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A REMARK ON COMPACT KÄHLER MANIFOLDS WITH NEF ANTICANONICAL BUNDLES AND ITS APPLICATIONS

JUNYAN CAO

ABSTRACT. Let (X, ω_X) be a compact Kähler manifold such that the anticanonical bundle $-K_X$ is nef. We prove that the slopes of the Harder-Narasimhan filtration of the tangent bundle with respect to a polarization of the form ω_X^{n-1} are semi-positive. As an application, we give a characterization of rationally connected compact Kähler manifolds with nef anticanonical bundles. As another application, we give a simple proof of the surjectivity of the Albanese map.

0. INTRODUCTION

Compact Kähler manifolds with semipositive anticanonical bundles have been studied in depth in [CDP12], where a rather general structure theorem for this type of manifolds has been obtained. It is a natural question to find some similar structure theorems for compact Kähler manifolds with nef anticanonical bundles. Obviously, we cannot hope the same structure theorem for this type of manifolds (cf. [CDP12, Remark 1.7]). It is conjectured that the Albanese map is a submersion and that the fibers exhibit no variation of their complex structure (cf. [CH13] for some special cases).

In relation with the structure of compact Kähler manifolds with nef anticanonical bundles, it is conjectured in [Pet12, Conj. 1.3] that the tangent bundles of projective manifolds with nef anticanonical bundles are generically nef. We first recall the notion of generically semipositive (resp. strictly positive) (cf. [Miy87, Section 6])

Definition 0.1. *Let X be a compact Kähler manifold and let E be a vector bundle on X . Let $\omega_1, \dots, \omega_{n-1}$ be Kähler classes. Let*

$$0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_s = E \quad (\text{resp. } \Omega_X^1)$$

be the Harder-Narasimhan semistable filtration with respect to $(\omega_1, \dots, \omega_{n-1})$. We say that E is generically $(\omega_1, \dots, \omega_{n-1})$ -semipositive (resp. strictly positive), if

$$\int_X c_1(\mathcal{E}_{i+1}/\mathcal{E}_i) \wedge \omega_1 \wedge \dots \wedge \omega_{n-1} \geq 0 \quad (\text{resp. } > 0) \quad \text{for all } i.$$

If $\omega_1 = \dots = \omega_{n-1}$, we write the polarization as ω_1^{n-1} for simplicity.

We rephrase [Pet12, Conj. 1.3] as follows

Conjecture 0.1. *Let X be a projective manifold with nef anticanonical bundle. Then T_X is generically (H_1, \dots, H_{n-1}) -semipositive for any $(n-1)$ -tuple of ample divisors H_1, \dots, H_{n-1} .*

In this article, we first give a partial positive answer to this conjecture. More precisely, we prove

Theorem 0.2. *Let X be a compact Kähler manifold with nef anticanonical bundle (resp. nef canonical bundle). Then T_X (resp. Ω_X^1) is generically ω_X^{n-1} -semipositive for any Kähler class ω_X .*

Remark 0.3. *If X is projective and K_X is nef, Theorem 0.2 is a special case of [Miy87, Cor. 6.4]. Here we prove it for arbitrary compact Kähler manifolds with nef canonical bundles. If $-K_X$ is nef, Theorem 0.2 is a new result even for algebraic manifolds.*

As an application, we give a characterization of rationally connected compact Kähler manifolds with nef anticanonical bundles.

Proposition 0.4. *Let X be a compact Kähler manifold with nef anticanonical bundle. Then the following four conditions are equivalent*

- (i): $H^0(X, (T_X^*)^{\otimes m}) = 0$ for all $m \geq 1$.
- (ii): X is rationally connected.
- (iii): T_X is generically ω_X^{n-1} -strictly positive for some Kähler class ω_X .
- (iv): T_X is generically ω_X^{n-1} -strictly positive for any Kähler class ω_X .

Remark 0.5. *Mumford has in fact stated the following conjecture which would generalize the first part of Proposition 0.4: for any compact Kähler manifold X , the variety X is rationally connected if and only if*

$$H^0(X, (T_X^*)^{\otimes m}) = 0 \quad \text{for all } m \geq 1.$$

We thus prove the conjecture of Mumford under the assumption that $-K_X$ is nef.

As another application, we study the pseudo-effectiveness of $c_2(T_X)$. It is conjectured by Kawamata that

Conjecture 0.6. *If X is a compact Kähler manifold with nef anticanonical bundle. Then*

$$\int_X (c_2(T_X)) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \geq 0,$$

for all nef classes $\omega_1, \dots, \omega_{n-2}$.

When $\dim X = 3$, this conjecture was partially solved by [Xie05]. Using Theorem 0.2 and an idea of A.Höring, we prove

Proposition 0.7. *Let (X, ω_X) be a compact Kähler manifold with nef anticanonical bundle. Then*

$$(1) \quad \int_X c_2(T_X) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} \geq 0$$

for $\epsilon > 0$ small enough. Moreover, if X is projective and the equality holds for some $\epsilon > 0$ small enough, then after a finite étale cover, X is either a torus or a smooth \mathbb{P}^1 -fibration over a torus.

As the last application, we study the Albanese map of compact Kähler manifolds with nef anticanonical bundles. It should be first mentioned that the surjectivity of the Albanese map has been studied in depth by several authors. If X is assumed to be projective, the surjectivity of the Albanese map was proved by Q.Zhang in [Zha96]. Still under the assumption that X is projective, [LTZZ10] proved that the Albanese map is equidimensional and all the fibres are reduced. Recently, M.Păun [Pău12] proved the surjectivity for arbitrary compact Kähler manifolds with nef

anticanonical bundles, as a corollary of a powerful method based on a direct image argument. Unfortunately, it is hard to get information for the singular fibers from his proof. Using Theorem 0.2, we give a new proof of the surjectivity for the Kähler case, and prove that the map is smooth outside a subvariety of codimension at least 2.

Proposition 0.8. *Let X be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map is surjective, and smooth outside a subvariety of codimension at least 2. In particular, the fibers of the Albanese map are connected and reduced in codimension 1.*

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1. PREPARATORY LEMMAS

The results in this section should be well known to experts. For the convenience of readers, we give an account of the proofs here.

Lemma 1.1. *Let (X, ω) be a compact Kähler manifold of dimension n and let E be a torsion free coherent sheaf. Let D_1, \dots, D_{n-1} be nef classes in $H^{1,1}(X, \mathbb{Q})$ and let A be a Kähler class. Let a be a sufficiently small positive number. Then the Harder-Narasimhan semistable filtration of E with respect to $(D_1 + a \cdot A, \dots, D_{n-1} + a \cdot A)$ is independent of a .*

Remark 1.2. *If A has rational coefficients, Lemma 1.1 is proved in [KMM04]. When A is not necessarily rational, the proof turns out to be a little bit more complicated. We begin with the following easy observation.*

Lemma 1.3. *In the situation of Lemma 1.1, let we take $k \in \{0, 1, 2, \dots, n\}$ arbitrary. Then we can find a basis $\{e_1, \dots, e_s\}$ of $H^{2k}(X, \mathbb{Q})$ depending only on A^k , such that*

$$(i): A^k = \sum_{i=1}^s \lambda_i \cdot e_i \text{ for some } \lambda_i > 0.$$

(ii): *Let \mathcal{F} be a torsion free coherent sheaf. Set*

$$D^t := \sum_{i_1 < i_2 < \dots < i_t} D_{i_1} \cdot D_{i_2} \cdot \dots \cdot D_{i_t} \quad \text{for any } t \in \mathbb{N},$$

and

$$(2) \quad a_i(\mathcal{F}) := c_1(\mathcal{F}) \cdot D^{n-k-1} \cdot e_i.$$

Then the subset S of the set \mathbb{Q} of rational numbers such that

$$S := \{a_i(\mathcal{F}) \mid \mathcal{F} \subset E, i \in \{1, \dots, s\}\}$$

is bounded from above, and the denominator (assumed positive) of all elements of S is uniformly bounded from above. Moreover, if $\{\mathcal{F}_t\}_t$ is a sequence of coherent subsheaves of E such that the set $\{c_1(\mathcal{F}_t) \cdot D^{n-k-1} \cdot A^k\}_t$ is bounded from below, then $\{c_1(\mathcal{F}_t) \cdot D^{n-k-1} \cdot A^k\}_t$ is a finite subset of \mathbb{Q} .

