Quasilinear and Hessian type equations with exponential reaction and measure data

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Contents

Abstract

We prove existence results concerning equations of the type $-\Delta_p u = P(u) + \mu$ for $p > 1$ and $F_k[-u] = P(u) + \mu$ with $1 \leq k < \frac{N}{2}$ in a bounded domain Ω, where μ is a positive Radon measure and $P(u) \sim e^{au^\beta}$ with $a > 0$ and $\beta \geq 1$. Sufficient conditions for existence are expressed in terms of the fractional maximal potential of μ. Two-sided estimates on the solutions are obtained in terms of some precise Wolff potentials of μ. Necessary conditions are obtained in terms of Orlicz capacities. We also establish existence results for a general Wolff potential equation under the form $u = \mathbf{W}^R_{\alpha,p}[P(u)] + f$.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be either a bounded domain or the whole $\mathbb{R}^N$, $p > 1$ and $k \in \{1, 2, ..., N\}$. We denote by

$$\Delta_p u := div\left(|\nabla u|^{p-2}\nabla u\right)$$

the p-Laplace operator and by

$$F_k[u] = \sum_{1\le j_1<j_2<\ldots<j_k\le N} \lambda_{j_1}\lambda_{j_2}...\lambda_{j_k}$$

the k-Hessian operator where $\lambda_1, ..., \lambda_N$ are the eigenvalues of the Hessian matrix D^2u. Let μ be a positive Radon measure in Ω; our aim is to study the existence of positive renormalized solutions to the following boundary value problems if Ω is bounded,

$$\begin{array}{rl} -\Delta_p u = P(u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{1.1}$$

and

$$\begin{array}{rl} F_k[-u] = P(u) + \mu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{array} \tag{1.2}$$

where P is an exponential function. If $\Omega = \mathbb{R}^N$, we consider the same equations, but the boundary conditions are replaced by $\inf_{\mathbb{R}^N} u = 0$. When $P(r) = r^q$ with $q > p - 1$, Phuc and Verbitsky published a seminal article [20] on the solvability of the corresponding problem (1.1). They obtained necessary and sufficient conditions involving Bessel capacities or Wolff potentials. For example, assuming that Ω is bounded, they proved that if μ has compact support in Ω it is equivalent to solve (1.1) with $P(r) = r^q$, or to have

$$\mu(E) \le cC_{p,\frac{p}{q+1-p}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{1.3}$$

where c is a suitable positive constant and $C_{p,\frac{p}{q+1-p}}$ a Bessel capacity, or to have

$$\int_B \left[\mathbf{W}^{2R}_{1,p}[\mu_B(x)]\right]^q dx \le C\mu(B) \qquad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \ne \emptyset, \tag{1.4}$$

where $R = \text{diam}(\Omega)$. Other conditions are expressed in terms of Riesz potentials and maximal fractional potentials. Their construction is based upon sharp estimates of solutions of the non-homogeneous problem

$$\begin{array}{rl} -\Delta_p u = \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{1.5}$$

for positive measures ω. We refer to [4, 5, 6, 7, 9, 13, 23] for the previous studies of these and other related results. Concerning the k-Hessian operator in a bounded $(k-1)$-convex domain Ω, they proved that if μ has compact support and $||\varphi||_{L^\infty(\partial\Omega)}$ is small enough, the corresponding problem (1.2) with $P(r) = r^q$ with $q > k$ admits a positive solution if and only if

$$\mu(E) \le cC_{2k,\frac{q}{q-k}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{1.6}$$

or equivalently

$$\int_B \left[\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}[\mu_B(x)]\right]^q dx \le C\mu(B) \qquad \text{for all ball } B \text{ s.t. } B\cap \text{supp}\mu \neq \emptyset. \tag{1.7}$$

The results concerning the linear case $p = 2$ and $k = 1$, can be found in [2, 3, 29]. The main tools in their proofs are derived from recent advances in potential theory for nonlinear elliptic equations obtained by Kilpelainen and Malý [15, 16], Trudinger and Wang [25, 26, 27], and Labutin [18] thanks to whom the authors first provide global pointwise estimates for solutions of the homogeneous Dirichlet problems in terms of Wolff potentials of suitable order.

For $s > 1$, $0 \le \alpha < \frac{N}{s}$, $\eta \ge 0$ and $0 < T \le \infty$, we recall that the *T-truncated Wolff potential* of a positive Radon measure μ is defined in $\mathbb{R}^N$ by

$$\mathbf{W}^T_{\alpha,s}[\mu](x) = \int_0^T \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t}, \tag{1.8}$$

the *T-truncated Riesz potential* of a positive Radon measure μ by

$$\mathbf{I}^T_\alpha[\mu](x) = \int_0^T \frac{\mu(B_t(x))}{t^{N-\alpha}} \frac{dt}{t} \tag{1.9}$$

and the *T-truncated η-fractional maximal potential* of μ by

$$\mathbf{M}^\eta_{\alpha,T}[\mu](x) = \sup\left\{\frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)} : 0 < t \le T\right\}, \tag{1.10}$$

where $h_\eta(t) = (-\ln t)^{-\eta}\chi_{(0,2^{-1}]}(t) + (\ln 2)^{-\eta}\chi_{[2^{-1},\infty)}(t)$. If $\eta = 0$, then $h_\eta = 1$ and we denote by $\mathbf{M}_{\alpha,T}[\mu]$ the corresponding *T-truncated fractional maximal potential* of μ. We also denote by $\mathbf{W}_{\alpha,s}[\mu]$ (resp $\mathbf{I}_\alpha[\mu]$, $\mathbf{M}_\alpha[\mu]$) the *∞-truncated Wolff potential (resp Riesz Potential, fractional maximal potential)* of μ. When the measures are only defined in an open subset $\Omega \subset \mathbb{R}^N$, they are naturally extended by 0 in Ω^c. For $l \in \mathbb{N}^*$, we define the *l-truncated exponential function*

$$H_l(r) = e^r - \sum_{j=0}^{l-1} \frac{r^j}{j!}, \tag{1.11}$$

and for $a > 0$ and $\beta \ge 1$, we set

$$P_{l,a,\beta}(r) = H_l(ar^\beta). \tag{1.12}$$

We put

$$Q_p(s) = \begin{cases} \sum\limits_{q=l}^{\infty} \frac{s^{\frac{\beta q}{p-1}}}{q^{\frac{\beta q}{p-1}} q!} & \text{if } p \neq 2, \\ H_l(s^\beta) & \text{if } p = 2, \end{cases} \tag{1.13}$$

$Q^*_p(r) = \max\{rs - Q_p(s) : s \ge 0\}$ is the complementary function to Q_p, and define the corresponding Bessel and Riesz capacities respectively by

$$cap_{G_p,Q^*_p}(E) = \inf\left\{\int_{\mathbb{R}^N} Q^*_p(f)dx : G_p * f \ge \chi_E, f \ge 0, Q^*_p(f) \in L^1(\mathbb{R}^N)\right\}, \tag{1.14}$$

and

$$cap_{I_p,Q_p^*}(E) = \inf\left\{\int_{\mathbb{R}^N} Q_p^*(f)dx : I_p * f \geq \chi_E, f \geq 0, Q_p^*(f) \in L^1(\mathbb{R}^N)\right\}, \tag{1.15}$$

where $G_p(x) = \mathcal{F}^{-1}\left((1+|.|^2)^{-\frac{p}{2}}\right)(x)$ is the Bessel kernel of order p and $I_p(x) = (N-p)^{-1}|x|^{-(N-p)}$.

The expressions $a \wedge b$ and $a \vee b$ stand for $\min\{a,b\}$ and $\max\{a,b\}$ respectively. Our main results are the following theorems.

Theorem 1.1 *Let $1 < p < N$, $a > 0$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p-1$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If μ is a nonnegative Radon measure in Ω, there exists $M > 0$ depending on N, p, l, a, β and diam (Ω) (the diameter of Ω) such that if*

$$||\mathbf{M}_{p,2\,diam\,(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

and $\omega = M||\mathbf{M}_{p,2\,diam(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$ with $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$, then $P_{l,a,\beta}\left(2c_pK_1\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\omega]\right)$ is integrable in Ω and the following Dirichlet problem

$$\begin{array}{ll} -\Delta_p u = P_{l,a,\beta}(u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{1.16}$$

admits a nonnegative renormalized solution u which satisfies

$$u(x) \leq 2c_pK_1\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\omega](x) \quad \forall x \in \Omega. \tag{1.17}$$

The role of $K_1 = K_1(N,p)$ will be made explicit in Theorem 3.5.

Conversely, if (1.16) admits a nonnegative renormalized solution u and $P_{l,a,\beta}(u)$ is integrable in Ω, then for any compact set $K \subset \Omega$, there exists a positive constant C depending on N, p, l, a, β and dist $(K, \partial\Omega)$ *such that*

$$\int_E P_{l,\alpha,\beta}(u)dx + \mu(E) \leq C cap_{G_p,Q_p^*}(E) \quad \text{for all Borel sets } E \subset K. \tag{1.18}$$

Furthermore, $u \in W_0^{1,p_1}(\Omega)$ for all $1 \leq p_1 < p$.

When $\Omega = \mathbb{R}^N$, we have a similar result provided μ has compact suppport.

Theorem 1.2 *Let $1 < p < N$, $a > 0$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > \frac{N(p-1)}{N-p}$ and $R > 0$. If μ is a nonnegative Radon measure in $\mathbb{R}^N$ with* supp $(\mu) \subset B_R(0)$ *there exists $M > 0$ depending on N, p, l, a, β and R such that if*

$$||\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

and $\omega = M||\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)} + \mu$, then $P_{l,a,\beta}\left(2c_pK_1\mathbf{W}_{1,p}[\omega]\right)$ is integrable in $\mathbb{R}^N$ and the following problem

$$\begin{array}{l} -\Delta_p u = P_{l,a,\beta}(u) + \mu \text{ in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u = 0, \end{array} \tag{1.19}$$

admits a p-superharmonic solution u which satisfies

$$u(x) \leq 2c_p K_1 \mathbf{W}_{1,p}[\omega](x) \quad \forall x \in \mathbb{R}^N, \tag{1.20}$$

(c_p and K_1 as in Theorem 1.1).

Conversely, if (1.19) *has a solution u and $P_{l,a,\beta}(u)$ is locally integrable in $\mathbb{R}^N$, then there exists a positive constant C depending on N, p, l, a, β such that*

$$\int_E P_{l,\alpha,\beta}(u)dx + \mu(E) \leq C cap_{I_p,Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.} \tag{1.21}$$

Furthermore, $u \in W^{1,p_1}_{loc}(\mathbb{R}^N)$ for all $1 \leq p_1 < p$.

