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Hung Nguyen Quoc, Laurent Veron. Quasilinear and Hessian type equations with exponential reaction and measure data. 2013. hal-00823874v4

**HAL Id: hal-00823874**

**<https://hal.science/hal-00823874v4>**

Preprint submitted on 12 Jun 2013 (v4), last revised 9 May 2014 (v6)

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# Quasilinear and Hessian type equations with exponential reaction and measure data

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## Contents

1	Introduction	2
2	Estimates on potentials and Wolff integral equations	8
3	Quasilinear Dirichlet problems	20
4	$p$ -superharmonic functions and quasilinear equations in $\mathbb{R}^N$	23
5	Hessian equations	24

## Abstract

We prove existence results concerning equations of the type  $-\Delta_p u = P(u) + \mu$  for  $p > 1$  and  $F_k[-u] = P(u) + \mu$  with  $1 \leq k < \frac{N}{2}$  in a bounded domain  $\Omega$ , where  $\mu$  is a positive Radon measure and  $P(u) \sim e^{au^\beta}$  with  $a > 0$  and  $\beta \geq 1$ . Sufficient conditions for existence are expressed in terms of the fractional maximal potential of  $\mu$ . Two-sided estimates on the solutions are obtained in terms of some precise Wolff potentials of  $\mu$ . Necessary conditions are obtained in terms of Orlicz capacities. We also establish existence results for a general Wolff potential equation under the form  $u = \mathbf{W}_{\alpha,p}^R[P(u)] + f$ .

*2010 Mathematics Subject Classification.* 31C15, 32 F10, 35J92, 35R06, 46E30.

*Key words:* quasilinear elliptic equations, Hessian equations, Wolff potential, maximal functions, Borel measures, Orlicz capacities.

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# 1 Introduction

Let  $\Omega \subset \mathbb{R}^N$  be either a bounded domain or the whole  $\mathbb{R}^N$ ,  $p > 1$  and  $k \in \{1, 2, \dots, N\}$ . We denote by

$$\Delta_p u := \operatorname{div} \left( |\nabla u|^{p-2} \nabla u \right)$$

the p-Laplace operator and by

$$F_k[u] = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq N} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k}$$

the k-Hessian operator where  $\lambda_1, \dots, \lambda_N$  are the eigenvalues of the Hessian matrix  $D^2 u$ . Let  $\mu$  be a positive Radon measure in  $\Omega$ ; our aim is to study the existence of positive renormalized solutions to the following boundary value problems if  $\Omega$  is bounded,

$$\begin{aligned} -\Delta_p u &= P(u) + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} F_k[-u] &= P(u) + \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (1.2)$$

where  $P$  is an exponential function. If  $\Omega = \mathbb{R}^N$ , we consider the same equations, but the boundary conditions are replaced by  $\inf_{\mathbb{R}^N} u = 0$ . When  $P(r) = r^q$  with  $q > p - 1$ , Phuc and Verbitsky published a seminal article [20] on the solvability of the corresponding problem (1.1). They obtained necessary and sufficient conditions involving Bessel capacities or Wolff potentials. For example, assuming that  $\Omega$  is bounded, they proved that if  $\mu$  has compact support in  $\Omega$  it is equivalent to solve (1.1) with  $P(r) = r^q$ , or to have

$$\mu(E) \leq c C_{p, \frac{p}{q+1-p}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.3)$$

where  $c$  is a suitable positive constant and  $C_{p, \frac{p}{q+1-p}}$  a Bessel capacity, or to have

$$\int_B [\mathbf{W}_{1,p}^{2R}[\mu_B(x)]]^q dx \leq C \mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \operatorname{supp} \mu \neq \emptyset, \quad (1.4)$$

where  $R = \operatorname{diam}(\Omega)$ . Other conditions are expressed in terms of Riesz potentials and maximal fractional potentials. Their construction is based upon sharp estimates of solutions of the non-homogeneous problem

$$\begin{aligned} -\Delta_p u &= \omega && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.5)$$

for positive measures  $\omega$ . We refer to [4, 5, 6, 7, 9, 13, 23] for the previous studies of these and other related results. Concerning the k-Hessian operator in a bounded  $(k-1)$ -convex domain  $\Omega$ , they proved that if  $\mu$  has compact support and  $\|\varphi\|_{L^\infty(\partial\Omega)}$  is small enough, the corresponding problem (1.2) with  $P(r) = r^q$  with  $q > k$  admits a positive solution if and only if

$$\mu(E) \leq c C_{2k, \frac{q}{q-k}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.6)$$

or equivalently

$$\int_B \left[ \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2R} [\mu_B(x)] \right]^q dx \leq C\mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \neq \emptyset. \quad (1.7)$$

The results concerning the linear case  $p = 2$  and  $k = 1$ , can be found in [2, 3, 29]. The main tools in their proofs are derived from recent advances in potential theory for nonlinear elliptic equations obtained by Kilpelainen and Malý [15, 16], Trudinger and Wang [25, 26, 27], and Labutin [18] thanks to whom the authors first provide global pointwise estimates for solutions of the homogeneous Dirichlet problems in terms of Wolff potentials of suitable order.

For  $s > 1$ ,  $0 \leq \alpha < \frac{N}{s}$ ,  $\eta \geq 0$  and  $0 < T \leq \infty$ , we recall that the  $T$ -truncated Wolff potential of a positive Radon measure  $\mu$  is defined in  $\mathbb{R}^N$  by

$$\mathbf{W}_{\alpha, s}^T[\mu](x) = \int_0^T \left( \frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t}, \quad (1.8)$$

the  $T$ -truncated Riesz potential of a positive Radon measure  $\mu$  by

$$\mathbf{I}_{\alpha}^T[\mu](x) = \int_0^T \frac{\mu(B_t(x))}{t^{N-\alpha}} \frac{dt}{t} \quad (1.9)$$

and the  $T$ -truncated  $\eta$ -fractional maximal potential of  $\mu$  by

$$\mathbf{M}_{\alpha, T}^{\eta}[\mu](x) = \sup \left\{ \frac{\mu(B_t(x))}{t^{N-\alpha h_{\eta}(t)}} : 0 < t \leq T \right\}, \quad (1.10)$$

where  $h_{\eta}(t) = (-\ln t)^{-\eta} \chi_{(0, 2^{-1})}(t) + (\ln 2)^{-\eta} \chi_{[2^{-1}, \infty)}(t)$ . If  $\eta = 0$ , then  $h_{\eta} = 1$  and we denote by  $\mathbf{M}_{\alpha, T}[\mu]$  the corresponding  $T$ -truncated fractional maximal potential of  $\mu$ . We also denote by  $\mathbf{W}_{\alpha, s}[\mu]$  (resp  $\mathbf{I}_{\alpha}[\mu]$ ,  $\mathbf{M}_{\alpha}[\mu]$ ) the  $\infty$ -truncated Wolff potential (resp Riesz Potential, fractional maximal potential) of  $\mu$ . When the measures are only defined in an open subset  $\Omega \subset \mathbb{R}^N$ , they are naturally extended by 0 in  $\Omega^c$ . For  $l \in \mathbb{N}^*$ , we define the  $l$ -truncated exponential function

$$H_l(r) = e^r - \sum_{j=0}^{l-1} \frac{r^j}{j!}, \quad (1.11)$$

and for  $a > 0$  and  $\beta \geq 1$ , we set

$$P_{l, a, \beta}(r) = H_l(ar^{\beta}). \quad (1.12)$$

We put

$$Q_p(s) = \begin{cases} \sum_{q=l}^{\infty} \frac{s^{\frac{\beta q}{p-1}}}{q^{\frac{\beta q}{p-1}} q!} & \text{if } p \neq 2, \\ H_l(s^{\beta}) & \text{if } p = 2, \end{cases} \quad (1.13)$$

$Q_p^*(r) = \max \{rs - Q_p(s) : s \geq 0\}$  is the complementary function to  $Q_p$ , and define the corresponding Bessel and Riesz capacities respectively by

$$\text{cap}_{G_p, Q_p^*}(E) = \inf \left\{ \int_{\mathbb{R}^N} Q_p^*(f) dx : G_p * f \geq \chi_E, f \geq 0, Q_p^*(f) \in L^1(\mathbb{R}^N) \right\}, \quad (1.14)$$

and

$$cap_{I_p, Q_p^*}(E) = \inf \left\{ \int_{\mathbb{R}^N} Q_p^*(f) dx : I_p * f \geq \chi_E, f \geq 0, Q_p^*(f) \in L^1(\mathbb{R}^N) \right\}, \quad (1.15)$$

where  $G_p(x) = \mathcal{F}^{-1} \left( (1 + |\cdot|^2)^{-\frac{p}{2}} \right) (x)$  is the Bessel kernel of order  $p$  and  $I_p(x) = (N - p)^{-1} |x|^{-(N-p)}$ .

The expressions  $a \wedge b$  and  $a \vee b$  stand for  $\min\{a, b\}$  and  $\max\{a, b\}$  respectively. Our main results are the following theorems.

**Theorem 1.1** *Let  $1 < p < N$ ,  $a > 0$ ,  $l \in \mathbb{N}$  and  $\beta \geq 1$  such that  $l\beta > p - 1$ . Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain. If  $\mu$  is a nonnegative Radon measure in  $\Omega$ , there exists  $M > 0$  depending on  $N, p, l, a, \beta$  and  $diam(\Omega)$  (the diameter of  $\Omega$ ) such that if*

$$\| \mathbf{M}_{p, 2 \, diam(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}} [\mu] \|_{L^\infty(\mathbb{R}^N)} \leq M,$$

and  $\omega = M \| \mathbf{M}_{p, 2 \, diam(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}} [1] \|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$  with  $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$ , then  $P_{l, a, \beta} \left( 2c_p K_1 \mathbf{W}_{1, p}^{2 \, diam(\Omega)} [\omega] \right)$  is integrable in  $\Omega$  and the following Dirichlet problem

$$\begin{aligned} -\Delta_p u &= P_{l, a, \beta}(u) + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.16)$$

admits a nonnegative renormalized solution  $u$  which satisfies

$$u(x) \leq 2c_p K_1 \mathbf{W}_{1, p}^{2 \, diam(\Omega)} [\omega](x) \quad \forall x \in \Omega. \quad (1.17)$$

The role of  $K_1 = K_1(N, p)$  will be made explicit in Theorem 3.5.