Proof. We can take a basis $\{e_i\}_{i=1}^s$ of $H^{2k}(X, \mathbb{Q})$ in a neighborhood of A^k , such that

$$A^k = \sum_{i=1}^s \lambda_i \cdot e_i \quad \text{for some } \lambda_i > 0$$

and $(e_i)^{k,k}$ can be represented by a smooth (k, k) -positive form on X (cf. [adg, Chapter 3, Def 1.1] for the definition of (k, k) -positivity), where $(e)^{k,k}$ is the projection of e in $H^{k,k}(X, \mathbb{R})$. We now check that $\{e_i\}_{i=1}^s$ satisfies the lemma. By construction, (i) is satisfied. As for (ii), since e_i and D_i are fixed and $c_1(\mathcal{F}) \in H^{1,1}(X, \mathbb{Z})$, the denominator of any elements in S is uniformly bounded from above. Thanks to (2), we know that S is bounded from above by using the same argument as in [Kob87, Lemma 7.16, Chapter 5]. For the last part of (ii), since

$$c_1(\mathcal{F}_t) \cdot D^{n-k-1} \cdot A^k = \sum_i a_i(\mathcal{F}_t) \cdot \lambda_i,$$

we obtain that $\{\sum_i a_i(\mathcal{F}_t) \cdot \lambda_i\}_t$ is uniformly bounded. Since $\lambda_i > 0$ and $a_i(\mathcal{F}_t)$ is uniformly upper bounded, we obtain that $a_i(\mathcal{F}_t)$ is uniformly bounded. Combining this with the fact already proved that the denominator of any elements in S is uniformly bounded, $\{c_1(\mathcal{F}_t) \cdot D^{n-k-1} \cdot A^k\}_t$ is thus finite. The lemma is proved. \square

Proof of Lemma 1.1. Let $\{a_p\}_{p=1}^{+\infty}$ be a decreasing positive sequence converging to 0. Let $\mathcal{F}_p \subset E$ be the first piece of the Harder-Narasimhan filtration of E with respect to $(D_1 + a_p \cdot A, \dots, D_{n-1} + a_p \cdot A)$. Set $D^k := \sum_{i_1 < i_2 < \dots < i_k} D_{i_1} \cdot D_{i_2} \cdot \dots \cdot D_{i_k}$.

Then

$$c_1(\mathcal{F}_p) \wedge (D_1 + a_p \cdot A) \wedge \dots \wedge (D_{n-1} + a_p \cdot A) = \sum_{k=0}^n (a_p)^k \cdot c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k.$$

By passing to a subsequence, we can suppose that $\text{rank } \mathcal{F}_p$ is constant. To prove Lemma 1.1, it is sufficient to prove that for any k , after passing to a subsequence, the intersection number $\{c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k\}_p$ is stationary when p is large enough¹.

We prove it by induction on k . Note first that, by [Kob87, Lemma 7.16, Chapter 5], the set $\{c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k\}_{p,k}$ is upper bounded. If $k = 0$, since

$$\begin{aligned} & c_1(\mathcal{F}_p) \wedge (D_1 + a_p \cdot A) \wedge \dots \wedge (D_{n-1} + a_p \cdot A) \\ & \geq \frac{\text{rank}(\mathcal{F}_p)}{\text{rank } E} c_1(E) \wedge (D_1 + a_p \cdot A) \wedge \dots \wedge (D_{n-1} + a_p \cdot A), \end{aligned}$$

and $\lim_{p \rightarrow +\infty} a_p = 0$, the upper boundedness of $\{c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k\}_{p,k}$ implies that the set $\{c_1(\mathcal{F}_p) \wedge D^{n-1}\}_{p=1}^{\infty}$ is bounded from below. Then (ii) of Lemma 1.3 implies that $\{c_1(\mathcal{F}_p) \wedge D^{n-1}\}_{p=1}^{\infty}$ is a finite set. By the pigeon hole principle, after passing to a subsequence, the set $\{c_1(\mathcal{F}_p) \wedge D^{n-1}\}_{p=1}^{\infty}$ is stationary. Now we suppose that

¹In fact, if $\mathcal{F} \subset E$ is always the first piece of semistable filtration with respect to the polarization $(D_1 + a_p \cdot A, \dots, D_{n-1} + a_p \cdot A)$ for a positive sequence $\{a_p\}_{p=0}^{+\infty}$ converging to 0, and $\mathcal{G} \subset E$ is always the first piece of semistable filtration for another sequence $\{b_p\}_{p=0}^{+\infty}$ converging to 0, the stability condition implies that

$$(3) \quad \text{rank}(\mathcal{G}) \cdot c_1(\mathcal{F}) \cdot D^k \cdot A^{n-k-1} = \text{rank}(\mathcal{F}) \cdot c_1(\mathcal{G}) \cdot D^k \cdot A^{n-k-1}$$

for any k . Therefore \mathcal{G} has the same slope as \mathcal{F} with respect to $(D_1 + a \cdot A, \dots, D_{n-1} + a \cdot A)$ for any $a > 0$. Then $\mathcal{F} = \mathcal{G}$.

$\{c_1(\mathcal{F}_p) \wedge D^{n-t-1} \wedge A^t\}_p$ is constant for $p \geq p_0$, where $t \in \{0, \dots, k-1\}$. Our aim is to prove that after passing to a subsequence,

$$\{c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k\}_{p=1}^\infty$$

is stationary. By definition, we have

$$\begin{aligned} & c_1(\mathcal{F}_p) \wedge (D_1 + a_p \cdot A) \wedge \dots \wedge (D_{n-1} + a_p \cdot A) \\ & \geq c_1(\mathcal{F}_{p_0}) \wedge (D_1 + a_p \cdot A) \wedge \dots \wedge (D_{n-1} + a_p \cdot A) \quad \text{for any } p \geq p_0. \end{aligned}$$

Since $\{c_1(\mathcal{F}_p) \wedge D^{n-t-1} \wedge A^t\}_p$ is constant for $p \geq p_0$ when $t \in \{0, \dots, k-1\}$, we obtain

$$(4) \quad \begin{aligned} & c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k + \sum_{i \geq 1} (a_p)^i \cdot c_1(\mathcal{F}_p) \wedge D^{n-k-1-i} \wedge A^{k+i} \geq \\ & c_1(\mathcal{F}_{p_0}) \wedge D^{n-k-1} \wedge A^k + \sum_{i \geq 1} (a_p)^i \cdot c_1(\mathcal{F}_{p_0}) \wedge D^{n-k-1-i} \wedge A^{k+i} \end{aligned}$$

for any $p \geq p_0$. Therefore the upper boundedness of $\{c_1(\mathcal{F}_p) \wedge D^{n-k-1-i} \wedge A^{k+i}\}_{p,i}$ implies that $\{c_1(\mathcal{F}^p) \wedge D^{n-k-1} \wedge A^k\}_{p=1}^{+\infty}$ is lower bounded. Therefore

$$\{c_1(\mathcal{F}^p) \wedge D^{n-k-1} \wedge A^k\}_{p=1}^{+\infty}$$

is uniformly bounded. Using (ii) of Lemma 1.3, $\{c_1(\mathcal{F}^p) \wedge D^{n-k-1} \wedge A^k\}_{p=1}^{+\infty}$ is a finite set. By the pigeon hole principle, after passing to a subsequence,

$$\{c_1(\mathcal{F}_p) \wedge D^{n-k-1} \wedge A^k\}_{p=1}^\infty$$

is stationary. The lemma is proved. \square

By the same argument as above, we can easily prove that

Lemma 1.4. *Let (X, ω) be a compact Kähler manifold and let E be a torsion free ω -stable coherent sheaf. Then E is also stable with respect to a small perturbation of ω .*

Proof. Using (ii) of Lemma 1.3 (by taking $D_i = 0$, $A = \omega$), we obtain that

$$\sup \left\{ \frac{1}{\text{rank}(\mathcal{F})} \int_X c_1(\mathcal{F}) \wedge \omega^{n-1} \mid \mathcal{F} \text{ a coherent subsheaf of } E \text{ with strictly smaller rank} \right\}$$

is strictly smaller than the slope of E . Therefore E is also stable with respect to a small perturbation of ω . \square

Remark 1.5. *If the Kähler metric $\omega \in H^2(X, \mathbb{Z})$, the lemma comes directly from the fact that the set*

$$\left\{ \int_X c_1(\mathcal{F}) \wedge \omega^{n-1} \mid \mathcal{F} \text{ a coherent subsheaf of } E \text{ with strictly smaller rank} \right\}$$

takes value in \mathbb{Z} .

We recall a regularization lemma proved in [Jac10, Prop. 3].