Concerning the k-Hessian operator we recall some notions notions introduced by Trudinger and Wang [25, 26, 27], and we follow their notations. For $k = 1, ..., N$ and $u \in C^2(\Omega)$ the k-Hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < ... < i_k \leq N} \lambda_{i_1} ... \lambda_{i_k}.$$

It is straightforward that

$$F_k[u] = \left[D^2u\right]_k,$$

where in general $[A]_k$ denotes the sum of the k-th principal minors of a matrix $A = (a_{ij})$. In order that there exists a smooth k-admissible function which vanishes on $\partial\Omega$, the boundary $\partial\Omega$ must satisfy a uniformly (k-1)-convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal. We also denote by $\Phi^k(\Omega)$ the class of upper-semicontinuous functions $\Omega \mapsto [-\infty, \infty)$ which are k-convex, or subharmonic in the Perron sense (see Definition 5.1). In this paper we prove the following theorem (in which expression $\mathbb{E}[q]$ is the largest integer less or equal to q)

Theorem 1.3 *Let $k \in \{1, 2, ..., \mathbb{E}[N/2]\}$ such that $2k < N$, $l \in \mathbb{N}^*$, $\beta \geq 1$ such that $l\beta > k$ and $a > 0$. Let Ω be a bounded uniformly (k-1)-convex domain in $\mathbb{R}^N$. Let φ be a nonnegative continuous function on $\partial\Omega$ and $\mu = \mu_1 + f$ be a nonnegative Radon measure where μ_1 has compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$. Let $K_2 = K_2(N, k)$ be the constant K_2 which appears in Theorem 5.3. Then, there exist positive constants b, M_1 and $M_2 > 0$ depending on N, k, l, a, β and diam (Ω) such that, if $\max_{\partial\Omega} \varphi \leq M_2$ and*

$$||\mathbf{M}^{\frac{k(\beta-1)}{\beta}}_{2k,2diam\,(\Omega)}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M_1,$$

then $P_{l,a,\beta}\left(2K_2\mathbf{W}^{2\,diam\,(\Omega)}_{\frac{2k}{k+1},k+1}[\mu]+b\right)$ *is integrable in* Ω *and the following Dirichlet problem*

$$\begin{array}{ll} F_k[-u]=P_{l,a,\beta}(u)+\mu & in\ \ \Omega,\\ u=\varphi & on\ \ \partial\Omega, \end{array} \tag{1.22}$$

admits a nonnegative solution u*, continuous near* $\partial\Omega$*, with* $-u\in\Phi^k(\Omega)$ *which satisfies*

$$u(x)\le 2K_2\mathbf{W}^{2diam\,\Omega}_{\frac{2k}{k+1},k+1}[\mu](x)+b\qquad\forall x\in\Omega. \tag{1.23}$$

Conversely, if (1.22) *admits a nonnegative solution* u*, continuous near* $\partial\Omega$*, such that* $-u\in\Phi^k(\Omega)$ *and* $P_{l,a,\beta}(u)$ *is integrable in* Ω*, then for any compact set* $K\subset\Omega$*, there exists a positive constant* C *depending on* N,k,l,a,β *and* $dist(K,\partial\Omega)$ *such that there holds*

$$\int_E P_{l,a,\beta}(u)dx+\mu(E)\le Ccap_{G_{2k},Q^*_{k+1}}(E)\qquad\forall E\subset K,E\ Borel, \tag{1.24}$$

where $Q_{k+1}(s)$ *is defined by (1.13) with* $p=k+1$*,* Q^*_{k+1} *is its complementary function and* $cap_{G_{2k},Q^*_{k+1}}(E)$ *is defined accordingly by (1.14).*

The following extension holds when $\Omega=\mathbb{R}^N$.

Theorem 1.4 *Let* $k\in\{1,2,...,\mathbb{E}[N/2]\}$ *such that* $2k<N$*,* $l\in\mathbb{N}^*$*,* $\beta\ge1$ *such that* $l\beta>\frac{Nk}{N-2k}$ *and* $a>0$*,* $R>0$*. If* μ *is a nonnegative Radon measure in* $\mathbb{R}^N$ *with* $\mathrm{supp}\,(\mu)\subset B_R(0)$ *there exists* $M>0$ *depending on* N,k,l,a,β *and* R *such that if*

$$||\mathbf{M}_{2k}^{\frac{k(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)}\le M.$$

and $\omega=M||\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)}+\mu$*, then* $P_{l,a\beta}\left(2K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega]\right)$ *is integrable in* $\mathbb{R}^N$ *(*K_2 *as in Theorem 1.3) and the following Dirichlet problem*

$$\begin{array}{l} F_k[-u]=P_{l,a,\beta}(u)+\mu\ \ \ in\ \ \mathbb{R}^N,\\ \inf_{\mathbb{R}^N}u=0, \end{array} \tag{1.25}$$

admits a nonnegative solution u *with* $-u\in\Phi^k(\mathbb{R}^N)$ *which satisfies*

$$u(x)\le 2K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega](x)\qquad\forall x\in\mathbb{R}^N. \tag{1.26}$$

Conversely, if (1.25) *admits a nonnegative solution* u *with* $-u\in\Phi^k(\mathbb{R}^N)$ *and* $P_{l,a,\beta}(u)$ *locally integrable in* $\mathbb{R}^N$*, then there exists a positive constant* C *depending on* N,k,l,a,β *and* R *such that there holds*

$$\int_E P_{l,a,\beta}(u)dx+\mu(E)\le Ccap_{I_{2k},Q^*_{k+1}}(E)\qquad\forall E\subset\mathbb{R}^N,E\ Borel. \tag{1.27}$$

where $cap_{I_{2k},Q^*_{k+1}}(E)$ *is defined accordingly by (1.15) with* $p=k+1$*.*

The four previous theorems are connected to the following results which deals with a class of nonlinear *Wolff integral equations*.

Theorem 1.5 *Let $\alpha > 0$, $p > 1$, $a > 0$, $\varepsilon > 0$, $R > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p-1$ and $0 < \alpha p < N$. Let f be a nonnegative measurable in $\mathbb{R}^N$ with the property that $\mu_1 = P_{l,a+\varepsilon,\beta}(f)$ is locally integrable in $\mathbb{R}^N$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$. There exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon$ and R such that if*

$$||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

then there exists a nonnegative function u such that $P_{l,a,\beta}(u)$ is locally integrable in $\mathbb{R}^N$ which satisfies

$$u = \mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u) + \mu] + f \quad in \quad \mathbb{R}^N, \tag{1.28}$$

and

$$u \leq F := 2c_p\mathbf{W}_{\alpha,p}^R[\omega_1] + 2c_p\mathbf{W}_{\alpha,p}^R[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1_{loc}(\mathbb{R}^N), \tag{1.29}$$

where $\omega_1 = M||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$ and $\omega_2 = M||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu_1$.

Conversely, if (1.28) admits a nonnegative solution u and $P_{l,a,\beta}(u)$ is locally integrable in $\mathbb{R}^N$, then there exists a positive constant C depending on $N, \alpha, p, l, a, \beta$ and R such that there holds

$$\int_E P_{l,a,\beta}(u)dx + \int_E P_{l,a+\varepsilon,\beta}(f)dx + \mu(E) \leq C cap_{G_{\alpha p},Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, \ E \ Borel. \tag{1.30}$$

When $R = \infty$ in the above theorem, we have a similar result provided f and μ have compact support in $\mathbb{R}^N$.

Theorem 1.6 *Let $\alpha > 0$, $p > 1$, $a > 0$, $\varepsilon > 0$, $R > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $0 < \alpha p < N$ and $l\beta > \frac{N(p-1)}{N-\alpha p}$. There exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon$ and R such that if f is a nonnegative measurable function in $\mathbb{R}^N$ with support in $B_R(0)$ such that that $\mu_1 = P_{l,a+\varepsilon,\beta}(f)$ is locally integrable in $\mathbb{R}^N$ and μ is a positive measure in $\mathbb{R}^N$ with support in $B_R(0)$ which verify*

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

then there exists a nonnegative function u such that $P_{l,a,\beta}(u)$ is integrable in $\mathbb{R}^N$ which satisfies

$$u = \mathbf{W}_{\alpha,p}[P_{l,a,\beta}(u) + \mu] + f \quad in \quad \mathbb{R}^N, \tag{1.31}$$

and

$$u \leq F := 2c_p\mathbf{W}_{\alpha,p}[\omega_1] + 2c_p\mathbf{W}_{\alpha,p}[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1(\mathbb{R}^N), \tag{1.32}$$

where $\omega_1 = M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)} + \mu$ and $\omega_2 = M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)} + \mu_1$.

Conversely, if (1.31) admits a nonnegative solution u such that $P_{l,a,\beta}(u)$ is integrable in $\mathbb{R}^N$, then there exists a positive constant C depending on $N, \alpha, p, l, a, \beta$ such that there holds

$$\int_E P_{l,a,\beta}(u)dx + \int_E P_{l,a,\beta}(f)dx + \mu(E) \leq C cap_{I_{\alpha p},Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, \ E \ Borel. \tag{1.33}$$

As an application of the Wolff integral equation we can notice that $\alpha = 1$, equation (1.31) is equivalent to

$$-\Delta_p(u - f) = P_{l,a,\beta}(u) \qquad \text{in } \mathbb{R}^N.$$

When $\alpha = \frac{2k}{k+1}$ and $p = k + 1$, it is equivalent to

$$F_k(-u + f) = P_{l,a,\beta}(u) \qquad \text{in } \mathbb{R}^N.$$

If $p = 2$ equation (1.31) becomes linear. If we set $\gamma = 2\alpha$, then

$$\begin{aligned}
\mathbf{W}_{\gamma,2}[\mu](x) &= \int_0^\infty \mu(B_t(x)) \frac{dt}{t^{N-\gamma-1}} \\
&= \int_{\mathbb{R}^N} \left(\int_{|x-y|}^\infty \frac{dt}{t^{N-\gamma-1}} \right) d\mu(y) \\
&= \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-\gamma}} \\
&= I_\gamma * \mu
\end{aligned}$$

where I_γ is the Riesz kernel of order γ. Thus (1.31) is equivalent to

$$(-\Delta)^\alpha(u - f) = P_{l,a,\beta}(u) \qquad \text{in } \mathbb{R}^N.$$

2 Estimates on potentials and Wolff integral equations

We denote by $B_r(a)$ the ball of center a and radius $r > 0$, $B_r = B_r(0)$ and by χ_E the characteristic function of a set E. The next estimates are crucial in the sequel.

