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Bose condensates in interaction with excitations - a kinetic model.

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Abstract. This paper deals with mathematical questions for Bose gases below the temperature T_{BEC} where Bose-Einstein condensation sets in. The model considered is of two-component type, consisting of a kinetic equation for the distribution function of a gas of (quasi-)particles interacting with a Bose condensate, which is described by a Gross-Pitaevskii equation. Existence results and moment estimates are proved in the space-homogeneous, isotropic case.

1 Preliminaries.

Starting from a detailed effective Hamiltonian for a Bose fluid, and motivated by considerations about the relevant physics, simplified models for classical fields can be derived in well-defined (formal) limits (cf [BCEP, S1]), or obtained by physics arguments for more direct approximations of the original potentials and operators. There for sufficiently low temperatures only interactions between thermally excited (quasi-)particles and condensate are of practical importance, as discussed in the papers [K, KK, HM, E, KD, ZNG, IT, ITG] and their references.
The present paper considers one such situation involving transfer of atoms between the two components, so that in particular no conservation laws for an individual component should be expected in the transient evolution. The model is based on the Beliaev-Popov approximation of the Bogoliubov Green-function description which ignores off-diagonal correlations, and the Thomas-Fermi approximation which neglects a quantum pressure term. The simplified two-component model obtained, consists of a kinetic equation for the distribution function of a gas of (quasi-)particles interacting with a Bose condensate, which in turn is described by a Gross-Pitaevskii equation (cf [PS]). In the local rest frame, the kinetic equation is ([ITG])

$$\frac{\partial f}{\partial t} + \nabla_p(E_p + v_s p) \cdot \nabla_x f - \nabla_x(E_p + v_s p) \cdot \nabla_p f = C(f, n_c). \tag{1.1}$$

Here v_s is the superfluid velocity, E_p the (Bogoliubov) excitation energy, and the collision term becomes

$$C(f, n_c)(p) = n_c \int |A|^2 \delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1) \tag{1.2}$$
$$-\delta(p - p_2) - \delta(p - p_3)]((1 + f_1)f_2 f_3 - f_1(1 + f_2)(1 + f_3))dp_1 dp_2 dp_3.$$

The transition probability kernel $|A|^2$ is given by the scattering amplitude

$$A := (u_3 - v_3)(u_1 u_2 + v_1 v_2) + (u_2 - v_2)(u_1 u_3 + v_1 v_3) - (u_1 - v_1)(u_2 v_3 + v_2 u_3).$$

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Here the Bose coherence factors u and v are

$$u_p^2 = \frac{\tilde{\epsilon}_p + E_p}{2E_p}, \quad u_p^2 - v_p^2 = 1,$$

with $\tilde{\epsilon}_p = \frac{p^2}{2m} + gn_c$, n_c the non-equilibrium density of the atoms in the condensate, m the atomic mass, and g a scattering length defined later.
The collision operator $C(f, n_c)$ can be formally obtained (cf [ST], [EMV], [N]) from the Nordheim-Uehling-Uhlenbeck collision operator

$$\tilde{C}_{NUU}(f)(p) = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} B\delta(p + p_* = p' + p'_*)\delta(E(p) + E(p_*) = E(p') + E(p'_*)) \\ \Big(f'f'_*(1+f)(1+f_*) - ff_*(1+f')(1+f'_*)\Big)dp_*dp'dp'_*$$

with

$$f_* = f(p_*), \quad f' = f(p'), \quad f'_* = f(p'_*).$$

Assuming that a condensate appears below the Bose-Einstein condensation temperature T_{BEC}, which splits the quantum gas distribution function into a condensate part $n_c\delta_{p=0}$ and an L^1-density part $f(t, x, p)$, we obtain

$$\tilde{C}_{NUU}(f + n_c\delta_{p=0}) = \tilde{C}_{NUU}(f) + C(f, n_c) + n_c^2\tilde{A} + n_c^3\tilde{B} + n_c\delta_{p=0}\tilde{D},$$

where a simple formal computation shows that $\tilde{A} = \tilde{B} = 0$. At the low temperatures considered, the number of excited (quasi-)particles is small, and the $\tilde{C}_{NUU}$ collision term can be neglected relative to the collision operator C.

The usual Gross-Pitaevskii (GP) equation for the wave function ψ (the order parameter) associated with a Bose condensate is

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2m}\Delta_x\psi + (U_{ext} + g|\psi|^2)\psi,$$

i.e. a Schrödinger equation complemented by a non-linear term accounting for two-body interactions. U_{ext} is an external potential, and with a the s-scattering length of the interaction potential and $g = \frac{4\pi a}{m}$. In the present context the GP equation is further generalized by letting the condensate move in a self-consistent Hartree-Fock mean field $2g\tilde{n} = 2g\int f(p)dp$ produced by the thermally excited atoms, together with a dissipative coupling term associated with the collisions. It is useful to split the equation for $\psi = \sqrt{n_c}e^{i\theta}$ into phase and amplitude variables (polar representation or the Madelung transform, cf [ZNG]), leading to

$$\frac{\partial n_c}{\partial t} + \nabla_x \cdot (n_c v_s) = -\int C(f, n_c)dp \tag{1.3}$$
$$\frac{\partial\theta}{\partial t} = -\mu_c - \frac{mv_s^2}{2},$$

with μ_c a local condensate chemical potential.

2 A space-homogeneous isotropic case; the mathematical setting.

This paper is the first part of a study of the Cauchy problem for the two component model (1.1), (1.3) of a kinetic gas of quasi-particles interacting with a GP condensate. The focus is on the space-homogeneous isotropic case and the superfluid rest frame (condensate velocity $v_s = \nabla_x \theta = 0$), i.e. the equations

$$\frac{\partial f}{\partial t} = C(f, n_c), \tag{2.1}$$

$$\frac{dn_c}{dt} = -\int C(f, n_c) dp, \tag{2.2}$$

with initial values

$$f(p, 0) = f_i(\mid p \mid), \quad n_c(0) = n_{ci}. \tag{2.3}$$

Here $f(p,t)$ is the density of the quasi-particles, $n_c(t)$ the mass of the condensate, and the collision operator C is given by (1.2).

The papers [E, ITG] consider the low temperature situation where the temperature is smaller than $0.4T_{BEC}$, with all $|p_i| << p_0$, i.e. where physically all quasi-particle momenta are much smaller than the characteristic momentum $p_0 = \sqrt{2mgn_c}$ for the crossover between the linear and quadratic parts of the Bogoliubov excitation energy of the quasi-particles;

$$E(p) := \sqrt{\frac{p^4}{4m^2} + \frac{gn_c}{m}p^2} \approx c|p|(1 + \frac{p^2}{8gmn_c}) = c|p|(1 + \frac{p^2}{4p_0^2}) \tag{2.4}$$

with $c := \sqrt{\frac{gn_c}{m}}$ the speed of Bogoliubov sound. Setting $m = \frac{1}{\sqrt{2}}$ gives $p_0 = c$.
The right hand side of (2.4) is usually taken as the value of $E(p)$ in applications with $|p| << p_0$. The Bose coherence factors can then be taken as

$$u_p = \sqrt{\frac{gn_c}{2E(p)}} + \frac{1}{2}\sqrt{\frac{E(p)}{2gn_c}}, \quad v_p = \sqrt{\frac{gn_c}{2E(p)}} - \frac{1}{2}\sqrt{\frac{E(p)}{2gn_c}}, \quad u_p^2 - v_p^2 = 1,$$

which gives

$$A = \frac{1}{2^{\frac{5}{2}}} \frac{\sqrt{E(p_*)E(p')E(p'_*)}}{(gn_c)^{\frac{3}{2}}} + \sqrt{\frac{gn_c}{2}}(\sqrt{\frac{E(p'_*)}{E(p_*)E(p')}} + \sqrt{\frac{E(p')}{E(p_*)E(p'_*)}} - \sqrt{\frac{E(p_*)}{E(p')E(p'_*)}}).$$

And so recalling that $E(p_*) = E(p') + E(p'_*)$, we obtain

$$A = \frac{1}{2^{\frac{5}{2}}} \frac{\sqrt{E(p_*)E(p')E(p'_*)}}{(gn_c)^{\frac{3}{2}}}.$$

With this A, the collision operator becomes

$$\begin{aligned} C(f, n_c)(p) = \int \chi \frac{E(p_1)E(p_2)E(p_3)}{g^3 n_c^2} \delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1) \\ -\delta(p - p_2) - \delta(p - p_3)]((1 + f_1)f_2 f_3 - f_1(1 + f_2)(1 + f_3)) dp_1 dp_2 dp_3, \end{aligned} \tag{2.5}$$

where χ denotes the truncation for $|p_i| \leq \lambda$, $1 \leq i \leq 3$. The choice of the positive constant λ will be discussed below.

The opposite limit of intermediate temperatures compared to T_{BEC}, and where all momenta $|p_i| >> p_0$, is considered in [E, ZNG] with the dominant excitation of Hartree-Fock single particle type. Expanding the square root definition of E in (2.4), we may approximate E_p by $\frac{p^2}{2m} + gn_c$ leading to a collision operator of the type

$$C(f, n_c) = kn_c \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \chi\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1)$$
$$-\delta(p - p_2) - \delta(p - p_3)]((1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3))dp_1dp_2dp_3 \tag{2.6}$$

(for the 'partial local equilibrium regime' of [ZNG], see also [ITG], with only collisions between excited particles and the condensate). Here χ is the characteristic function of the set of (p, p_1, p_2, p_3) with $|p|$, $|p_1|$, $|p_2|$, $|p_3| \geq \alpha$ for a given positive constant α.

