$  between two vertices composing a representative pair of  $G_F$  arose from the modifications. Therefore, the equivalence between nae-satisfying  $F$  and finding a good orientation of  $G_F^c$  is still correct. We claim, as in the case  $k = 2$ , that there is an orientation of the edges we just added so that every non-representative pair of vertices of  $G_F$  is joined by a  $k$ -dipath.

To see that claim, define an arbitrary ordering  $\sigma = (v_1, \dots, v_{|V(G_F^c)|})$  over all vertices of  $G_F^c$ , and consider the following partial orientation. First, for every vertex  $v$  of  $G_F^c$ , let  $\overrightarrow{v s_v^1}$  and  $\overrightarrow{p_v^1 v}$  be arcs. Then, for every level  $i \in \{1, \dots, x\}$  of  $G_v$ , let  $\overrightarrow{s_v^i p_v^i}$  be an arc. Next, for every  $i \in \{1, \dots, x-1\}$ , add the arcs  $\overrightarrow{s_v^i s_v^{i+1}}$ ,  $\overrightarrow{p_v^{i+1} p_v^i}$ ,  $\overrightarrow{p_v^i s_v^{i+1}}$  and  $\overrightarrow{p_v^{i+1} s_v^i}$  to the partial orientation. Now, on the one hand, if  $k$  is even, then, for every shortcut vertex  $e$  of  $G_F$ , add the arcs  $\overrightarrow{s_v^x e}$  and  $\overrightarrow{e p_v^x}$ . Next, for every  $i < j$  consider  $\overrightarrow{s_{v_i}^x s_{v_j}^x}$ ,  $\overrightarrow{s_{v_i}^x p_{v_j}^x}$ ,  $\overrightarrow{p_{v_i}^x s_{v_j}^x}$  and  $\overrightarrow{p_{v_i}^x p_{v_j}^x}$  as arcs of the partial orientation. Additionally, if  $\{v_i, v_j\}$  is a representative pair, then let  $\overrightarrow{s_{v_j}^{x-1} e_{v_i, v_j}}$ ,  $\overrightarrow{e_{v_i, v_j} p_{v_i}^{x-1}}$ ,  $\overrightarrow{s_{v_i}^{x-1} e_{v_i, v_j}}$ , and  $\overrightarrow{e_{v_i, v_j} p_{v_j}^{x-1}}$  be arcs. On the other hand, if  $k$  is odd, then let  $\overrightarrow{s_{v_i}^x z}$  and  $\overrightarrow{z p_{v_i}^x}$  be arcs. Finally, if  $\{v_i, v_j\}$  is representative, then let  $\overrightarrow{s_{v_i}^x p_{v_j}^x}$  and  $\overrightarrow{s_{v_j}^x p_{v_i}^x}$  be arcs. Orient all the remaining edges arbitrarily.

Note that, under the partial orientation given above, any vertex from a gadget can directly access the upper or lower level of the same gadget. Besides, any such non-root vertex can also reach or be reached by any other vertex from another gadget by means of a dipath with length at most  $k$ . As in the case  $k = 2$ , such a path typically goes up across a first gadget, then exits the first gadget to enter the second one (either directly from the  $x^{th}$  level or via  $z$ ), and finally goes down across the second gadget. Because the gadgets have  $x = \lfloor \frac{k}{2} \rfloor$  levels, the length of such a path does not exceed  $k$ . Finally observe that if  $\{u, v\}$  is representative, then there is no path with length at most  $k$  joining  $u$  and  $v$  passing across the

gadgets. On the contrary, if  $\{u, v\}$  is non-representative, then there is a path with length exactly  $k$  joining  $u$  and  $v$ . This path necessarily passes through the shortcut between  $G_u$  and  $G_v$ .

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