Theorem 2.1 ***1.*** *There exists a positive constant c_1, depending only on N, α, p such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > p - 1$, $0 < R \le \infty$ we have*

$$(c_1 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} \left(\mathbf{I}^R_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx \le \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \le (c_1 q)^q \int_{\mathbb{R}^N} \left(\mathbf{I}^R_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx, \tag{2.1}$$

2. *Let $R > 0$. There exists a positive constant c_2, depending only on N, α, p, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > p - 1$ we have*

$$(c_2 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} \left(\mathbf{G}_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx \le \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \le (c_2 q)^q \int_{\mathbb{R}^N} \left(\mathbf{G}_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx, \tag{2.2}$$

where $\mathbf{G}_{\alpha p}[\mu] := G_{\alpha p} * \mu$ denotes the Bessel potential of order αp of μ.

3. *There exists a positive constant c_3 depending only on N, α, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > 1$ we have*

$$c_3^{-q} \int_{\mathbb{R}^N} \left(\mathbf{G}_\alpha[\mu][\mu](x)\right)^q dx \le \int_{\mathbb{R}^N} \left(\mathbf{I}^R_\alpha[\mu](x)\right)^q dx \le c_3^q \int_{\mathbb{R}^N} \left(\mathbf{G}_\alpha[\mu](x)\right)^q dx. \tag{2.3}$$

Proof. Note that $W^R_{\alpha/2,2}[\mu] = I^R_\alpha[\mu]$. We can find proof of (2.3) in [8, Step 3, Theorem 2.3]. By [8, Step 2, Theorem 2.3], there is $c_4 > 0$ such that

$$\int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \geq c_4^q \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}} dx \ \ \forall q > 0,\ 0 < R \leq \infty \text{ and } \ \mu \in \mathfrak{M}^+(\mathbb{R}^N). \tag{2.4}$$

We recall that $\mathbf{M}_{\alpha p,R}[\mu] = \mathbf{M}^0_{\alpha p,R}[\mu]$ by (1.10). Next we show that for all $q > 0$, $0 < R \leq \infty$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ there holds

$$\int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}} dx \ \geq (c_5 q)^{-q} \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx, \tag{2.5}$$

for some positive constant c_5 depending on N, α, p, q. Indeed, we denote μ_n by $\chi_{B_n}\mu$ for $n \in \mathbb{N}^*$. By [17, Theorem 1.2] or [8, Proposition 2.2], there exist constants $c_6 = c_6(N,\alpha,p) > 0$, $a = a(\alpha,p) > 0$ and $\varepsilon_0 = \varepsilon(N,\alpha,p)$ such that for all $n \in \mathbb{N}^*$, $t > 0$, $0 < R \leq \infty$ and $0 < \varepsilon < \varepsilon_0$, there holds

$$\left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > 3t\right\}\right| \leq c_6 \exp\left(-a\varepsilon^{-1}\right) \left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > t\right\}\right| + \left|\left\{(\mathbf{M}_{\alpha p,R}\mu_n)^{\frac{1}{p-1}} > \varepsilon t\right\}\right|.$$

Multiplying by qt^{q-1} and integrating over $(0,\infty)$, we obtain

$$\begin{aligned}\int_0^\infty qt^{q-1} \left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > 3t\right\}\right| dt \leq c_6 \exp\left(-a\varepsilon^{-1}\right) \int_0^\infty qt^{q-1} \left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > t\right\}\right| dt \\ + \int_0^\infty qt^{q-1} \left|\left\{(\mathbf{M}_{\alpha p,R}\mu_n)^{\frac{1}{p-1}} > \varepsilon t\right\}\right| dt,\end{aligned}$$

which implies

$$\varepsilon^q \left(3^{-q} - c_6 \exp\left(-a\varepsilon^{-1}\right)\right) \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu_n](x)\right)^q dx \leq \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}\mu_n)^{\frac{q}{p-1}} dx.$$

We see that $\sup_{0<\varepsilon<\varepsilon_0} \varepsilon^q \left(3^{-q} - c_6 \exp\left(-a\varepsilon^{-1}\right)\right) \geq (c_7 q)^{-q}$ for some constant c_7 which does not depend on q. Therefore (2.5) follows by Fatou's lemma. Hence, it is easy to obtain (2.1) from (2.4) and (2.5). At end, we obtain (2.2) from (2.1) and (2.3). ∎

The next result is proved in [8].

Theorem 2.2 *Let $\alpha > 0$, $p > 1$, $0 \leq \eta < p-1$, $0 < \alpha p < N$ and $L > 0$. Set $\delta = \frac{1}{2}\left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \log(2)$. Then there exists $C(L) > 0$, depending on N, α, p, η and L such that for any $R \in (0,\infty]$, $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$, any $a \in \mathbb{R}^N$ and $0 < r \leq L$, there holds*

$$\frac{1}{|B_{2r}(a)|} \int_{B_{2r}(a)} \exp\left(\delta \frac{\left(\mathbf{W}^R_{\alpha,p}[\mu_{B_r(a)}](x)\right)^{\frac{p-1}{p-1-\eta}}}{||\mathbf{M}^\eta_{\alpha p,R}[\mu_{B_r}(a)]||^{\frac{1}{p-1-\eta}}_{L^\infty(B_r(a))}}\right) dx \leq C(L), \tag{2.6}$$

where $\mu_{B_r(a)} = \chi_{B_r(a)}\mu$. Furthermore, if $\eta = 0$, C is independent of L.

Theorem 2.3 *Let $\alpha > 0$, $p > 1$ with $0 < \alpha p < N$, $\beta \geq 1$ and $R > 0$. Assume $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ satisfies*

$$||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq 1, \tag{2.7}$$

and set $\omega = ||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$. Then there exist positive constants C, δ_0 and c independent on μ such that $\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\beta\right)$ is locally integrable in $\mathbb{R}^N$,

$$\left\|\mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\beta\right)\right]\right\|_{L^\infty(\mathbb{R}^N)} \leq C \tag{2.8}$$

and

$$\mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\beta\right)\right] \leq c\mathbf{W}^R_{\alpha,p}[\omega] \quad \text{in } \mathbb{R}^N. \tag{2.9}$$

Proof. Let δ be as in Theorem 2.2. From (2.7), we have

$$||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]||_{L^\infty(\mathbb{R}^N)} \leq 2.$$

Let $x \in \mathbb{R}^N$. Since $\omega(B_t(y)) \leq 2t^{N-\alpha p}h_{\frac{(p-1)(\beta-1)}{\beta}}(t)$, for all $r \in (0,R)$ and $y \in \mathbb{R}^N$ we have

$$\begin{aligned}
\mathbf{W}^R_{\alpha,p}[\omega](y) &= \mathbf{W}^r_{\alpha,p}[\omega](y) + \int_r^R \left(\frac{\omega(B_t(y))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \\
&\leq \mathbf{W}^r_{\alpha,p}[\omega](y) + 2^{\frac{1}{p-1}}\int_{r\wedge 2^{-1}}^{2^{-1}}(-\ln t)^{-\frac{\beta-1}{\beta}}\frac{dt}{t} + 2^{\frac{1}{p-1}}\int_{2^{-1}}^{R\vee 2^{-1}}(-\ln t)^{-\frac{\beta-1}{\beta}}\frac{dt}{t} \\
&\leq \mathbf{W}^r_{\alpha,p}[\omega](y) + c_8(-\ln(r\wedge 2^{-1}))^{\frac{1}{\beta}} + c_8.
\end{aligned}$$

Thus,

$$\left(\mathbf{W}^R_{\alpha,p}[\omega](y)\right)^\beta \leq 3^{\beta-1}\left(\mathbf{W}^r_{\alpha,p}[\omega](y)\right)^\beta + c_9\ln\left(\frac{1}{r\wedge 2^{-1}}\right) + c_9. \tag{2.10}$$

Let $\theta \in (0,1]$, since $\exp\left(\frac{a+b}{2}\right) \leq \exp(a) + \exp(b)$ for all $a, b \in \mathbb{R}$, we get from (2.10)

$$\begin{aligned}
\exp\left(\theta\delta 3^{-\beta}\left(\mathbf{W}^R_{\alpha,p}[\omega](y)\right)^\beta\right) &\leq \exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}[\omega](y)\right)^\beta\right) + c_{10}\exp\left(\theta c_{11}\ln\left(\frac{1}{r\wedge 2^{-1}}\right)\right) \\
&\leq \exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}[\omega](y)\right)^\beta\right) + c_{10}\left(r\wedge 2^{-1}\right)^{-\theta c_{11}}.
\end{aligned} \tag{2.11}$$

For $r > 0$, $0 < t \leq r$, $y \in B_r(x)$ there holds $B_t(y) \subset B_{2r}(x)$. Thus, $\mathbf{W}^r_{\alpha,p}[\omega] = \mathbf{W}^r_{\alpha,p}[\omega_{B_{2r}(x)}]$ in $B_r(x)$. Then, using (2.6) in Theorem 2.2 with $L = 2R$ we get

$$\int_{B_r(x)}\exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}[\omega]\right)^\gamma\right) = \int_{B_r(x)}\exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}\left[\omega_{B_{2r}(x)}\right]\right)^\gamma\right) \leq c_{12}r^N.$$

Therefore, taking $\theta = 1 \wedge \frac{\alpha p}{2c_{11}}$, we deduce from (2.11)

$$\begin{aligned}\mathbf{W}^R_{\alpha,p}\left[\exp\left(\theta\delta 3^{-\beta}\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^{\gamma}\right)\right](x) &\leq \int_0^R \left(c_{12}r^{\alpha p} + c_{13}\left(r\wedge 2^{-1}\right)^{-\theta c_{11}} r^{\alpha p}\right)^{\frac{1}{p-1}}\frac{dr}{r}\\ &\leq \int_0^R \left(c_{12}r^{\alpha p} + c_{13}\left(r\wedge 2^{-1}\right)^{-\frac{\alpha p}{2}} r^{\alpha p}\right)^{\frac{1}{p-1}}\frac{dr}{r}\\ &\leq c_{14}.\end{aligned}$$

Hence, we get (2.8) with $\delta_0 = \left(1\wedge\frac{\alpha p}{2c_{11}}\right)\delta 3^{-\beta}$; we also get (2.9) since $\mathbf{W}^R_{\alpha,p}[\omega] \geq c_{15}$ for some positive constant $c_{15} > 0$. ∎

We recall that H_l and $P_{l,a,\beta}$ have been defined in (1.11) and (1.12).