In the general case, the collision operator is

$$C(f, n_c)(p) = n_c \int |A|^2\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1)$$
$$-\delta(p - p_2) - \delta(p - p_3)]((1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3))dp_1dp_2dp_3, \tag{2.7}$$

with the excitation energy E defined by

$$E(p) = |p|\sqrt{\frac{p^2}{4m^2} + \frac{gn_c}{m}}.$$

As follows from the definitions of A and $E(p)$ above, the kernel $|A|^2$ is bounded by a multiple of

$$|\bar{A}|^2 := \Big(\frac{|p_1|}{\sqrt{n_c}} \wedge 1\Big)\Big(\frac{|p_2|}{\sqrt{n_c}} \wedge 1\Big)\Big(\frac{|p_3|}{\sqrt{n_c}} \wedge 1\Big),$$

in the physically interesting cases when asymptotically all $|p_i| << p_0$, all $|p_i| >> p_0$, or one $|p_i| << p_0$ and the others $>> p_0$. The three cases are relevant for low respectively intermediate temperatures compared to T_{BEC}, and (the third case) for the collision of low temperature phonons with high temperature excitations (atoms). The asymptotic situation of two $|p_i| << p_0$ and one $p_i >> p_0$ (with unbounded A) is excluded by the energy condition.
In the main part of this paper we shall use $|\bar{A}|^2$ as the kernel in the collision operator and prove the following result.

Theorem 2.1 *Let $n_{ci} > 0$ and $f_i(p) = f_i(|p|) \in L^1$ be given with f_i nonnegative and $f_i(p)|p|^{2+\gamma} \in L^1$ for some $\gamma > 0$. For the collision operator (2.7) with the transition probability kernel $|\bar{A}|^2$, there exists a nonnegative solution $(f, n_c) \in C^1([0, \infty); L^1_+) \times C^1([0, \infty))$ to the initial value problem (2.1-3). The condensate density n_c is locally bounded away from zero for $t > 0$. The excitation density f has energy locally bounded in time. Total mass $M_0 = n_{ci} + \int f_i(p)dp$ is conserved, and the moment $\int |p|^{2+\gamma} f dp$ is locally bounded in time.*

In the low temperature case with the collision operator taken as (2.5), if the mathematical condition corresponding to the physics requirement $|p| << p_0$ is taken as $|p| \leq p_0^2 := \lambda$, the proof of Theorem 2.1 simplifies. It holds that

Theorem 2.2 *Let $n_{ci} > 0$ and $f_i(p) = f_i(|p|) \in L^1$ be given with f_i nonnegative. There exists a nonnegative solution $(f, n_c) \in C^1([0, \infty); L^1_+) \times C^1([0, \infty))$ to the initial value problem (2.1-3) for the collision operator (2.5). The condensate density n_c is locally bounded away from zero for $t > 0$. The excitation density f has energy bounded globally in time. Total mass $M_0 = n_{ci} + \int f_i(p)dp$ is conserved.*

An existence result was obtained in [N] for (2.1-3) in the intermediate temperature case, with the collision operator (2.6) without the cut-off function χ, and considering the excitation density f in measure sense. For (2.6) with the cut-off function χ included, existence also holds in the present L^1-setting.

Theorem 2.3 *Let $n_{ci} > 0$ and $f_i(p) = f_i(|p|) \in L^1$ be given with f_i nonnegative and $f_i(p)|p|^{2+\gamma} \in L^1$ for some $\gamma > 0$. There exists a nonnegative solution $(f, n_c) \in C^1([0,\infty); L^1_+) \times C^1([0,\infty))$ to the initial value problem (2.1-3) for the collision operator (2.6). The condensate density n_c is locally bounded away from zero for $t > 0$. Total mass $M_0 = n_{ci} + \int f_i(p)dp$ is conserved together with the integral $\int \frac{p^2}{2m} f_i(p)dp + \frac{1}{2} g n_{ci} \int f_i(p)dp + gM_0(\int f_i(p)dp + \frac{1}{2} n_c)$ of energy type. The moment $\int |p|^{2+\gamma} f dp$ is locally bounded in time.*

Also the recent paper [S2] considers the spatially homogeneous and isotropic kinetic regime of weakly interacting bosons with s-wave scattering. It has a focus on post-nucleation self-similar solutions. Another recent paper, [EPV], studies linearized space homogeneous kinetic problems in settings related to but not identical to the ones discussed here, and with a focus on large time behaviour.

3 Proof of the main theorem.

The collision operator in the general case is

$$C(f, n_c)(p) = n_c \int |\bar{A}|^2 \delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1)$$
$$-\delta(p - p_2) - \delta(p - p_3)]((1 + f_1)f_2 f_3 - f_1(1 + f_2)(1 + f_3))dp_1 dp_2 dp_3, \tag{3.1}$$

with the kernel

$$|\bar{A}|^2 := \Big(\frac{|p_1|}{\sqrt{n_c}} \wedge 1\Big)\Big(\frac{|p_2|}{\sqrt{n_c}} \wedge 1\Big)\Big(\frac{|p_3|}{\sqrt{n_c}} \wedge 1\Big),$$

and the excitation energy E defined by

$$E(p, n_c) = E(p) = E_p = |p|\sqrt{\frac{p^2}{4m^2} + \frac{g n_c}{m}}.$$

The constants m and g are taken as $\frac{1}{2}$ in the rest of this section. Adding (2.2) and (2.1) integrated with respect to p, it follows that $n_c(t) + \int f(t, |p|)dp = M_0$, i.e. total mass is conserved. It holds that

$$\int \varphi(p) C(f, n_c)(p)dp = n_c \int |\bar{A}^2|(\varphi(p_1) - \varphi(p_2) - \varphi(p_3))\delta(p_1 - p_2 - p_3)$$
$$\delta(E(p_1) - E(p_2) - E(p_3))(f_2 f_3 - f_1(1 + f_2 + f_3))dp_1 dp_2 dp_3.$$

The energy (resp. the condensate density) is bounded from above (resp. from below) locally in time as follows.

Lemma 3.1 *Let the initial data (f_i, n_{ci}) satisfy $0 < n_{ci} < M_0$ and $n_{ci} + \int f_i(|p|)dp = M_0$. Then there is $T_0 > 0$ such that $n_c(t) \geq \frac{n_{ci}}{2}$ and $\int E(p, n_c) f(t, p)dp$ is bounded from above on $[0, T_0]$ for any nonnegative solution (f, n_c) to (2.1-3).*

Proof of Lemma 3.1

It follows from (2.2) and (3.1) that for any nonnegative solution (f, n_c) to (2.1-3),

$$|\frac{n_c'}{n_c}(t)| \leq \int (\frac{|p_1|}{\sqrt{n_c}} \wedge 1)(\frac{|p_2|}{\sqrt{n_c}} \wedge 1)(\frac{|p_3|}{\sqrt{n_c}} \wedge 1)\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)$$
$$(f_2 f_3 + f_1(2f_2 + 1))dp_1 dp_2 dp_3 =: X_1 + X_2 + X_3.$$

Using spherical coordinates for p_2 and p_3, with axis directed by p_2 and azimuthal angle φ_3 for p_3, setting $|p| = r$, and then performing the change of variables $\varphi_3 \to s = cos\varphi_3$,

$$X_1 \leq 2k \int_0^\infty r_2^2(\frac{r_2}{\sqrt{n_c}} \wedge 1) f(t, r_2) \int_0^{r_2} r_3^2(\frac{r_3}{\sqrt{n_c}} \wedge 1) f(t, r_3) Y_1 dr_3 dr_2,$$

where

$$Y_1 = \int_{-1}^1 \delta(F_1(s))ds,$$
$$F_1(s) := \sqrt{(r_2^2 + r_3^2 + 2r_2r_3s)^2 + n_c(r_2^2 + r_3^2 + 2r_2r_3s)} - S_1,$$
$$S_1 := \sqrt{r_2^4 + n_c r_2^2} + \sqrt{r_3^4 + n_c r_3^2}.$$

F_1 vanishes for a single value s_1 of s. Straightforward computations show that $|s_1| \leq 1$. Moreover,

$$F_1'(s) = r_2 r_3 \frac{2(r_2^2 + r_3^2 + 2r_2r_3s) + n_c}{\sqrt{(r_2^2 + r_3^2 + 2r_2r_3s)^2 + n_c(r_2^2 + r_3^2 + 2r_2r_3s)}} \geq r_2 r_3 \frac{\sqrt{(r_2^2 + r_3^2 + 2r_2r_3s) + n_c}}{\sqrt{r_2^2 + r_3^2 + 2r_2r_3s}}.$$

And so,

$$Y_1 \leq \frac{1}{r_2 r_3}.$$

Hence,

$$X_1 \leq 2k \int_0^\infty r_2(\frac{r_2}{\sqrt{n_c}} \wedge 1) f(t, r_2) \int_0^{r_2} r_3(\frac{r_3}{\sqrt{n_c}} \wedge 1) f(t, r_3) dr_3 dr_2$$
$$\leq \frac{2k}{n_c}\Big(\int f(t, p)dp\Big)^2.$$

