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The complexity of deciding whether a graph admits an orientation with fixed weak diameter

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Abstract. An orientation of an undirected graph G has weak diameter k if, for every pair $\{u, v\}$ of vertices of G , there is a directed path with length at most k joining u and v in either direction. In this work, we show that deciding whether a graph admits an orientation with weak diameter k is NP-complete whenever $k \geq 2$.

1 Introduction

Let G be a simple undirected graph with vertex (resp. edge) set $V(G)$ (resp. $E(G)$). By *orienting* every edge uv of G , either from u to v or from v to u , one obtains an *orientation* \vec{G} of G . This oriented graph \vec{G} has the same vertex set as G , i.e. $V(\vec{G}) = V(G)$, and, for every edge $uv \in E(G)$, we have either $\vec{uv} \in E(\vec{G})$ or $\vec{vu} \in E(\vec{G})$ depending on the orientation assigned to uv .

The *distance* $dist(G, u, v)$ from u to v in G is the minimal length of a path joining u and v . We refer to the maximum distance between two vertices of G as its *diameter*, and denote it $diam(G)$. These definitions can be naturally adapted to the context of oriented graphs. A *k-dipath* $\vec{v_1 v_2 \dots v_{k+1}}$ of \vec{G} is a sequence of $k + 1$ distinct vertices such that $\vec{v_i v_{i+1}} \in E(\vec{G})$ for every $i \in \{1, \dots, k\}$. The *directed distance* $dist(\vec{G}, u, v)$ from u to v in \vec{G} is the minimal length of a dipath starting from u and ending at v . Note that, contrary to the undirected case, we may have $dist(\vec{G}, u, v) \neq dist(\vec{G}, v, u)$. Therefore, two definitions of the oriented diameter can be adopted.

On the one hand, one can consider that two vertices u and v of \vec{G} are close when one of $dist(\vec{G}, u, v)$ and $dist(\vec{G}, v, u)$ is small. In other words, we consider that two vertices are close if there is a dipath with small length joining them in \vec{G} regardless of its direction. Let $dist_w(\vec{G}, u, v)$ be $\min\{dist(\vec{G}, u, v), dist(\vec{G}, v, u)\}$. The *weak diameter* of \vec{G} , denoted $diam_w(\vec{G})$, is the maximum $dist_w(\vec{G}, u, v)$ taken over all pairs $\{u, v\}$ of vertices of \vec{G} .

On the other hand, note that the existence of a dipath with small length from u to v in \vec{G} does not guarantee that there is a short way from v to u . For this reason, the weak diameter of \vec{G} may not be representative of its oriented distances. Let $dist_s(\vec{G}, u, v) = \max\{dist(\vec{G}, u, v), dist(\vec{G}, v, u)\}$. The *strong diameter* of \vec{G} , denoted $diam_s(\vec{G})$, is the maximum $dist_s(\vec{G}, u, v)$ regarding all possible pairs $\{u, v\}$ of vertices of \vec{G} .

An orientation of G is k -weak (resp. k -strong) if it has weak (resp. strong) diameter at most k . We only deal with k -weak orientations in this paper, mainly with the case $k = 2$ which is related to some graph colouring notions as explained below. A k -colouring of G is a partition of $V(G)$ into k parts such that two vertices of a same colour class are not adjacent. The least number of colours used by a colouring of G is referred to as the *chromatic number* of G , denoted $\chi(G)$. Clearly, we have $\chi(G) = |V(G)|$ if and only if G is a complete graph.

Now consider similar notions but for oriented graphs. An *oriented k -colouring* of \vec{G} is a partition of $V(\vec{G})$ into k parts such that two arcs between two colour classes have the same direction. Note that this property implies that any two vertices from a same colour class cannot be adjacent. As usually, the least number of colours used by an oriented colouring of \vec{G} is defined as the *oriented chromatic number* of \vec{G} , denoted $\chi_o(\vec{G})$. Contrary to the undirected case, an oriented graph whose oriented chromatic number is exactly its order is not necessarily a tournament. As an illustration of that claim, remark that $\chi_o(\vec{C}_5) = 5$, where \vec{C}_5 is the circuit on 5 vertices.

The observation above led to the introduction of *oriented cliques* (or *o-clique* for short), which are the analogues of cliques regarding oriented colouring. In other words, o-cliques are those oriented graphs whose oriented chromatic number is exactly their order. O-cliques have been mainly studied regarding the extremal theory point of view. The interested reader may refer to [1, ?, ?, ?, ?] for interesting results regarding the order or the size of o-cliques.