Theorem 2.4 *Let $\alpha > 0$, $p > 1$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $0 < \alpha p < N$, $l\beta > \frac{N(p-1)}{N-\alpha p}$ and $R > 0$. Assume that $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ has support in B_R and verifies*

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq 1, \tag{2.12}$$

and set $\omega = ||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu$. Then there exist $C = C(N,\alpha,p,l,\beta,R) > 0$ and $\delta_1 = \delta_1(N,\alpha,p,l,\beta,R) > 0$ such that $H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right)$ is integrable in $\mathbb{R}^N$ and

$$\mathbf{W}_{\alpha,p}\left[H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right)\right](x) \leq C\mathbf{W}_{\alpha,p}[\omega](x) \quad \forall\ x\in\mathbb{R}^N. \tag{2.13}$$

Proof. We have from (2.12)

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]||_{L^\infty(\mathbb{R}^N)} \leq 2. \tag{2.14}$$

Let $\delta_1 > 0$ and $x \in \mathbb{R}^N$ fixed. We split the Wolff potential $\mathbf{W}_{\alpha,p}[\omega]$ into lower and upper parts defined by

$$\mathbf{L}^t_{\alpha,p}[\omega](x) = \int_t^{+\infty}\left(\frac{\omega(B_r(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dr}{r},$$

and

$$\mathbf{W}^t_{\alpha,p}[\omega](x) = \int_0^{t}\left(\frac{\omega(B_r(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dr}{r}.$$

Using the convexity we have

$$H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right) \leq H_l\left(\delta_1 2^\beta\left(\mathbf{L}^t_{\alpha,p}[\omega]\right)^\beta\right) + H_l\left(\delta_1 2^\beta\left(\mathbf{W}^t_{\alpha,p}[\omega]\right)^\beta\right).$$

Thus,

$$\mathbf{W}_{\alpha,p}\left[H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right)\right](x) \leq c_{16}\int_0^{+\infty}\left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} + c_{16}\int_0^{+\infty}\left(\frac{\omega^2_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t},$$

where $d\omega_t^1 = H_l\left(\delta_1 2^\beta \left(\mathbf{L}_{\alpha,p}^t[\omega]\right)^\beta\right)dx$ and $d\omega_t^2 = H_l\left(\delta_1 2^\beta \left(\mathbf{W}_{\alpha,p}^t[\omega]\right)^\beta\right)dx$. Inequality (2.13) will follows from the two inequalities below,

$$\int_0^{+\infty}\left(\frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \le c_{17}\mathbf{W}_{\alpha,p}[\omega](x), \tag{2.15}$$

and

$$\omega_t^2(B_t(x)) \le c_{18}\omega(B_{4t}(x)). \tag{2.16}$$

Step 1: Proof of (2.15). Since $B_r(y) \subset B_{2r}(x)$ for $y \in B_t(x)$ and $r \ge t$, there holds

$$\mathbf{L}_{\alpha,p}^t[\omega](y) \le \int_t^{+\infty}\left(\frac{\omega(B_{2r}(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dr}{r} = 2^{\frac{N-\alpha p}{p-1}}\mathbf{L}_{\alpha,p}^{t/2}[\omega](x).$$

It follows

$$\omega_t^1(B_t(x)) \le |B_1(0)|t^N H_l\left(\delta_1 c_{19}\left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x)\right)^\beta\right).$$

Thus,

$$\int_0^{+\infty}\left(\frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \le c_{20}\int_0^\infty A_t(x)dt, \tag{2.17}$$

where

$$A_t(x) = \left(t^{\alpha p}H_l\left(\delta_1 c_{19}\left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x)\right)^\beta\right)\right)^{\frac{1}{p-1}}\frac{1}{t}.$$

Since $H_l(s) \le s^l \exp(s)$ for all $s \ge 0$,

$$\begin{aligned} A_t(x) &\le c_{21}\left(t^{\alpha p}\left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x)\right)^{l\beta}\exp\left(\delta_1 c_{19}\left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x)\right)^\beta\right)\right)^{\frac{1}{p-1}}\frac{1}{t} \\ &= c_{21}t^{\frac{\alpha p}{p-1}-1}\left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x)\right)^{\frac{l\beta-p+1}{p-1}}\exp\left(\delta_1 c_{22}\left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x)\right)^\beta\right)\mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \end{aligned}$$

Now we estimate $\mathbf{L}_{\alpha,p}^{t/2}[\omega]$.

Case 1: $t \in (0,1)$. From (2.14) we deduce

$$\begin{aligned} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) &= \int_{t/2}^{1/2}\left(\frac{\omega(B_s(x))}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{ds}{s} + \int_{1/2}^{\infty}\left(\frac{\omega(B_s(x))}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\ &\le c_{23}\int_{t/2}^{1/2}(-ln(s))^{-1+\frac{1}{\beta}}\frac{ds}{s} + \int_{1/2}^{\infty}\left(\frac{\omega(B_R)}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{ds}{s} \\ &\le c_{24}\left(-\ln(t/2)\right)^{\frac{1}{\beta}}, \end{aligned}$$

which implies

$$\begin{aligned} A_t(x) &\le c_{25}t^{\frac{\alpha p}{p-1}-1}\left(-\ln(t/2)\right)^{\frac{l\beta-p+1}{\beta(p-1)}}\exp\left(\delta_1 c_{26}(-\ln(t/2))\right)\mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \\ &\le c_{27}t^{\frac{\alpha p}{p-1}-1}\left(-\ln(t/2)\right)^{\frac{l\beta-p+1}{\beta(p-1)}}t^{-\delta_1 c_{26}}\mathbf{L}_{\alpha,p}^{t/2}[\omega](x). \end{aligned}$$

We take $\delta_1 \leq \frac{1}{2c_{26}}\left(\frac{\alpha p}{p-1}-1\right)$ and obtain

$$A_t(x) \leq c_{28}\mathbf{L}^{t/2}_{\alpha,p}[\omega](x) \ \ \forall t \in (0,1). \tag{2.18}$$

Case 2: $t \geq 1$. We have

$$\mathbf{L}^{t/2}_{\alpha,p}[\omega](x) \leq \int_{t/2}^{\infty}\left(\frac{\omega(B_R)}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{ds}{s} = c_{29}t^{-\frac{N-\alpha p}{p-1}},$$

thus

$$\begin{aligned} A_t(x) &\leq c_{30}t^{\frac{\alpha p}{p-1}-1}t^{-\frac{(l\beta-p+1)(N-\alpha p)}{(p-1)^2}}\exp\left(\delta_1 c_{31}t^{-\frac{N-\alpha p}{p-1}}\right)\mathbf{L}^{t/2}_{\alpha,p}[\omega](x) \\ &\leq c_{32}t^{-1-\gamma}\mathbf{L}^{t/2}_{\alpha,p}[\omega](x), \end{aligned}$$

where $\gamma = \frac{1}{p-1}\left(\frac{l\beta(N-\alpha p)}{p-1}-N\right) > 0$.
Therefore, $A_t(x) \leq c_{33}(t\vee 1)^{-1-\gamma}\mathbf{L}^{t/2}_{\alpha,p}[\omega](x)$ for all $t>0$. Therefore, from (2.17)

$$\int_0^{+\infty}\left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \leq c_{33}\int_0^{\infty}(t\vee 1)^{-1-\gamma}\mathbf{L}^{t/2}_{\alpha,p}[\omega](x)dt.$$

Using Fubini Theorem we get

$$\begin{aligned} \int_0^{+\infty}\left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} &\leq c_{34}\int_0^{\infty}\int_0^t(s\vee 1)^{-1-\gamma}ds\left(\frac{\omega(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \\ &\leq c_{35}\int_0^{\infty}\left(\frac{\omega(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \\ &= c_{35}\mathbf{W}_{\alpha,p}[\mu](x), \end{aligned}$$

which follows (2.15).
Step 2: Proof of (2.16). For $t>0$, $r\leq t$ and $y\in B_t(x)$ we have $B_r(y)\subset B_{2t}(x)$, thus

$$\omega^2_t(B_t(x)) = \int_{B_t(x)} H_l\left(\delta_1 2^{\beta}\left(\mathbf{W}^t_{\alpha,p}[\omega_{B_{2t}(x)}](y)\right)^{\beta}\right)dy.$$

By Theorem 2.2 there exists $c_{36}>0$ such that for $0<\delta_1\leq c_{36}$, $0<t<2R$, $x\in\mathbb{R}^N$,

$$\int_{B_{4t}(x)}\exp\left(\delta_1 2^{\beta}\left(\mathbf{W}_{\alpha,p}[\omega_{B_{2t}(x)}](y)\right)^{\beta}\right)dy \leq c_{37}t^N. \tag{2.19}$$

We take $0<\delta_1\leq c_{36}$.
Case 1: $x\in\overline{B}_R(0)$. If $0<t<2R$, from (2.19) we get

$$\omega^2_t(B_t(x)) \leq c_{37}t^N \leq c_{38}\omega(B_{4t}(x)).$$

If $t \geq 2R$, since for any $|y| \geq 2R$,

$$\mathbf{W}_{\alpha,p}[\omega](y) = \int_{|y|/2}^{\infty} \left(\frac{\omega(B_t(y))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{39} \int_{|y|/2}^{\infty} t^{-1-\frac{N-\alpha p}{p-1}} dt \leq c_{40} |y|^{-\frac{N-\alpha p}{p-1}},$$

and thanks to (2.19) we have

$$\begin{aligned}
\omega_t^2(B_t(x)) &\leq \int_{B_{2R}(0)} \exp\left(\delta_1 2^\beta \left(\mathbf{W}_{\alpha,p}[\omega_{B_R(0)}](y) \right)^\beta \right) dy + \int_{B_{2R}(0)^c} H_l\left(\delta_1 2^\beta \left(\mathbf{W}_{\alpha,p}[\omega](y) \right)^\beta \right) dy \\
&\leq c_{41} R^N + \int_{B_{2R}(0)^c} H_l\left(c_{42} |y|^{-\frac{\beta(N-\alpha p)}{p-1}} \right) dy \\
&\leq c_{43} + c_{43} \int_{B_{2R}(0)^c} |y|^{-\frac{l\beta(N-\alpha p)}{p-1}} dy = c_{43} + c_{44} R^{N-\frac{l\beta(N-\alpha p)}{p-1}} \\
&\leq c_{45} |B_{4t}(x) \cap B_R| \leq c_{46} \omega(B_{4t}(x)).
\end{aligned}$$

From this we also have $H_l\left(\delta_1 \left(\mathbf{W}_{\alpha,p}[\omega] \right)^\beta \right) \in L^1(\mathbb{R}^N)$.
Case 2: $x \in B_R(0)^c$. If $|x| > R + t$ then $\omega_t^2(B_t(x)) = 0$. Next we consider the case $R < |x| \leq R + t$. If $0 < t < 2R$, we have $B_t((R - \frac{t}{2})\frac{x}{|x|}) \subset B_{4t}(x) \cap B_R$; thus from (2.19) we get

$$\omega_t^2(B_t(x)) \leq c_{47} t^N = c_{48} \left| B_t\left((R - \frac{t}{2}) \frac{x}{|x|} \right) \right| \leq c_{48} \left| B_{4t}(x) \cap B_R \right| \leq c_{49} \omega(B_{4t}(x)).$$

If $t \geq 2R$, as in Case 1 we also obtain $\omega_t^2(B_t(x)) \leq c_{50} \omega(B_{4t}(x))$. Hence, we get (2.16). Therefore, the result follows with $\delta_1 = \frac{1}{2c_{26}} \left(\frac{\alpha p}{p-1} - 1 \right) \wedge c_{36}$. ∎

In the next result we obtain estimate on a sequence of solutions of Wolff integral inequations obtained by induction.