Lemma 1.6. *Let E be a vector bundle on a compact complex manifold X and \mathcal{F} be a subsheaf of E with torsion free quotient. Then after a finite number of blowups $\pi : \tilde{X} \rightarrow X$, there exists a holomorphic subbundle F of $\pi^*(E)$ containing $\pi^*(\mathcal{F})$ with a holomorphic quotient bundle, such that $\pi_*(F) = \mathcal{F}$ in codimension 1.*

We need another lemma which is proved in full generality in [DPS94, Prop. 1.15]. For completeness, we give the proof here in an over simplified case, but the idea is the same.

Lemma 1.7. *Let (X, ω) be a compact Kähler manifold. Let E be an extension of two vector bundles E_1, E_2*

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0.$$

We suppose that there exist two smooth metrics h_1, h_2 on E_1 and E_2 , such that

$$(5) \quad \frac{i\Theta_{h_1}(E_1) \wedge \omega^{n-1}}{\omega^n} \geq c_1 \cdot \text{Id}_{E_1} \quad \text{and} \quad \frac{i\Theta_{h_2}(E_2) \wedge \omega^{n-1}}{\omega^n} \geq c_2 \cdot \text{Id}_{E_2}$$

pointwise. Then for any $\epsilon > 0$, there exists a smooth metric h_ϵ on E such that

$$\frac{i\Theta_{h_\epsilon}(E) \wedge \omega^{n-1}}{\omega^n} \geq (\min(c_1, c_2) - \epsilon) \cdot \text{Id}_E,$$

and

$$(6) \quad \|i\Theta_{h_\epsilon}(E)\|_{L^\infty} \leq C \cdot (\|i\Theta_{h_1}(E_1)\|_{L^\infty} + \|i\Theta_{h_2}(E_2)\|_{L^\infty})$$

for some uniform constant C independent of ϵ .

Proof. Let $[E] \in H^1(X, \text{Hom}(E_2, E_1))$ be the element representing E in the extension group. Let E_s be another extension of E_1 and E_2 , such that $[E_s] = s \cdot [E]$, where $s \in \mathbb{C}^*$. Then there exists an isomorphism between these two vector bundles (cf. [adg, Remark 14.10, Chapter V]). We denote the isomorphism by

$$\varphi_s : E \rightarrow E_s.$$

Thanks to (5), if $|s|$ is small enough with respect to ϵ , we can find a smooth metric h_s on E_s satisfying

$$(7) \quad \frac{i\Theta_{h_s}(E_s) \wedge \omega^{n-1}}{\omega^n} \geq (\min(c_1, c_2) - \epsilon) \cdot \text{Id}_{E_s}$$

and

$$(8) \quad \|i\Theta_{h_s}(E_s)\|_{L^\infty} \leq C \cdot (\|i\Theta_{h_1}(E_1)\|_{L^\infty} + \|i\Theta_{h_2}(E_2)\|_{L^\infty})$$

for some uniform constant C (cf. [adg, Prop 14.9, Chapter V]). Let $h = \varphi_s^*(h_s)$ be the induced metric on E . Then for any $\alpha \in E$,

$$(9) \quad \begin{aligned} \frac{\langle i\Theta_h(E)\alpha, \alpha \rangle_h}{\langle \alpha, \alpha \rangle_h} &= \frac{\langle \varphi_s^{-1} \circ i\Theta_{h_s}(E_s)\varphi_s(\alpha), \alpha \rangle_h}{\langle \alpha, \alpha \rangle_h} \\ &= \frac{\langle i\Theta_{h_s}(E_s)\varphi_s(\alpha), \varphi_s(\alpha) \rangle_{h_s}}{\langle \varphi_s(\alpha), \varphi_s(\alpha) \rangle_{h_s}}. \end{aligned}$$

Combining this with (7), we get

$$\frac{\langle i\Theta_h(E)\alpha, \alpha \rangle_h \wedge \omega^{n-1}}{\langle \alpha, \alpha \rangle_h \cdot \omega^n} \geq (\min(c_1, c_2) - \epsilon) \cdot \text{Id}_E.$$

Moreover, (9) implies also (6). The lemma is proved. \square

We recall the following well-known equality in Kähler geometry.

Proposition 1.8. *Let (X, ω_X) be a Kähler manifold of dimension n , R be the curvature tensor and Ric be the Ricci tensor (cf. the definition of [Zhe00, Section 7.5]). Let $i\Theta_{\omega_X}(T_X)$ be the curvature of T_X induced by ω_X . We have*

$$\left\langle \frac{i\Theta_{\omega_X}(T_X) \wedge \omega_X^{n-1}}{\omega_X^n} u, v \right\rangle_{\omega_X} = \text{Ric}(u, \bar{v}).$$

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of T_X with respect to ω_X . By definition, we have

$$\left\langle \frac{i\Theta_{\omega_X}(T_X) \wedge \omega_X^{n-1}}{\omega_X^n} u, v \right\rangle_{\omega_X} = \sum_{1 \leq i \leq n} \langle i\Theta_{\omega_X}(T_X)u, v \rangle(e_i, \bar{e}_i) = \sum_{1 \leq i \leq n} R(e_i, \bar{e}_i, u, \bar{v}).$$

By definition of the Ricci curvature (cf. [Zhe00, Page 180]), we have

$$\text{Ric}(u, \bar{v}) = \sum_{1 \leq i \leq n} R(u, \bar{v}, e_i, \bar{e}_i).$$

Combining this with the First Bianchi equality

$$\sum_{1 \leq i \leq n} R(e_i, \bar{e}_i, u, \bar{v}) = \sum_{1 \leq i \leq n} R(u, \bar{v}, e_i, \bar{e}_i),$$

the proposition is proved. \square

2. MAIN THEOREM

We first prove Theorem 0.2 in the case when $-K_X$ is nef.

Theorem 2.1. *Let (X, ω) be a compact n -dimensional Kähler manifold with nef anticanonical bundle. Let*

$$(10) \quad 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = T_X$$

be a filtration of torsion-free subsheaves such that $\mathcal{E}_{i+1}/\mathcal{E}_i$ is an ω -stable torsion-free subsheaf of T_X/\mathcal{E}_i of maximal slope². Let

$$\mu(\mathcal{E}_{i+1}/\mathcal{E}_i) = \frac{1}{\text{rank}(\mathcal{E}_{i+1}/\mathcal{E}_i)} \int_X c_1(\mathcal{E}_{i+1}/\mathcal{E}_i) \wedge \omega^{n-1}$$

be the slope of $\mathcal{E}_{i+1}/\mathcal{E}_i$ with respect to ω^{n-1} . Then

$$\mu(\mathcal{E}_{i+1}/\mathcal{E}_i) \geq 0 \quad \text{for all } i.$$

Proof. We first consider a simplified case.

Case 1 : (10) is regular, i.e., all $\mathcal{E}_i, \mathcal{E}_{i+1}/\mathcal{E}_i$ are vector bundles.

By the stability condition, to prove the theorem, it is sufficient to prove that

$$(11) \quad \int_X c_1(T_X/\mathcal{E}_i) \wedge \omega^{n-1} \geq 0 \quad \text{for any } i.$$

Thanks to the nefness of $-K_X$, for any $\epsilon > 0$, there exists a Kähler metric ω_ϵ in the same class of ω such that (cf. the proof of [DPS93, Thm. 1.1])

$$(12) \quad \text{Ric}_{\omega_\epsilon} \geq -\epsilon\omega_\epsilon,$$

²Using (ii) of Lemma 1.3 (by taking $D_i = 0$ and $A = \omega$), one can prove the existence of such a filtration by a standard argument [HN75].

where $\text{Ric}_{\omega_\epsilon}$ is the Ricci curvature with respect to the metric ω_ϵ . Thanks to Proposition 1.8, we have

$$\left\langle \frac{i\Theta_{\omega_\epsilon}(T_X) \wedge \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \alpha, \alpha \right\rangle_{\omega_\epsilon} = \text{Ric}_{\omega_\epsilon}(\alpha, \bar{\alpha}).$$

Then (12) implies a pointwise estimate

$$(13) \quad \frac{i\Theta_{\omega_\epsilon}(T_X) \wedge \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \geq -\epsilon \cdot \text{Id}_{T_X}.$$

Taking the induced metric on T_X/\mathcal{E}_i (we also denote it by ω_ϵ), we get (cf. [adg, Chapter V])

$$(14) \quad \frac{i\Theta_{\omega_\epsilon}(T_X/\mathcal{E}_i) \wedge \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \geq -\epsilon \cdot \text{Id}_{T_X/\mathcal{E}_i}.$$

Therefore

$$\int_X c_1(T_X/\mathcal{E}_i) \wedge \omega_\epsilon^{n-1} \geq -\text{rank}(T_X/\mathcal{E}_i) \cdot \epsilon \int_X \omega_\epsilon^n.$$

Combining this with the fact that $[\omega_\epsilon] = [\omega]$, we get

$$(15) \quad \int_X c_1(T_X/\mathcal{E}_i) \wedge \omega^{n-1} = \int_X c_1(T_X/\mathcal{E}_i) \wedge \omega_\epsilon^{n-1} \geq -C\epsilon,$$

for some constant C . Letting $\epsilon \rightarrow 0$, (11) is proved.