By now extending the notion of oriented chromatic number to undirected graphs, finding 2-weak orientations of graphs becomes of some interest. If G is an undirected graph, then $\chi_o(G)$ is defined as

$$\chi_o(G) = \max\{\chi_o(\vec{G}), \vec{G} \text{ is an orientation of } G\}.$$

Hence, we have $\chi_o(G) = |V(G)|$ if and only if G admits an orientation which is an o-clique. An important result due to Klostermeyer and MacGillivray states that an oriented graph is an o-clique if and only if it has weak diameter at most 2 [4]. Therefore, we get that $\chi_o(G) = |V(G)|$ if and only if G admits a 2-weak orientation.

From the algorithmic point of view, the complexity of deciding whether an undirected graph admits a 2-weak orientation was still unknown. Note that the analogous question for 2-strong orientations was settled down by Chvátal and Thomassen, who showed this problem to be NP-complete [2]. We here settle this question by studying the following decision problem.

ORIENTATION WITH WEAK DIAMETER k - k -OWD

Instance: A graph G .

Question: Does G admit a k -weak orientation?

Clearly, a graph admits a 1-weak orientation if and only if it is a complete graph. Therefore, 1-OWD is in P. In this work, we show that k -OWD is NP-complete for every $k \geq 2$. For this purpose, we first give the proof for the case $k = 2$ in Section 2. We then explain how to generalize this result in Section 3.

2 Main result

We prove the following complexity result.

Theorem 1. *2-OWD is NP-complete.*

Proof. k -OWD is in NP for every k since one can, given an orientation \vec{G} of G , check whether $\text{diam}_w(\vec{G}) \leq k$. For this purpose, one just has to check, for every possible pair $\{u, v\}$ of distinct vertices of G , whether either u and v are adjacent, or there exist $k' \leq k - 1$ vertices $w_1, \dots, w_{k'}$ such that $\overrightarrow{uw_1 \dots w_{k'} v}$ or $\overrightarrow{vw_1 \dots w_{k'} u}$ is a dipath of \vec{G} . This witness algorithm runs in polynomial time with respect to the order of G .

We now show that 2-OWD is NP-hard by reduction from the following NP-complete problem.

NOT-ALL-EQUAL 3-SAT - NAE-3SAT

Instance: A 3CNF formula F over variables $\{x_1, \dots, x_n\}$ and clauses $\{C_1, \dots, C_m\}$.

Question: Is F *nae-satisfiable*, i.e. is there a truth assignment of the variables of F such that no clause of F has its three literals having the same truth value?

The interesting property of NAE-3SAT is that it remains NP-complete even if none of its clauses contains a negated variable (MONOTONE NAE-3SAT). Note further that if a clause of F is of the form $(x_i \vee x_i \vee x_i)$, then F cannot be nae-satisfied. It is thus understood that F meets these two properties throughout this proof. From F , we produce a graph G_F such that F is nae-satisfiable if and only if G_F admits a 2-weak orientation \vec{G}_F . This reduction is achieved in polynomial time regarding the size of F .

For every $i \in \{1, \dots, n\}$ we denote by n_i the number of distinct clauses that contain the variable x_i in F , and by c_j the number of distinct variables in the clause C_j for every $j \in \{1, \dots, m\}$. Clearly, we have $n_i \geq 1$ and $c_j \in \{2, 3\}$ for every such integers.

We first describe the *core* G_F^c of G_F , i.e. the subgraph of G_F from which the equivalence with F follows. The subgraph G_F^c does not have diameter 2, but G_F will be augmented later so that it has diameter 2, and this without altering the equivalence. With each variable x_i of F , we associate $n_i + 2$ vertices u_i, u'_i , and $v_{i,j_1}, \dots, v_{i,j_{n_i}}$ in G_F^c , where j_1, \dots, j_{n_i} are the distinct indices of the clauses of F that contain x_i . These vertices are joined in G_F^c in such a way that they form a star whose central vertex is u'_i . Now, with every clause C_j in F , we associate a vertex w_j in G_F^c . In the case where $c_j = 3$, we also add a vertex w'_j to G_F^c . Finally, whenever a variable x_i belongs to a clause C_j in F , we add an edge linking $v_{i,j}$ and w_j in G_F^c , and, if $c_j = 3$, an edge linking $v_{i,j}$ and w'_j .