Conversely, if (1.16) admits a nonnegative renormalized solution  $u$  and  $P_{l, a, \beta}(u)$  is integrable in  $\Omega$ , then for any compact set  $K \subset \Omega$ , there exists a positive constant  $C$  depending on  $N, p, l, a, \beta$  and  $\text{dist}(K, \partial\Omega)$  such that

$$\int_E P_{l, a, \beta}(u) dx + \mu(E) \leq C cap_{G_p, Q_p^*}(E) \quad \text{for all Borel sets } E \subset K. \quad (1.18)$$

Furthermore,  $u \in W_0^{1, p_1}(\Omega)$  for all  $1 \leq p_1 < p$ .

When  $\Omega = \mathbb{R}^N$ , we have a similar result provided  $\mu$  has compact support.

**Theorem 1.2** *Let  $1 < p < N$ ,  $a > 0$ ,  $l \in \mathbb{N}$  and  $\beta \geq 1$  such that  $l\beta > \frac{N(p-1)}{N-p}$  and  $R > 0$ . If  $\mu$  is a nonnegative Radon measure in  $\mathbb{R}^N$  with  $\text{supp}(\mu) \subset B_R(0)$  there exists  $M > 0$  depending on  $N, p, l, a, \beta$  and  $R$  such that if*

$$\| \mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}} [\mu] \|_{L^\infty(\mathbb{R}^N)} \leq M,$$

and  $\omega = M \| \mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}} [\chi_{B_R(0)}] \|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R(0)} + \mu$ , then  $P_{l, a, \beta} (2c_p K_1 \mathbf{W}_{1, p} [\omega])$  is integrable in  $\mathbb{R}^N$  and the following problem

$$\begin{aligned} -\Delta_p u &= P_{l, a, \beta}(u) + \mu \text{ in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u &= 0, \end{aligned} \quad (1.19)$$

admits a  $p$ -superharmonic solution  $u$  which satisfies

$$u(x) \leq 2c_p K_1 \mathbf{W}_{1,p}[\omega](x) \quad \forall x \in \mathbb{R}^N, \quad (1.20)$$

( $c_p$  and  $K_1$  as in Theorem 1.1).

Conversely, if (1.19) has a solution  $u$  and  $P_{l,\alpha,\beta}(u)$  is locally integrable in  $\mathbb{R}^N$ , then there exists a positive constant  $C$  depending on  $N, p, l, \alpha, \beta$  such that

$$\int_E P_{l,\alpha,\beta}(u) dx + \mu(E) \leq C \text{cap}_{I_p, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.} \quad (1.21)$$

Furthermore,  $u \in W_{loc}^{1,p_1}(\mathbb{R}^N)$  for all  $1 \leq p_1 < p$ .

Concerning the  $k$ -Hessian operator we recall some notions introduced by Trudinger and Wang [25, 26, 27], and we follow their notations. For  $k = 1, \dots, N$  and  $u \in C^2(\Omega)$  the  $k$ -Hessian operator  $F_k$  is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where  $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$  denotes the eigenvalues of the Hessian matrix of second partial derivatives  $D^2u$  and  $S_k$  is the  $k$ -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}.$$

It is straightforward that

$$F_k[u] = [D^2u]_k,$$

where in general  $[A]_k$  denotes the sum of the  $k$ -th principal minors of a matrix  $A = (a_{ij})$ . In order that there exists a smooth  $k$ -admissible function which vanishes on  $\partial\Omega$ , the boundary  $\partial\Omega$  must satisfy a uniformly  $(k-1)$ -convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant  $c_0$ , where  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$  denote the principal curvatures of  $\partial\Omega$  with respect to its inner normal. We also denote by  $\Phi^k(\Omega)$  the class of upper-semicontinuous functions  $\Omega \mapsto [-\infty, \infty)$  which are  $k$ -convex, or subharmonic in the Perron sense (see Definition 5.1). In this paper we prove the following theorem (in which expression  $\mathbb{E}[q]$  is the largest integer less or equal to  $q$ )

**Theorem 1.3** *Let  $k \in \{1, 2, \dots, \mathbb{E}[N/2]\}$  such that  $2k < N$ ,  $l \in \mathbb{N}^*$ ,  $\beta \geq 1$  such that  $l\beta > k$  and  $a > 0$ . Let  $\Omega$  be a bounded uniformly  $(k-1)$ -convex domain in  $\mathbb{R}^N$ . Let  $\varphi$  be a nonnegative continuous function on  $\partial\Omega$  and  $\mu = \mu_1 + f$  be a nonnegative Radon measure where  $\mu_1$  has compact support in  $\Omega$  and  $f \in L^q(\Omega)$  for some  $q > \frac{N}{2k}$ . Let  $K_2 = K_2(N, k)$  be the constant  $K_2$  which appears in Theorem 5.3. Then, there exist positive constants  $b, M_1$  and  $M_2 > 0$  depending on  $N, k, l, \alpha, \beta$  and  $\text{diam}(\Omega)$  such that, if  $\max_{\partial\Omega} \varphi \leq M_2$  and*

$$\| \mathbf{M}_{2k, 2\text{diam}(\Omega)}^{\frac{k(\beta-1)}{\beta}}[\mu] \|_{L^\infty(\mathbb{R}^N)} \leq M_1,$$

then  $P_{l,a,\beta} \left( 2K_2 \mathbf{W}_{\frac{2k}{k+1},k+1}^{2diam(\Omega)} [\mu] + b \right)$  is integrable in  $\Omega$  and the following Dirichlet problem

$$\begin{aligned} F_k[-u] &= P_{l,a,\beta}(u) + \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (1.22)$$

admits a nonnegative solution  $u$ , continuous near  $\partial\Omega$ , with  $-u \in \Phi^k(\Omega)$  which satisfies

$$u(x) \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1},k+1}^{2diam\Omega} [\mu](x) + b \quad \forall x \in \Omega. \quad (1.23)$$

Conversely, if (1.22) admits a nonnegative solution  $u$ , continuous near  $\partial\Omega$ , such that  $-u \in \Phi^k(\Omega)$  and  $P_{l,a,\beta}(u)$  is integrable in  $\Omega$ , then for any compact set  $K \subset \Omega$ , there exists a positive constant  $C$  depending on  $N, k, l, a, \beta$  and  $\text{dist}(K, \partial\Omega)$  such that there holds

$$\int_E P_{l,a,\beta}(u) dx + \mu(E) \leq C \text{cap}_{G_{2k}, Q_{k+1}^*}(E) \quad \forall E \subset K, E \text{ Borel}, \quad (1.24)$$

where  $Q_{k+1}(s)$  is defined by (1.13) with  $p = k + 1$ ,  $Q_{k+1}^*$  is its complementary function and  $\text{cap}_{G_{2k}, Q_{k+1}^*}(E)$  is defined accordingly by (1.14).

The following extension holds when  $\Omega = \mathbb{R}^N$ .

**Theorem 1.4** Let  $k \in \{1, 2, \dots, \mathbb{E}[N/2]\}$  such that  $2k < N$ ,  $l \in \mathbb{N}^*$ ,  $\beta \geq 1$  such that  $l\beta > \frac{Nk}{N-2k}$  and  $a > 0$ ,  $R > 0$ . If  $\mu$  is a nonnegative Radon measure in  $\mathbb{R}^N$  with  $\text{supp}(\mu) \subset B_R(0)$  there exists  $M > 0$  depending on  $N, k, l, a, \beta$  and  $R$  such that if

$$\|\mathbf{M}_{2k}^{\frac{k(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M.$$

and  $\omega = M \|\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R(0)} + \mu$ , then  $P_{l,a,\beta} \left( 2K_2 \mathbf{W}_{\frac{2k}{k+1},k+1}[\omega] \right)$  is integrable in  $\mathbb{R}^N$  ( $K_2$  as in Theorem 1.3) and the following Dirichlet problem

$$\begin{aligned} F_k[-u] &= P_{l,a,\beta}(u) + \mu && \text{in } \mathbb{R}^N, \\ \inf_{\mathbb{R}^N} u &= 0, \end{aligned} \quad (1.25)$$

admits a nonnegative solution  $u$  with  $-u \in \Phi^k(\mathbb{R}^N)$  which satisfies

$$u(x) \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1},k+1}[\omega](x) \quad \forall x \in \mathbb{R}^N. \quad (1.26)$$

Conversely, if (1.25) admits a nonnegative solution  $u$  with  $-u \in \Phi^k(\mathbb{R}^N)$  and  $P_{l,a,\beta}(u)$  locally integrable in  $\mathbb{R}^N$ , then there exists a positive constant  $C$  depending on  $N, k, l, a, \beta$  and  $R$  such that there holds

$$\int_E P_{l,a,\beta}(u) dx + \mu(E) \leq C \text{cap}_{I_{2k}, Q_{k+1}^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}. \quad (1.27)$$

where  $\text{cap}_{I_{2k}, Q_{k+1}^*}(E)$  is defined accordingly by (1.15) with  $p = k + 1$ .

The four previous theorems are connected to the following results which deals with a class of nonlinear *Wolff integral equations*.

