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Abstract: The analysis of plates can be performed applying three- or derived two-dimensional equations. An alternative approach can be based on a priori two-dimensional equations (direct approach). Such a theory is mathematically and physically so strong as the three-dimensional continuum mechanics, but the stiffness properties must be identified by some special techniques. Here such a theory will be applied to plates composed of functionally graded materials (FGM).

Keywords: functionally graded materials, effective properties concept.

1 Introduction

The Kirchhoff's plate theory was worked out for thin plates made of homogeneous isotropic materials. This theory yields very simple results in applications based on two fruitful assumptions: instead of the stress tensor components stress resultants (averaged stresses in the thickness direction) are introduced and the mechanical properties can be presented by an effective property - the bending stiffness D, which combines the Young's modulus E, the Poisson's ratio ν and the thickness h

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

For the Kirchhoff's theory one observes very rough limitations which are not helpful in modern applications. Various improvements of the classical theory were suggested beginning from the 40th of the last century.

Summarizing the state of the art one can conclude that there are different possibilities to introduce the two-dimensional plate equations [1]. The advantages/disadvantages are obliviously. The direct approach is related to the a priori two-dimensional equations, which are physically-based and so exact like the three-dimensional continuum mechanics equations. The constitutive equations must be identified separately, which is a non-trivial problem. The concept of effective properties can support the handling of the identification, which will be demonstrated for plates composed of FGM.

2 Based on the Direct Approach Governing Equations

The direct approach is based on the ideas of the brothers Cosserat in continuum mechanics. Considering this model one has force and moments as primary variables [1, 2]. From this it follows that both the translations and the rotations are independently. Now the two-dimensional equations can be introduced by a natural way: the geometrical relations (kinematics), the material independent balances, and the material dependent equations. Finally, one needs boundary and, maybe, initial conditions.

2.1 Material-independent Equations

The two-dimensional plate theory which allows to model homogeneous and inhomogeneous plates can be presented as follows:
Euler's equations of motion and kinematic equations

$$\nabla \cdot \boldsymbol{T} + \boldsymbol{q} = \rho \ddot{\boldsymbol{u}} + \rho \boldsymbol{\Theta}_1 \cdot \ddot{\boldsymbol{\varphi}}, \qquad \nabla \cdot \boldsymbol{M} + \boldsymbol{T}_\times + \boldsymbol{m} = \rho \boldsymbol{\Theta}_1^{\mathrm{T}} \cdot \ddot{\boldsymbol{u}} + \rho \boldsymbol{\Theta}_2 \cdot \ddot{\boldsymbol{\varphi}} \quad (1)$$

$$\boldsymbol{\mu} = \frac{1}{2}\left[\nabla \boldsymbol{u} \cdot \boldsymbol{a} + (\nabla \boldsymbol{u} \cdot a)^{\mathrm{T}}\right], \qquad \boldsymbol{\gamma} = \nabla \boldsymbol{u} \cdot \boldsymbol{n} + \boldsymbol{c} \cdot \varphi, \qquad \boldsymbol{\kappa} = \nabla \boldsymbol{\varphi} \quad (2)$$

Here $\boldsymbol{T}$, $\boldsymbol{M}$ are the tensors of forces and moments, $\boldsymbol{q}$, $\boldsymbol{m}$ are the surface loads (forces and moments), $T_\times$ is the vector invariant of the force tensor, ∇ is the nabla operator, $u, \boldsymbol{\varphi}$ are the vectors of the displacements and the rotations, $\boldsymbol{\Theta}_1, \Theta_2$ are the first and the second tensor of inertia, ρ is the density, $(\ldots)^{\mathrm{T}}$ denotes transposed and $(\dot{\ldots})$ is the time derivative. $\boldsymbol{a}$ is the first metric tensor, $\boldsymbol{n}$ is the unit normal vector, $c = -\boldsymbol{a} \times n$ is the discriminant tensor, $\boldsymbol{\mu}$, γ and $\boldsymbol{\kappa}$ are the tensor of the in-plane strains, the vector of the transverse shear strains and the tensor of the out-of-plane strains, respectively.

2.2 Two-dimensional Constitutive Equations

Limiting our discussion to the elastic behavior and small strains the following statements for the constitutive modeling can be done. At first, the strain energy can be expended in a Taylor series limited by quadratic terms. In addition, we assume that the eigen-stresses can be neglected. At second, the positive definiteness is guarantied. In this case one gets the following constitutive equations

$$\begin{aligned} T \cdot a &= A \cdot\cdot\, \boldsymbol{\mu} + B \cdot\cdot\, \kappa + \gamma \cdot \boldsymbol{\Gamma}_1, \\ T \cdot \boldsymbol{n} &= \Gamma \cdot \boldsymbol{\gamma} + \boldsymbol{\Gamma}_1 \cdot\cdot\, \mu + \boldsymbol{\Gamma}_2 \cdot\cdot\, \kappa, \\ \boldsymbol{M}^{\mathrm{T}} &= \mu \cdot\cdot\, \boldsymbol{B} + \boldsymbol{C} \cdot\cdot\, \boldsymbol{\kappa} + \gamma \cdot \boldsymbol{\Gamma}_2 \end{aligned}$$

$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are 4th rank tensors, $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2$ are 3rd rank tensors, Γ is a 2nd rank tensor of the effective stiffness properties. They depend on the material properties and the cross section geometry. In the general case these tensors contain 36 different values - a reduction is possible assuming some symmetries concerning the material behavior and the geometry.