Case 2: The general case

By Lemma 1.6, there exists a desingularization $\pi : \tilde{X} \rightarrow X$, such that $\pi^*(T_X)$ admits a filtration:

$$(16) \quad 0 \subset E_1 \subset E_2 \subset \cdots \subset \pi^*(T_X),$$

where $E_i, E_i/E_{i-1}$ are vector bundles and $\pi_*(E_i) = \mathcal{E}_i$ outside an analytic subset of codimension at least 2. Let $\tilde{\mu}$ be the slope with respect to $\pi^*(\omega)$. Then

$$(17) \quad \tilde{\mu}(E_i/E_{i-1}) = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$$

(cf. [Jac10, Lemma 2]), and E_i/E_{i-1} is a $\pi^*(\omega)$ -stable subsheaf of $\pi^*(T_X)/E_{i-1}$ of maximal slope (cf. Remark 2.2 after the proof).

We now prove that $\tilde{\mu}(E_i/E_{i-1}) \geq 0$. Thanks to (13), for any $\epsilon > 0$ small enough, we have

$$\frac{i\Theta_{\pi^*\omega_\epsilon}(\pi^*(T_X)) \wedge (\pi^*\omega_\epsilon)^{n-1}}{(\pi^*\omega_\epsilon)^n} \geq -\epsilon \cdot \text{Id}_{\pi^*(T_X)},$$

which implies that

$$(18) \quad \frac{i\Theta_{\pi^*\omega_\epsilon}(\pi^*(T_X)/E_i) \wedge (\pi^*\omega_\epsilon)^{n-1}}{(\pi^*\omega_\epsilon)^n} \geq -\epsilon \cdot \text{Id}_{\pi^*(T_X)/E_i}.$$

By the same argument as in Case 1, (18) and the maximal slope condition of E_{i+1}/E_i in $\pi^*(T_X)/E_i$ implies that

$$\tilde{\mu}(E_{i+1}/E_i) = \frac{1}{\text{rank}(E_{i+1}/E_i)} \int_{\tilde{X}} c_1(E_{i+1}/E_i) \wedge \pi^*\omega^{n-1} \geq -C\epsilon$$

for some constant C independent of ϵ . Letting $\epsilon \rightarrow 0$, we get $\tilde{\mu}(E_{i+1}/E_i) \geq 0$. Combining this with (17), the theorem is proved. \square

Remark 2.2. *In the situation of (16) in Theorem 2.1, we would like to prove that E_i/E_{i-1} is also stable for $\pi^*\omega + \epsilon\omega_{\tilde{X}}$ for any $\epsilon > 0$ small enough.*

Proof of Remark 2.2. Let \mathcal{F} be any coherent sheaf satisfying

$$(19) \quad E_{i-1} \subset \mathcal{F} \subset E_i \quad \text{and} \quad \text{rank } \mathcal{F} < \text{rank } E_i.$$

It is sufficient to prove that

$$(20) \quad \frac{1}{\text{rank } \mathcal{F}/E_{i-1}} \int_{\tilde{X}} c_1(\mathcal{F}/E_{i-1}) \wedge (\pi^* \omega + \epsilon \omega_{\tilde{X}})^{n-1} < \tilde{\mu}(E_i/E_{i-1})$$

for a uniform $\epsilon > 0$, where $\tilde{\mu}$ is the slope with respect to $\pi^*(\omega)$ as defined in Theorem 2.1.

Thanks to the formula

$$\int_{\tilde{X}} c_1(\mathcal{F}) \wedge \pi^*(\omega)^{n-1} = \int_X c_1(\pi_*(\mathcal{F})) \wedge \omega^{n-1},$$

Lemma 1.4 and the stability condition of $\mathcal{E}_i/\mathcal{E}_{i-1}$ imply that the upper bound of the set

$$\left\{ \frac{1}{\text{rank } \mathcal{F}/E_{i-1}} \int_{\tilde{X}} c_1(\mathcal{F}/E_{i-1}) \wedge \pi^* \omega^{n-1} \mid \mathcal{F} \text{ satisfies (19)} \right\}$$

is strictly smaller than $\tilde{\mu}(E_i/E_{i-1})$. Combining this with the fact that

$$\int_{\tilde{X}} c_1(\mathcal{F}) \wedge \omega_{\tilde{X}}^s \wedge \pi^*(\omega)^{n-s-1}$$

is uniformly bounded from above for any s , (20) is proved. \square

We now prove Theorem 0.2 in the case when K_X is nef.

Theorem 2.3. *Let (X, ω) be a compact Kähler manifold with nef canonical bundle. Let*

$$(21) \quad 0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = \Omega_X^1$$

be a filtration of torsion-free subsheaves such that $\mathcal{E}_{i+1}/\mathcal{E}_i$ is an ω -stable torsion-free subsheaf of T_X/\mathcal{E}_i of maximal slope. Then

$$\int_X c_1(\mathcal{E}_{i+1}/\mathcal{E}_i) \wedge \omega^{n-1} \geq 0 \quad \text{for all } i.$$

Proof. The proof is almost the same as Theorem 2.1. First of all, since K_X is nef, for any $\epsilon > 0$, there exists a smooth function ψ_ϵ on X , such that

$$\text{Ric}_\omega + i\partial\bar{\partial}\psi_\epsilon \leq \epsilon\omega.$$

By solving the Monge-Ampère equation

$$(22) \quad (\omega + i\partial\bar{\partial}\varphi_\epsilon)^n = \omega^n \cdot e^{-\psi_\epsilon - \epsilon\varphi_\epsilon},$$

we can construct a new Kähler metric ω_ϵ in the cohomology class of ω :

$$\omega_\epsilon := \omega + i\partial\bar{\partial}\varphi_\epsilon.$$

Thanks to (22), we have

$$\begin{aligned} \text{Ric}_{\omega_\epsilon} &= \text{Ric}_\omega + i\partial\bar{\partial}\psi_\epsilon + \epsilon i\partial\bar{\partial}\varphi_\epsilon \\ &\leq \epsilon\omega + \epsilon i\partial\bar{\partial}\varphi_\epsilon = \epsilon\omega_\epsilon. \end{aligned}$$

We first suppose that (21) is regular, i.e., \mathcal{E}_i and $\mathcal{E}_{i+1}/\mathcal{E}_i$ are free for all i . Let $\alpha \in \Omega_{X,x}^1$ for some point $x \in X$ with norm $\|\alpha\|_{\omega_\epsilon} = 1$ and let α^* be the dual of α with respect to ω_ϵ . Then we have also a pointwise estimate at x :

$$\left\langle \frac{i\Theta_{\omega_\epsilon}(\Omega_X) \wedge \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \alpha, \alpha \right\rangle = \left\langle -\frac{i\Theta_{\omega_\epsilon}(T_X) \wedge \omega_\epsilon^{n-1}}{\omega_\epsilon^n} \alpha^*, \alpha^* \right\rangle$$

$$= -\operatorname{Ric}_{\omega_\epsilon}(\alpha^*, \alpha^*) \geq -\epsilon.$$

By the same proof as in Theorem 2.1, $\int_X c_1(\mathcal{E}_{i+1}/\mathcal{E}_i) \wedge \omega^{n-1}$ is semi-positive for any i . For the general case, the proof follows exactly the same line as in Theorem 2.1. \square

3. APPLICATIONS

As an application, we give a characterization of rationally connected compact Kähler manifolds with nef anticanonical bundles.

Proposition 3.1. *Let X be a compact Kähler manifold with nef anticanonical bundle. Then the following four conditions are equivalent*

- (i): $H^0(X, (T_X^*)^{\otimes m}) = 0$ for all $m \geq 1$.
- (ii): X is rationally connected.
- (iii): T_X is generically ω_X^{n-1} strictly positive for some Kähler class ω_X .
- (iv): T_X is generically ω_X^{n-1} strictly positive for any Kähler class ω_X .

Proof. The implications (iv) \Rightarrow (iii), (ii) \Rightarrow (i) are obvious. For the implication (iii) \Rightarrow (ii), we first note that (iii) implies (i) by Bochner technique. Therefore X is projective and any Kähler class can be approximated by rational Kähler classes. Using [BM01, Theorem 0.1], (iii) implies (ii).