**Theorem 1.5** *Let  $\alpha > 0$ ,  $p > 1$ ,  $a > 0$ ,  $\varepsilon > 0$ ,  $R > 0$ ,  $l \in \mathbb{N}^*$  and  $\beta \geq 1$  such that  $l\beta > p-1$  and  $0 < \alpha p < N$ . Let  $f$  be a nonnegative measurable in  $\mathbb{R}^N$  with the property that  $\mu_1 = P_{l,a+\varepsilon,\beta}(f)$  is locally integrable in  $\mathbb{R}^N$  and  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ . There exists  $M > 0$  depending on  $N, \alpha, p, l, a, \beta, \varepsilon$  and  $R$  such that if*

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

then there exists a nonnegative function  $u$  such that  $P_{l,a,\beta}(u)$  is locally integrable in  $\mathbb{R}^N$  which satisfies

$$u = \mathbf{W}_{\alpha, p}^R[P_{l,a,\beta}(u) + \mu] + f \quad \text{in } \mathbb{R}^N, \quad (1.28)$$

and

$$u \leq F := 2c_p \mathbf{W}_{\alpha, p}^R[\omega_1] + 2c_p \mathbf{W}_{\alpha, p}^R[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1_{loc}(\mathbb{R}^N), \quad (1.29)$$

where  $\omega_1 = M \|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$  and  $\omega_2 = M \|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu_1$ .

Conversely, if (1.28) admits a nonnegative solution  $u$  and  $P_{l,a,\beta}(u)$  is locally integrable in  $\mathbb{R}^N$ , then there exists a positive constant  $C$  depending on  $N, \alpha, p, l, a, \beta$  and  $R$  such that there holds

$$\int_E P_{l,a,\beta}(u) dx + \int_E P_{l,a+\varepsilon,\beta}(f) dx + \mu(E) \leq C \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}. \quad (1.30)$$

When  $R = \infty$  in the above theorem, we have a similar result provided  $f$  and  $\mu$  have compact support in  $\mathbb{R}^N$ .

**Theorem 1.6** *Let  $\alpha > 0$ ,  $p > 1$ ,  $a > 0$ ,  $\varepsilon > 0$ ,  $R > 0$ ,  $l \in \mathbb{N}^*$  and  $\beta \geq 1$  such that  $0 < \alpha p < N$  and  $l\beta > \frac{N(p-1)}{N-\alpha p}$ . There exists  $M > 0$  depending on  $N, \alpha, p, l, a, \beta, \varepsilon$  and  $R$  such that if  $f$  is a nonnegative measurable function in  $\mathbb{R}^N$  with support in  $B_R(0)$  such that that  $\mu_1 = P_{l,a+\varepsilon,\beta}(f)$  is locally integrable in  $\mathbb{R}^N$  and  $\mu$  is a positive measure in  $\mathbb{R}^N$  with support in  $B_R(0)$  which verify*

$$\|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

then there exists a nonnegative function  $u$  such that  $P_{l,a,\beta}(u)$  is integrable in  $\mathbb{R}^N$  which satisfies

$$u = \mathbf{W}_{\alpha, p}[P_{l,a,\beta}(u) + \mu] + f \quad \text{in } \mathbb{R}^N, \quad (1.31)$$

and

$$u \leq F := 2c_p \mathbf{W}_{\alpha, p}[\omega_1] + 2c_p \mathbf{W}_{\alpha, p}[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1(\mathbb{R}^N), \quad (1.32)$$

where  $\omega_1 = M \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R(0)} + \mu$  and  $\omega_2 = M \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R(0)} + \mu_1$ .

Conversely, if (1.31) admits a nonnegative solution  $u$  such that  $P_{l,a,\beta}(u)$  is integrable in  $\mathbb{R}^N$ , then there exists a positive constant  $C$  depending on  $N, \alpha, p, l, a, \beta$  such that there holds

$$\int_E P_{l,a,\beta}(u) dx + \int_E P_{l,a,\beta}(f) dx + \mu(E) \leq C \text{cap}_{I_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}. \quad (1.33)$$



As an application of the Wolff integral equation we can notice that  $\alpha = 1$ , equation (1.31) is equivalent to

$$-\Delta_p(u - f) = P_{l,\alpha,\beta}(u) \quad \text{in } \mathbb{R}^N.$$

When  $\alpha = \frac{2k}{k+1}$  and  $p = k + 1$ , it is equivalent to

$$F_k(-u + f) = P_{l,\alpha,\beta}(u) \quad \text{in } \mathbb{R}^N.$$

If  $p = 2$  equation (1.31) becomes linear. If we set  $\gamma = 2\alpha$ , then

$$\begin{aligned} \mathbf{W}_{\gamma,2}[\mu](x) &= \int_0^\infty \mu(B_t(x)) \frac{dt}{t^{N-\gamma-1}} \\ &= \int_{\mathbb{R}^N} \left( \int_{|x-y|}^\infty \frac{dt}{t^{N-\gamma-1}} \right) d\mu(y) \\ &= \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-\gamma}} \\ &= I_\gamma * \mu \end{aligned}$$

where  $I_\gamma$  is the Riesz kernel of order  $\gamma$ . Thus (1.31) is equivalent to

$$(-\Delta)^\alpha(u - f) = P_{l,\alpha,\beta}(u) \quad \text{in } \mathbb{R}^N.$$

## 2 Estimates on potentials and Wolff integral equations

We denote by  $B_r(a)$  the ball of center  $a$  and radius  $r > 0$ ,  $B_r = B_r(0)$  and by  $\chi_E$  the characteristic function of a set  $E$ . The next estimates are crucial in the sequel.

**Theorem 2.1 1.** *There exists a positive constant  $c_1$ , depending only on  $N, \alpha, p$  such that for all  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  and  $q > p - 1$ ,  $0 < R \leq \infty$  we have*

$$(c_1 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} (\mathbf{I}_{\alpha p}^R[\mu](x))^{\frac{q}{p-1}} dx \leq \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx \leq (c_1 q)^q \int_{\mathbb{R}^N} (\mathbf{I}_{\alpha p}^R[\mu](x))^{\frac{q}{p-1}} dx, \quad (2.1)$$

**2.** *Let  $R > 0$ . There exists a positive constant  $c_2$ , depending only on  $N, \alpha, p, R$  such that for all  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  and  $q > p - 1$  we have*

$$(c_2 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}} dx \leq \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx \leq (c_2 q)^q \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}} dx, \quad (2.2)$$

where  $\mathbf{G}_{\alpha p}[\mu] := G_{\alpha p} * \mu$  denotes the Bessel potential of order  $\alpha p$  of  $\mu$ .

**3.** *There exists a positive constant  $c_3$  depending only on  $N, \alpha, R$  such that for all  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  and  $q > 1$  we have*

$$c_3^{-q} \int_{\mathbb{R}^N} (\mathbf{G}_\alpha[\mu][\mu](x))^q dx \leq \int_{\mathbb{R}^N} (\mathbf{I}_\alpha^R[\mu](x))^q dx \leq c_3^q \int_{\mathbb{R}^N} (\mathbf{G}_\alpha[\mu](x))^q dx. \quad (2.3)$$

**Proof.** Note that  $W_{\alpha/2,2}^R[\mu] = I_\alpha^R[\mu]$ . We can find proof of (2.3) in [8, Step 3, Theorem 2.3]. By [8, Step 2, Theorem 2.3], there is  $c_4 > 0$  such that

$$\int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx \geq c_4^q \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}} dx \quad \forall q > 0, 0 < R \leq \infty \text{ and } \mu \in \mathfrak{M}^+(\mathbb{R}^N). \quad (2.4)$$

We recall that  $\mathbf{M}_{\alpha p,R}[\mu] = \mathbf{M}_{\alpha p,R}^0[\mu]$  by (1.10). Next we show that for all  $q > 0, 0 < R \leq \infty$  and  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  there holds

$$\int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}} dx \geq (c_5 q)^{-q} \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx, \quad (2.5)$$

for some positive constant  $c_5$  depending on  $N, \alpha, p, q$ . Indeed, we denote  $\mu_n$  by  $\chi_{B_n} \mu$  for  $n \in \mathbb{N}^*$ . By [17, Theorem 1.2] or [8, Proposition 2.2], there exist constants  $c_6 = c_6(N, \alpha, p) > 0$ ,  $a = a(\alpha, p) > 0$  and  $\varepsilon_0 = \varepsilon_0(N, \alpha, p)$  such that for all  $n \in \mathbb{N}^*, t > 0, 0 < R \leq \infty$  and  $0 < \varepsilon < \varepsilon_0$ , there holds

$$|\{\mathbf{W}_{\alpha,p}^R \mu_n > 3t\}| \leq c_6 \exp(-a\varepsilon^{-1}) |\{\mathbf{W}_{\alpha,p}^R \mu_n > t\}| + \left| \left\{ (\mathbf{M}_{\alpha p,R} \mu_n)^{\frac{1}{p-1}} > \varepsilon t \right\} \right|.$$

Multiplying by  $qt^{q-1}$  and integrating over  $(0, \infty)$ , we obtain

$$\begin{aligned} \int_0^\infty qt^{q-1} |\{\mathbf{W}_{\alpha,p}^R \mu_n > 3t\}| dt &\leq c_6 \exp(-a\varepsilon^{-1}) \int_0^\infty qt^{q-1} |\{\mathbf{W}_{\alpha,p}^R \mu_n > t\}| dt \\ &\quad + \int_0^\infty qt^{q-1} \left| \left\{ (\mathbf{M}_{\alpha p,R} \mu_n)^{\frac{1}{p-1}} > \varepsilon t \right\} \right| dt, \end{aligned}$$

which implies

$$\varepsilon^q (3^{-q} - c_6 \exp(-a\varepsilon^{-1})) \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu_n](x))^q dx \leq \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R} \mu_n)^{\frac{q}{p-1}} dx.$$

We see that  $\sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^q (3^{-q} - c_6 \exp(-a\varepsilon^{-1})) \geq (c_7 q)^{-q}$  for some constant  $c_7$  which does not depend on  $q$ . Therefore (2.5) follows by Fatou's lemma. Hence, it is easy to obtain (2.1) from (2.4) and (2.5). At end, we obtain (2.2) from (2.1) and (2.3).  $\blacksquare$

The next result is proved in [8].