Theorem 2.5 *Assume that the assumptions on α, p, a, l, β, ε, f, μ_1 and μ of Theorem 1.5 are fulfilled and R, K are positive real numbers. Suppose that $\{u_m\}$ is a sequence of nonnegative measurable functions in $\mathbb{R}^N$ that satisfies*

$$\begin{aligned}
u_{m+1} &\leq K \mathbf{W}_{\alpha,p}^R [P_{l,a,\beta}(u_m) + \mu] + f \quad \forall m \in \mathbb{N}, \\
u_0 &\leq K \mathbf{W}_{\alpha,p}^R [\mu] + f.
\end{aligned} \tag{2.20}$$

Then there exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon, K$ and R such that if

$$||\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}} [\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}} [\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$P_{l,a,\beta}\left(4c_p K \mathbf{W}_{\alpha,p}^R[\omega_1] + 4c_p K \mathbf{W}_{\alpha,p}^R[\omega_2] + f \right) \in L^1_{loc}(\mathbb{R}^N), \tag{2.21}$$

and

$$u_m \leq 2c_p K \mathbf{W}_{\alpha,p}^R[\omega_1] + 2c_p K \mathbf{W}_{\alpha,p}^R[\omega_2] + f \quad \forall m \in \mathbb{N}, \tag{2.22}$$

where

$$\omega_1 = M||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu, \tag{2.23}$$

$$\omega_2 = M||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu_1 \tag{2.24}$$

and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$.

Proof. The proof is based upon Theorems 2.3 and 2.4. Set $c_{a,\varepsilon} = 2\left(1 - \left(\frac{a}{a+\varepsilon}\right)^{1/\beta}\right)^{-1}$ and $\overline{a} = a\left(4c_{a,\varepsilon}c_pK\right)^\beta$. If $0 < M \leq 1$ we define ω_1 and ω_2 by (2.23) and (2.24) respectively. We now assume

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M.$$

We prove first that

$$\mathbf{W}^R_{\alpha,p}\left[H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_{1,2}]\right)^\beta\right)\right] \leq \mathbf{W}^R_{\alpha,p}[\omega_{1,2}]. \tag{2.25}$$

By Theorem 2.3, there exist $c, \delta_0 > 0$ independent on μ such that $\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}\left[M^{-1}\omega_{1,2}\right]\right)^\beta\right)$ is locally integrable in $\mathbb{R}^N$ and

$$\mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}\left[M^{-1}\omega_{1,2}\right]\right)^\beta\right)\right] \leq c\mathbf{W}^R_{\alpha,p}[M^{-1}\omega_{1,2}] \quad \text{in } \mathbb{R}^N.$$

Since $\theta^{-l}H_l(s) \leq H_l(\theta^{-1}s)$ for all $s \geq 0$ and $0 < \theta \leq 1$, it follows

$$\begin{aligned}
\mathbf{W}^R_{\alpha,p}\left[M^{-\frac{1}{2}\left(\frac{\beta l}{p-1}+1\right)}H_l\left(\delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)}\left(\mathbf{W}^R_{\alpha,p}[\omega_{1,2}]\right)^\beta\right)\right] &\leq \mathbf{W}^R_{\alpha,p}\left[H_l\left(\delta_0 M^{-\frac{\beta}{p-1}}\left(\mathbf{W}^R_{\alpha,p}[\omega_{1,2}]\right)^\beta\right)\right] \\
&\leq \mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[M^{-1}\omega_{1,2}]\right)^\beta\right)\right] \\
&\leq cM^{-\frac{1}{p-1}}\mathbf{W}^R_{\alpha,p}[\omega_{1,2}].
\end{aligned}$$

Hence,

$$\mathbf{W}^R_{\alpha,p}\left[H_l\left(\delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)}\left(\mathbf{W}^R_{\alpha,p}[\omega_{1,2}]\right)^\beta\right)\right] \leq cM^{\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)}\mathbf{W}^R_{\alpha,p}[\omega_{1,2}].$$

Therefore (2.25) is achieved if we prove

$$\overline{a} \leq \delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)} \quad \text{and} \quad cM^{\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)} \leq 1,$$

which is equivalent to

$$M \leq \left(\delta_0\overline{a}^{-1}\right)^{\left(\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)\right)^{-1}}.$$

Thus, we choose $M = 1 \wedge \left(\delta_0\overline{a}^{-1}\right)^{\left(\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)\right)^{-1}}$; we obtain (2.25) and the fact that $H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_{1,2}]\right)^\beta\right) \in L^1_{loc}(\mathbb{R}^N)$.

Now, we prove (2.22) by induction. Clearly, (2.22) holds with $m = 0$. Next we assume that (2.22) holds with $m = n$, and we claim that

$$u_{n+1} \leq 2c_pK\mathbf{W}^R_{\alpha,p}[\omega_1] + 2c_pK\mathbf{W}^R_{\alpha,p}[\omega_2] + f. \tag{2.26}$$

In fact, since (2.22) holds with $m = n$ and $P_{l,a,\beta}$ is convex, we have

$$\begin{aligned} P_{l,a,\beta}(u_n) &\leq P_{l,a,\beta}\left(4c_pK\mathbf{W}^R_{\alpha,p}[\omega_1] + 4c_pK\mathbf{W}^R_{\alpha,p}[\omega_2] + f\right) \\ &\leq P_{l,a,\beta}\left(4c_{a,\varepsilon}c_pK\mathbf{W}^R_{\alpha,p}[\omega_1]\right) + P_{l,\varepsilon,a}\left(4c_{a,\varepsilon}c_pK\mathbf{W}^R_{\alpha,p}[\omega_2]\right) + P_{l,a,\beta}\left(\left(1+\frac{\varepsilon}{a}\right)^{1/\beta} f\right) \\ &= H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_1]\right)^{\beta}\right) + H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_2]\right)^{\beta}\right) + P_{l,a+\varepsilon,\beta}(f). \end{aligned}$$

From this we derive (2.21). By the definition of u_{n+1} and the sub-additive property of $\mathbf{W}^R_{\alpha,p}[.]$, we obtain

$$\begin{aligned} u_{n+1} &\leq K\mathbf{W}^R_{\alpha,p}\left[H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_1]\right)^{\beta}\right) + H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_2]\right)^{\beta}\right) + P_{l,a+\varepsilon,\beta}(f) + \mu\right] + f \\ &\leq c_pK\mathbf{W}^R_{\alpha,p}\left[H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_1]\right)^{\beta}\right)\right] + c_pK\mathbf{W}^R_{\alpha,p}\left[H_l\left(\overline{a}\left(\mathbf{W}^R_{\alpha,p}[\omega_2]\right)^{\beta}\right)\right] \\ &\qquad + c_pK\mathbf{W}^R_{\alpha,p}\left[P_{l,a+\varepsilon,\beta}(f)\right] + c_pK\mathbf{W}^R_{\alpha,p}\left[\mu\right] + f. \end{aligned}$$

Hence follows (2.26) from (2.25). This completes the proof of the theorem. ∎

The next result is obtained by an easy adaptation of the proof Theorem 2.5.

Theorem 2.6 *Assume that the assumptions on α, p, a, l, β, ε, f, μ_1 and μ of Theorem 1.6 are fulfilled and R, K are positive real numbers. Suppose that $\{u_m\}$ is a sequence of nonnegative measurable functions in $\mathbb{R}^N$ that satisfies*

$$\begin{aligned} u_{m+1} &\leq K\mathbf{W}_{\alpha,p}[P_{l,a,\beta}(u_m) + \mu] + f \quad \forall m \in \mathbb{N}, \\ u_0 &\leq K\mathbf{W}_{\alpha,p}[\mu] + f. \end{aligned} \tag{2.27}$$

Then there exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon, K$ and R such that if

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$P_{l,a,\beta}\left(4c_pK\mathbf{W}_{\alpha,p}[\omega_3] + 4c_pK\mathbf{W}_{\alpha,p}[\omega_4] + f\right) \in L^1(\mathbb{R}^N), \tag{2.28}$$

and

$$u_m \leq 2c_pK\mathbf{W}_{\alpha,p}[\omega_3] + 2c_pK\mathbf{W}_{\alpha,p}[\omega_4] + f \quad \forall m \in \mathbb{N}, \tag{2.29}$$

where

$$\omega_3 = M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu \tag{2.30}$$

and

$$\omega_4 = M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu. \tag{2.31}$$

Let $P \in C(\mathbb{R}^+)$ be a decreasing positive function. The (α, P)-Orlicz-Bessel capacity of a Borel set $E \subset \mathbb{R}^N$ is defined by (see [1, Sect 2.6])

$$cap_{G_\alpha,P}(E) = \inf\left\{\int_{\mathbb{R}^N} P(f) : G_\alpha * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N)\right\},$$

and the (α, P)-Orlicz-Riesz capacity

$$cap_{I_\alpha,P}(E) = \inf\left\{\int_{\mathbb{R}^N} P(f) : I_\alpha * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N)\right\}.$$

Theorem 2.7 *Let $\alpha > 0$, $p > 1$, $a > 0$, $c > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p - 1$ and $0 < \alpha p < N$. Let $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$.*
1. *Let $0 < R \leq \infty$. If u is a nonnegative Borel function in $\mathbb{R}^N$ such that $P_{l,a,\beta}(u)$ is locally integrable in $\mathbb{R}^N$ and*

$$u(x) \geq c\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u) + \mu](x) \qquad \forall x \in \mathbb{R}^N, \tag{2.32}$$

then the following statements holds.
(i) If $R < \infty$, there exists a positive constant C_1 depending on $N, \alpha, p, l, a, \beta, c$ and R such that

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \leq C_1 cap_{G_{\alpha p},Q_p^*}(E) \qquad \forall E \subset \mathbb{R}^N, \ E \text{ Borel.} \tag{2.33}$$

(ii) If $R = \infty$, there exists a positive constant C_2 depending on $N, \alpha, p, l, a, \beta, c$ such that

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \leq C_2 cap_{I_{\alpha p},Q_p^*}(E) \qquad \forall E \subset \mathbb{R}^N, \ E \text{ Borel.} \tag{2.34}$$

2. *Let Ω be a bounded domain in $\mathbb{R}^N$, $\mu \in \mathfrak{M}^+(\Omega)$ and $\delta \in (0, 1)$. If u is a nonnegative Borel function in Ω such that $P_{l,a,\beta}(u)$ is locally integrable in Ω and*

$$u(x) \geq c\mathbf{W}^{\delta d(x,\partial\Omega)}_{\alpha,p}[P_{l,a,\beta}(u) + \mu](x) \qquad \forall x \in \Omega, \tag{2.35}$$

then, for any compact set $K \subset \Omega$, there exists a positive constant C_3 depending on $N, \alpha, p, l, a, \beta, c, \delta$ and $dist(K, \partial\Omega)$ such that

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \leq C_3 cap_{G_{\alpha p},Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel}, \tag{2.36}$$

where Q_p^* is the complementary function to Q_p.