This construction is depicted in Figure 1 (a). A pair $\{u, v\}$ of distinct vertices of G_F^c is *representative* if it matches one of the following forms.

1. $\{u, v\} = \{u_i, v_{i,j}\}$ where $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$.
2. $\{u, v\} = \{u'_i, w_j\}$ where $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, and $x_i \in C_j$.
3. $\{u, v\} = \{v_{i_1,j}, v_{i_2,j}\}$ where $i_1 \neq i_2$, $j \in \{1, \dots, m\}$, and $x_{i_1}, x_{i_2} \in C_j$.

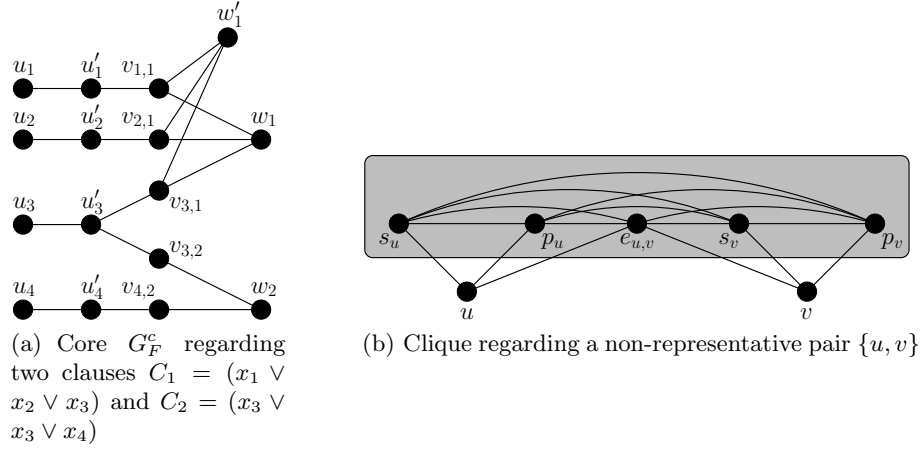


Fig. 1. Subgraphs obtained while constructing the reduced graph G_F

An orientation of G_F^c is *good* if two vertices forming a representative pair are linked by a 2-dipath in either direction. Note that, in this definition, there is no requirement on the oriented distance between two vertices which are at distance at least 3. A representative pair is a pair of vertices which are not adjacent in G_F , and for which there are only a few paths with length at most 2 between them. All of these paths belong to G_F^c so that the existence of a 2-weak orientation of G_F depends on the existence of a good orientation of G_F^c .

We claim that we have an equivalence between nae-satisfying F and finding a good orientation of G_F^c . This relies on the following three observations.

Claim 1. *Suppose x_i belongs to the clauses $C_{j_1}, \dots, C_{j_{n_i}}$ of F . Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c , $\overrightarrow{u_i u'_i v_{i,j} w_j}$ is a 3-dipath for every $j \in \{j_1, \dots, j_{n_i}\}$, or $\overrightarrow{w_j v_{i,j} u'_i u_i}$ is a 3-dipath for every $j \in \{j_1, \dots, j_{n_i}\}$.*

Proof. Suppose $\overrightarrow{u_i u'_i} \in E(\overrightarrow{G_F^c})$. Because u_i is at distance 2 from the $v_{i,j}$'s in G_F^c and u'_i is the only neighbour of u_i , we have to add $\overrightarrow{u'_i v_{i,j_1}}, \dots, \overrightarrow{u'_i v_{i,j_{n_i}}}$ to $E(\overrightarrow{G_F^c})$ so that there are 2-dipaths joining u_i and the $v_{i,j}$'s in $\overrightarrow{G_F^c}$. Similarly, because $v_{i,j}$ is the only vertex joining u'_i and w_j in G_F^c , and $\overrightarrow{u'_i v_{i,j}}$ belongs to $E(\overrightarrow{G_F^c})$ for every $j \in \{j_1, \dots, j_{n_i}\}$, we have to add $\overrightarrow{v_{i,j_1} w_{j_1}}, \dots, \overrightarrow{v_{i,j_{n_i}} w_{j_{n_i}}}$ to $E(\overrightarrow{G_F^c})$ so that there are 2-dipaths joining u'_i and the w_j 's. The claim follows similarly from the assumption $\overrightarrow{u'_i u_i} \in E(\overrightarrow{G_F^c})$.