**Theorem 2.2** *Let  $\alpha > 0, p > 1, 0 \leq \eta < p - 1, 0 < \alpha p < N$  and  $L > 0$ . Set  $\delta = \frac{1}{2} \left( \frac{p-1-\eta}{12(p-1)} \right)^{\frac{p-1}{p-1-\eta}} \alpha p \log(2)$ . Then there exists  $C(L) > 0$ , depending on  $N, \alpha, p, \eta$  and  $L$  such that for any  $R \in (0, \infty], \mu \in \mathfrak{M}^+(\mathbb{R}^N)$ , any  $a \in \mathbb{R}^N$  and  $0 < r \leq L$ , there holds*

$$\frac{1}{|B_{2r}(a)|} \int_{B_{2r}(a)} \exp \left( \delta \frac{(\mathbf{W}_{\alpha,p}^R[\mu_{B_r(a)}](x))^{\frac{p-1}{p-1-\eta}}}{\|\mathbf{M}_{\alpha p,R}^\eta[\mu_{B_r(a)}]\|_{L^\infty(B_r(a))}^{\frac{1}{p-1-\eta}}} \right) dx \leq C(L), \quad (2.6)$$

where  $\mu_{B_r(a)} = \chi_{B_r(a)} \mu$ . Furthermore, if  $\eta = 0$ ,  $C$  is independent of  $L$ .

**Theorem 2.3** *Let  $\alpha > 0$ ,  $p > 1$  with  $0 < \alpha p < N$ ,  $\beta \geq 1$  and  $R > 0$ . Assume  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  satisfies*

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1, \quad (2.7)$$

and set  $\omega = \|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$ . Then there exist positive constants  $C$ ,  $\delta_0$  and  $c$  independent on  $\mu$  such that  $\exp\left(\delta_0(\mathbf{W}_{\alpha, p}^R[\omega])^\beta\right)$  is locally integrable in  $\mathbb{R}^N$ ,

$$\left\| \mathbf{W}_{\alpha, p}^R \left[ \exp\left(\delta_0(\mathbf{W}_{\alpha, p}^R[\omega])^\beta\right) \right] \right\|_{L^\infty(\mathbb{R}^N)} \leq C \quad (2.8)$$

and

$$\mathbf{W}_{\alpha, p}^R \left[ \exp\left(\delta_0(\mathbf{W}_{\alpha, p}^R[\omega])^\beta\right) \right] \leq c \mathbf{W}_{\alpha, p}^R[\omega] \quad \text{in } \mathbb{R}^N. \quad (2.9)$$

**Proof.** Let  $\delta$  be as in Theorem 2.2. From (2.7), we have

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq 2.$$

Let  $x \in \mathbb{R}^N$ . Since  $\omega(B_t(y)) \leq 2t^{N-\alpha p} h_{\frac{(p-1)(\beta-1)}{\beta}}(t)$ , for all  $r \in (0, R)$  and  $y \in \mathbb{R}^N$  we have

$$\begin{aligned} \mathbf{W}_{\alpha, p}^R[\omega](y) &= \mathbf{W}_{\alpha, p}^r[\omega](y) + \int_r^R \left( \frac{\omega(B_t(y))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha, p}^r[\omega](y) + 2^{\frac{1}{p-1}} \int_{r \wedge 2^{-1}}^{2^{-1}} (-\ln t)^{-\frac{\beta-1}{\beta}} \frac{dt}{t} + 2^{\frac{1}{p-1}} \int_{2^{-1}}^{R \vee 2^{-1}} (-\ln t)^{-\frac{\beta-1}{\beta}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha, p}^r[\omega](y) + c_8 (-\ln(r \wedge 2^{-1}))^{\frac{1}{\beta}} + c_8. \end{aligned}$$

Thus,

$$\left(\mathbf{W}_{\alpha, p}^R[\omega](y)\right)^\beta \leq 3^{\beta-1} \left(\mathbf{W}_{\alpha, p}^r[\omega](y)\right)^\beta + c_9 \ln\left(\frac{1}{r \wedge 2^{-1}}\right) + c_9. \quad (2.10)$$

Let  $\theta \in (0, 1]$ , since  $\exp\left(\frac{a+b}{2}\right) \leq \exp(a) + \exp(b)$  for all  $a, b \in \mathbb{R}$ , we get from (2.10)

$$\begin{aligned} \exp\left(\theta \delta 3^{-\beta} \left(\mathbf{W}_{\alpha, p}^R[\omega](y)\right)^\beta\right) &\leq \exp\left(\delta \left(\mathbf{W}_{\alpha, p}^r[\omega](y)\right)^\beta\right) + c_{10} \exp\left(\theta c_{11} \ln\left(\frac{1}{r \wedge 2^{-1}}\right)\right) \\ &\leq \exp\left(\delta \left(\mathbf{W}_{\alpha, p}^r[\omega](y)\right)^\beta\right) + c_{10} (r \wedge 2^{-1})^{-\theta c_{11}}. \end{aligned} \quad (2.11)$$

For  $r > 0$ ,  $0 < t \leq r$ ,  $y \in B_r(x)$  there holds  $B_t(y) \subset B_{2r}(x)$ . Thus,  $\mathbf{W}_{\alpha, p}^r[\omega] = \mathbf{W}_{\alpha, p}^r[\omega_{B_{2r}(x)}]$  in  $B_r(x)$ . Then, using (2.6) in Theorem 2.2 with  $L = 2R$  we get

$$\int_{B_r(x)} \exp\left(\delta \left(\mathbf{W}_{\alpha, p}^r[\omega]\right)^\gamma\right) = \int_{B_r(x)} \exp\left(\delta \left(\mathbf{W}_{\alpha, p}^r[\omega_{B_{2r}(x)}]\right)^\gamma\right) \leq c_{12} r^N.$$

Therefore, taking  $\theta = 1 \wedge \frac{\alpha p}{2c_{11}}$ , we deduce from (2.11)

$$\begin{aligned} \mathbf{W}_{\alpha,p}^R \left[ \exp \left( \theta \delta 3^{-\beta} (\mathbf{W}_{\alpha,p}^R [\omega])^\gamma \right) \right] (x) &\leq \int_0^R \left( c_{12} r^{\alpha p} + c_{13} (r \wedge 2^{-1})^{-\theta c_{11}} r^{\alpha p} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\leq \int_0^R \left( c_{12} r^{\alpha p} + c_{13} (r \wedge 2^{-1})^{-\frac{\alpha p}{2}} r^{\alpha p} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\leq c_{14}. \end{aligned}$$

Hence, we get (2.8) with  $\delta_0 = \left( 1 \wedge \frac{\alpha p}{2c_{11}} \right) \delta 3^{-\beta}$ ; we also get (2.9) since  $\mathbf{W}_{\alpha,p}^R [\omega] \geq c_{15}$  for some positive constant  $c_{15} > 0$ .  $\blacksquare$

We recall that  $H_l$  and  $P_{l,a,\beta}$  have been defined in (1.11) and (1.12).

**Theorem 2.4** *Let  $\alpha > 0$ ,  $p > 1$ ,  $l \in \mathbb{N}^*$  and  $\beta \geq 1$  such that  $0 < \alpha p < N$ ,  $l\beta > \frac{N(p-1)}{N-\alpha p}$  and  $R > 0$ . Assume that  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  has support in  $B_R$  and verifies*

$$\| \mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}} [\mu] \|_{L^\infty(\mathbb{R}^N)} \leq 1, \quad (2.12)$$

and set  $\omega = \| \mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}} [\chi_{B_R}] \|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R} + \mu$ . Then there exist  $C = C(N, \alpha, p, l, \beta, R) > 0$  and  $\delta_1 = \delta_1(N, \alpha, p, l, \beta, R) > 0$  such that  $H_l \left( \delta_1 (\mathbf{W}_{\alpha,p}[\omega])^\beta \right)$  is integrable in  $\mathbb{R}^N$  and

$$\mathbf{W}_{\alpha,p} \left[ H_l \left( \delta_1 (\mathbf{W}_{\alpha,p}[\omega])^\beta \right) \right] (x) \leq C \mathbf{W}_{\alpha,p}[\omega](x) \quad \forall x \in \mathbb{R}^N. \quad (2.13)$$

**Proof.** We have from (2.12)

$$\| \mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}} [\omega] \|_{L^\infty(\mathbb{R}^N)} \leq 2. \quad (2.14)$$

Let  $\delta_1 > 0$  and  $x \in \mathbb{R}^N$  fixed. We split the Wolff potential  $\mathbf{W}_{\alpha,p}[\omega]$  into lower and upper parts defined by

$$\mathbf{L}_{\alpha,p}^t[\omega](x) = \int_t^{+\infty} \left( \frac{\omega(B_r(x))}{r^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r},$$

and

$$\mathbf{W}_{\alpha,p}^t[\omega](x) = \int_0^t \left( \frac{\omega(B_r(x))}{r^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Using the convexity we have

$$H_l \left( \delta_1 (\mathbf{W}_{\alpha,p}[\omega])^\beta \right) \leq H_l \left( \delta_1 2^\beta (\mathbf{L}_{\alpha,p}^t[\omega])^\beta \right) + H_l \left( \delta_1 2^\beta (\mathbf{W}_{\alpha,p}^t[\omega])^\beta \right).$$

Thus,

$$\mathbf{W}_{\alpha,p} \left[ H_l \left( \delta_1 (\mathbf{W}_{\alpha,p}[\omega])^\beta \right) \right] (x) \leq c_{16} \int_0^{+\infty} \left( \frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} + c_{16} \int_0^{+\infty} \left( \frac{\omega_t^2(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t},$$

where  $d\omega_t^1 = H_l \left( \delta_1 2^\beta (\mathbf{L}_{\alpha,p}^t[\omega])^\beta \right) dx$  and  $d\omega_t^2 = H_l \left( \delta_1 2^\beta (\mathbf{W}_{\alpha,p}^t[\omega])^\beta \right) dx$ . Inequality (2.13) will follow from the two inequalities below,

$$\int_0^{+\infty} \left( \frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{17} \mathbf{W}_{\alpha,p}[\omega](x), \quad (2.15)$$

and

$$\omega_t^2(B_t(x)) \leq c_{18} \omega(B_{4t}(x)). \quad (2.16)$$

*Step 1: Proof of (2.15).* Since  $B_r(y) \subset B_{2r}(x)$  for  $y \in B_t(x)$  and  $r \geq t$ , there holds

$$\mathbf{L}_{\alpha,p}^t[\omega](y) \leq \int_t^{+\infty} \left( \frac{\omega(B_{2r}(x))}{r^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = 2^{\frac{N-\alpha p}{p-1}} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x).$$

It follows

$$\omega_t^1(B_t(x)) \leq |B_1(0)| t^N H_l \left( \delta_1 c_{19} \left( \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^\beta \right).$$

Thus,

$$\int_0^{+\infty} \left( \frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{20} \int_0^\infty A_t(x) dt, \quad (2.17)$$

where

$$A_t(x) = \left( t^{\alpha p} H_l \left( \delta_1 c_{19} \left( \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^\beta \right) \right)^{\frac{1}{p-1}} \frac{1}{t}.$$

Since  $H_l(s) \leq s^l \exp(s)$  for all  $s \geq 0$ ,

$$\begin{aligned} A_t(x) &\leq c_{21} \left( t^{\alpha p} \left( \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^{l\beta} \exp \left( \delta_1 c_{19} \left( \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^\beta \right) \right)^{\frac{1}{p-1}} \frac{1}{t} \\ &= c_{21} t^{\frac{\alpha p}{p-1}-1} \left( \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^{\frac{l\beta-p+1}{p-1}} \exp \left( \delta_1 c_{22} \left( \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^\beta \right) \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \end{aligned}$$

Now we estimate  $\mathbf{L}_{\alpha,p}^{t/2}[\omega]$ .