We now prove that (i) \Rightarrow (iv). Let ω be any Kähler class. Let

$$(23) \quad 0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = T_X$$

be the Harder-Narasimhan semistable filtration with respect to ω^{n-1} . To prove (iv), it is sufficient to prove

$$\int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge \omega^{n-1} > 0.$$

Recall that Theorem 2.1 implies already that

$$\int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge \omega^{n-1} \geq 0.$$

We suppose by contradiction that

$$(24) \quad \int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge \omega^{n-1} = 0.$$

Let $\alpha \in H^{1,1}(X, \mathbb{R})$. We define new Kähler metrics $\omega_\epsilon = \omega + \epsilon\alpha$ for $|\epsilon|$ small enough. Thanks to [Miy87, Cor. 2.3], the ω_ϵ^{n-1} -semistable filtration of T_X is a refinement of (23). Therefore, Theorem 2.1 implies that

$$\int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge (\omega + \epsilon\alpha)^{n-1} \geq 0$$

for $|\epsilon|$ small enough. Then (24) implies that

$$\int_X c_1(T_X/\mathcal{E}_{s-1}) \wedge \omega^{n-2} \wedge \alpha = 0 \quad \text{for any } \alpha \in H^{1,1}(X, \mathbb{R}).$$

By the Hodge index theorem, we obtain that $c_1(T_X/\mathcal{E}_{s-1}) = 0$. By duality, there exists a subsheaf $\mathcal{F} \subset \Omega_X^1$, such that

$$(25) \quad c_1(\mathcal{F}) = 0 \quad \text{and} \quad \det \mathcal{F} \subset (T_X^*)^{\otimes \operatorname{rank} \mathcal{F}}.$$

Observing that $H^1(X, \mathcal{O}_X) = 0$ by assumption, i.e., the group $\text{Pic}^0(X)$ is trivial, hence (25) implies the existence of an integer m such that $(\det \mathcal{F})^{\otimes m}$ is a trivial line bundle. Observing moreover that $(\det \mathcal{F})^{\otimes m} \subset (T_X^*)^{\otimes m \cdot \text{rank } \mathcal{F}}$, then

$$H^0(X, (T_X^*)^{\otimes m \cdot \text{rank } \mathcal{F}}) \neq 0,$$

which contradicts with (i). The implication (i) \Rightarrow (iv) is proved. \square

Remark 3.2. *One can also prove the implication (iii) \Rightarrow (ii) without using the profound theorem of [BM01]. We give the proof in Appendix A.*

The above results lead to the following question about rationally connected manifolds with nef anticanonical bundles.

Question 3.3. *Let X be a smooth compact manifold. Then T_X is generically ω^{n-1} -strictly positive for any Kähler metric ω if and only if X is rationally connected manifold with nef anticanonical bundle.*

As a second application, we study a Conjecture of Y.Kawamata (cf. [Miy87, Thm. 1.1] for the dual case and [Xie05] for dimension 3.)

Conjecture 3.4. *If X is a compact Kähler manifold with nef anticanonical bundle. Then*

$$\int_X c_2(T_X) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \geq 0$$

for all nef classes $\omega_1, \dots, \omega_{n-1}$.

Using Theorem 2.1 and a more refined argument as in [Miy87, Thm.6.1], we can prove

Proposition 3.5. *Let (X, ω_X) be a compact Kähler manifold with nef anticanonical bundle and let ω_X be a Kähler metric. Then*

$$(26) \quad \int_X c_2(T_X) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-2} \geq 0$$

for any $\epsilon > 0$ small enough.

Proof. Let nd be the numerical dimension of $-K_X$. If $\text{nd} = 0$, then $c_1(K_X) = 0$. It is well know that (26) holds in this case (cf. [Miy87] or [Kob87]). From now on, we suppose that $\text{nd} \geq 1$. Let

$$(27) \quad 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_l = T_X$$

be a stable filtration of the Harder-Narasimhan semistable filtration of T_X with respect to the polarization $(c_1(-K_X) + \epsilon \omega_X)^{n-1}$ for some small $\epsilon > 0$. By Lemma 1.1, the filtration (27) is independent of ϵ when ϵ is sufficiently small. By Theorem 2.1, we have

$$(28) \quad c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (c_1(-K_X) + \epsilon \omega_X)^{n-1} \geq 0$$

for any i and $\epsilon > 0$ sufficiently small. Then

$$c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X)^{\text{nd}} \wedge (\omega_X)^{n-1-\text{nd}} \geq 0 \quad \text{for any } i.$$

Since

$$\sum_i c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X)^{\text{nd}} \wedge (\omega_X)^{n-1-\text{nd}} = (-K_X)^{\text{nd}+1} \wedge (\omega_X)^{n-1-\text{nd}} = 0,$$

we obtain

$$(29) \quad c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X)^{\text{nd}} \wedge (\omega_X)^{n-1-\text{nd}} = 0 \quad \text{for any } i.$$

Combining (29) with (28), we obtain

$$(30) \quad c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X)^{\text{nd}-1} \wedge (\omega_X)^{n-\text{nd}} \geq 0 \quad \text{for any } i.$$

Combining this with the stability condition of the filtration, we can find an integer $k \geq 1$ such that

$$(31) \quad c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X)^{\text{nd}-1} \wedge (\omega_X)^{n-\text{nd}} = a_i > 0 \quad \text{for } i \leq k,$$

and

$$c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X)^{\text{nd}-1} \wedge (\omega_X)^{n-\text{nd}} = 0 \quad \text{for } i > k.$$

We begin to prove (26). Set $r_i := \text{rank}(\mathcal{F}_i/\mathcal{F}_{i-1})$. By Lübke's inequality (cf. the proof of [Miy87, Thm. 6.1]), we have

$$(32) \quad \begin{aligned} & c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2} \\ & \geq (c_1(-K_X)^2 - \sum_i \frac{1}{r_i} c_1(\mathcal{F}_i/\mathcal{F}_{i-1})^2) (-K_X + \epsilon\omega_X)^{n-2}. \end{aligned}$$

There are three cases.

Case (1): $\sum_{i \leq k} r_i \geq 2$ and $\text{nd} \geq 2$. Using the Hodge index theorem, we have ³

$$(33) \quad \begin{aligned} & (\alpha^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}) ((-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}) \\ & \leq (\alpha \wedge (-K_X) \wedge (-K_X + \epsilon\omega_X)^{n-2})^2, \end{aligned}$$

for any $\alpha \in H^{1,1}(X, \mathbb{R})$. If we take $\alpha = c_1(\mathcal{F}_i/\mathcal{F}_{i-1})$ in (33) and use (32), we obtain

$$(34) \quad \begin{aligned} & c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2} \geq \\ & c_1(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2} - \sum_{i \leq k} \frac{1}{r_i} \frac{(c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X) \wedge (-K_X + \epsilon\omega_X)^{n-2})^2}{(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}} \end{aligned}$$

Now we estimate the two terms in the right hand side of (34). Using (31), we have

$$c_1(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2} = \left(\sum_{i \leq k} a_i \right) \epsilon^{n-\text{nd}} + O(\epsilon^{n-\text{nd}})$$

and

$$\begin{aligned} & \sum_{i \leq k} \frac{1}{r_i} \frac{(c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X) \wedge (-K_X + \epsilon\omega_X)^{n-2})^2}{(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}} \\ & = \frac{1}{\sum_{i \leq k} a_i} \left(\sum_{i \leq k} \frac{a_i^2}{r_i} \right) \cdot \epsilon^{n-\text{nd}} + O(\epsilon^{n-\text{nd}}). \end{aligned}$$

Since $\sum_{i \leq k} r_i \geq 2$, we have

$$\sum_{i \leq k} a_i > \frac{1}{\sum_{i \leq k} a_i} \left(\sum_{i \leq k} \frac{a_i^2}{r_i} \right).$$

Therefore $c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2}$ is strictly positive when $\epsilon > 0$ is small enough.

³It is important that $\alpha^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}$ maybe negative.