Proof. Set $d\omega = P_{l,a,\beta}(u)dx + d\mu$.
1. We have

$$P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega]\right)dx \leq d\omega \quad \text{in } \mathbb{R}^N.$$

Let M_ω denote the centered Hardy-Littlewood maximal function which is defined for any $f \in L^1_{loc}(\mathbb{R}^N, d\omega)$ by

$$M_\omega f(x) = \sup_{t>0} \frac{1}{\omega(B_t(x))} \int_{B_t(x)} |f| d\omega.$$

If $E \subset \mathbb{R}^N$ is a Borel set, we have

$$\int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega]\right) dx \leq \int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

Since M_ω is bounded on $L^s(\mathbb{R}^N, d\omega)$, $s > 1$, we deduce from Fefferman's result [11] that

$$\int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega]\right) dx \leq c_{51}\omega(E),$$

for some constant c_{51} only depends on N and $\frac{l\beta}{p-1}$. Since $M_\omega \chi_E \leq 1$, we derive

$$\begin{aligned}(M_\omega \chi_E(x))^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega](x)\right) &\geq P_{l,a,\beta}\left(c\left(M_\omega \chi_E(x)\right)^{\frac{1}{p-1}} \mathbf{W}^R_{\alpha,p}[\omega](x)\right) \\ &\geq P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega_E](x)\right),\end{aligned}$$

where $\omega_E = \chi_E \omega$. Thus

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega_E]\right) dx \leq c_{51}\omega(E) \qquad \forall E \subset \mathbb{R}^N, E \text{ Borel.} \tag{2.37}$$

From (2.1), (2.2) and (2.3) we get

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega_E](x)\right) dx \geq \int_{\mathbb{R}^N} Q_p\left(c_{52}\mathbf{G}_{\alpha p}[\omega_E](x)\right) dx \quad \text{if } R < \infty,$$

and

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega_E](x)\right) dx \geq \int_{\mathbb{R}^N} Q_p\left(c_{53}\mathbf{I}_{\alpha p}[\omega_E](x)\right) dx \quad \text{if } R = \infty,$$

where Q_p is defined by (1.13) and $c_{52} = (c_2\beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$ if $p \neq 2$, $c_{52} = c_3^{-1} a^{\frac{1}{\beta}} c$ if $p = 2$ (the constant c_2 defined in (2.2), depends on R, therefore $c_{52} = c_{52}(r_K)$) and $c_{53} = (c_1\beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$ if $p \neq 2$, $c_{53} = a^{\frac{1}{\beta}} c$ if $p = 2$. Thus, from (2.37) we obtain that for all Borel set $E \subset \mathbb{R}^N$ there holds

$$\int_{\mathbb{R}^N} Q_p\left(c_{52}\mathbf{G}_{\alpha p}[\omega_E](x)\right) dx \leq c_{51}\omega(E) \text{ if } R < \infty,$$

and

$$\int_{\mathbb{R}^N} Q_p\left(c_{53}\mathbf{I}_{\alpha p}[\omega_E](x)\right) dx \leq c_{51}\omega(E) \text{ if } p = \infty.$$

We recall that $Q_p^*(s) = \sup_{t>0}\{st - Q_p(t)\}$ satisfies the sub-additivity Δ_2-condition (see Chapter 2 in [19]).

(i) We assume $R < \infty$. For every $f \geq 0$, $Q_p^*(f) \in L^1(\Omega)$ such that $G_{\alpha p} * f \geq \chi_E$, we have

$$\begin{aligned}\omega(E) &\leq \int_{\mathbb{R}^N} G_{\alpha p} * f d\omega_E = (2c_{51})^{-1} \int_{\mathbb{R}^N} \left(c_{52}\mathbf{G}_{\alpha p}\left[\omega_E\right]\right)\left(2c_{51}c_{52}^{-1} f\right) dx \\ &\leq (2c_{51})^{-1} \int_{\mathbb{R}^N} Q_p\left(c_{52}\mathbf{G}_{\alpha p}\left[\omega_E\right]\right) dx + (2c_{51})^{-1} \int_{\mathbb{R}^N} Q_p^*\left(2c_{51}c_{52}^{-1} f\right) dx \\ &\leq 2^{-1}\omega(E) + c_{54} \int_{\mathbb{R}^N} Q_p^*\left(f\right) dx,\end{aligned}$$

the last inequality following from the Δ_2-condition. Notice that c_{54}, as well as the next constant c_{55}, depends on r_K. Thus,

$$\omega(E) \le 2c_{54} \int_{\mathbb{R}^N} Q_p^*(f)\, dx.$$

Then, we get

$$\omega(E) \le c_{55} cap_{G_{\alpha p}, Q_p^*}(E) \qquad \forall E \subset \mathbb{R}^N, E \text{ Borel}.$$

Which implies (2.33).
(ii) We assume $R = \infty$. For every $f \ge 0$, $Q_p^*(f) \in L^1(\Omega)$ such that $I_{\alpha p} * f \ge \chi_E$, since $I_{\alpha p} * \omega_E = I_{\alpha p}[\omega_E]$, as above we have

$$\begin{aligned} \omega(E) &\le \int_{\mathbb{R}^N} I_{\alpha p} * f d\omega_E = \int_{\mathbb{R}^N} (\mathbf{I}_{\alpha p} * \omega_E)\, f dx = \int_{\mathbb{R}^N} \mathbf{I}_{\alpha p}[\omega_E]\, f dx \\ &\le 2^{-1}\omega(E) + c_{56} \int_{\mathbb{R}^N} Q_p^*(f)\, dx, \end{aligned}$$

Then, it follows (2.34).
2. Let $K \subset \Omega$ be compact. Set $r_K = dist(K, \partial\Omega)$ and $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$. We have

$$P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega]\right) dx \le d\omega \quad \text{in } \Omega.$$

Thus, for any Borel set $E \subset K$

$$\int_\Omega (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega]\right) dx \le \int_\Omega (M_\omega \chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

As above we get

$$\int_\Omega P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x)\right) dx \le c_{51}\omega(E) \quad \forall E \subset K, E \text{ Borel}. \tag{2.38}$$

Note that if $x \in \Omega$ and $d(x, \partial\Omega) \le r_K/8$, then $B_t(x) \subset \Omega \backslash \Omega_K$ for all $t \in (0, \delta d(x, \partial\Omega))$; indeed, for all $y \in B_t(x)$

$$d(y, \partial\Omega) \le d(x, \partial\Omega) + |x - y| < (1 + \delta) d(x, \partial\Omega) < \frac{1}{4} r_K,$$

thus

$$d(y, K) \ge d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4} r_K > \frac{1}{2} r_K,$$

which implies $y \notin \Omega_K$. We deduce that

$$\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \ge \mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K}[\omega_E](x) \qquad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K}[\omega_E](x) = 0 \qquad \forall x \in \Omega^c.$$

Hence we obtain from (2.38),

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K}[\omega_E](x)\right) dx \le c_{51}\omega(E) \qquad \forall E \subset K, E \text{ Borel}. \tag{2.39}$$

As above we also obtain

$$\omega(E) \le c_{57} cap_{G_{\alpha p}, Q_p^*}(E) \qquad \forall E \subset K,\ E \text{ Borel},$$

where the positive constant c_{57} depends on r_K. Inequality (2.36) follows and this completes the proof of the Theorem. ∎

Proof of Theorem 1.5. Consider the sequence $\{u_m\}_{m\ge 0}$ of nonnegative functions defined by $u_0 = f$ and

$$u_{m+1} = \mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u_m)] + f \quad \text{in } \mathbb{R}^N \quad \forall m \ge 0.$$

By Theorem 2.5, $\{u_m\}_{m\ge 0}$ is well defined and (2.21) and (2.22) are satisfied. It is easy to see that $\{u_m\}$ is nondecreasing. Hence, thanks to the dominated convergence theorem, we obtain that $u(x) = \lim_{m\to\infty} u_m(x)$ is a solution of equation (1.28) which satisfies (1.29).
Conversely, we obtain (1.33) directly from Theorem 2.7. ∎

Proof of Theorem 1.6. The proof is similar to the previous one by using Theorem 2.7, Part 1, (ii). ∎

3 Quasilinear Dirichlet problems

Let Ω be a bounded domain in $\mathbb{R}^N$. If $\mu \in \mathfrak{M}_b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative parts in the Jordan decomposition. We denote by $\mathfrak{M}_0(\Omega)$ the space of measures in Ω which are absolutely continuous with respect to the $c_{1,p}^\Omega$-capacity defined on a compact set $K \subset \Omega$ by

$$c_{1,p}^\Omega(K) = \inf\left\{\int_\Omega |\nabla\varphi|^p dx : \varphi \ge \chi_K, \varphi \in C_c^\infty(\Omega)\right\}.$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c_{1,p}^\Omega$-capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. It is well known that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ can be written under the form $\mu_0 = f - div\ g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega)$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. If u is a measurable function defined in Ω, finite a.e. and such that $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u|\le k} v$ a.e. in Ω and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [10].