*Case 1:  $t \in (0, 1)$ .* From (2.14) we deduce

$$\begin{aligned} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) &= \int_{t/2}^{1/2} \left( \frac{\omega(B_s(x))}{s^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} + \int_{1/2}^\infty \left( \frac{\omega(B_s(x))}{s^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\leq c_{23} \int_{t/2}^{1/2} (-\ln(s))^{-1+\frac{1}{\beta}} \frac{ds}{s} + \int_{1/2}^\infty \left( \frac{\omega(B_R)}{s^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\leq c_{24} (-\ln(t/2))^{\frac{1}{\beta}}, \end{aligned}$$

which implies

$$\begin{aligned} A_t(x) &\leq c_{25} t^{\frac{\alpha p}{p-1}-1} (-\ln(t/2))^{\frac{l\beta-p+1}{\beta(p-1)}} \exp(\delta_1 c_{26} (-\ln(t/2))) \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \\ &\leq c_{27} t^{\frac{\alpha p}{p-1}-1} (-\ln(t/2))^{\frac{l\beta-p+1}{\beta(p-1)}} t^{-\delta_1 c_{26}} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x). \end{aligned}$$

We take  $\delta_1 \leq \frac{1}{2c_{26}} \left( \frac{\alpha p}{p-1} - 1 \right)$  and obtain

$$A_t(x) \leq c_{28} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \quad \forall t \in (0, 1). \quad (2.18)$$

*Case 2:  $t \geq 1$ .* We have

$$\mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \leq \int_{t/2}^{\infty} \left( \frac{\omega(B_R)}{s^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} = c_{29} t^{-\frac{N-\alpha p}{p-1}},$$

thus

$$\begin{aligned} A_t(x) &\leq c_{30} t^{\frac{\alpha p}{p-1}-1} t^{-\frac{(l\beta-p+1)(N-\alpha p)}{(p-1)^2}} \exp\left(\delta_1 c_{31} t^{-\frac{N-\alpha p}{p-1}}\right) \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \\ &\leq c_{32} t^{-1-\gamma} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x), \end{aligned}$$

where  $\gamma = \frac{1}{p-1} \left( \frac{l\beta(N-\alpha p)}{p-1} - N \right) > 0$ .

Therefore,  $A_t(x) \leq c_{33} (t \vee 1)^{-1-\gamma} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x)$  for all  $t > 0$ . Therefore, from (2.17)

$$\int_0^{+\infty} \left( \frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{33} \int_0^{+\infty} (t \vee 1)^{-1-\gamma} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) dt.$$

Using Fubini Theorem we get

$$\begin{aligned} \int_0^{+\infty} \left( \frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} &\leq c_{34} \int_0^{+\infty} \int_0^t (s \vee 1)^{-1-\gamma} ds \left( \frac{\omega(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq c_{35} \int_0^{+\infty} \left( \frac{\omega(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= c_{35} \mathbf{W}_{\alpha,p}[\mu](x), \end{aligned}$$

which follows (2.15).

*Step 2: Proof of (2.16).* For  $t > 0$ ,  $r \leq t$  and  $y \in B_t(x)$  we have  $B_r(y) \subset B_{2t}(x)$ , thus

$$\omega_t^2(B_t(x)) = \int_{B_t(x)} H_l \left( \delta_1 2^\beta (\mathbf{W}_{\alpha,p}^t[\omega_{B_{2t}(x)}](y))^\beta \right) dy.$$

By Theorem 2.2 there exists  $c_{36} > 0$  such that for  $0 < \delta_1 \leq c_{36}$ ,  $0 < t < 2R$ ,  $x \in \mathbb{R}^N$ ,

$$\int_{B_{4t}(x)} \exp\left(\delta_1 2^\beta (\mathbf{W}_{\alpha,p}[\omega_{B_{2t}(x)}](y))^\beta\right) dy \leq c_{37} t^N. \quad (2.19)$$

We take  $0 < \delta_1 \leq c_{36}$ .

*Case 1:  $x \in \overline{B_R}(0)$ .* If  $0 < t < 2R$ , from (2.19) we get

$$\omega_t^2(B_t(x)) \leq c_{37} t^N \leq c_{38} \omega(B_{4t}(x)).$$

If  $t \geq 2R$ , since for any  $|y| \geq 2R$ ,

$$\mathbf{W}_{\alpha,p}[\omega](y) = \int_{|y|/2}^{\infty} \left( \frac{\omega(B_t(y))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{39} \int_{|y|/2}^{\infty} t^{-1-\frac{N-\alpha p}{p-1}} dt \leq c_{40} |y|^{-\frac{N-\alpha p}{p-1}},$$

and thanks to (2.19) we have

$$\begin{aligned} \omega_t^2(B_t(x)) &\leq \int_{B_{2R}(0)} \exp\left(\delta_1 2^\beta (\mathbf{W}_{\alpha,p}[\omega_{B_R(0)}](y))^\beta\right) dy + \int_{B_{2R}(0)^c} H_l \left(\delta_1 2^\beta (\mathbf{W}_{\alpha,p}[\omega](y))^\beta\right) dy \\ &\leq c_{41} R^N + \int_{B_{2R}(0)^c} H_l \left(c_{42} |y|^{-\frac{\beta(N-\alpha p)}{p-1}}\right) dy \\ &\leq c_{43} + c_{43} \int_{B_{2R}(0)^c} |y|^{-\frac{l\beta(N-\alpha p)}{p-1}} dy = c_{43} + c_{44} R^{N-\frac{l\beta(N-\alpha p)}{p-1}} \\ &\leq c_{45} |B_{4t}(x) \cap B_R| \leq c_{46} \omega(B_{4t}(x)). \end{aligned}$$

From this we also have  $H_l \left(\delta_1 (\mathbf{W}_{\alpha,p}[\omega])^\beta\right) \in L^1(\mathbb{R}^N)$ .

*Case 2:*  $x \in B_R(0)^c$ . If  $|x| > R+t$  then  $\omega_t^2(B_t(x)) = 0$ . Next we consider the case  $R < |x| \leq R+t$ . If  $0 < t < 2R$ , we have  $B_t\left((R-\frac{t}{2})\frac{x}{|x|}\right) \subset B_{4t}(x) \cap B_R$ ; thus from (2.19) we get

$$\omega_t^2(B_t(x)) \leq c_{47} t^N = c_{48} \left| B_t \left( \left( R - \frac{t}{2} \right) \frac{x}{|x|} \right) \right| \leq c_{48} |B_{4t}(x) \cap B_R| \leq c_{49} \omega(B_{4t}(x)).$$

If  $t \geq 2R$ , as in Case 1 we also obtain  $\omega_t^2(B_t(x)) \leq c_{50} \omega(B_{4t}(x))$ . Hence, we get (2.16). Therefore, the result follows with  $\delta_1 = \frac{1}{2c_{26}} \left( \frac{\alpha p}{p-1} - 1 \right) \wedge c_{36}$ .  $\blacksquare$

In the next result we obtain estimate on a sequence of solutions of Wolff integral inequations obtained by induction.

**Theorem 2.5** *Assume that the assumptions on  $\alpha, p, a, l, \beta, \varepsilon, f, \mu_1$  and  $\mu$  of Theorem 1.5 are fulfilled and  $R, K$  are positive real numbers. Suppose that  $\{u_m\}$  is a sequence of nonnegative measurable functions in  $\mathbb{R}^N$  that satisfies*

$$\begin{aligned} u_{m+1} &\leq K \mathbf{W}_{\alpha,p}^R [P_{l,a,\beta}(u_m) + \mu] + f \quad \forall m \in \mathbb{N}, \\ u_0 &\leq K \mathbf{W}_{\alpha,p}^R [\mu] + f. \end{aligned} \tag{2.20}$$

Then there exists  $M > 0$  depending on  $N, \alpha, p, l, a, \beta, \varepsilon, K$  and  $R$  such that if

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$P_{l,a,\beta} (4c_p K \mathbf{W}_{\alpha,p}^R [\omega_1] + 4c_p K \mathbf{W}_{\alpha,p}^R [\omega_2] + f) \in L_{loc}^1(\mathbb{R}^N), \tag{2.21}$$

and

$$u_m \leq 2c_p K \mathbf{W}_{\alpha,p}^R [\omega_1] + 2c_p K \mathbf{W}_{\alpha,p}^R [\omega_2] + f \quad \forall m \in \mathbb{N}, \tag{2.22}$$

where

$$\omega_1 = M \|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}} [1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu, \quad (2.23)$$

$$\omega_2 = M \|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}} [1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu_1 \quad (2.24)$$

and  $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$ .