Similarly,

$$X_2 \leq 2k \int_0^\infty r_1^2(\frac{r_1}{\sqrt{n_c}} \wedge 1) f(t, r_1) \int_0^{r_1} r_2^2(\frac{r_2}{\sqrt{n_c}} \wedge 1) f(t, r_2) Y_2 dr_1 dr_2,$$

where

$$Y_2 = \int_{-1}^1 \delta(F_2(s))ds,$$
$$F_2(s) := \sqrt{(r_1^2 + r_2^2 + 2r_1r_2s)^2 + n_c(r_1^2 + r_2^2 + 2r_1r_2s)} - S_2,$$
$$S_2 := \sqrt{r_1^4 + n_c r_1^2} - \sqrt{r_2^4 + n_c r_2^2}.$$

F_2 vanishes for a single value s_2 of s. Straightforward computations show that $|s_2| \leq 1$. Moreover,

$$F_2'(s) = r_1 r_2 \frac{2(r_1^2 + r_2^2 + 2r_1r_2s) + n_c}{\sqrt{(r_1^2 + r_2^2 + 2r_1r_2s)^2 + n_c(r_1^2 + r_2^2 + 2r_1r_2s)}} \geq r_1 r_2 \frac{\sqrt{(r_1^2 + r_2^2 + 2r_1r_2s) + n_c}}{\sqrt{r_2^1 + r_3^2 + 2r_1r_2s}}.$$

And so,

$$Y_2 \le \frac{1}{r_1 r_2},$$
$$X_2 \le \frac{2k}{n_c}\Big(\int f(t,p)dp\Big)^2.$$

Finally,

$$X_3 \le k\int r_1(\frac{r_1}{\sqrt{n_c}}\wedge 1)f(t,r_1)\int_0^{r_1} r_2(\frac{r_2}{\sqrt{n_c}}\wedge 1)dr_2dr_1$$
$$= \frac{k}{\sqrt{n_c}}\int_0^{\sqrt{n_c}} r_1^2 f(t,r_1)\int_0^{r_1}\frac{r_2^2}{\sqrt{n_c}}dr_2dr_1 + k\int_{\sqrt{n_c}}^{+\infty} r_1 f(t,r_1)\Big(\int_0^{\sqrt{n_c}}\frac{r_2^2}{\sqrt{n_c}}dr_2 + \int_{\sqrt{n_c}}^{r_1} r_2 dr_2\Big)dr_1$$
$$\le \frac{k}{n_c}\int_0^{\sqrt{n_c}} r_1^5 f(t,r_1)dr_1 + \frac{k}{2}\int_{\sqrt{n_c}}^{+\infty} r_1^3 f(t,r_1)dr_1$$
$$\le k\sqrt{n_c}\int f(t,p)dp + \frac{k}{2\sqrt{n_c}}\int p^2 f(t,p)dp \le M_0 k\sqrt{n_c} + \frac{k}{2\sqrt{n_c}}\int p^2 f(t,p)dp.$$

And so,

$$|n_c'(t)| \le k(4M_0^2 + M_0 n_c^{\frac{3}{2}} + \sqrt{n_c}\int p^2 f(t,p)dp). \tag{3.2}$$

Denote by $G(t,n) = \int E(p,n)f(t,p)dp$. Then

$$\frac{\partial G}{\partial n} = \int \frac{|p|}{2\sqrt{p^2+n}}f(t,p)dp \in [0, \frac{M_0}{2}].$$

Hence,

$$G(t,n_c(t)) \le G(t,n_{ci}) + M_0^2.$$

Moreover,

$$\frac{d}{dt}G(t,n_{ci}) = \int |p|\sqrt{p^2+n_{ci}}C(f,n_c)(t,p)dp$$
$$= n_c\int (|\frac{|p_1|}{\sqrt{n_c}}|\wedge 1)(|\frac{|p_2|}{\sqrt{n_c}}|\wedge 1)(|\frac{|p_3|}{\sqrt{n_c}}|\wedge 1)$$
$$\Big(|p_1|\sqrt{p_1^2+n_{ci}} - |p_2|\sqrt{p_2^2+n_{ci}} - |p_3|\sqrt{p_3^2+n_{ci}}\Big)$$
$$\delta(p_1-p_2-p_3)\delta(|p_1|\sqrt{p_1^2+n_c(t)} - |p_2|\sqrt{p_2^2+n_c(t)} - |p_3|\sqrt{p_3^2+n_c(t)})$$
$$(f_2f_3 - f_1(1+f_2+f_3)dp_1dp_2dp_3$$
$$= n_c\int |\bar{A}|^2\Big(|p_1|(\sqrt{p_1^2+n_{ci}} - \sqrt{p_1^2+n_c(t)}) - |p_2|(\sqrt{p_2^2+n_{ci}} - \sqrt{p_2^2+n_c(t)})$$
$$-|p_3|(\sqrt{p_3^2+n_{ci}} - \sqrt{p_3^2+n_c(t)})\Big)$$
$$\delta(p_1-p_2-p_3)\delta(|p_1|\sqrt{p_1^2+n_c(t)} - |p_2|\sqrt{p_2^2+n_c(t)} - |p_3|\sqrt{p_3^2+n_c(t)})$$
$$(f_2f_3 - f_1(1+f_2+f_3)dp_1dp_2dp_3.$$

It follows from

$$|p||\sqrt{p^2+n_{ci}} - \sqrt{p^2+n_c(t)}\,| \le |\,n_{ci} - n_c(t)\,| \le M_0, \quad p \in I\!R^3,$$

and similar computations as in the control of X_1, X_2 and X_3 above, that

$$|\frac{d}{dt}G(t,n_{ci})| \leq 2kM_0 n_c(t)\Big(\frac{4M_0^2}{n_c(t)} + 2M_0\sqrt{n_c(t)} + \frac{1}{\sqrt{n_c}}\int p^2 f(t,p)dp\Big)$$
$$\leq 2kM_0 n_c(t)\Big(\frac{4M_0^2}{n_c(t)} + 2M_0\sqrt{n_c(t)} + \frac{G(t,n_{ci})}{\sqrt{n_c}}\Big). \tag{3.3}$$

And so,

$$G(t,n_c(t)) \leq M_0^2 + G(0,n_{ci})\exp(2M_0^{\frac{3}{2}}t) + k(8M_0^3 + 2M_0^4 + 2M_0^{\frac{5}{2}})(\exp(2M_0^{\frac{3}{2}}t) - 1). \tag{3.4}$$

The lemma follows. ■

If the solution exists on $[0,T[$, then it follows from a refinement of (3.2) in the proof of Lemma 3.1 that $\inf_{[0,T[} n_c(t) > 0$. For a contradiction assume that $\liminf_{t\to T} n_c(t) = 0$, which implies $\lim_{t\to T} n_c(t) = 0$. By (3.4), the energy is uniformly bounded on $[0,T[$. By (2.2) and (3.1)

$$\frac{n_c'}{n_c}(t) = \int (\frac{|p_1|}{\sqrt{n_c}} \wedge 1)(\frac{|p_2|}{\sqrt{n_c}} \wedge 1)(\frac{|p_3|}{\sqrt{n_c}} \wedge 1)\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)$$
$$(f_2 f_3 - f_1(2f_2 + 1))dp_1 dp_2 dp_3.$$

This gives

$$\frac{n_c'}{n_c}(t) = 16\pi^2 \int_0^\infty y^2(\frac{y}{\sqrt{n_c}} \wedge 1)f(t,y)\int_0^y z^2(\frac{z}{\sqrt{n_c}} \wedge 1)f(t,z)$$
$$\Big(\frac{1}{|F_1'(s_1)|}(\sqrt{\frac{y^2+z^2+2yzs_1}{n_c}} \wedge 1) - \frac{1}{|F_2'(s_2)|}(\sqrt{\frac{y^2+z^2+2yzs_2}{n_c}} \wedge 1)\Big)dzdy$$
$$-\int (\frac{|p_1|}{\sqrt{n_c}} \wedge 1)(\frac{|p_2|}{\sqrt{n_c}} \wedge 1)(\frac{|p_3|}{\sqrt{n_c}} \wedge 1)\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)f_1 dp_1 dp_2 dp_3. \tag{3.5}$$

But $s_1 \geq s_2$, since

$$2(y^2 + z^2 + 2yzs_1) = \sqrt{n_c^2 + 4(E(y) + E(z))^2} - n_c \geq \sqrt{n_c^2 + 4(E(y) - E(z))^2} - n_c = 2(y^2 + z^2 + 2yzs_2).$$

Moreover, $F_1' = F_2'$ is a non-increasing function. And so, for $(y,z) \neq (0,0)$,

$$\frac{1}{|F_1'(s_1)|}(\sqrt{\frac{y^2+z^2+2yzs_1}{n_c}} \wedge 1) - \frac{1}{|F_2'(s_2)|}(\sqrt{\frac{y^2+z^2+2yzs_2}{n_c}} \wedge 1)$$

is positive. Thus, the first term in the r.h.s. of (3.5) is continuous and positive for $t \in [0,T[$. Hence for some $k > 0$,

$$n_c'(t) \geq k - 2M_0 n_c^{\frac{3}{2}} - \sqrt{n_c}\int p^2 f(t,p)dp$$

for $t < T$, with the integral $\int p^2 f(t,p)dp$ bounded. This contradicts $\lim_{t\to T} n_c(t) = 0$.