Let us consider an orthotropic material behavior and a plane mid-surface. In this case instead of the general form of the effective stiffness tensors, one gets

$$\begin{aligned}
\boldsymbol{A} &= A_{11}\boldsymbol{a}_1\boldsymbol{a}_1 + A_{12}(\boldsymbol{a}_1\boldsymbol{a}_2 + \boldsymbol{a}_2\boldsymbol{a}_1) + A_{22}\boldsymbol{a}_2\boldsymbol{a}_2 + A_{44}\boldsymbol{a}_4\boldsymbol{a}_4,\\
\boldsymbol{B} &= B_{13}a_1a_3 + B_{14}\boldsymbol{a}_1\boldsymbol{a}_4 + B_{23}\boldsymbol{a}_2a_3 + B_{24}\boldsymbol{a}_2a_4 + B_{42}a_4a_2,\\
\boldsymbol{C} &= C_{22}\boldsymbol{a}_2\boldsymbol{a}_2 + C_{33}\boldsymbol{a}_3\boldsymbol{a}_3 + C_{34}(\boldsymbol{a}_3\boldsymbol{a}_4 + \boldsymbol{a}_4\boldsymbol{a}_3) + C_{44}\boldsymbol{a}_4\boldsymbol{a}_4,\\
\boldsymbol{\Gamma} &= \Gamma_1 a_1 + \Gamma_2\boldsymbol{a}_2, \qquad \Gamma_1 = 0, \qquad \Gamma_2 = \boldsymbol{0}
\end{aligned}$$

with $\boldsymbol{a}_1 = a = \boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2e_2$, $\boldsymbol{a}_2 = e_1e_1 - \boldsymbol{e}_2\boldsymbol{e}_2$, $\boldsymbol{a}_3 = c = e_1e_2 - \boldsymbol{e}_2\boldsymbol{e}_1$, $\boldsymbol{a}_4 = \boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1$; $\boldsymbol{e}_1, \boldsymbol{e}_2$ are unit basic vectors.

3 Stiffness Values

The individuality of each class of plates in the framework of the direct approach is expressed by the effective properties (stiffness, density, inertia terms, etc.). Let us focus our attention on the stiffness expressions. The identification of the effective stresses should be performed on the base of the properties of the real material. Let us assume Hooke's Law with material properties which depend on z. The identification of the effective properties can be performed with the help of static boundary value problems (two-dimensional, three-dimensional) and the comparison of the forces and moments (in the sense of averaged stresses or stress resultants). Finally, we get the following expressions for the classical stiffness tensor components [3]

$$\begin{aligned}
(A_{11}; -B_{13}; C_{33}) &= \frac{1}{4}\left\langle \frac{E_1 + E_2 + 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}}(1; z; z^2)\right\rangle,\\
(A_{22}; B_{24}; C_{44}) &= \frac{1}{4}\left\langle \frac{E_1 + E_2 - 2E_1\nu_{21}}{1 - \nu_{12}\nu_{21}}(1; z; z^2)\right\rangle,\\
(A_{12}; -B_{23} = B_{14}; -C_{34}) &= \frac{1}{4}\left\langle \frac{E_1 - E_2}{1 - \nu_{12}\nu_{21}} z(1; z; z^2)\right\rangle,\\
(A_{44}; -B_{42}; C_{22}) &= < G_{12}(1; z; z^2) >,
\end{aligned} \tag{3}$$

where $< \ldots >$ is the integral over the plate thickness h. In addition, two non-classical stiffness are obtained [3]

$$\Gamma_1 = \frac{1}{2}(\lambda^2 + \eta^2)\frac{A_{44}C_{22} - B_{42}^2}{A_{44}}, \quad \Gamma_2 = \frac{1}{2}(\eta^2 - \lambda^2)\frac{A_{44}C_{22} - B_{42}^2}{A_{44}} \tag{4}$$

Here η^2 and λ^2 are the minimal eigen-values of Sturm-Liouville problems

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(G_{1n}\frac{\mathrm{d}Z}{\mathrm{d}z}\right) + \eta^2 G_{12}Z = 0, \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(G_{2n}\frac{\mathrm{d}Z}{\mathrm{d}z}\right) + \lambda^2 G_{12}Z = 0, \quad \left.\frac{\mathrm{d}Z}{\mathrm{d}z}\right|_{|z|=\frac{h}{2}} = 0$$

Described above approach was applied to a FGM plate made of metal foams with nonhomogeneous distribution of porosity [3].

4 Summary and Outlook

A plate theory based an Zhilin's direct approach method in the theory of shells for FGM is introduced. It is clearly shown that based on the assumption of linear elastic behavior the identification for different foams can be realized. It was helpful that the concept of effective properties can be applied.

References

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HOLM ALTENBACH
Lehrstuhl fur Technische Mechanik
Zentrum fur Ingenieurwissenschaften
Martin-Luther-Universitt Halle-Wittenberg
D-06099 Halle (Saale)
Germany

e-mail: holm.altenbach@iw.uni-halle.de

VICTOR A. EREMEYEV
South Scientific Center of RASci & South Federal University
8a, Milchakova St.
344090 Rostov on Don
Russian Federation

e-mail: eremeyev@math.rsu.ru