**Proof.** The proof is based upon Theorems 2.3 and 2.4. Set  $c_{a,\varepsilon} = 2 \left(1 - \left(\frac{a}{a+\varepsilon}\right)^{1/\beta}\right)^{-1}$  and  $\bar{a} = a(4c_{a,\varepsilon}c_p K)^\beta$ . If  $0 < M \leq 1$  we define  $\omega_1$  and  $\omega_2$  by (2.23) and (2.24) respectively. We now assume

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}} [\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}} [\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M.$$

We prove first that

$$\mathbf{W}_{\alpha, p}^R \left[ H_l \left( \bar{a} (\mathbf{W}_{\alpha, p}^R [\omega_{1,2}])^\beta \right) \right] \leq \mathbf{W}_{\alpha, p}^R [\omega_{1,2}]. \quad (2.25)$$

By Theorem 2.3, there exist  $c, \delta_0 > 0$  independent on  $\mu$  such that  $\exp\left(\delta_0 (\mathbf{W}_{\alpha, p}^R [M^{-1}\omega_{1,2}])^\beta\right)$  is locally integrable in  $\mathbb{R}^N$  and

$$\mathbf{W}_{\alpha, p}^R \left[ \exp\left(\delta_0 (\mathbf{W}_{\alpha, p}^R [M^{-1}\omega_{1,2}])^\beta\right) \right] \leq c \mathbf{W}_{\alpha, p}^R [M^{-1}\omega_{1,2}] \quad \text{in } \mathbb{R}^N.$$

Since  $\theta^{-l} H_l(s) \leq H_l(\theta^{-1}s)$  for all  $s \geq 0$  and  $0 < \theta \leq 1$ , it follows

$$\begin{aligned} \mathbf{W}_{\alpha, p}^R \left[ M^{-\frac{1}{2}\left(\frac{\beta l}{p-1}+1\right)} H_l \left( \delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)} (\mathbf{W}_{\alpha, p}^R [\omega_{1,2}])^\beta \right) \right] &\leq \mathbf{W}_{\alpha, p}^R \left[ H_l \left( \delta_0 M^{-\frac{\beta}{p-1}} (\mathbf{W}_{\alpha, p}^R [\omega_{1,2}])^\beta \right) \right] \\ &\leq \mathbf{W}_{\alpha, p}^R \left[ \exp\left(\delta_0 (\mathbf{W}_{\alpha, p}^R [M^{-1}\omega_{1,2}])^\beta\right) \right] \\ &\leq c M^{-\frac{1}{p-1}} \mathbf{W}_{\alpha, p}^R [\omega_{1,2}]. \end{aligned}$$

Hence,

$$\mathbf{W}_{\alpha, p}^R \left[ H_l \left( \delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)} (\mathbf{W}_{\alpha, p}^R [\omega_{1,2}])^\beta \right) \right] \leq c M^{\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)} \mathbf{W}_{\alpha, p}^R [\omega_{1,2}].$$

Therefore (2.25) is achieved if we prove

$$\bar{a} \leq \delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)} \quad \text{and} \quad c M^{\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)} \leq 1,$$

which is equivalent to

$$M \leq (\delta_0 \bar{a}^{-1})^{\left(\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)\right)^{-1}}.$$

Thus, we choose  $M = 1 \wedge (\delta_0 \bar{a}^{-1})^{\left(\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)\right)^{-1}}$ ; we obtain (2.25) and the fact that  $H_l \left( \bar{a} (\mathbf{W}_{\alpha, p}^R [\omega_{1,2}])^\beta \right) \in L_{loc}^1(\mathbb{R}^N)$ .



Now, we prove (2.22) by induction. Clearly, (2.22) holds with  $m = 0$ . Next we assume that (2.22) holds with  $m = n$ , and we claim that

$$u_{n+1} \leq 2c_p K \mathbf{W}_{\alpha,p}^R[\omega_1] + 2c_p K \mathbf{W}_{\alpha,p}^R[\omega_2] + f. \quad (2.26)$$

In fact, since (2.22) holds with  $m = n$  and  $P_{l,a,\beta}$  is convex, we have

$$\begin{aligned} P_{l,a,\beta}(u_n) &\leq P_{l,a,\beta}(4c_p K \mathbf{W}_{\alpha,p}^R[\omega_1] + 4c_p K \mathbf{W}_{\alpha,p}^R[\omega_2] + f) \\ &\leq P_{l,a,\beta}(4c_{a,\varepsilon} c_p K \mathbf{W}_{\alpha,p}^R[\omega_1]) + P_{l,\varepsilon,a}(4c_{a,\varepsilon} c_p K \mathbf{W}_{\alpha,p}^R[\omega_2]) + P_{l,a,\beta}\left(\left(1 + \frac{\varepsilon}{a}\right)^{1/\beta} f\right) \\ &= H_l\left(\bar{a} (\mathbf{W}_{\alpha,p}^R[\omega_1])^\beta\right) + H_l\left(\bar{a} (\mathbf{W}_{\alpha,p}^R[\omega_2])^\beta\right) + P_{l,a+\varepsilon,\beta}(f). \end{aligned}$$

From this we derive (2.21). By the definition of  $u_{n+1}$  and the sub-additive property of  $\mathbf{W}_{\alpha,p}^R[\cdot]$ , we obtain

$$\begin{aligned} u_{n+1} &\leq K \mathbf{W}_{\alpha,p}^R\left[H_l\left(\bar{a} (\mathbf{W}_{\alpha,p}^R[\omega_1])^\beta\right) + H_l\left(\bar{a} (\mathbf{W}_{\alpha,p}^R[\omega_2])^\beta\right) + P_{l,a+\varepsilon,\beta}(f) + \mu\right] + f \\ &\leq c_p K \mathbf{W}_{\alpha,p}^R\left[H_l\left(\bar{a} (\mathbf{W}_{\alpha,p}^R[\omega_1])^\beta\right)\right] + c_p K \mathbf{W}_{\alpha,p}^R\left[H_l\left(\bar{a} (\mathbf{W}_{\alpha,p}^R[\omega_2])^\beta\right)\right] \\ &\quad + c_p K \mathbf{W}_{\alpha,p}^R[P_{l,a+\varepsilon,\beta}(f)] + c_p K \mathbf{W}_{\alpha,p}^R[\mu] + f. \end{aligned}$$

Hence follows (2.26) from (2.25). This completes the proof of the theorem.  $\blacksquare$

The next result is obtained by an easy adaptation of the proof Theorem 2.5.

**Theorem 2.6** *Assume that the assumptions on  $\alpha, p, a, l, \beta, \varepsilon, f, \mu_1$  and  $\mu$  of Theorem 1.6 are fulfilled and  $R, K$  are positive real numbers. Suppose that  $\{u_m\}$  is a sequence of nonnegative measurable functions in  $\mathbb{R}^N$  that satisfies*

$$\begin{aligned} u_{m+1} &\leq K \mathbf{W}_{\alpha,p}[P_{l,a,\beta}(u_m) + \mu] + f \quad \forall m \in \mathbb{N}, \\ u_0 &\leq K \mathbf{W}_{\alpha,p}[\mu] + f. \end{aligned} \quad (2.27)$$

Then there exists  $M > 0$  depending on  $N, \alpha, p, l, a, \beta, \varepsilon, K$  and  $R$  such that if

$$\|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$P_{l,a,\beta}(4c_p K \mathbf{W}_{\alpha,p}[\omega_3] + 4c_p K \mathbf{W}_{\alpha,p}[\omega_4] + f) \in L^1(\mathbb{R}^N), \quad (2.28)$$

and

$$u_m \leq 2c_p K \mathbf{W}_{\alpha,p}[\omega_3] + 2c_p K \mathbf{W}_{\alpha,p}[\omega_4] + f \quad \forall m \in \mathbb{N}, \quad (2.29)$$

where

$$\omega_3 = M \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R} + \mu \quad (2.30)$$

and

$$\omega_4 = M \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R} + \mu. \quad (2.31)$$

Let  $P \in C(\mathbb{R}^+)$  be a decreasing positive function. The  $(\alpha, P)$ -Orlicz-Bessel capacity of a Borel set  $E \subset \mathbb{R}^N$  is defined by (see [1, Sect 2.6])

$$\text{cap}_{G_{\alpha}, P}(E) = \inf \left\{ \int_{\mathbb{R}^N} P(f) : G_{\alpha} * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N) \right\},$$

and the  $(\alpha, P)$ -Orlicz-Riesz capacity

$$\text{cap}_{I_{\alpha}, P}(E) = \inf \left\{ \int_{\mathbb{R}^N} P(f) : I_{\alpha} * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N) \right\}.$$

**Theorem 2.7** *Let  $\alpha > 0$ ,  $p > 1$ ,  $a > 0$ ,  $c > 0$ ,  $l \in \mathbb{N}^*$  and  $\beta \geq 1$  such that  $l\beta > p - 1$  and  $0 < \alpha p < N$ . Let  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ .*

**1.** *Let  $0 < R \leq \infty$ . If  $u$  is a nonnegative Borel function in  $\mathbb{R}^N$  such that  $P_{l, a, \beta}(u)$  is locally integrable in  $\mathbb{R}^N$  and*

$$u(x) \geq c \mathbf{W}_{\alpha, p}^R [P_{l, a, \beta}(u) + \mu](x) \quad \forall x \in \mathbb{R}^N, \quad (2.32)$$

*then the following statements holds.*

*(i) If  $R < \infty$ , there exists a positive constant  $C_1$  depending on  $N, \alpha, p, l, a, \beta, c$  and  $R$  such that*

$$\int_E P_{l, a, \beta}(u) dx + \mu(E) \leq C_1 \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.} \quad (2.33)$$

*(ii) If  $R = \infty$ , there exists a positive constant  $C_2$  depending on  $N, \alpha, p, l, a, \beta, c$  such that*

$$\int_E P_{l, a, \beta}(u) dx + \mu(E) \leq C_2 \text{cap}_{I_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.} \quad (2.34)$$