Claim 2. *Suppose C_j has only two distinct variables x_{i_1} and x_{i_2} in F . Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c , either $\overrightarrow{v_{i_1,j} w_j v_{i_2,j}}$ or $\overrightarrow{v_{i_2,j} w_j v_{i_1,j}}$ is a 2-dipath.*

Proof. The claim follows from the fact that $v_{i_1,j} w_j v_{i_2,j}$ is the only path with length at most 2 joining $v_{i_1,j}$ and $v_{i_2,j}$ in G_F^c .

Claim 3. Suppose C_j has three distinct variables x_{i_1} , x_{i_2} and x_{i_3} in F . Then, in any good orientation $\overrightarrow{G_F^c}$ of G_F^c , the three edges $v_{i_1,j}w_j$, $v_{i_2,j}w_j$ and $v_{i_3,j}w_j$ cannot be all directed from or towards w_j .

Proof. Note that there are only two paths with length at most 2 joining any two of $v_{i_1,j}$, $v_{i_2,j}$, and $v_{i_3,j}$. These pass through w_j and w'_j , respectively. Note thus that if one of the two situations of the claim occurs, then there is no 2-dipath of $\overrightarrow{G_F^c}$ joining two of these vertices which passes by w_j . So there must be three 2-dipaths joining these vertices passing by w'_j , but this is impossible.

Remark that if two of the edges $v_{i_1,j}w_j$, $v_{i_2,j}w_j$ and $v_{i_3,j}w_j$ have the same direction while the third one is oriented in the opposite direction, e.g. suppose that $\overrightarrow{v_{i_1,j}w_j}$, $\overrightarrow{v_{i_2,j}w_j}$ and $\overrightarrow{w_jv_{i_3,j}}$ are edges of $\overrightarrow{G_F^c}$, then we can obtain three 2-dipaths joining any two of $v_{i_1,j}$, $v_{i_2,j}$, and $v_{i_3,j}$. So far, note that there are two 2-dipaths starting from $v_{i_1,j}$ and $v_{i_2,j}$, respectively, and ending at $v_{i_3,j}$. The last 2-dipath starting from $v_{i_1,j}$ and ending at $v_{i_2,j}$ can be obtained by orienting the edges incident with w'_j , e.g. by adding $\overrightarrow{v_{i_1,j}w'_j}$ and $\overrightarrow{w'_jv_{i_2,j}}$ to $E(\overrightarrow{G_F^c})$.

According to Claims 1, 2 and 3, we have an equivalence between nae-satisfying F and finding a good orientation of G_F^c . Indeed, assume that adding $\overrightarrow{u_iu'_i}$ (resp. $\overrightarrow{u'_iu_i}$) to $E(\overrightarrow{G_F^c})$ simulates an assignment of the variable x_i to true (resp. false). Claim 1 reflects the fact that, in a truth assignment of the variables of F , the variable x_i has the same truth value in every clause it appears in. Claims 2 and 3 depict the fact that all variables of a clause cannot have the same truth value. Thus, from a satisfying truth assignment of the variables of F we can deduce a good orientation of G_F^c , and vice-versa.

We now augment G_F with additional vertices so that there is a path with length 2 joining every two non-adjacent vertices of G_F^c that do not form a representative pair. This is done in such a way that there is an orientation of the edges of $E(G_F) - E(G_F^c)$ so that every two vertices of G_F that do not form a representative pair are joined by a 2-dipath. In this way, the existence of a 2-weak orientation of G_F only relies on the existence of a good orientation of G_F^c .

First, for every vertex v of G_F^c , add two new vertices s_v and p_v , and the edges vs_v and vp_v . Then, for every non-representative pair $\{u, v\}$ of G_F^c , add a new vertex $e_{u,v}$, and the edges $ue_{u,v}$ and $ve_{u,v}$. Finally, turn the subgraph of G_F induced by $\bigcup_{v \in V(G_F^c)} \{s_v, p_v\} \cup \bigcup_{\{u,v\} \in NR} \{e_{u,v}\}$ into a clique, where NR denotes the set of non-representative pairs of G_F^c . This construction is depicted in Figure 1 (b).