Lemma 3.2

$$|\int C(f,n_c)(p)dp| \leq k\Big(\int f(p)dp\Big)^2 + k\sqrt{n_c}\int p^2 f(p)dp.$$

Proof of Lemma 3.2.

$$|\int C(f,n_c)(p)dp| \leq n_c \int (\frac{|p_1|}{\sqrt{n_c}} \wedge 1)(\frac{|p_2|}{\sqrt{n_c}} \wedge 1)(\frac{|p_3|}{\sqrt{n_c}} \wedge 1)$$
$$\delta(p_1-p_2-p_3)\delta(E(p_1)-E(p_2)-E(p_3))(f_2f_3+f_1(1+f_2+f_3))dp_1dp_2dp_3. \tag{3.6}$$

To control the f_2f_3 term of (3.6), use spherical coordinates for p_2 and p_3 with the axis for p_3 directed by p_2 and denote by φ_3 the azimuthal angle related to p_3. Then

$$\int (\frac{|p_1|}{\sqrt{n_c}} \wedge 1)(\frac{|p_2|}{\sqrt{n_c}} \wedge 1)(\frac{|p_3|}{\sqrt{n_c}} \wedge 1)\delta(p_1-p_2-p_3)\delta(E(p_1)-E(p_2)-E(p_3))f_2f_3dp_1dp_2dp_3 \leq$$
$$k\int r_2^2(\frac{r_2}{\sqrt{n_c}} \wedge 1)f_2 \int r_3^2(\frac{r_3}{\sqrt{n_c}} \wedge 1)f_3Y_1dr_2dr_3,$$

where

$$Y_1 = \int_{-1}^{1} \delta\Big(\sqrt{(r_2^2+r_3^2+2r_2r_3s)^2+n(r_2^2+r_3^2+2r_2r_3s)} - E_2 - E_3\Big)ds \leq \frac{1}{r_2r_3}.$$

And so,

$$n_c\int (\frac{|p_1|}{\sqrt{n_c}} \wedge 1)(\frac{|p_2|}{\sqrt{n_c}} \wedge 1)(\frac{|p_3|}{\sqrt{n_c}} \wedge 1)\delta(p_1-p_2-p_3))\delta(E(p_1)-E(p_2)-E(p_3))f(p_2)f(p_3)dp_1dp_2dp_3$$
$$\leq k(\int f(p)dp)^2.$$

In the same way the other quadratic terms of (3.6) can be bounded by $k(\int f(p)dp)^2$. Finally,

$$n_c\int (\frac{|p_1|}{\sqrt{n_c}} \wedge 1)(\frac{|p_2|}{\sqrt{n_c}} \wedge 1)(\frac{|p_3|}{\sqrt{n_c}} \wedge 1)\delta(p_1-p_2-p_3)\delta(E(p_1)-E(p_2)-E(p_3))f_1dp_1dp_2dp_3$$
$$\leq kn_c\int r_1f_1\int_0^{r_1}(\frac{r_2}{\sqrt{n_c}} \wedge 1)r_2dr_2dr_1 \leq k\sqrt{n_c}\int p^2f(p)dp.$$

■

Lemma 3.3 *Given $0 < n_* < M_0$, there is a constant k such that for any $n \in [n_*, M_0]$ and isotropic functions $(f,g) \in L^1_+(\mathbb{R}^3) \times L^1_+(\mathbb{R}^3)$ with L^1 norm bounded by M_0,*

$$\int |(C(f,n)-C(g,n))(p)|dp \leq k\int (1+\sqrt{n}p^2)|(f-g)(p)|dp \tag{3.7}$$

with k independent of n, f, g.

Proof of Lemma 3.3.
Denote by $\mu(f) = (f_2f_3 - f_1(1+f_2+f_3))$. Then by computations similar to those used in the proof of Lemma 3.2,

$$\int |(C(f,n)-C(g,n))(p)|dp$$
$$\leq kn\int |p_1| \wedge 1|p_2| \wedge 1|p_3| \wedge 1\delta(p_1-p_2-p_3)\delta(E_1-E_2-E_3)|\mu(f)-\mu(g)|dp_1dp_2dp_3$$
$$\leq k(\int fdp + \int gdp)\int |f(p)-g(p)|dp + k\sqrt{n}\int p^2|f(p)-g(p)|dp.$$

■

Lemma 3.4 *For any* $\gamma \in [0,1]$,

$$\int |p|^{2+2\gamma} f(t,p)dp \le \int |p|^{2+2\gamma} f_i(p)dp + M_0 t \sup_{s\le t}(\int (1+p^2) f(s,p)dp)^2.$$

<u>Proof of Lemma 3.4.</u>
Let $\gamma \in [0,1]$ be fixed. Multiplying the equation satisfied by f by $|p|^{2+2\gamma}$ and integrating it on $(0,t)\times\mathbb{R}^3$ leads to

$$\begin{aligned}
&\int |p|^{2+2\gamma} f(t,p)dp + \int_0^t n_c(s) \int |\bar{A}|^2 (|p_1|^{2+2\gamma} - |p_2|^{2+2\gamma} \\
&\quad -|p_3|^{2+2\gamma}) f_1 \delta(p_1 = p_2+p_3)\delta(E_1 = E_2+E_3) dp_1 dp_2 dp_3 ds \\
= &\int |p|^{2+2\gamma} f_i(p)dp + \int_0^t n_c(s) \int |\bar{A}|^2 (|p_1|^{2+2\gamma} - |p_2|^{2+2\gamma} - |p_3|^{2+2\gamma})(f_2 f_3 \\
&\quad - f_1(f_2+f_3))\delta(p_1 = p_2+p_3)\delta(E_1 = E_2+E_3) dp_1 dp_2 dp_3 ds. \qquad (3.8)
\end{aligned}$$

It is sufficient to prove that there is a positive constant $\tilde{K}$ such that

$$0 \le r_1^{2+2\gamma} - r_2^{2+2\gamma} - r_3^{2+2\gamma} \le \tilde{K}(1+r_2^2)(1+r_3^2),$$

when $E_1 = E_2 + E_3$. Indeed, the second term in the left member of (3.8) will then be nonnegative, whereas the second term in the right member will be bounded from above by $M_0 t \sup_{s\le t}(\int (1+p^2) f(s,p)dp)^2$. Since

$$r_i^2 = \frac{\sqrt{n^2+4E_i^2} - n}{2}, \quad 1 \le i \le 3,$$

(3.8) holds if there is a positive constant K such that

$$\begin{aligned}
0 &\le \Big(\sqrt{n^2+4(E_2+E_3)^2} - n\Big)^{1+\gamma} - \Big(\sqrt{n^2+4E_2^2} - n\Big)^{1+\gamma} - \Big(\sqrt{n^2+4E_3^2} - n\Big)^{1+\gamma} \\
&\le K\Big(\sqrt{n^2+4E_2^2} - n + 2\Big)\Big(\sqrt{n^2+4E_3^2} - n + 2\Big), \quad (E_2,E_3) \in (\mathbb{R}_+)^2. \qquad (3.9)
\end{aligned}$$

To prove the left inequality of (3.9), consider the function

$$g(x,E_3) := \Big(\sqrt{n^2+4(x+E_3)^2} - n\Big)^{1+\gamma} - \Big(\sqrt{n^2+4x^2} - n\Big)^{1+\gamma} - \Big(\sqrt{n^2+4E_3^2} - n\Big)^{1+\gamma}, \quad x \ge 0.$$

Then,

$$\frac{\sqrt{n^2+4(x+E_3)^2}\sqrt{n^2+4x^2}}{4(1+\gamma)} \frac{\partial g}{\partial x}(x,E_3) =: C,$$

where

$$C = (x+E_3)\sqrt{n^2+4x^2}\Big(\sqrt{n^2+4(x+E_3)^2} - n\Big)^{\gamma} - x\sqrt{n^2+4(x+E_3)^2}\Big(\sqrt{n^2+4x^2} - n\Big)^{\gamma}$$

is nonnegative if

$$\frac{x+E_3}{x}\sqrt{\frac{n^2+4x^2}{n^2+4(x+E_3)^2}} \ge 1,$$

which is true, since

$$\Big(\frac{x+E_3}{x}\Big)^2 \geq \frac{n^2+4(x+E_3)^2}{n^2+4x^2}$$

is equivalent to

$$2n^2E_3x + n^2E_3^2 \geq 0.$$

And so, the function g is non-decreasing in x. It follows from $g(0,E_3)=0$ that g is non-negative. For the right inequality of (3.9), it is by symmetry enough to consider $E_3 \leq x$. The inequality is obtained by proving that g is bounded from above by a multiple of the function h defined by

$$h(x,E_3) := \Big(\sqrt{n^2+4x^2}+2-n\Big)\Big(\sqrt{n^2+4E_3^2}+2-n\Big), \quad x \geq 0.$$

For h we notice that $(\sqrt{n^2+4y^2}-n+2) \geq 2$ $(\geq y)$ for $y \leq n$ $(y \geq n)$. It follows that $h(x,E_3) \geq 4$ for $x, E_3 \leq n$, $h(x,E_3) \geq 2\cdot x$ for $x \geq n, E_3 \leq n$, and $h(x,E_3) \geq x \cdot E_3$ for $x, E_3 \geq n$.
For $x, E_3 \leq n$, $g(x,E_3)$ is positive and bounded from above by some constant c_1. We thus require $K \geq \frac{c_1}{4}$. For $x, E_3 \geq n$ we start from

$$\Big(\sqrt{n^2+4y^2}-n\Big)^{1+\gamma} = (2y)^{1+\gamma}\Big(\sqrt{\frac{n^2}{4y^2}+1}-\frac{n}{2y}\Big)^{1+\gamma} = (2y)^{1+\gamma}\Big(1+\frac{n^2}{8y^2}(\theta\frac{n^2}{4y^2}+1)^{-\frac{1}{2}}-\frac{n}{2y}\Big)^{1+\gamma},$$

which gives

$$(2y)^{1+\gamma}-(1+\gamma)\frac{15}{32}ny^\gamma \geq \Big(\sqrt{n^2+4y^2}-n\Big)^{1+\gamma} \geq (2y)^{1+\gamma}-(1+\gamma)2^\gamma ny^\gamma.$$