**2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $\mu \in \mathfrak{M}^+(\Omega)$  and  $\delta \in (0, 1)$ . If  $u$  is a nonnegative Borel function in  $\Omega$  such that  $P_{l, a, \beta}(u)$  is locally integrable in  $\Omega$  and*

$$u(x) \geq c \mathbf{W}_{\alpha, p}^{\delta d(x, \partial\Omega)} [P_{l, a, \beta}(u) + \mu](x) \quad \forall x \in \Omega, \quad (2.35)$$

*then, for any compact set  $K \subset \Omega$ , there exists a positive constant  $C_3$  depending on  $N, \alpha, p, l, a, \beta, c, \delta$  and  $\text{dist}(K, \partial\Omega)$  such that*

$$\int_E P_{l, a, \beta}(u) dx + \mu(E) \leq C_3 \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel,} \quad (2.36)$$

*where  $Q_p^*$  is the complementary function to  $Q_p$ .*

**Proof.** Set  $d\omega = P_{l, a, \beta}(u) dx + d\mu$ .

**1.** We have

$$P_{l, a, \beta} (c \mathbf{W}_{\alpha, p}^R [\omega]) dx \leq d\omega \quad \text{in } \mathbb{R}^N.$$

Let  $M_{\omega}$  denote the centered Hardy-Littlewood maximal function which is defined for any  $f \in L^1_{loc}(\mathbb{R}^N, d\omega)$  by

$$M_{\omega} f(x) = \sup_{t>0} \frac{1}{\omega(B_t(x))} \int_{B_t(x)} |f| d\omega.$$

If  $E \subset \mathbb{R}^N$  is a Borel set, we have

$$\int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta} (c \mathbf{W}_{\alpha,p}^R[\omega]) dx \leq \int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

Since  $M_\omega$  is bounded on  $L^s(\mathbb{R}^N, d\omega)$ ,  $s > 1$ , we deduce from Fefferman's result [11] that

$$\int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta} (c \mathbf{W}_{\alpha,p}^R[\omega]) dx \leq c_{51} \omega(E),$$

for some constant  $c_{51}$  only depends on  $N$  and  $\frac{l\beta}{p-1}$ . Since  $M_\omega \chi_E \leq 1$ , we derive

$$\begin{aligned} (M_\omega \chi_E(x))^{\frac{l\beta}{p-1}} P_{l,a,\beta} (c \mathbf{W}_{\alpha,p}^R[\omega](x)) &\geq P_{l,a,\beta} \left( c (M_\omega \chi_E(x))^{\frac{1}{p-1}} \mathbf{W}_{\alpha,p}^R[\omega](x) \right) \\ &\geq P_{l,a,\beta} (c \mathbf{W}_{\alpha,p}^R[\omega_E](x)), \end{aligned}$$

where  $\omega_E = \chi_E \omega$ . Thus

$$\int_{\mathbb{R}^N} P_{l,a,\beta} (c \mathbf{W}_{\alpha,p}^R[\omega_E]) dx \leq c_{51} \omega(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.} \quad (2.37)$$

From (2.1), (2.2) and (2.3) we get

$$\int_{\mathbb{R}^N} P_{l,a,\beta} (c \mathbf{W}_{\alpha,p}^R[\omega_E](x)) dx \geq \int_{\mathbb{R}^N} Q_p (c_{52} \mathbf{G}_{\alpha p}[\omega_E](x)) dx \quad \text{if } R < \infty,$$

and

$$\int_{\mathbb{R}^N} P_{l,a,\beta} (c \mathbf{W}_{\alpha,p}^R[\omega_E](x)) dx \geq \int_{\mathbb{R}^N} Q_p (c_{53} \mathbf{I}_{\alpha p}[\omega_E](x)) dx \quad \text{if } R = \infty,$$

where  $Q_p$  is defined by (1.13) and  $c_{52} = (c_2 \beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$  if  $p \neq 2$ ,  $c_{52} = c_3^{-1} a^{\frac{1}{\beta}} c$  if  $p = 2$  (the constant  $c_2$  defined in (2.2), depends on  $R$ , therefore  $c_{52} = c_{52}(r_K)$ ) and  $c_{53} = (c_1 \beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$  if  $p \neq 2$ ,  $c_{53} = a^{\frac{1}{\beta}} c$  if  $p = 2$ . Thus, from (2.37) we obtain that for all Borel set  $E \subset \mathbb{R}^N$  there holds

$$\int_{\mathbb{R}^N} Q_p (c_{52} \mathbf{G}_{\alpha p}[\omega_E](x)) dx \leq c_{51} \omega(E) \quad \text{if } R < \infty,$$

and

$$\int_{\mathbb{R}^N} Q_p (c_{53} \mathbf{I}_{\alpha p}[\omega_E](x)) dx \leq c_{51} \omega(E) \quad \text{if } p = \infty.$$

We recall that  $Q_p^*(s) = \sup_{t>0} \{st - Q_p(t)\}$  satisfies the sub-additivity  $\Delta_2$ -condition (see Chapter 2 in [19]).

(i) We assume  $R < \infty$ . For every  $f \geq 0$ ,  $Q_p^*(f) \in L^1(\Omega)$  such that  $G_{\alpha p} * f \geq \chi_E$ , we have

$$\begin{aligned} \omega(E) &\leq \int_{\mathbb{R}^N} G_{\alpha p} * f d\omega_E = (2c_{51})^{-1} \int_{\mathbb{R}^N} (c_{52} \mathbf{G}_{\alpha p}[\omega_E]) (2c_{51} c_{52}^{-1} f) dx \\ &\leq (2c_{51})^{-1} \int_{\mathbb{R}^N} Q_p (c_{52} \mathbf{G}_{\alpha p}[\omega_E]) dx + (2c_{51})^{-1} \int_{\mathbb{R}^N} Q_p^* (2c_{51} c_{52}^{-1} f) dx \\ &\leq 2^{-1} \omega(E) + c_{54} \int_{\mathbb{R}^N} Q_p^*(f) dx, \end{aligned}$$

the last inequality following from the  $\Delta_2$ -condition. Notice that  $c_{54}$ , as well as the next constant  $c_{55}$ , depends on  $r_K$ . Thus,

$$\omega(E) \leq 2c_{54} \int_{\mathbb{R}^N} Q_p^*(f) dx.$$

Then, we get

$$\omega(E) \leq c_{55} \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.}$$

Which implies (2.33).

(ii) We assume  $R = \infty$ . For every  $f \geq 0$ ,  $Q_p^*(f) \in L^1(\Omega)$  such that  $I_{\alpha p} * f \geq \chi_E$ , since  $I_{\alpha p} * \omega_E = I_{\alpha p}[\omega_E]$ , as above we have

$$\begin{aligned} \omega(E) &\leq \int_{\mathbb{R}^N} I_{\alpha p} * f d\omega_E = \int_{\mathbb{R}^N} (\mathbf{I}_{\alpha p} * \omega_E) f dx = \int_{\mathbb{R}^N} \mathbf{I}_{\alpha p}[\omega_E] f dx \\ &\leq 2^{-1}\omega(E) + c_{56} \int_{\mathbb{R}^N} Q_p^*(f) dx, \end{aligned}$$

Then, it follows (2.34).

**2.** Let  $K \subset \Omega$  be compact. Set  $r_K = \text{dist}(K, \partial\Omega)$  and  $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$ . We have

$$P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega] \right) dx \leq d\omega \quad \text{in } \Omega.$$

Thus, for any Borel set  $E \subset K$

$$\int_{\Omega} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega] \right) dx \leq \int_{\Omega} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

As above we get

$$\int_{\Omega} P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \right) dx \leq c_{51}\omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.38)$$

Note that if  $x \in \Omega$  and  $d(x, \partial\Omega) \leq r_K/8$ , then  $B_t(x) \subset \Omega \setminus \Omega_K$  for all  $t \in (0, \delta d(x, \partial\Omega))$ ; indeed, for all  $y \in B_t(x)$

$$d(y, \partial\Omega) \leq d(x, \partial\Omega) + |x - y| < (1 + \delta)d(x, \partial\Omega) < \frac{1}{4}r_K,$$

thus

$$d(y, K) \geq d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4}r_K > \frac{1}{2}r_K,$$

which implies  $y \notin \Omega_K$ . We deduce that

$$\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \geq \mathbf{W}_{\alpha,p}^{\frac{5}{8}r_K}[\omega_E](x) \quad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\alpha,p}^{\frac{5}{8}r_K}[\omega_E](x) = 0 \quad \forall x \in \Omega^c.$$

Hence we obtain from (2.38),

$$\int_{\mathbb{R}^N} P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\frac{5}{8}r_K}[\omega_E](x) \right) dx \leq c_{51}\omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.39)$$

As above we also obtain

$$\omega(E) \leq c_{57} \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel},$$

where the positive constant  $c_{57}$  depends on  $r_K$ . Inequality (2.36) follows and this completes the proof of the Theorem.  $\blacksquare$

**Proof of Theorem 1.5.** Consider the sequence  $\{u_m\}_{m \geq 0}$  of nonnegative functions defined by  $u_0 = f$  and

$$u_{m+1} = \mathbf{W}_{\alpha, p}^R [P_{l, a, \beta}(u_m)] + f \quad \text{in } \mathbb{R}^N \quad \forall m \geq 0.$$

By Theorem 2.5,  $\{u_m\}_{m \geq 0}$  is well defined and (2.21) and (2.22) are satisfied. It is easy to see that  $\{u_m\}$  is nondecreasing. Hence, thanks to the dominated convergence theorem, we obtain that  $u(x) = \lim_{m \rightarrow \infty} u_m(x)$  is a solution of equation (1.28) which satisfies (1.29).