Note that, thanks to our modifications, there is an orientation of the edges of G_F such that every two vertices u and v of G_F that do not form a representative pair of G_F^c are joined by a 2-dipath. This may be obtained for instance as follows. Define an arbitrary ordering $\sigma = (v_1, \dots, v_{|V(G_F^c)|})$ of the vertices of G_F^c . Now for every vertex v_i of G_F^c , let $\overrightarrow{v_i s_{v_i}}$ and $\overrightarrow{p_{v_i} v_i}$ be arcs. Moreover, for every vertex v_j such that $i < j$ (if any), let $\overrightarrow{s_{v_i} s_{v_j}}$, $\overrightarrow{s_{v_i} p_{v_j}}$, $\overrightarrow{p_{v_i} s_{v_j}}$ and $\overrightarrow{p_{v_i} p_{v_j}}$ be arcs. If the pair

$\{v_i, v_j\}$ is non-representative, then let also $\overrightarrow{v_i e_{v_i, v_j}}$ and $\overrightarrow{e_{v_i, v_j} v_j}$ be arcs of the partial orientation. Finally, for every other vertex v_k such that $i < j, k$ (resp. $j, k < i$) and $\{v_j, v_k\}$ is a non-representative pair, let $\overrightarrow{s_{v_i} e_{v_j, v_k}}$ and $\overrightarrow{p_{v_i} e_{v_j, v_k}}$ (resp. $\overrightarrow{e_{v_j, v_k} s_{v_i}}$ and $\overrightarrow{e_{v_j, v_k} p_{v_i}}$) be arcs. Any other edge can be oriented arbitrarily.

It is easy to check that, under the orientation given above, there is a dipath with length at most 2 joining every two non-adjacent vertices u and v that do not form a representative pair of G_F . First suppose that $u = v_i$ and $v = v_j$ are such vertices. Then, note that $\overrightarrow{u e_{u, v}}$ belongs to the orientation. Now, if $u = v_i \in V(G_F^c)$ and $v \notin V(G_F^c)$, then v is a vertex from the clique and is adjacent to some vertex $v_j \neq v_i$ of G_F^c . Then, note that either $\overrightarrow{u s_u v}$ or $\overrightarrow{v p_u u}$ is a 2-dipath of G_F assuming that $i < j$ or $i > j$, respectively.

Finally remark that, for every representative pair $\{u, v\}$ of G_F , there is no new path with length 2 joining u and v passing through new vertices resulting from the modification. Indeed, such a path would have to start from u , enter the clique, cross the clique, and end at v . Thus, such a path has length at least 3. It follows that the distances between vertices from representative pairs of G_F are not altered by the modifications, and thus that the equivalence between nae-satisfying F and finding a good orientation of G_F^c is preserved. Therefore, F is nae-satisfiable if and only if G_F admits a 2-weak orientation.

3 Generalisation

Although our motivations for restricting our intention on 2-OWD are supported by a colouring problem, one could naturally ask about the complexity of any problem k -OWD with $k \geq 3$.

In what follows, we explain how to generalize the reduction we gave for the case $k = 2$ to any fixed $k \geq 3$. From a 3CNF formula F , we first produce the core G_F^c of G_F and define an orientation of G_F^c to be good if and only if some (representative) pairs of vertices of G_F^c are linked by a dipath with length exactly k in either direction. The structure of G_F^c is representative of the structure of F so that a good orientation of G_F^c implies a nae-assignment of F , and vice-versa. We then augment G_F with additional gadgets so that there is an orientation of the new edges under which every two vertices from a non-representative pair of G_F are joined by a dipath with length at most k in either direction. This is done in such a way that the existence of a good orientation of G_F^c does not depend of the edges resulting from the modifications.

This augmentation consists mainly in *rooting* a gadget G_v in each vertex v of G_F^c . All gadgets of G_F are connected in a specific way. In the case where $\{u, v\}$ is not a representative pair, we add a *shortcut* between G_u and G_v , i.e. an alternative shorter path for joining two vertices of G_u and G_v . Typically, every vertex u' of G_u is at distance at most k from any vertex v' of G_v , unless $u' = u$, $v' = v$ and $\{u, v\}$ is a representative pair. In the proof of Theorem 1, i.e. for the case $k = 2$, the gadget G_v is the triangle induced by v , s_v , and p_v , and, assuming $\{u, v\}$ is not representative, the shortcut between G_u and G_v is the vertex $e_{u, v}$.

Typically, the regular path between u and v , i.e. the one passing through the clique, has length 3, while the path passing by the shortcut $e_{u,v}$ has length 2.