Definition 3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist λ_k^+ and λ_k^- belonging to $\mathfrak{M}_b^+ \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on

the sets $u = k$ and $u = -k$, with the property that $\mu_k^+ \rightharpoonup \mu_s^+$, $\mu_k^- \rightharpoonup \lambda_s^-$ in the narrow topology of measures and such that

$$\int_{\{|u|<k\}} |\nabla u|^{p-2} \nabla u.\nabla\varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_\Omega \varphi d\lambda_k^+ - \int_\Omega \varphi d\lambda_k^-,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2 *We recall that if u is a renormalized solution to problem (3.1), then $\frac{|\nabla u|^p}{(|u|+1)^r} \in L^1(\Omega)$ for all $r > 1$. From this it follows by Hölder's inequality that $u \in W_0^{1,p_1}(\Omega)$ for all $1 \le p_1 < p$ provided $e^{a|u|} \in L^1(\Omega)$ for some $a > 0$. Furthermore, $u \ge 0$ a.e. in Ω if $\mu \in \mathfrak{M}_b^+(\Omega)$.*

The following general stability result has been proved in [10, Th 4.1, Sec 5.1].

Theorem 3.3 *If $\{\mu_n\}$ is a bounded sequence in $\mathfrak{M}_b(\Omega)$ which converges in the weak sense of measures to some $\mu \in \mathfrak{M}_b(\Omega)$, and $\{u_n\}$ a sequence of renormalized solutions of problem*

$$\begin{aligned} -\Delta_p u_n &= \mu_n && \text{in } \Omega, \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

then, up to a subsequence, $\{u_n\}$ converges a.e. to a solution of equation $-\Delta_p u = \mu$ in $\mathcal{D}'(\Omega)$, and such that $\frac{1}{k}\int_\Omega |\nabla T_k(u)|^p dx \le M$ for every $k > 0$.

The next result is a sharp extension of the stability Theorem 3.3.

Theorem 3.4 *Let $\mu = \mu_0 + \mu_s^+ - \mu_s^-$, with $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$ and μ_s^+, μ_s^- belonging to $\mathfrak{M}_s^+(\Omega)$. Let $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$ with $F_n \in L^1(\Omega)$, $g_n \in (L^{p'}(\Omega))^N$ and ρ_n, η_n belonging to $\mathfrak{M}_b^+(\Omega)$. Assume that $\{F_n\}$ converges to F weakly in $L^1(\Omega)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(\Omega))^N$ and $(\operatorname{div} g_n)$ is bounded in $\mathfrak{M}_b(\Omega)$; assume also that $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ to μ_s^- in the narrow topology. If $\{u_n\}$ is a sequence of renormalized solutions of (3.2), then, up to a subsequence, it converges a.e. in Ω to a renormalized solution u of problem (3.1). Furthermore $T_k(u_n)$ converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$ for any $k > 0$.*

We also recall the following estimate [20, Th 2.1].

Theorem 3.5 *Let Ω be a bounded domain of $\mathbb{R}^N$. Then there exists a constant $K_1 > 0$, depending on p and N such that if $\mu \in \mathfrak{M}_b^+(\Omega)$ and u is a nonnegative renormalized solution of problem (3.1) with data μ, there holds*

$$\frac{1}{K_1}\mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \le u(x) \le K_1 \mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\mu](x) \quad \forall x \in \Omega, \tag{3.3}$$

where the positive constant K_1 only depends on N, p.

Proof of Theorem 1.1. Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of nonnegative renormalized solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu && \text{in } \Omega, \\ u_0 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu && \text{in } \Omega, \\ u_{m+1} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By Theorem 3.5 we have

$$\chi_\Omega u_0 \le K_1 \mathbf{W}^R_{1,p}[\mu],$$
$$\chi_\Omega u_{m+1} \le K_1 \mathbf{W}^R_{1,p}[P_{l,a,\beta}(u_m) + \mu] \quad \forall m \in \mathbb{N},$$

where $R = 2\,diam\,(\Omega)$. Thus, by Theorem 2.5 with $f \equiv 0$, there exists $M > 0$ depending on N, p, l, a, β, K_1 and R such that $P_{l,a,\beta}(4c_p K_1 \mathbf{W}^R_{1,p}[\omega]) \in L^1(\Omega)$ and

$$u_m(x) \le 2c_p K_1 \mathbf{W}^R_{1,p}[\omega](x) \quad \forall x \in \Omega, m \in \mathbb{N}, \tag{3.4}$$

provided that

$$||M_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M,$$

where $\omega = M||\mathbf{M}_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$. This implies that $\{u_m\}$ is well defined; by Theorem 3.3 it contains a subsequence that we still denote by $\{u_m\}$ which converges a.e in Ω to function u which satisfies (1.17) in Ω. Furthermore, we deduce from (3.4) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 3.4 we obtain that u is a renormalized solution of (1.16).
Conversely, assume that (1.16) admits a nonnegative renormalized solution u. By Theorem 3.5 there holds

$$u(x) \ge \frac{1}{K_1} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \Omega.$$

Hence, we achieve (1.18) from Theorem 2.7. ∎

Applications. We consider the case $p = 2$, $\beta = 1$. Then $l = 2$ and

$$P_{l,a,\beta}(r) = e^{ar} - 1 - ar.$$

If Ω is a bounded domain in $\mathbb{R}^N$, there exists $M > 0$ such that if μ is a positive Radon measure in Ω which satisfies

$$\mu(B_t(x)) \le M t^{N-2} \qquad \forall t > 0 \text{ and almost all } x \in \Omega,$$

there exists a positive solution u to the following problem

$$\begin{aligned} -\Delta u &= e^{au} - 1 - au + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Furthermore

$$u(x) \le K(N) \int_0^{2\,diam\,\Omega} \frac{\omega(B_t(x))}{t^{N-1}} dt = K(N) \int_0^{2\,diam\,(\Omega)} \frac{\mu(B_t(x))}{t^{N-1}} dt + b \qquad \forall x \in \Omega.$$

where $b = 2K(N)M||\mathbf{M}_{p,2\,diam\,(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)}(diam\,\Omega)^2$. In the case $N = 2$ this result has already been proved by Richard and Véron [22, Prop 2.4].

4 p-superharmonic functions and quasilinear equations in $\mathbb{R}^N$

We recall some definitions and properties of p-superharmonic functions.

Definition 4.1 *A function u is said to be p-harmonic in $\mathbb{R}^N$ if $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $-\Delta_p u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$. A function u is called a p-supersolution in $\mathbb{R}^N$ if $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ and $-\Delta_p u \geq 0$ in $\mathcal{D}'(\mathbb{R}^N)$.*

Definition 4.2 *A lower semicontinuous (l.s.c) function $u : \mathbb{R}^N \to (-\infty, \infty]$ is called p-superharmonic if u is not identically infinite and if, for all open $D \subset\subset \mathbb{R}^N$ and all $v \in C(\overline{D})$, p-harmonic in D, $v \leq u$ on ∂D implies $v \leq u$ in D.*

Let u be a p-superharmonic in $\mathbb{R}^N$. It is well known that $u \wedge k \in W^{1,p}_{loc}(\mathbb{R}^N)$ is a p-supersolution for all $k > 0$ and $u < \infty$ a.e in $\mathbb{R}^N$, thus, u has a gradient (see the previous section). We also have $|\nabla u|^{p-1} \in L^q_{loc}(\mathbb{R}^N)$, $\frac{|\nabla u|^p}{(|u|+1)^r} \in L^1_{loc}(\mathbb{R}^N)$ and $u \in L^s_{loc}(\mathbb{R}^N)$ for $1 \leq q < \frac{N}{N-1}$ and $r > 1$, $1 \leq s < \frac{N(p-1)}{N-p}$ (see [14, Theorem 7.46]). In particular, if $e^{a|u|} \in L^1_{loc}(\mathbb{R}^N)$ for some $a > 0$, then $u \in W^{1,p_1}_{loc}(\mathbb{R}^N)$ for all $1 \leq p_1 < p$ by Hölder's inequality. Thus by the dominated convergence theorem,

$$\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lim_{k \to \infty} \int_{\mathbb{R}^N} |\nabla (u \wedge k)|^{p-2} \nabla (u \wedge k) \nabla \varphi \geq 0.$$

Hence, by the Riesz Representation Theorem we conclude that there is a nonnegative Radon measure denoted by $\mu[u]$, called Riesz measure, such that $-\Delta_p u = \mu[u]$ in $\mathcal{D}'(\mathbb{R}^N)$.

The following weak convergence result for Riesz measures proved in [28] will be used to prove the existence of p-superharmonic solutions to quasilinear equations.

Theorem 4.3 *Suppose that $\{u_n\}$ is a sequence of nonnegative p-superharmonic functions in $\mathbb{R}^N$ that converges a.e to a p-superharmonic function u. Then the sequence of measures $\{\mu[u_n]\}$ converges to $\mu[u]$ in the weak sense of measures.*

The next theorem is proved in [20]

Theorem 4.4 *Let μ be a measure in $\mathfrak{M}^+(\mathbb{R}^N)$. Suppose that $\mathbf{W}_{1,p}[\mu] < \infty$ a.e. Then there exists a nonnegative p-superharmonic function u in $\mathbb{R}^N$ such that $-\Delta_p u = \mu$ in $\mathcal{D}'(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} u = 0$ and*

$$\frac{1}{K_1} \mathbf{W}_{1,p}[\mu](x) \leq u(x) \leq K_1 \mathbf{W}_{1,p}[\mu](x), \tag{4.1}$$

for all x in $\mathbb{R}^N$, where the constant K_1 is as in Theorem 3.5. Furthermore any p-superharmonic function u in $\mathbb{R}^N$, such that $\inf_{\mathbb{R}^N} u = 0$ satisfies (4.1) with $\mu = -\Delta_p u$.