It follows that

$$\begin{aligned} g(x,E_3) &\leq 2^{1+\gamma}\Big((x+E_3)^{1+\gamma}-x^{1+\gamma}-E_3^{1+\gamma}\Big)+(1+\gamma)2^\gamma n(x^\gamma+E_3^\gamma) \\ &= 2^{1+\gamma}\Big(x((x+E_3)^\gamma-x^\gamma)+E_3((x+E_3)^\gamma-E_3^\gamma)\Big)+(1+\gamma)2^\gamma n(x^\gamma+E_3^\gamma) \\ &\leq 2^{2+2\gamma}E_3x^\gamma+2^{2+\gamma}nx^\gamma, \end{aligned}$$

which is bounded by a multiple of $h(x,E_3)$ in this domain. This also holds for $x \geq n, E_3 \leq n$ with an analogous proof. ■

<u>Proof of Theorem 2.1.</u>
By Lemma 3.1, for any fixed initial data f_i, given $n_{ci} > 0$, there is a positive time T_0 and $n_* > 0$ such that any solution $n(t)$ of (2.1-3) is bounded from below by n_* on $[0,T_0]$. Let $\mathcal{K}$ denote the closed and convex subset of $C([0,T_0])$ consisting of the functions n in $C([0,T_0])$ such that $n(0)=n_{ci}$ and $\frac{n_*}{2} \leq n(t) \leq M_0$, $t \in [0,T_0]$. A local in time solution to equations (2.1-3) is found as a fixed point of the following map. Let a (large) truncation value P be defined for the linear part of the collision operator. Let Φ be the map defined on $\mathcal{K}$ by $\Phi(n)=m$, where

$$m(t) = M_0 - \int f(t,p)dp, \quad t \in [0,T_0],$$

and f is the mild solution in $C([0,\tau_0];L^1)$ for some τ_0 defined below with $0<\tau_0 \leq T_0$, to

$$\begin{aligned} &\frac{\partial f}{\partial t} = C^P(f,n), \\ &f(0,p) = f_i(p), \end{aligned} \tag{3.10}$$

with

$$C^P(f,n) = n\int |\bar{A}|^2\delta(p_1-p_2-p_3)\delta(E_1-E_2-E_3)[\delta(p-p_1)$$
$$-\delta(p-p_2)-\delta(p-p_3)](f_2f_3-f_1(f_2+f_3))dp_1dp_2dp_3$$
$$-n\int \chi_{|p|<P}|\bar{A}|^2\delta(p_1-p_2-p_3)\delta(E_1-E_2-E_3)[\delta(p-p_1)-\delta(p-p_2)-\delta(p-p_3)]f_1dp_1dp_2dp_3.$$

Here $\chi_{|p|<P}$ is the characteristic function of the set where $|p|<P$. We notice that the discussion after Lemma 3.1 also holds for (3.10). Writing the equation (3.10) in exponential form and estimating the solution from below by the term containing the initial value, it follows that the bound of n from below, n_*, can be taken independent of P because the first term to the right in (3.5) can in this way be bounded from below uniformly in P on any time interval.
Given n, a mild solution f for (3.10) can be constructed as the limit of the nonnegative sequence (f_j), defined by $f_0 = f_i$ and

$$\frac{\partial f_{j+1}}{\partial t} + f_{j+1}C_l^P(f_j,n) = C_g^P(f_j,n), \tag{3.11}$$
$$f_{j+1}(0,p) = f_i(p).$$

The collision frequency is

$$C_l^P(f,n) = n[2\int |\bar{A}|^2\delta(p-p_2-p_3)\delta(E_p-E_2-E_3)f_2dp_2dp_3$$
$$+2\int |\bar{A}|^2\delta(p_1-p_2-p)\delta(E_1-E_2-E_p)f_2dp_1dp_2$$
$$+\int \chi_{|p|<P}|\bar{A}|^2\delta(p-p_2-p_3)\delta(E_p-E_2-E_3)dp_2dp_3],$$

which preserves positivity together with the gain term $C_g^P(f,n) = C^P(f,n) + fC_l^P(f,n)$. For any nonnegative functions $f,g \in L^1$ and any $n \in [0,M_0]$, it holds that

$$\int |(C^P(f,n)-C^P(g,n))(p)|dp \le kn(P^2 + \int f(p)dp + \int g(p)dp)\int |(f-g)(p)|dp. \tag{3.12}$$

For $\tau_0 > 0$ smaller than $\frac{c}{\int f_i(p)dp+P^2}$, where c is a suitable constant, the sequence (f_j) is uniformly bounded by $2\int f_i(p)dp$ and converges in $C([0,\tau_0];L^1)$ to a mild solution f of (3.10) (using (3.12) and induction), since

$$\sup_{t\in[0,\tau_0]} |(f_{j+1}-f_j)(t,\cdot)|_{L^1} \le k\tau_0 \sup_{t\in[0,\tau_0]} |(f_j-f_{j-1})(t,\cdot)|_{L^1}, \quad j\in\mathbb{N}.$$

The nonnegative solution f is unique in $C([0,\tau_0];L^1)$ by the L^1-Lipschitz property (3.12) of $C^P(\cdot,n)$. The time-interval $[0,\tau_0]$ can be so chosen that $m(t) = \Phi(n)(t) \ge \frac{1}{2}n_*$ uniformly for $n\in\mathcal{K}$ and $0\le t\le\tau_0$.
The map Φ is continuous. Indeed, let $(n,\tilde{n}) \in \mathcal{K}\times\mathcal{K}$ and $m = \Phi(n)$ resp. $\tilde{m} = \Phi(\tilde{n})$. Then for $t \le \tau_0$,

$$\int |(f-\tilde{f})(t,p)|dp \le kt \parallel n-\tilde{n}\parallel_\infty | + k\int_0^t\int |(f-\tilde{f})(s,p)|dpds.$$

Consequently, for τ small

$$\sup_{t\in[0,\tau]}\int |(f-\tilde{f})(t,p)|dp \le k\tau \parallel n-\tilde{n}\parallel_\infty, \tag{3.13}$$

and so

$$\| m - \tilde{m} \|_\infty \leq k\tau \| n - \tilde{n} \|_\infty .$$

The continuity of Φ on $[0, \tau_0]$ follows. Moreover, the map Φ is compact by Arzela-Ascoli. Indeed $\Phi(\mathcal{K})$ is bounded on $[0, \tau_0]$, since $\frac{1}{2}n_* \leq \Phi(n)(t) \leq M_0$ for $t \in [0, \tau_0]$, $n \in \mathcal{K}$. Besides it is equicontinuous, since

$$\begin{aligned}
|\Phi(n)(t_1) - \Phi(n)(t_2)| = |\int f(t_1, p)dp - \int f(t_2, p)dp| &\leq |t_1 - t_2| \sup_{t\in[0,\tau_0]} \int |C(f, n)(t, p)|dp \\
&\leq k\Big(\sup_{t\in[0,\tau_0]} \int f(t, p)dp(P^2 + \sup_{t\in[0,\tau_0]} \int f(t, p)dp)\Big)|t_1 - t_2| \\
&\leq k\Big(2 \int f_i(p)dp(P^2 + 2 \int f_i(p)dp)\Big)|t_1 - t_2|, \quad n \in K.
\end{aligned}$$

Consequently, there is a pair of functions $(f^P, n_c^P) \in C([0, \tau_0], L^1) \times C([0, \tau_0])$, such that f^P is nonnegative and satisfies (3.10) in mild form with a truncation for $| p |> P$ in the linear part of the collision operator, and n_c^P being a fixed point of Φ, satisfies the corresponding equation (2.2) in mild form. Since $\int C^P(f^P, n_c^P)dp$ is continuous in t, the solution n_c^P is continuously differentiable in t and satisfies (2.2) in strong form. Also f^P satisfies (2.1), (2.3) for C^P in strong form since

Lemma 3.5 *The family $(C^P(f^P, n^P)(t))$, $t \in [0, T_0]$, with values in L^1, is t-continuous in the L^1-norm, uniformly with respect to P and t.*

<u>Proof of Lemma 3.5</u>
Let us first discuss the time difference in the linear term from $C^P(f^P, n^P)(t + h) - C^P(f^P, n^P)(t)$, i.e.

$$\begin{aligned}
\int | \int dp_1 dp_2 dp_3 \chi_{|p|<P} |\bar{A}|^2 \delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1) - \delta(p - p_2) \\
-\delta(p - p_3)]f_1^P(t + h) - \int dp_1 dp_2 dp_3 \chi_{|p|<P} |\bar{A}|^2 \delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1) \\
-\delta(p - p_2) - \delta(p - p_3)]f_1^P(t)|dp. \qquad (3.14)
\end{aligned}$$