Case (2): $\sum_{i \leq k} r_i = 1$ and $nd \geq 2$. In this case, we obtain immediately that $r_1 = 1$ and $k = 1$. Moreover, (31) in this case means that

$$c_1(\mathcal{F}_1) \wedge (-K_X)^{nd-1} \wedge (\omega_X)^{n-nd} > 0,$$

and

$$(35) \quad c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X)^{nd-1} \wedge (\omega_X)^{n-nd} = 0 \quad \text{for } i \geq 2.$$

Set s be the smallest integer such that

$$c_1(\mathcal{F}_2/\mathcal{F}_1) \wedge (-K_X)^{nd-s} \wedge (\omega_X)^{n-nd+s-1} > 0.$$

Taking $\alpha = c_1(\mathcal{F}_i/\mathcal{F}_{i-1})$ in (33) for any $i \geq 2$, we get

$$(36) \quad \begin{aligned} & c_1(\mathcal{F}_i/\mathcal{F}_{i-1})^2 \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} \\ & \leq \frac{(c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X) \wedge (-K_X + \epsilon\omega_X)^{n-2})^2}{(-K_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}} \\ & \leq \frac{(\epsilon^{n+s-nd-1})^2}{\epsilon^{n-nd}} (1 + O(1)) = \epsilon^{2s+n-nd-2} + O(\epsilon^{2s+n-nd-2}) \text{ for } i \geq 2. \end{aligned}$$

Similarly, if we take $\alpha = \sum_{i \geq 2} c_1(\mathcal{F}_i/\mathcal{F}_{i-1})$ in (33), we obtain

$$(37) \quad \left(\sum_{i \geq 2} c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \right)^2 \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} \leq \epsilon^{2s+n-nd-2}.$$

Combining (36), (37) with (32), we obtain

$$\begin{aligned} & c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2} \\ & \geq (c_1(-K_X))^2 - \sum_{i \geq 2} \frac{1}{r_i} c_1(\mathcal{F}_i/\mathcal{F}_{i-1})^2 - (c_1(-K_X) - \sum_{i \geq 2} c_1(\mathcal{F}_i/\mathcal{F}_{i-1}))^2 (-K_X + \epsilon\omega_X)^{n-2} \\ & = 2c_1(-K_X) \wedge \left(\sum_{i \geq 2} c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \right) \wedge (-K_X + \epsilon\omega_X)^{n-2} \\ & \quad - \left(\sum_{i \geq 2} \frac{1}{r_i} c_1(\mathcal{F}_i/\mathcal{F}_{i-1})^2 + \left(\sum_{i \geq 2} c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \right)^2 \right) \wedge (-K_X + \epsilon\omega_X)^{n-2} \\ & \geq \epsilon^{n-nd+s-1} - \epsilon^{n-nd+2s-2}. \end{aligned}$$

Let us observe that by (35) we have $s \geq 2$. Therefore $c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2}$ is strictly positive for $\epsilon > 0$ small enough.

Case (3): $nd = 1$. Using (32), we have

$$c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2} \geq - \sum_i \frac{1}{r_i} c_1(\mathcal{F}_i/\mathcal{F}_{i-1})^2 (-K_X + \epsilon\omega_X)^{n-2}.$$

By the Hodge index theorem, we obtain

$$\begin{aligned} & c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2} \\ & \geq \lim_{t \rightarrow 0^+} - \sum_i \frac{1}{r_i} \frac{(c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X + t\omega_X) \wedge (-K_X + \epsilon\omega_X)^{n-2})^2}{(-K_X + t\omega_X)^2 \wedge (-K_X + \epsilon\omega_X)^{n-2}}. \end{aligned}$$

Let us observe that by (29) we have

$$c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (-K_X) \wedge (\omega_X)^{n-2} = 0 \quad \text{for any } i.$$

Then

$$c_2(T_X) \wedge (-K_X + \epsilon\omega_X)^{n-2}$$

$$\geq \lim_{t \rightarrow 0^+} - \sum_i \frac{1}{r_i} \cdot \frac{(t\epsilon^{n-2}c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge \omega_X^{n-1})^2}{t^2\epsilon^{n-2}\omega_X^n + t\epsilon^{n-2}(-K_X)\omega_X^{n-1}} = 0.$$

□

It is interesting to study the case when the equality holds in (26) of Proposition 3.5. We will prove that in this case, X is either a torus or a smooth \mathbb{P}^1 -fibration over a torus. Before proving this result, we first prove an auxiliary lemma.

Lemma 3.6. *Let (X, ω_X) be a compact Kähler manifold with nef anticanonical bundle. Let*

$$(38) \quad 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_l = T_X$$

be a stable filtration of the Harder-Narasimhan semistable filtration of T_X with respect to $(c_1(-K_X) + \epsilon\omega_X)^{n-1}$ as in (27). If X is not a torus and

$$(39) \quad \int_X c_2(T_X) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0$$

for some $\epsilon > 0$ small enough, we have

- (i) $\text{nd}(-K_X) = 1$.
- (ii) $(\mathcal{F}_i/\mathcal{F}_{i-1})^{**}$ is projectively flat for all i , i.e., $(\mathcal{F}_i/\mathcal{F}_{i-1})^{**}$ is locally free and there exists a smooth metric h on it such that $i_{\Theta_h}(\mathcal{F}_i/\mathcal{F}_{i-1})^{**} = \alpha_i \cdot \text{Id}$ for some $\alpha_i \in H^{1,1}(X, \mathbb{Z})$. Moreover, α_i is nef and proportional to $c_1(-K_X)$.
- (iii) $c_2(\mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for all i , and (38) is regular outside a subvariety of codimension at least 3.

Remark 3.7. *We first remark that for a vector bundle V of rank k supported on a subvariety $j : Z \subset X$ of codimension r , by the Grothendieck-Riemann-Roch theorem, we have*

$$c_r(j_*(V)) = (-1)^{r-1}(r-1)!k[Z].$$

*Therefore for any torsion free sheaf \mathcal{E} , we have $c_2(\mathcal{E}) \geq c_2(\mathcal{E}^{**})$ and the equality holds if and only if $\mathcal{E} = \mathcal{E}^{**}$ outside a subvariety of codimension at least 3.*

Proof. If $\text{nd}(-K_X) = 0$, then K_X is numerically trivial. In this case, we know that (39) implies that X is a torus (cf. for instance, [Tia00, Thm 2.13]). From now on, we suppose that $\text{nd}(-K_X) \geq 1$. By the proof of Proposition 3.5, the equality (39) implies that the filtration (38) is in the case (3), i.e., $\text{nd}(-K_X) = 1$. Then

$$(40) \quad \int_X c_1(-K_X)^2 \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0.$$

By the proof of Proposition 3.5, the equality (39) implies also that the filtration (38) also satisfies:

$$(41) \quad \int_X c_1(\mathcal{F}_i/\mathcal{F}_{i-1})^2 \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0.$$

Moreover, (39) implies that the equality holds in (32), which implies that, (cf. the proof of [Miy87, Thm 6.1, Cor 4.7])

$$(42) \quad \int_X c_2((\mathcal{F}_i/\mathcal{F}_{i-1})^{**}) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = \int_X c_2(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0.$$

We now check (ii) and (iii). By [BS94, Cor 3], (42) and (41) imply that $(\mathcal{F}_i/\mathcal{F}_{i-1})^{**}$ is locally free and projectively flat. The first part of (ii) is proved.

For the second part of (ii), since $(\mathcal{F}_i/\mathcal{F}_{i-1})^{**}$ is projectively flat, there exists an $\alpha_i \in H^{1,1}(X, \mathbb{Z})$, such that $c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) = \text{rank}(\mathcal{F}_i/\mathcal{F}_{i-1}) \cdot \alpha_i$. By (29), we have

$$\int_X c_1(-K_X) \wedge c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0.$$

Combining this with (40) and (41), by the Hodge index theorem⁴, we obtain that $c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) = a_i \cdot c_1(-K_X)$ for some $a_i \in \mathbb{Q}$. By Theorem 2.1, we have

$$c_1(\mathcal{F}_i/\mathcal{F}_{i-1}) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-1} \geq 0.$$

Therefore $a_i \geq 0$ and α_i is nef. (ii) is proved. As for (iii), by (i) and (ii), we obtain that $c_2((\mathcal{F}_i/\mathcal{F}_{i-1})^{**}) = 0$. Combining this with (42) and the Remark 3.7, we get (iii). \square

Using an idea of A.Höring, we finally prove that

Proposition 3.8. *Let (X, ω_X) be a projective manifold with nef anticanonical bundle. We suppose that $\int_X c_2(X) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0$ for some $\epsilon > 0$ small enough. Then after a finite étale cover, X is either a torus or a smooth \mathbb{P}^1 -fibration over a torus.*

Proof. If X is a torus, the proposition is proved. From now on, we suppose that X is not a torus. Let

$$(43) \quad 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_l = T_X$$

a stable filtration of the Harder-Narasimhan semistable filtration of T_X with respect to $(c_1(-K_X) + \epsilon\omega_X)^{n-1}$ as in (27). Let Z be the locus where (43) is not regular. By Lemma 3.6, we have $\text{codim}_X Z \geq 3$ and $\text{nd}(-K_X) = 1$. Moreover, (ii) of Lemma 3.6 implies that T_X is nef on any smooth projective curve in $X \setminus Z$.