Let $k \geq 3$ be fixed. The core G_F^c has the following vertices. Similarly as in the case $k = 2$, let $u_i, u'_i, v_{i,j}$ be vertices for every $i \in \{1, \dots, n\}$ and $j \in \{i_1, \dots, i_{n_i}\}$, where i_1, \dots, i_{n_i} are the distinct indices of the clauses of F that contain the variable x_i . The remaining vertices of G_F^c depend on the parity of k . If k is even, then, for every $j \in \{1, \dots, m\}$, add the vertex w_j , and additionally w'_j when $c_j = 3$, as in the case $k = 2$. Otherwise, if k is odd, then, for every $j \in \{1, \dots, m\}$, proceed as follows. First, if $c_j = 2$, then just add an edge $a_j b_j$ to G_F^c . Otherwise, if $c_j = 3$, then add two cycles $a_j w_j b_j a_j$ and $a'_j w'_j b'_j a'_j$ with length 3 to G_F^c .

By joining two vertices u and v by a path, we mean that we identify the endvertices of a new path with u and v , respectively. We now link all the vertices of G_F^c by means of several vertex-disjoint paths. First, join every pair $\{u_i, u'_i\}$ by a path with length $\lfloor \frac{k}{2} \rfloor$. Then, join every pair $\{u'_i, v_{i,j}\}$ by a path with length $\lceil \frac{k}{2} \rceil$. Now, for every $j \in \{1, \dots, m\}$, add the following paths to G_F^c .

- If $c_j = 2$, and x_{i_1} and x_{i_2} are the distinct variables of C_j , then join each of $\{v_{i_1,j}, w_j\}$ and $\{v_{i_2,j}, w_j\}$ by a path with length $\frac{k}{2}$ if k is even, or join each of $\{v_{i_1,j}, a_j\}$ and $\{v_{i_2,j}, b_j\}$ by a path with length $\lfloor \frac{k}{2} \rfloor$ otherwise.
- If $c_j = 3$, and x_{i_1}, x_{i_2} and x_{i_3} are the variables of C_j , then, if k is even, join every vertex among $\{v_{i_1,j}, v_{i_2,j}, v_{i_3,j}\}$ and each vertex in $\{w_j, w'_j\}$ by means of a path with length $\frac{k}{2}$. Otherwise, if k is odd, then join $v_{i_1,j}$ and both a_j and a'_j , $v_{i_2,j}$ and both w_j and w'_j , and $v_{i_3,j}$ and both b_j and b'_j by a path with length $\lfloor \frac{k}{2} \rfloor$ otherwise.

A pair of vertices of G_F^c is representative whenever it is of the form $\{u_i, v_{i,j}\}$, $\{u'_i, s\}$, where s is of the form a_j, b_j , or w_j such that there is a path with length $\lfloor \frac{k}{2} \rfloor$ between a vertex $v_{i,j}$ and s , or $\{v_{i_1,j}, v_{i_2,j}\}$. An orientation of G_F^c is good if every two vertices of G_F^c composing a representative pair are joined by a k -dipath in either direction. Note that, by construction, every two such vertices are at distance exactly k in G_F^c , and that the possibilities for joining them by a k -dipath are quite limited. For these reasons, Claims 1, 2 and 3 introduced in the proof of Theorem 1 can be derived for the general case. We get that nae-satisfying F is equivalent to finding a good orientation of G_F^c .

We finally augment G_F so that every two of its vertices are at distance at most k . This is done by adding a gadget G_v to G_F for every vertex v of G_F^c as explained above. Set $x = \lfloor \frac{k}{2} \rfloor$. For every $i \in \{1, \dots, x\}$, add two new vertices s_v^i and p_v^i to G_v . These two vertices form the i^{th} level of G_v , and are said to be i -vertices. Next, for every $i \in \{1, \dots, x-1\}$, add all possible edges between the i - and $(i+1)$ -vertices of G_v so that two consecutive levels of G_v form a clique on 4 vertices. Finally, add an edge between v and every 1-vertex of G_v .

We finish the construction of G_F by adding some connection between the gadgets. We distinguish two cases depending on the parity of k .