We move the difference through, one factor at a time. We recall that $\int(1 + |p|^2)f^P(p, t)dp$ is uniformly in P and t bounded, and that $n^P(t)$ is t-continuous uniformly in P. When the factor $f_1^P(t + h) - f_1^P(t)$ appears, it is written as an integral $\int_t^{t+h} C^P(f^P, n^P)(s)ds$, using the mild form of the equation. Here $C^P(f^P, n^P)$ can be estimated similarly to the proof of Lemma 3.1. Consider again the linear term

$$\begin{aligned}
\int_t^{t+h} ds \int dp \int dp_1 dp_2 dp_3 \chi_{|p|<P} |\bar{A}|^2 \delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)\delta(p - p_1) \\
\int \chi_{|p_1|<P} |\bar{A}|^2 \delta(p_1' - p_2' - p_3')\delta(E_1' - E_2' - E_3')\delta(p_1 - p_1')f(p_1', s)dp_1' dp_2' dp_3'. \qquad (3.15)
\end{aligned}$$

Use the computation of X_3 in the proof of Lemma 3.1 to estimate the outer integral with respect to $dpdp_1dp_2dp_3$. This gives a bound

$$\begin{aligned}
k \int_0^\infty dr_1 r_1 \int_0^{r_1} r_2 dr_2 \int |\bar{A}|^2 \delta(p_1' - p_2' - p_3')\delta(E_1' - E_2' - E_3')\delta(p_1 - p_1')f(p_1', s)dp_1' dp_2' dp_3' \\
= k \int_0^\infty dr_1 r_1^3 \int |\bar{A}|^2 \delta(p_1' - p_2' - p_3')\delta(E_1' - E_2' - E_3')\delta(p_1 - p_1')f(p_1', s)dp_1' dp_2' dp_3' \\
= k \int |\bar{A}|^2 \delta(p_1' - p_2' - p_3')\delta(E_1' - E_2' - E_3')f(p_1', s)dp_1' dp_2' dp_3' \int |p_1|\delta(p_1 - p_1')dp_1.
\end{aligned}$$

Again using the X_3-computation for the remaining integral gives the bound

$$k\int_0^\infty dr_1' {r_1'}^2 f(r_1',s)\int_0^{r_1'} dr_2' r_2' \le k\int dp_1'|p_1'|^2 f(p_1',s).$$

And so the final integration $\int_t^{t+h} ds$ gives the bound $hk\sup_{[t,t+h]}\int p^2 f(p,s)dp$. It follows that (3.15), uniformly in P and t, L^1-converges to zero when $h\to 0$. That holds in a similar way for the other terms in (3.14), and for the non-linear terms of $C^P(f^P,n^P)$. ■

The conservation of total mass follows from the fixed point property. The boundedness of the energy of f^P follows from the proof of Lemma 3.1. Using (3.4) the integrals $\int(1+p^2)f^P dp$ are also uniformly in P bounded. Moreover, the estimate of Lemma 3.4 holds uniformly in P for f^P. Observing that n_* is so chosen that for any P, $n_c^P \ge n_*$ on any subinterval $[0,T_0']$ of $[0,T_0]$ where n_c^P exists, the result can for each P be extended by iteration to the whole interval of time $[0,T_0]$.

The f^P's are also the limits in $C([0,T_0];L^1_{1+p^2})$ of increasing sequences $(\tilde f_j^P)$. It will be used in the study of $lim_{P\to\infty}f^P$ below, that such $\tilde f_j^P$'s share with the f^P's any uniform bound for $(2+\gamma)$-moments. The increasing sequences of approximations are defined by $\tilde f_0^P=0$ and

$$\frac{\partial \tilde f_{j+1}^P}{\partial t}+\tilde f_{j+1}^P\Big(n_c^P\int|\bar A|^2\chi_{|p|<P}\delta(p-p_2-p_3)\delta(E_p-E_2-E_3)dp_2dp_3+\tilde C_l(f^P,n_c^P)\Big)=\tilde C_g(\tilde f_j^P,n_c^P)$$
$$+2n_c^P\int|\bar A|^2\chi_{|p|<P}\delta(p_1-p_2-p)\delta(E_1-E_2-E_p)f_1^P dp_1dp_2, \quad (3.16)$$
$$\tilde f_{j+1}^P(0,p)=f_i(p)\chi_{|p|<P}.$$

Here $\tilde C_l$ is defined by

$$\tilde C_l(f,n)=2n[\int|\bar A|^2\delta(p-p_2-p_3)\delta(E_p-E_2-E_3)f_2dp_2dp_3$$
$$+\int|\bar A|^2\delta(p_1-p_2-p)\delta(E_1-E_2-E_p)f_2dp_1dp_2],$$

and

$$\tilde C_g(f,n)=n[\int|\bar A|^2\delta(p-p_2-p_3)\delta(E_p-E_2-E_3)f_2f_3dp_2dp_3$$
$$+2\int|\bar A|^2\delta(p_1-p_2-p)\delta(E_1-E_2-E_p)f_1f_2dp_1dp_2$$
$$+2f\int|\bar A|^2\delta(p_1-p_2-p)\delta(E_1-E_2-E_p)f_1dp_1dp_2].$$

Let us prove by induction that $(\tilde f_j^P)$, $j\in\mathbb{N}$, is an increasing sequence of nonnegative functions bounded by f^P. First, $0\le\tilde f_0^P\le f^P$ and $\tilde f_0^P\le\tilde f_1^P$, by definition of $\tilde f_0^P$ and the nonnegativity of f^P and $\tilde f_1^P$. Then, assuming that $\tilde f_{j-1}^P\le\tilde f_j^P$ and $\tilde f_{j-1}^P\le f^P$, the function $\tilde f_{j+1}^P-\tilde f_j^P$ (resp. $f^P-\tilde f_j^P$) satisfies

$$\frac{\partial(\tilde f_{j+1}^P-\tilde f_j^P)}{\partial t}+(\tilde f_{j+1}^P-\tilde f_j^P)(n_c^P\int|\bar A|^2\chi_{|p|<P}\delta(p-p_2-p_3)\delta(E_p-E_2-E_3)dp_2dp_3+\tilde C_l(f^P,n_c^P))$$
$$=\tilde C_g(\tilde f_j^P,n_c^P)-\tilde C_g(\tilde f_{j-1}^P,n_c^P),$$
$$(\tilde f_{j+1}^P-\tilde f_j^P)(0,p)=0,$$

(resp.

$$\frac{\partial(f^P-\tilde f_j^P)}{\partial t}+(f^P-\tilde f_j^P)(n_c^P\int|\bar A|^2\chi_{|p|<P}\delta(p-p_2-p_3)\delta(E_p-E_2-E_3)dp_2dp_3+\tilde C_l(f^P,n_c^P))$$
$$=\tilde C_g(f^P,n_c^P)-\tilde C_g(\tilde f_{j-1}^P,n_c^P),$$
$$(f^P-\tilde f_j^P)(0,p)=0.)$$

Writing $(\tilde f_{j+1}^P-\tilde f_j^P)(t,p)$ (resp. $(f^P-\tilde f_j^P)(t,p)$) in exponential form, and deducing from the induction assumption that $\tilde C_g(\tilde f_j^P,n_c^P)-\tilde C_g(\tilde f_{j-1}^P,n_c^P)$ (resp. $\tilde C_g(\tilde f_j^P,n_c^P)-\tilde C_g(\tilde f_{j-1}^P,n_c^P)$) is nonnegative, leads to the nonnegativity of $(\tilde f_{j+1}^P-\tilde f_j^P)(t,p)$ (resp. $(\tilde f^P-\tilde f_j^P)(t,p)$). Its limit $\tilde f^P$ satisfies the same equation as f^P. The collision operator for the equation satisfied by $f^P-\tilde f^P$ is Lipschitz continuous similarly to (3.12), hence $f^P=\tilde f^P$.

It remains to prove that a subsequence of (f^P,n_c^P) converges in $C([0,T_0],L^1_{1+p^2})\times C([0,T_0])$, and that its limit solves the system (2.1-3). Using Arzela-Ascoli as above, the sequence (n^P) is compact in $C([0,T_0])$.

Lemma 3.6 *Given $t\in[0,T_0]$, the family $(g^P(t)):=(\int|\bar A|^2\chi_{|p|<P}\delta(p_1-p_2-p)\delta(E_1-E_2-E_p)f_1^P(t)dp_1dp_2)$ is compact in L^1. This also holds for the family $(\tilde C_l(f^P,n_c^P)(t))$.*

<u>Proof of Lemma 3.6.</u>
By definition,

$$g^P(p)=\chi_{|p|<P}\int|\bar A|^2f_1^P\delta(p_1=p_2+p)\delta(E_1=E_2+E_p)dp_1dp_2$$
$$=\chi_{|p|<P}\int|\bar A|^2r_1^2f_1^P(r_1)\int_{-1}^1\delta(F(s)=0)dsdr_1,$$

where

$$F(s):=\sqrt{(2|p|r_1s-r_1^2-p^2)^2-n(2|p|r_1s-r_1^2-p^2)}+E_p-E_1.$$

Then,

$$F(s)=0\text{ iif }(2|p|r_1s-r_1^2-p^2)^2-n(2|p|r_1s-r_1^2-p^2)-(E_p-E_1)^2=0\text{ and }r_1\geq|p|$$
$$\text{iif }2|p|r_1s-r_1^2-p^2=\frac{n-\sqrt{n^2+4(E_p-E_1)^2}}{2}=:2|p|r_1\tilde s-r_1^2-p^2\text{ and }r_1\geq|p|.$$