By the condition $\text{nd}(-K_X) = 1$, we know that K_X is not nef. Then there exists a Mori contraction $\varphi: X \rightarrow Y$. Since $\text{nd}(-K_X) = 1$ and $-K_X$ is ample on all the φ -fibres, we see that all the φ -fibres have dimension at most one. By Ando's theorem [And85] we know that φ is either a blow-up along a smooth subvariety of codimension two or a conic bundle. There are three cases.

1st case: $\varphi: X \rightarrow Y$ is a blow-up along a smooth subvariety S of Y and $\text{codim}_Y S = 2$. Let E be the exceptional divisor. Then a general fiber F of $\varphi: E \rightarrow S$ does not meet Z , since $\text{codim}_X Z \geq 3$. Therefore $T_X|_F$ is nef. On the other hand, since $F = \mathbb{P}^1$, we have a direct decomposition

$$T_X|_F = T_E|_F \oplus N_{E/X}|_F = T_E|_F \oplus [E]|_F.$$

Since $[-E]|_F$ is strictly positive, $T_X|_F$ must contain a strictly negative part. We get a contradiction.

2nd case: $\varphi: X \rightarrow Y$ is a conic bundle with a non-empty discriminant $\Delta \subset Y$. Let $C \subset X$ be an irreducible component of a general fiber of $\varphi: \varphi^{-1}(\Delta) \rightarrow \Delta$. Then $T_X|_C$ is not nef (cf. [And85, Thm 3.1, Lemma 1.5]). On the other hand, it is known that $\text{codim}_Y \Delta = 1$ (cf. for instance, [Sar82]). Since C is a general fiber over Δ , we obtain that $C \cap Z = \emptyset$. Therefore $T_X|_C$ is nef. We get a contradiction.

⁴In fact, let $Q(\alpha, \beta) = \int_X \alpha \wedge \beta \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2}$. Then Q is of index $(1, m)$. Let V be the subspace of $H^{1,1}(X, \mathbb{R})$ where Q is negative definite. If $Q(\alpha_1, \alpha_1) = Q(\alpha_2, \alpha_2) = Q(\alpha_1, \alpha_2) = 0$ for some non trivial α_1, α_2 , then both α_1 and α_2 are not contained in V . Therefore we can find a $t \in \mathbb{R}$, such that $(\alpha_1 - t\alpha_2) \in V$. Since $Q(\alpha_1 - t\alpha_2, \alpha_1 - t\alpha_2) = 0$, we get $\alpha_1 - t\alpha_2 = 0$. Therefore α_1 is proportional to α_2 .

3rd case: $\varphi: X \rightarrow Y$ is a smooth \mathbb{P}^1 -bundle. By [Miy83, 4.11] we have

$$\varphi_*(K_X^2) = -4K_Y,$$

so $K_X^2 = 0$ implies that $K_Y \equiv 0$. By the condition

$$\int_X c_2(X) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0,$$

we obtain that $c_2(Y) = 0$. Therefore, after a finite étale cover, Y is a torus. The proposition is proved. \square

Remark 3.9. *In general, if $\int_X c_2(X) \wedge (c_1(-K_X) + \epsilon\omega_X)^{n-2} = 0$, We cannot hope that X can be covered by a torus. In fact, the example [DPS94, Example 3.3] satisfies the equality $c_2(X) = 0$ and X can not be decomposed as direct product of torus with \mathbb{P}^1 . Using [DHP08], we know that X cannot be covered by torus. Therefore we propose the following conjecture, which is a mild modification of the question of Yau:*

Conjecture 3.10. *Let (X, ω_X) be a compact Kähler manifold with nef anticanonical bundle. Then $\int_X c_2(T_X) \wedge \omega_X^{n-2} \geq 0$. If the equality holds for some Kähler metric, then X is either a torus or a smooth \mathbb{P}^1 -fibration over a torus.*

Remark 3.11. *If one could prove that T_X is generically nef with respect to the polarizations $(c_1(-K_X) + \epsilon\omega, \omega, \dots, \omega)$ for any $\epsilon > 0$ small enough, using the same argument as in this section, one could prove this conjecture.*

As the last application of Theorem 2.1, we give a new proof of the surjectivity of Albanese map when X is a compact Kähler manifold with nef anticanonical bundle.

Proposition 3.12. *Let (X, ω) be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map is surjective, and smooth outside a subvariety of codimension at least 2. In particular, the fibers of the Albanese map are connected and reduced in codimension 1.*

Proof. Let

$$(44) \quad 0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_s = T_X$$

be a filtration of torsion-free subsheaves such that $\mathcal{E}_{i+1}/\mathcal{E}_i$ is an ω -stable torsion-free subsheaf of T_X/\mathcal{E}_i of maximal slope.

Case 1: (44) is regular, i.e., all \mathcal{E}_i and $\mathcal{E}_i/\mathcal{E}_{i-1}$ are locally free

In this case, we can prove that the Albanese map is submersive. Let $\tau \in H^0(X, T_X^*)$ be a nontrivial element. To prove that the Albanese map is submersive, it is sufficient to prove that τ is non vanishing everywhere. Thanks to Theorem 2.1 and the stability condition of $\mathcal{E}_i/\mathcal{E}_{i-1}$, we can find a smooth metric h_i on $\mathcal{E}_i/\mathcal{E}_{i-1}$ such that

$$\frac{i\Theta_{h_i}(\mathcal{E}_i/\mathcal{E}_{i-1}) \wedge \omega^{n-1}}{\omega^n} = \lambda_i \cdot \text{Id}_{\mathcal{E}_i/\mathcal{E}_{i-1}}$$

for some constant $\lambda_i \geq 0$. Thanks to the construction of $\{h_i\}$ and Lemma 1.7, for any $\epsilon > 0$, there exists a smooth metric h_ϵ on T_X , such that

$$(45) \quad \frac{i\Theta_{h_\epsilon}(T_X) \wedge \omega^{n-1}}{\omega^n} \geq -\epsilon \cdot \text{Id}_{T_X},$$

and the matrix valued $(1, 1)$ -form $i\Theta_{h_\epsilon}(T_X)$ is uniformly bounded. Let h_ϵ^* be the dual metric on T_X^* . Then the closed $(1, 1)$ -current

$$T_\epsilon = \frac{i}{2\pi} \partial \bar{\partial} \ln \|\tau\|_{h_\epsilon^*}^2$$

satisfies

$$(46) \quad T_\epsilon \geq -\frac{\langle i\Theta_{h_\epsilon^*}(T_X^*)\tau, \tau \rangle_{h_\epsilon^*}}{\|\tau\|_{h_\epsilon^*}^2}.$$

Since $-\Theta_{h_\epsilon^*}(T_X^*) = {}^t\Theta_{h_\epsilon}(T_X)$, (45) and (46) imply a pointwise estimate

$$(47) \quad T_\epsilon \wedge \omega^{n-1} \geq -\epsilon \omega^n.$$

We suppose by contradiction that $\tau(x) = 0$ for some point $x \in X$. By Lemma 1.7, $i\Theta_{h_\epsilon}(T_X)$ is uniformly lower bounded. Therefore, there exists a constant C such that $T_\epsilon + C\omega$ is a positive current for any ϵ . After replacing by a subsequence, we can thus suppose that T_ϵ converge weakly to a current T , and $T + C\omega$ is a positive current. Since $\tau(x) = 0$, we have

$$\nu(T_\epsilon + C\omega, x) \geq 1 \quad \text{for any } \epsilon,$$

where $\nu(T_\epsilon + C\omega, x)$ is the Lelong number of the current $T_\epsilon + C\omega$ at x . Using the main theorem in [Siu74], we obtain that $\nu(T + C\omega, x) \geq 1$. Therefore there exists a constant $C_1 > 0$ such that

$$\int_{B_x(r)} (T + C\omega) \wedge \omega^{n-1} \geq C_1 \cdot r^{2n-2} \quad \text{for } r \text{ small enough,}$$

where $B_x(r)$ is the ball of radius r centered at x . Then

$$\int_{U_x} T \wedge \omega^{n-1} > 0$$

for some neighborhood U_x of x . Therefore

$$(48) \quad \lim_{\epsilon \rightarrow 0} \int_{U_x} T_\epsilon \wedge \omega^{n-1} > 0.$$

Combining (47) with (48), we obtain

$$\lim_{\epsilon \rightarrow 0} \int_X T_\epsilon \wedge \omega^{n-1} > 0.$$

We get a contradiction by observing that all T_ϵ are exact forms.