Proof of Theorem 1.2. Let $\{u_m\}_{m \in \mathbb{N}}$ be a sequence of p-superharmonic solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_0 &= 0, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu \ \text{ in } \ \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_{m+1} &= 0. \end{aligned}$$

By Theorem 4.4 we have

$$\begin{aligned} u_0 &\le K_1 \mathbf{W}_{1,p}[\mu], \\ u_{m+1} &\le K_1 \mathbf{W}_{1,p}[P_{l,a,\beta}(u_m) + \mu] \ \ \forall m \in \mathbb{N}. \end{aligned}$$

Thus, by Theorem 2.5 with $f \equiv 0$, there exists $M > 0$ depending on N, p, l, a, β, K_1 and R such that $P_{l,a,\beta}(4c_p K_1 \mathbf{W}_{1,p}[\omega]) \in L^1(\mathbb{R}^N)$ and

$$u_m \le 2c_p K_1 \mathbf{W}_{1,p}[\omega] \quad \forall m \in \mathbb{N}, \tag{4.2}$$

provided that

$$||M_p^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M,$$

where $\omega = M||\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)} + \mu$. This implies that $\{u_m\}$ is well defined; by [15, Theorem 1.17] it contains a subsequence that we still denote by $\{u_m\}$ which converges a.e in $\mathbb{R}^N$ to a p-superharmonic function u which satisfies (1.20) in Ω. Furthermore, we deduce from (4.2) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\mathbb{R}^N)$. Finally, by Theorem 4.3 we conclude that u is a p-superharmonic solution of (1.19).
Conversely, assume that (1.19) admits a nonnegative renormalized solution u. By Theorem 4.4 there holds

$$u(x) \ge \frac{1}{K_1}\mathbf{W}_{1,p}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \mathbb{R}^N.$$

Hence, we obtain (1.21) from Theorem 2.7. ∎

5 Hessian equations

In this section $\Omega \subset \mathbb{R}^N$ is either a bounded domain with a C^2 boundary or the whole $\mathbb{R}^N$. For $k = 1, .., N$ and $u \in C^2(\Omega)$ the k-hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2 u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivative D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \le i_1 < ... < i_k \le N} \lambda_{i_1} ... \lambda_{i_k}.$$

We can see that

$$F_k[u] = \left[D^2 u\right]_k,$$

where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the k-th principal minors. We assume that $\partial\Omega$ is uniformly (k-1)-convex, that is

$$S_{k-1}(\kappa) \ge c_0 > 0 \ on \ \ \partial\Omega,$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal.

Definition 5.1 *An upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is k-convex (k-subharmonic) if, for every open set $\Omega' \subset \overline{\Omega}' \subset \Omega$ and for every function $v \in C^2(\Omega') \cap C(\overline{\Omega})$ satisfying $F_k[v] \leq 0$ in Ω', the following implication is true*

$$u \leq v \text{ on } \partial\Omega' \quad \Longrightarrow \quad u \leq v \text{ in } \Omega'.$$

We denote by $\Phi^k(\Omega)$ the class of all k-subharmonic functions in Ω which are not identically equal to $-\infty$.

The following weak convergence result for k-Hessian operators proved in [26] is fundamental in our study.

Theorem 5.2 *Let Ω be either a bounded uniformly (k-1)-convex in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$. For each $u \in \Phi^k(\Omega)$, there exist a nonnegative Radon measure $\mu_k[u]$ in Ω such that*

1 *$\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$.*

2 *If $\{u_n\}$ is a sequence of k-convex functions which converges a.e to u, then $\mu_k[u_n] \rightharpoonup \mu_k[u]$ in the weak sense of measures.*

As in the case of quasilinear equations with measure data, precise estimates of solutions of k-Hessian equations with measures data are expressed in terms of Wolff potentials. The next results are proved in [26, 18, 20].

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2, uniformly (k-1)-convex domain. Let φ be a nonnegative continuous function on $\partial\Omega$ and μ be a nonnegative Radon measure. Suppose that μ can be decomposed under the form*

$$\mu = \mu_1 + f$$

where μ_1 is a measure with compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$ if $k \leq \frac{N}{2}$, or $p = 1$ if $k > \frac{N}{2}$. Then there exists a nonnegative function u in Ω such that $-u \in \Phi^k(\Omega)$, continuous near $\partial\Omega$ and u is a solution of the problem

$$\begin{aligned} F_k[-u] &= \mu \qquad \text{in } \Omega, \\ u &= \varphi \qquad \text{on } \partial\Omega. \end{aligned}$$

Furthermore, any nonnegative function u such that $-u \in \Phi^k(\Omega)$ which is continuous near $\partial\Omega$ and is a solution of above equation, satisfies

$$\frac{1}{K_2}\mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[\mu] \leq u(x) \leq K_2\left(\mathbf{W}^{2diam\,\Omega}_{\frac{2k}{k+1},k+1}[\mu](x) + \max_{\partial\Omega}\varphi\right), \tag{5.1}$$

where K_2 is a positive constant independent of x, u and Ω.

Theorem 5.4 *Let μ be a measure in $\mathfrak{M}^+(\mathbb{R}^N)$ and $2k < N$. Suppose that $\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu] < \infty$ a.e. Then there exists u, $-u \in \Phi^k(\mathbb{R}^N)$ such that $\inf_{\mathbb{R}^N} u = 0$ and $F_k[-u] = \mu$ in $\mathbb{R}^N$ and*

$$\frac{1}{K_2}\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu](x) \leq u(x) \leq K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu](x) \tag{5.2}$$

for all x in $\mathbb{R}^N$, where the constant K_2 is the one of the previous Theorem. Furthermore, if u is a nonnegative function such that $-u \in \Phi^k(\mathbb{R}^N)$, then (5.2) holds with $\mu = F_k[-u]$.

Proof of Theorem 1.3. We defined a sequence of nonnegative functions u_m, continuous near $\partial\Omega$ and such that $-u_m \in \Phi^k(\Omega)$, by the following iterative scheme

$$\begin{array}{rll} F_k[-u_0] = \mu & \text{in} & \Omega, \\ u_0 = \varphi & \text{on} & \partial\Omega, \end{array} \tag{5.3}$$

and, for $m \geq 0$,

$$\begin{array}{rll} F_k[-u_{m+1}] = P_{l,a,\beta}(u_m) + \mu & \text{in} & \Omega, \\ u_{m+1} = \varphi & \text{on} & \partial\Omega. \end{array} \tag{5.4}$$

By Theorem 5.3 we have

$$\begin{array}{l} \chi_\Omega u_0 \leq K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu] + b_0, \\ \chi_\Omega u_{m+1} \leq K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u_m) + \mu] + b_0, \end{array} \tag{5.5}$$

where $b_0 = K_2 \max_{\partial\Omega} \varphi$ and $R = 2\,diam\,(\Omega)$. We set $\omega_1 = M||\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$, $\omega_2 = M||\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + P_{l,2a,\beta}(b_0)$ and $g = 4c_p K \mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega_2] + b_0$. Since ω_2 is constant, g has the same property and actually $g = 2c_p K (c_N \omega_2)^{\frac{1}{k}} R^2 + b_0$. Then, by Theorem 2.5 with $f = b_0$ and $\varepsilon = a$, there exists $M_1 > 0$ depending on N, k, l, a, β and R such that $P_{l,a,\beta}\left(4K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega_1] + g\right) \in L^1(\Omega)$ and

$$u_m(x) \leq 2K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega_1](x) + g \qquad \forall x \in \Omega,\, \forall m \geq 0, \tag{5.6}$$

provided that

$$||M^{\frac{k(\beta-1)}{\beta}}_{2k,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M_1 \;\text{ and }\; ||M^{\frac{k(\beta-1)}{\beta}}_{2k,R}[P_{l,2a,\beta}(b_0)]||_{L^\infty(\mathbb{R}^N)} \leq M_1.$$

It follows that one can find constants M_2 depending on N, k, l, a, β, R and M_1 such that if $\max_{\partial\Omega} \varphi \leq M_2$, then $||M^{\frac{k(\beta-1)}{\beta}}_{2k,R}[P_{l,2a,\beta}(b_0)]||_{L^\infty(\mathbb{R}^N)} \leq M_1$. Hence, we deduce from (5.6) that $P_{l,a,\beta}\left(2K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu] + b_2\right) \in L^1(\Omega)$ and

$$u_m(x) \leq 2K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu](x) + b_2 \qquad \forall x \in \Omega,\, \forall m \geq 0, \tag{5.7}$$

for some constant b_2 $(= g)$ depending on N, k, l, a, β, R and M_1. Note that because we can write

$$\omega = P_{l,a,\beta}(u_m) + \mu = (\mu_1 + \chi_{\Omega_\delta} P_{l,a,\beta}(u_m)) + ((1 - \chi_{\Omega_\delta}) P_{l,a,\beta}(u_m) + f)\,,$$

where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$ and $\delta > 0$ is small enough and since u_m is continuous near $\partial\Omega$, then ω satisfies the assumptions of the data in Theorem 5.3. Therefore the sequence $\{u_m\}$ is well defined. Since $-u_m$ is k-subharmonic and the sequence $\{u_m\}$ is bounded in $L^1(\Omega)$,

the sequence $\{u_m\}$ is also bounded in $W^{1,1}_{loc}(\Omega)$ (see e.g. [26]); we can find a subsequence, still denoted by the index m, such that u_m converges a.e in Ω to a function u for which (1.23) is satisfied in Ω with the constant $b_2 = g$. Furthermore, we deduce from (5.6) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 5.2, we obtain that u satisfies (1.22) and (1.23).
Conversely, assume that (1.22) admits nonnegative solution u, continuous near $\partial\Omega$, such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$. Then by Theorem 5.3 we have

$$u(x) \geq \frac{1}{K_2}\mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u)+\mu](x) \quad \text{for all } x \in \Omega.$$

Using Theorem 2.7, we conclude that (1.24) holds. ∎

Proof of Theorem 1.4. We define a sequence of nonnegative functions u_m, continuous near $\partial\Omega$ and such that $-u_m \in \Phi^k(\Omega)$, by the following iterative scheme

$$\begin{array}{ll} F_k[-u_0] = \mu & \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_0 = 0, & \end{array} \tag{5.8}$$

and, for $m \geq 0$,

$$\begin{array}{ll} F_k[-u_{m+1}] = P_{l,a,\beta}(u_m) + \mu & \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_{m+1} = 0. & \end{array} \tag{5.9}$$

By Theorem 5.4, we have

$$\begin{aligned} u_0 &\leq K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu], \\ u_{m+1} &\leq K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u_m)+\mu]. \end{aligned} \tag{5.10}$$

Thus, by Theorem 2.5 with $f \equiv 0$, there exists $M > 0$ depending on N, k, l, a, β and R such that $P_{l,a,\beta}\left(2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega]\right) \in L^1(\mathbb{R}^N)$,

$$u_m \leq 2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega] \qquad \forall m \geq 0, \tag{5.11}$$

provided that $||M^{\frac{k(\beta-1)}{\beta}}_{2k,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M$, where $\omega = M||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{2k}[\chi_{B_R}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R} + \mu$.
Therefore the sequence $\{u_m\}$ is well defined. By arguing as in the proof of theorem 1.3 we obtain that u satisfies (1.25) and (1.26).
Conversely, assume that (1.25) admits anonnegative solution u and $-u \in \Phi^k(\mathbb{R}^N)$ such that $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$, then by Theorem 5.4 we have

$$u \geq \frac{1}{K_2}\mathbf{W}_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u)+\mu].$$

Using Theorem 2.7, we conclude that (1.27) holds. ∎

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