Moreover,

$$F'(s)=\frac{|p|r_1\Big(2(2|p|r_1s-r_1^2-p^2)-n\Big)}{F(s)-(E_p-E_1)},$$

so that

$$|F'(\tilde s)|=\frac{|p|r_1\sqrt{n^2+4(E_p-E_1)^2}}{|E_p-E_1|}.$$

And so, up to an angular factor,

$$g^P(p)=\frac{\chi_{|p|<P}}{|p|}\int_{|p|}^{+\infty}|\bar A|^2r_1f_1^P\frac{|E_p-E_1|}{\sqrt{n^2+4(E_p-E_1)^2}}\chi_{|\tilde s|\leq1}dr_1$$
$$=\frac{\chi_{|p|<P}}{|p|}\int_{|p|}^{+\infty}|\bar A|^2r_1f_1^P\frac{|E_p-E_1|}{\sqrt{n^2+4(E_p-E_1)^2}}\chi_{|2r_1^2+2p^2+n-\sqrt{n^2+4(E_p-E_1)^2}|\leq4|p|r_1}dr_1.$$

Moreover,

$$|2r_1^2 + 2p^2 + n - \sqrt{n^2 + 4(E_p - E_1)^2}| \leq 4|p|r_1$$

if and only if

$$\sqrt{n^2 + 4(E_p - E_1)^2} \leq 2r_1^2 + 4rr_1 + 2r^2 + n,$$

and

$$2r_1^2 - 4rr_1 + 2r^2 + n \leq \sqrt{n^2 + 4(E_p - E_1)^2}.$$

Straightforward computations show that the first inequality is satisfied for any (n, p, p_1). The second inequality is equivalent to

$$r_1^4 - 3rr_1^3 + (4r^2 + \frac{3}{4}n)r_1^2 - 3r(r^2 + \frac{n}{2})r_1 + r^4 + \frac{3}{4}nr^2 \geq 0,$$

which is also true for any (n, p_1, p), since its left hand side is equal to

$$(r_1 - r)^4 + (r_1 - r)^2(r_1 r + \frac{3}{4}n).$$

And so,

$$g^P(p) = \frac{\chi_{|p|<P}}{|p|} \int_{|p|}^{+\infty} |\bar{A}|^2(p_1, p_1 - p, p) r_1 f_1^P \frac{|E_p - E_1|}{\sqrt{n^2 + 4(E_p - E_1)^2}} dr_1.$$

We next prove the L^1-compactness of $(g^P)(t)$. First,

$$\lim_{K \to +\infty} \int_{|p|>K} g^P(p) dp = 0.$$

Indeed,

$$\begin{aligned} \int_{|p|>K} g^P(p) dp &\leq k \int_K^{+\infty} r_1 f^P(r_1) \int_K^{r_1} r_3 dr_3 dr_1 \\ &\leq k \int_K^{+\infty} r_1^3 f^P(r_1) dr_1 \leq \frac{k}{K} \int p^2 f^P(p) dp, \end{aligned}$$

which uniformly in P and t tends to zero when $K \to +\infty$.
Let us prove that for fixed $K > 0$,

$$\lim_{h \to 0} \int |g^P(p+h)\chi_{|p+h|<K} - g^P(p)\chi_{|p|<K}| dp = 0, \tag{3.17}$$

uniformly with respect to P and t. It holds that

$$\begin{aligned} \int_{|p|<K+1} |g^P(p+h) - g^P(p)| dp &\leq M_0 \int_{|p|<K+1} |\frac{\chi_{|p+h|<P}}{|p+h|} - \frac{\chi_{|p|<P}}{|p|}| \frac{dp}{|p|} \\ &+ \int f_1^P \int_{|p|<K+1} |\bar{A}^2(p_1, p_1 - p - h, p + h)| \frac{|E_{p+h} - E_1|}{\sqrt{n^2 + 4(E_{p+h} - E_1)^2}} \\ &- |\bar{A}^2(p_1, p_1 - p, p)| \frac{|E_p - E_1|}{\sqrt{n^2 + 4(E_p - E_1)^2}} |\frac{dp}{|p|} dp_1 + O(h). \end{aligned}$$

The inner p-integral in the last term of the previous inequality tends to zero when h tends to zero, uniformly with respect to p_1, and so (3.17) follows.
For the family $\tilde{C}_l(f^P, n_c^P)$ it is enough to consider a sequence (P_j) tending to infinity, for which $(n_c^{P_j})$ is uniformly in t convergent. From there the proof is similar to the previous case, since

$$\tilde{C}_l(f^P, n_c^P)(p) = n^P\Big(\frac{2}{|p|}\int_0^{|p|} |\bar{A}|^2 r_2 f_2^P \frac{|E_p - E_2|}{\sqrt{n^2 + 4(E_p - E_2)^2}} dr_2 + \frac{2}{|p|}\int_0^{+\infty} |\bar{A}|^2 r_2 f_2^P \frac{E_2 + E_p}{\sqrt{n^2 + 4(E_2 + E_p)^2}} dr_2\Big).$$

■

We can now take a subsequence $(g^{P_l}, \tilde{C}_l(f^{P_l}, n_c^{P_l}), n_c^{P_l})$ with P_l tending to infinity, converging for rational t to a limit (g, h, n_c), and will for such t prove that (f^{P_l}) is a Cauchy sequence in L^1. The Cauchy property for irrational t then follows using Lemma 3.5.
To prove that (f^{P_l}) is a Cauchy sequence in L^1, split $f^{P_{l'}} - f^{P_{l''}}$ into

$$f^{P_{l'}} - f^{P_{l''}} = (f^{P_{l'}} - \tilde{f}_j^{P_{l'}}) + (\tilde{f}_j^{P_{l'}} - \tilde{f}_j^{P_{l''}}) + (\tilde{f}_j^{P_{l''}} - f^{P_{l''}}).$$

By the estimate of Lemma 3.4 it is enough to prove the convergence on compact p-sets. On any such set the factor $\int |\bar{A}|^2 \chi_{|p|<P} \delta(p - p_2 - p_3)\delta(E_p - E_2 - E_3) dp_2 dp_3$ in the left hand side of (3.16) does not depend on P for P large enough.
It follows from the equations (3.10) and (3.16) for $f^{P_{l'}} - \tilde{f}_{j+1}^{P_{l'}}$, and from the cancelation of the inhomogeneous term in the right hand side of the equation, that

$$\lim_{j\to+\infty} (f^{P_{l'}} - \tilde{f}_j^{P_{l'}}) = 0,$$

uniformly with respect to l' and t. It remains to prove that for J given,

$$\lim_{l'\to+\infty, l''\to+\infty} (\tilde{f}_J^{P_{l'}} - \tilde{f}_J^{P_{l''}}) = 0$$

in L^1-sense. Consider first the case $J = 1$. In (3.16) for $\tilde{f}_1^{P_{l'}}$, the term

$$\tilde{f}_1^{P_{l'}} n_c^{P_{l'}} \int |\bar{A}|^2 \chi_{|p|<P} \delta(p - p_2 - p_3)\delta(E_p - E_2 - E_3) dp_2 dp_3$$

to the left can be written for n_c, plus a term in $n_c - n_c^{P_{l'}}$, which tends to zero in L^1, when $l' \to \infty$. The L^1 limit of the right hand side is $n_c g$. So using the L^1-convergence of the $\tilde{C}_{l'}$-term, (3.16) gives that $\tilde{f}_1^{P_{l'}}$ converges in L^1 to some $\tilde{f}_1$. It follows that $\tilde{C}_g(\tilde{f}_1^{P_l}, n_c^{P_1})$ converges in L^1 to $\tilde{C}_g(\tilde{f}_1, n_c)$. We can now repeat the convergence argument for $\tilde{f}_2^{P_{l'}}$. By finite induction for $j = 1, ..., J$, the desired convergence holds. Also $f = \lim f^{P_l} \in L^1$, and

$$\frac{\partial f}{\partial t} + f(n_c \int |\bar{A}|^2 \chi \delta(p - p_2 - p_3)\delta(E_p - E_2 - E_3) dp_2 dp_3 + h) = \tilde{C}_g(f, n_c) + 2n_c g,$$
$$f(0, p) = f_i(p).$$

But the L^1-convergence of (f^{P_l}) and Lemma 3.4 imply that

$$g = \lim_{l\to\infty} g^{P_l} = \lim_{l\to\infty} \int |\bar{A}|^2 \chi \chi_{|p|<P} \delta(p_1 - p_2 - p)\delta(E_1 - E_2 - E_p) f_1^P dp_1 dp_2$$
$$= \int |\bar{A}|^2 \chi \delta(p_1 - p_2 - p)\delta(E_1 - E_2 - E_p) f_1 dp_1 dp_2,$$
$$h = \lim_{l\to\infty} \tilde{C}_l(f^{P_l}, n_c^{P_l}) = \tilde{C}_l(f, n_c).$$

And so (f, n_c) satisfies (2.1-3) on $[0, T_0]$ in mild form with the $|p|^{2+\gamma}$-moment of f bounded in L^1. This can be continued up to any time T, since by (3.5) $\inf_{t\in[0,T[} n_c(t) > 0$ for any time interval $[0, T[$ where (f, n_c) exists. The C^1-properties of f, n_c with respect to time, follow as above for f^P, n_c^P. ■

4 The special cases $|p| << p_0$ and $|p| >> p_0$.

In the case $|p| << p_0$ the collision operator becomes

$$C(f, n_c) = kn_c \int_{|p_i|\le\lambda} \frac{E(p_1)E(p_2)E(p_3)}{(gn_c)^3} \delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1)$$
$$-\delta(p - p_2) - \delta(p - p_3)]((1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3))dp_1dp_2dp_3$$
$$= kn_c^{-\frac{1}{2}} \int_{|p_i|\le\lambda} |p_1|(1 + \frac{p_1^2}{4p_0^2})|p_2|(1 + \frac{p_2^2}{4p_0^2})|p_3|(1 + \frac{p_3^2}{4p_0^2})\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1)$$
$$-\delta(p - p_2) - \delta(p - p_3)](f_2f_3 - f_1(1 + f_2 + f_3))dp_1dp_2dp_3$$
$$= kn_c^{-\frac{1}{2}} \int_{|p_i|\le\lambda} \tilde{E}_1\tilde{E}_2\tilde{E}_3\delta(p_1 - p_2 - p_3)\delta(c(\tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3))[\delta(p - p_1)$$
$$-\delta(p - p_2) - \delta(p - p_3)](f_2f_3 - f_1(1 + f_2 + f_3))dp_1dp_2dp_3,$$

with $\tilde{E}(|p|) = |p|(1 + \frac{p^2}{4p_0^2})$.
Global conservation of total mass and of momentum are obtained as in the general case. And so energy is globally bounded, since the p-domain is bounded.