- If k is even, then we turn the subgraph induced by all x -vertices of G_F into a clique. Next, for every non-representative pair $\{u, v\}$ of G_F^c , add a shortcut

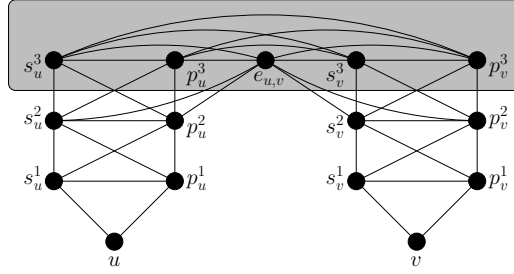


Fig. 2. The gadgets G_u and G_v obtained for a non-representative pair $\{u, v\}$ and $k = 6$

- vertex $e_{u,v}$ to the clique constructed just before. Finally, add every edge between $e_{u,v}$ and the vertices from the $(x-1)^{th}$ levels of G_u and G_v .
- Otherwise, if k is odd, then add a new vertex z to G_F , and add all possible edges between z and x -vertices. For every non-representative pair $\{u, v\}$ of G_F^c , also add the shortcut edges $p_u^x s_v^x$ and $s_u^x p_v^x$ to G_F .

This construction is depicted in Figure 2. Note that no new path with length at most k between two vertices composing a representative pair of G_F arose from the modifications. Therefore, the equivalence between nae-satisfying F and finding a good orientation of G_F^c is still correct. We claim, as in the case $k = 2$, that there is an orientation of the edges we just added so that every non-representative pair of vertices of G_F is joined by a k -dipath.

To see that claim, define an arbitrary ordering $\sigma = (v_1, \dots, v_{|V(G_F^c)|})$ over all vertices of G_F^c , and consider the following partial orientation. First, for every vertex v of G_F^c , let $\overrightarrow{v s_v^1}$ and $\overrightarrow{p_v^1 v}$ be arcs. Then, for every level $i \in \{1, \dots, x\}$ of G_v , let $\overrightarrow{s_v^i p_v^i}$ be an arc. Next, for every $i \in \{1, \dots, x-1\}$, add the arcs $\overrightarrow{s_v^i s_v^{i+1}}$, $\overrightarrow{p_v^{i+1} p_v^i}$, $\overrightarrow{p_v^i s_v^{i+1}}$ and $\overrightarrow{p_v^{i+1} s_v^i}$ to the partial orientation. Now, on the one hand, if k is even, then, for every shortcut vertex e of G_F , add the arcs $\overrightarrow{s_v^x e}$ and $\overrightarrow{e p_v^x}$. Next, for every $i < j$ consider $\overrightarrow{s_{v_i}^x s_{v_j}^x}$, $\overrightarrow{s_{v_i}^x p_{v_j}^x}$, $\overrightarrow{p_{v_i}^x s_{v_j}^x}$ and $\overrightarrow{p_{v_i}^x p_{v_j}^x}$ as arcs of the partial orientation. Additionally, if $\{v_i, v_j\}$ is a representative pair, then let $\overrightarrow{s_{v_j}^{x-1} e_{v_i, v_j}}$, $\overrightarrow{e_{v_i, v_j} p_{v_i}^{x-1}}$, $\overrightarrow{s_{v_i}^{x-1} e_{v_i, v_j}}$, and $\overrightarrow{e_{v_i, v_j} p_{v_j}^{x-1}}$ be arcs. On the other hand, if k is odd, then let $\overrightarrow{s_{v_i}^x z}$ and $\overrightarrow{z p_{v_i}^x}$ be arcs. Finally, if $\{v_i, v_j\}$ is representative, then let $\overrightarrow{s_{v_i}^x p_{v_j}^x}$ and $\overrightarrow{s_{v_j}^x p_{v_i}^x}$ be arcs. Orient all the remaining edges arbitrarily.

Note that, under the partial orientation given above, any vertex from a gadget can directly access the upper or lower level of the same gadget. Besides, any such non-root vertex can also reach or be reached by any other vertex from another gadget by means of a dipath with length at most k . As in the case $k = 2$, such a path typically goes up across a first gadget, then exits the first gadget to enter the second one (either directly from the x^{th} level or via z), and finally goes down across the second gadget. Because the gadgets have $x = \lfloor \frac{k}{2} \rfloor$ levels, the length of such a path does not exceed k . Finally observe that if $\{u, v\}$ is representative, then there is no path with length at most k joining u and v passing across the

gadgets. On the contrary, if $\{u, v\}$ is non-representative, then there is a path with length exactly k joining u and v . This path necessarily passes through the shortcut between G_u and G_v .

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