Case 2: General case

By Lemma 1.6, there exists a desingularization $\pi : \tilde{X} \rightarrow X$, such that $\pi^*(T_X)$ admits a filtration:

$$0 \subset E_1 \subset E_2 \subset \cdots \subset \pi^*(T_X)$$

satisfying that $E_i, E_i/E_{i-1}$ are vector bundles and $\pi_*(E_i) = \mathcal{E}_i$ on $X \setminus Z$, where Z is an analytic subset of codimension at least 2. Let $\tau \in H^0(X, T_X^*)$ be a nontrivial element. Our aim is to prove that τ is non vanishing outside Z .

Let $x \in \tilde{X} \setminus \pi^{-1}(Z)$. Let U_x be a small neighborhood of x such that $U_x \subset \tilde{X} \setminus \pi^{-1}(Z)$. We suppose by contradiction that $\pi^*(\tau)(x) = 0$. By [BS94], there exists Hermitian-Einstein metrics $h_{\epsilon, i}$ on E_i/E_{i-1} with respect to $\pi^*\omega + \epsilon\omega_{\tilde{X}}$, and

$\{i\Theta_{h_{\epsilon,i}}(E_i/E_{i-1})\}_\epsilon$ is uniformly bounded on U_x ⁵. Combining this with Lemma 1.7, we can construct a smooth metric h_ϵ on $\pi^*(T_X)$ such that

$$(49) \quad \frac{i\Theta_{h_\epsilon}(\pi^*(T_X)) \wedge (\pi^*\omega + \epsilon\omega_{\tilde{X}})^{n-1}}{(\pi^*\omega + \epsilon\omega_{\tilde{X}})^n} \geq -2C\epsilon \cdot \text{Id}_{\pi^*(T_X)},$$

and $i\Theta_{h_\epsilon}(\pi^*(T_X))$ is uniformly bounded on U_x . Let $T_\epsilon = \frac{i}{2\pi}\partial\bar{\partial}\ln\|\pi^*(\tau)\|_{h_\epsilon}^2$. By the same argument as in Case 1, the uniform boundedness of $i\Theta_{h_\epsilon}(\pi^*(T_X))$ in a neighborhood of x implies the existence of a neighborhood U'_x of x and a constant $c > 0$, such that

$$\lim_{\epsilon \rightarrow 0} \int_{U'_x} T_\epsilon \wedge (\pi^*(\omega) + \epsilon\omega_{\tilde{X}})^{n-1} \geq c.$$

Combining this with (49), we get

$$\lim_{\epsilon \rightarrow 0} \int_{\tilde{X}} T_\epsilon \wedge (\pi^*(\omega) + \epsilon\omega_{\tilde{X}})^{n-1} \geq c,$$

which contradicts with the fact that all T_ϵ are exact. Therefore τ is non vanishing outside Z . Proposition 3.12 is proved. \square

APPENDIX A. A BOCHNER TECHNIQUE PROOF

We would like to give a proof of the implication $(iii) \Rightarrow (ii)$ in Proposition 3.1 without using [BM01, Thm 0.1].

Proof. By [CDP12, Criterion 1.1], to prove the implication, it is sufficient to prove that for some ample line bundle F on X , there exists a constant $C_F > 0$, such that

$$(50) \quad H^0(X, (T_X^*)^{\otimes m} \otimes F^{\otimes k}) = 0 \quad \text{for all } m, k \text{ satisfying } m \geq C_F \cdot k.$$

Thanks to the condition (iii) , there exists a Kähler class A , such that for the Harder-Narasimhan semistable filtration with respect to A

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = T_X,$$

we have

$$\mu_A(\mathcal{F}_i/\mathcal{F}_{i-1}) \geq c \quad \text{for all } i,$$

for some constant $c > 0$. Moreover, for the Harder-Narasimhan filtration of $(T_X)^{\otimes m}$ with respect to A , $m \cdot c$ is also a lower bound of the minimal slope with respect to the filtration.

We now prove (50) by a basic Bochner technique. After replacing by a more refined filtration, we can suppose that

$$(51) \quad 0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_s = (T_X)^{\otimes m}$$

is a filtration of torsion-free subsheaves such that $\mathcal{E}_{i+1}/\mathcal{E}_i$ is an ω -stable torsion-free subsheaf of T_X/\mathcal{E}_i of maximal slope for simplicity. Let ω be a positive $(1,1)$ -form representing $c_1(A)$.

⁵In fact, [BS94] proved that $h_{\epsilon,i}$ and $h_{\epsilon,i}^{-1}$ are $C^{1,\alpha}$ -uniform bounded in U_x . Since U_x is in $X \setminus Z$, $\omega_\epsilon := \pi^*\omega + \epsilon\omega_{\tilde{X}}$ is uniformly strict positive on U_x . By [Kob87, Chapter I, (14.16)] and Hermitian-Einstein condition, we obtain that $\Delta_{\omega_\epsilon}(h_{\epsilon,i})_{j,k}$ is uniformly C^α bounded on U_x , where Δ_{ω_ϵ} is the Laplacian with respect to ω_ϵ and $(h_{\epsilon,i})_{j,k} := h_{\epsilon,i}(e_j, e_k)$ for a fixed base $\{e_k\}$ of E_i/E_{i-1} . The standard elliptic estimates gives the uniform boundedness of $i\Theta_{h_{\epsilon,i}}(E_i/E_{i-1})$ on U_x .

If (51) is regular, then there exists a Hermitian-Einstein metric on every quotient. Since $\mu_A(\mathcal{E}_i/\mathcal{E}_{i-1}) \geq c \cdot m$, thanks to Lemma 1.7, we can construct a smooth metric h on $(T_X)^{\otimes m}$, such that

$$(52) \quad \frac{i\Theta_h(T_X^{\otimes m}) \wedge \omega^{n-1}}{\omega^n} \geq \frac{m \cdot c}{2} \text{Id}.$$

Let $\tau \in H^0(X, (T_X^*)^{\otimes m} \otimes F^{\otimes k})$. We have

$$(53) \quad \Delta_\omega(\|\tau\|_{h^*}^2) = \|D'_h \tau\|^2 - \frac{\langle i\Theta_{h^*}((T_X^*)^{\otimes m} \otimes F^{\otimes k})\tau, \tau \rangle \wedge \omega^{n-1}}{\omega^n}.$$

Let C_F be a constant large enough with respect to c . If $m \geq C_F \cdot k$, (52) implies that

$$\int_X \|D'_h \tau\|^2 \omega^n - \langle i\Theta_{h^*}((T_X^*)^{\otimes m} \otimes F^{\otimes k})\tau, \tau \rangle \wedge \omega^{n-1} \geq c_1 \|\tau\|_{h^*}^2$$

for some constant $c_1 > 0$. Observing moreover

$$\int_X \Delta_\omega(\|\tau\|_{h^*}^2) \omega^n = 0,$$

we get $\tau = 0$.

If (51) is not necessary regular, by Lemma 1.6, we can find a resolution $\pi : \tilde{X} \rightarrow X$ such that there exists a regular filtration

$$0 \subset E_1 \subset E_2 \subset \cdots \subset \pi^*(T_X)$$

and

$$\mu_{\pi^*(A)}(E_i/E_{i-1}) = \mu_A(\mathcal{E}_i/\mathcal{E}_{i-1}) \geq c \cdot m.$$

Thanks to the strict positivity of c , if ϵ is small enough,

$$(54) \quad \mu_\epsilon(E_i/E_{i-1}) \geq \frac{c \cdot m}{2} \quad \text{for any } i,$$

where μ_ϵ is the slope with respect to $\pi^*(A) + \epsilon\omega_{\tilde{X}}$. Thanks to Remark 2.2, E_i/E_{i-1} are also stable for $\pi^*(A) + \epsilon\omega_{\tilde{X}}$ when ϵ small enough. Therefore there exists a smooth Hermitian-Einstein metric on every quotient E_i/E_{i-1} . Using Lemma 1.7, (54) implies the existence of a smooth metric h_ϵ on $\pi^*(T_X)^{\otimes m}$, such that

$$(55) \quad \frac{i\Theta_{h_\epsilon}(\pi^*(T_X)^{\otimes m}) \wedge (\pi^*(\omega) + \epsilon\omega_{\tilde{X}})^{n-1}}{(\pi^*(\omega) + \epsilon\omega_{\tilde{X}})^n} \geq \frac{m \cdot c}{4} \text{Id}$$

for ϵ small enough. Using the same Bochner technique as in (52) and (53), applied to $\pi^*(T_X)$ with respect to $\pi^*(A) + \epsilon\omega_{\tilde{X}}$, we get

$$H^0(\tilde{X}, \pi^*((T_X^*)^{\otimes m} \otimes F^{\otimes k})) = 0 \quad \text{for all } m, k \text{ satisfying } m \geq C_F \cdot k.$$

(50) is thus proved. \square

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