Lemma 4.1

$$|\int C(f, n_c)(p)dp| \le kn_c(\int f(p)dp)(1 + \int f(p)dp).$$

<u>Proof of Lemma 4.1</u>

$$|\int C(f, n_c)(p)dp| \le kn_c{}^{-2} \int_{|p_i|<\lambda} E(p_1)E(p_2)E(p_3)$$
$$\delta(p_1 - p_2 - p_3)\delta(E(p_1) - E(p_2) - E(p_3))(f_2f_3 + f_1(1 + f_2 + f_3))dp_1dp_2dp_3$$
$$= kn_c^{-1} \int_{|p_i|\le\lambda} \tilde{E}_1\tilde{E}_2\tilde{E}_3\delta(p_1 - p_2 - p_3)\delta(\tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3)(f_2f_3 + f_1(1 + f_2 + f_3))dp_1dp_2dp_3. \quad (4.1)$$

For controlling the f_2f_3 term of (4.1), use spherical coordinates for p_2 and p_3 with the axis for p_3 directed by p_2 and denote by φ_3 the azimuthal angle related to p_3. Then

$$n_c^{-1} \int \chi\tilde{E}(p_1)\tilde{E}(p_2)\tilde{E}(p_3)\delta(p_1 - p_2 - p_3))\delta(\tilde{E}(p_1) - \tilde{E}(p_2) - \tilde{E}(p_3))f(p_2)f(p_3)dp_1dp_2dp_3$$
$$= kn_c^{-1} \int_0^\lambda r_2^2\tilde{E}_2f(p_2) \int_0^{r_2} r_3^2\tilde{E}_3(\tilde{E}_2 + \tilde{E}_3)f(p_3)Xdr_2dr_3,$$

where

$$X = \int_{-1}^{1} \chi\delta\Big(\tilde{E}(\sqrt{r_2^2 + r_3^2 + 2r_2r_3s}) - \tilde{E}_2 - \tilde{E}_3\Big)ds \le \frac{kn_c}{r_2r_3},$$

since

$$\delta\Big(\tilde{E}(\sqrt{r_2^2 + r_3^2 + 2r_2r_3s}) - \tilde{E}_2 - \tilde{E}_3\Big) = \frac{\sqrt{r_2^2 + r_3^2 + 2r_2r_3s}}{r_2r_3\tilde{E}'(\sqrt{r_2^2 + r_3^2 + 2r_2r_3s})}\delta(s = \tilde{s})$$
$$= \frac{4p_0^2\sqrt{r_2^2 + r_3^2 + 2r_2r_3s}}{r_2r_3(4p_0^2 + 3(r_2^2 + r_3^2 + 2r_2r_3s))}\delta(s = \tilde{s}),$$

and

$$\frac{4p_0^2 r}{4p_0^2 + 3r^2} \le kr \le kn_c, \quad 0 \le r \le \lambda.$$

And so,

$$kn_c^{-1}\int \chi\tilde{E}(p_1)\tilde{E}(p_2)\tilde{E}(p_3)\delta(p_1 - p_2 - p_3))\delta(\tilde{E}(p_1) - \tilde{E}(p_2) - \tilde{E}(p_3))f(p_2)f(p_3)dp_1dp_2dp_3$$
$$\le k\int_0^\lambda r_2\tilde{E}_2^2 f_2 dr_2 \int_0^\lambda r_3\tilde{E}_3 f_3 dr_3 \le kn_c(\int f(p)dp)^2.$$

In the same way the other terms of (4.1) can be bounded by $kn_c\int f(p)dp$ or $kn_c(\int f(p)dp)^2$. ■

Lemma 4.2 *Given $0 < n_* < M_0$, there is a constant k such that for any $n \in [n_*, M_0]$ and isotropic functions $(f, g) \in L^1_+(\mathbb{R}^3) \times L^1_+(\mathbb{R}^3)$ with L^1 norm bounded by M_0,*

$$\int |(C(f,n) - C(g,n))(p)|dp \le k\int |(f-g)(p)|dp \tag{4.2}$$

with k independent of n, f, g.

<u>Proof of Lemma 4.2</u>
Denote by $\mu(f) = (f_2f_3 - f_1(1 + f_2 + f_3))$. Then by computations similar to those used in the proof of Lemma 4.1,

$$\int |(C(f,n) - C(g,n))(p)|dp$$
$$\le \frac{k}{n}\int \chi\tilde{E}_1\tilde{E}_2\tilde{E}_3\delta(p_1 - p_2 - p_3)\delta(\tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3)|\mu(f) - \mu(g)|dp_1dp_2dp_3$$
$$\le kn(1 + \int_{|p|\le\lambda} fdp + \int_{|p|\le\lambda} gdp)\int_{|p|\le\lambda} |f(p) - g(p)|dp.$$ ■

Lemma 4.3 *Given $T > 0$, consider the problem (2.1-3) on the interval $[0, T]$ with $0 < n_{ci} < M_0$ and $n_{ci} + \int f_i(|p|)dp = M_0$. Then there is $n_* > 0$ such that $n(t) \ge n_*$ on $[0, T]$ for any solution (f, n_c).*

Proof of Lemma 4.3
We know from Lemma 4.1 that

$$|\int C(f,n_c)(t,p)dp| \leq kn_c(t)(\int f(t,p)dp)(1+\int f(t,p)dp) \leq kM_0(1+M_0)n_c(t).$$

It follows that

$$\frac{dn_c}{dt} = -n_c \cdot \frac{1}{n_c}\int C(f,n_c)dp,$$

with $\frac{1}{n_c}\int C(f,n_c)dp$ bounded on $[0,T]$. And so n_c is at most exponentially decreasing with respect to time. The lemma follows. ■

After these preparations the proof of Theorem 2.2 becomes a simplified version of the proof of Theorem 2.1.

For the intermediate temperature case of Theorem 2.3,we take

$$|\bar{A}|^2 = 1 \text{ for } |p| \geq \alpha, |p_1| \geq \alpha, |p_2| \geq \alpha, |p_3| \geq \alpha,$$
$$|\bar{A}|^2 = 0 \quad \text{otherwise},$$

with the corresponding collision operator

$$C(f,n_c) = n_c \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |\bar{A}|^2\delta(p_1-p_2-p_3)\delta(E_1-E_2-E_3)[\delta(p-p_1)$$
$$-\delta(p-p_2)-\delta(p-p_3)]((1+f_1)f_2f_3 - f_1(1+f_2)(1+f_3))dp_1dp_2dp_3 \tag{4.3}$$

Global conservation of total mass and of momentum are obtained as in the general case. Moreover, multiplying (2.1) by p^2 and integrating, leads to

$$\frac{d}{dt}\int p^2 f(t,p)dp = n_c \int \chi\delta(p_1-p_2-p_3)\delta(p_1^2 = p_2^2+p_3^2+n_c)(|p_1|^2-|p_2|^2-|p_3|^2)$$
$$(f_2f_3 - f_1(1+f_2+f_3))dp_1dp_2dp_3 = -2mgn_c\int C(f,n_c)(t,p)dp = 2mgn_cn_c'.$$

Hence,

$$\int p^2 f(t,p)dp = \int p^2 f_i(p)dp + mgn_c^2(t) - mgn_{ci}^2. \tag{4.4}$$

This implies that the integral

$$\int p^2 f(t,p)dp + mgn_c(t)\int f(t,p)dp + mgM_0(2\int f(t,p)dp + n_c(t)$$

with kinetic and interaction energy components, is conserved.
The proof of Lemma 3.1 simplifies accordingly, as do the proofs of Lemma 3.2 and Lemma 3.3, now leading to the estimates

$$|\int C(f,n_c)(p)dp| \leq k\Big(\int f(p)dp\Big)^2 + \frac{kn_c}{\alpha}\int p^2 f(p)dp,$$

respectively

$$\int |(C(f,n) - C(g,n))(p)|dp \leq k \int (1 + \frac{n}{\alpha}p^2)|(f-g)(p)|dp.$$

Also the proofs of Lemma 3.4 and Lemma 3.5 simplify. From here the proof of Theorem 2.3 again becomes a simplified version of the proof of Theorem 2.1.

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