Conversely, we obtain (1.33) directly from Theorem 2.7.  $\blacksquare$

**Proof of Theorem 1.6.** The proof is similar to the previous one by using Theorem 2.7, Part 1, (ii).  $\blacksquare$

### 3 Quasilinear Dirichlet problems

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ . If  $\mu \in \mathfrak{M}_b(\Omega)$ , we denote by  $\mu^+$  and  $\mu^-$  respectively its positive and negative parts in the Jordan decomposition. We denote by  $\mathfrak{M}_0(\Omega)$  the space of measures in  $\Omega$  which are absolutely continuous with respect to the  $c_{1,p}^\Omega$ -capacity defined on a compact set  $K \subset \Omega$  by

$$c_{1,p}^\Omega(K) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega) \right\}.$$

We also denote  $\mathfrak{M}_s(\Omega)$  the space of measures in  $\Omega$  with support on a set of zero  $c_{1,p}^\Omega$ -capacity. Classically, any  $\mu \in \mathfrak{M}_b(\Omega)$  can be written in a unique way under the form  $\mu = \mu_0 + \mu_s$  where  $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$  and  $\mu_s \in \mathfrak{M}_s(\Omega)$ . It is well known that any  $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$  can be written under the form  $\mu_0 = f - \text{div } g$  where  $f \in L^1(\Omega)$  and  $g \in L^p(\Omega)$ .

For  $k > 0$  and  $s \in \mathbb{R}$  we set  $T_k(s) = \max\{\min\{s, k\}, -k\}$ . If  $u$  is a measurable function defined in  $\Omega$ , finite a.e. and such that  $T_k(u) \in W_{loc}^{1,p}(\Omega)$  for any  $k > 0$ , there exists a measurable function  $v : \Omega \rightarrow \mathbb{R}^N$  such that  $\nabla T_k(u) = \chi_{|u| \leq k} v$  a.e. in  $\Omega$  and for all  $k > 0$ . We define the gradient  $\nabla u$  of  $u$  by  $v = \nabla u$ . We recall the definition of a renormalized solution given in [10].

**Definition 3.1** Let  $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$ . A measurable function  $u$  defined in  $\Omega$  and finite a.e. is called a renormalized solution of

$$\begin{aligned} -\Delta_p u &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

if  $T_k(u) \in W_0^{1,p}(\Omega)$  for any  $k > 0$ ,  $|\nabla u|^{p-1} \in L^r(\Omega)$  for any  $0 < r < \frac{N}{N-1}$ , and  $u$  has the property that for any  $k > 0$  there exist  $\lambda_k^+$  and  $\lambda_k^-$  belonging to  $\mathfrak{M}_b^+ \cap \mathfrak{M}_0(\Omega)$ , respectively concentrated on

the sets  $u = k$  and  $u = -k$ , with the property that  $\mu_k^+ \rightharpoonup \mu_s^+$ ,  $\mu_k^- \rightharpoonup \lambda_s^-$  in the narrow topology of measures and such that

$$\int_{\{|u|<k\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda_k^+ - \int_{\Omega} \varphi d\lambda_k^-,$$

for every  $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ .

**Remark 3.2** We recall that if  $u$  is a renormalized solution to problem (3.1), then  $\frac{|\nabla u|^p}{(|u|+1)^r} \in L^1(\Omega)$  for all  $r > 1$ . From this it follows by Hölder's inequality that  $u \in W_0^{1,p_1}(\Omega)$  for all  $1 \leq p_1 < p$  provided  $e^{a|u|} \in L^1(\Omega)$  for some  $a > 0$ . Furthermore,  $u \geq 0$  a.e. in  $\Omega$  if  $\mu \in \mathfrak{M}_b^+(\Omega)$ .

The following general stability result has been proved in [10, Th 4.1, Sec 5.1].

**Theorem 3.3** If  $\{\mu_n\}$  is a bounded sequence in  $\mathfrak{M}_b(\Omega)$  which converges in the weak sense of measures to some  $\mu \in \mathfrak{M}_b(\Omega)$ , and  $\{u_n\}$  a sequence of renormalized solutions of problem

$$\begin{aligned} -\Delta_p u_n &= \mu_n && \text{in } \Omega, \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.2)$$

then, up to a subsequence,  $\{u_n\}$  converges a.e. to a solution of equation  $-\Delta_p u = \mu$  in  $\mathcal{D}'(\Omega)$ , and such that  $\frac{1}{k} \int_{\Omega} |\nabla T_k(u)|^p dx \leq M$  for every  $k > 0$ .

The next result is a sharp extension of the stability Theorem 3.3.

**Theorem 3.4** Let  $\mu = \mu_0 + \mu_s^+ - \mu_s^-$ , with  $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$  and  $\mu_s^+$ ,  $\mu_s^-$  belonging to  $\mathfrak{M}_s^+(\Omega)$ . Let  $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$  with  $F_n \in L^1(\Omega)$ ,  $g_n \in (L^{p'}(\Omega))^N$  and  $\rho_n, \eta_n$  belonging to  $\mathfrak{M}_b^+(\Omega)$ . Assume that  $\{F_n\}$  converges to  $F$  weakly in  $L^1(\Omega)$ ,  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(\Omega))^N$  and  $(\operatorname{div} g_n)$  is bounded in  $\mathfrak{M}_b(\Omega)$ ; assume also that  $\{\rho_n\}$  converges to  $\mu_s^+$  and  $\{\eta_n\}$  to  $\mu_s^-$  in the narrow topology. If  $\{u_n\}$  is a sequence of renormalized solutions of (3.2), then, up to a subsequence, it converges a.e. in  $\Omega$  to a renormalized solution  $u$  of problem (3.1). Furthermore  $T_k(u_n)$  converges to  $T_k(u)$  in  $W_0^{1,p}(\Omega)$  for any  $k > 0$ .

We also recall the following estimate [20, Th 2.1].

**Theorem 3.5** Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ . Then there exists a constant  $K_1 > 0$ , depending on  $p$  and  $N$  such that if  $\mu \in \mathfrak{M}_b^+(\Omega)$  and  $u$  is a nonnegative renormalized solution of problem (3.1) with data  $\mu$ , there holds

$$\frac{1}{K_1} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \leq u(x) \leq K_1 \mathbf{W}_{1,p}^{2 \operatorname{diam}(\Omega)}[\mu](x) \quad \forall x \in \Omega, \quad (3.3)$$

where the positive constant  $K_1$  only depends on  $N, p$ .

**Proof of Theorem 1.1.** Let  $\{u_m\}_{m \in \mathbb{N}}$  be a sequence of nonnegative renormalized solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu && \text{in } \Omega, \\ u_0 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and, for  $m \in \mathbb{N}$ ,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu && \text{in } \Omega, \\ u_{m+1} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By Theorem 3.5 we have

$$\begin{aligned} \chi_\Omega u_0 &\leq K_1 \mathbf{W}_{1,p}^R[\mu], \\ \chi_\Omega u_{m+1} &\leq K_1 \mathbf{W}_{1,p}^R[P_{l,a,\beta}(u_m) + \mu] \quad \forall m \in \mathbb{N}, \end{aligned}$$

where  $R = 2 \operatorname{diam}(\Omega)$ . Thus, by Theorem 2.5 with  $f \equiv 0$ , there exists  $M > 0$  depending on  $N, p, l, a, \beta, K_1$  and  $R$  such that  $P_{l,a,\beta}(4c_p K_1 \mathbf{W}_{1,p}^R[\omega]) \in L^1(\Omega)$  and

$$u_m(x) \leq 2c_p K_1 \mathbf{W}_{1,p}^R[\omega](x) \quad \forall x \in \Omega, m \in \mathbb{N}, \quad (3.4)$$

provided that

$$\|M_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

where  $\omega = M \|\mathbf{M}_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$  and  $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$ . This implies that  $\{u_m\}$  is well defined; by Theorem 3.3 it contains a subsequence that we still denote by  $\{u_m\}$  which converges a.e in  $\Omega$  to function  $u$  which satisfies (1.17) in  $\Omega$ . Furthermore, we deduce from (3.4) and the dominated convergence theorem that  $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$  in  $L^1(\Omega)$ . Finally, by Theorem 3.4 we obtain that  $u$  is a renormalized solution of (1.16).

Conversely, assume that (1.16) admits a nonnegative renormalized solution  $u$ . By Theorem 3.5 there holds

$$u(x) \geq \frac{1}{K_1} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \Omega.$$

Hence, we achieve (1.18) from Theorem 2.7. ■

**Applications.** We consider the case  $p = 2, \beta = 1$ . Then  $l = 2$  and

$$P_{l,a,\beta}(r) = e^{ar} - 1 - ar.$$

If  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ , there exists  $M > 0$  such that if  $\mu$  is a positive Radon measure in  $\Omega$  which satisfies

$$\mu(B_t(x)) \leq Mt^{N-2} \quad \forall t > 0 \text{ and almost all } x \in \Omega,$$

there exists a positive solution  $u$  to the following problem

$$\begin{aligned} -\Delta u &= e^{au} - 1 - au + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Furthermore

$$u(x) \leq K(N) \int_0^{2 \operatorname{diam} \Omega} \frac{\omega(B_t(x))}{t^{N-1}} dt = K(N) \int_0^{2 \operatorname{diam}(\Omega)} \frac{\mu(B_t(x))}{t^{N-1}} dt + b \quad \forall x \in \Omega.$$

where  $b = 2K(N)M \|\mathbf{M}_{p,2 \operatorname{diam}(\Omega)}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} (\operatorname{diam} \Omega)^2$ . In the case  $N = 2$  this result has already been proved by Richard and Véron [22, Prop 2.4].

## 4 $p$ -superharmonic functions and quasilinear equations in $\mathbb{R}^N$

We recall some definitions and properties of  $p$ -superharmonic functions.

**Definition 4.1** A function  $u$  is said to be  $p$ -harmonic in  $\mathbb{R}^N$  if  $u \in W_{loc}^{1,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$  and  $-\Delta_p u = 0$  in  $\mathcal{D}'(\mathbb{R}^N)$ . A function  $u$  is called a  $p$ -supersolution in  $\mathbb{R}^N$  if  $u \in W_{loc}^{1,p}(\mathbb{R}^N)$  and  $-\Delta_p u \geq 0$  in  $\mathcal{D}'(\mathbb{R}^N)$ .

**Definition 4.2** A lower semicontinuous (l.s.c) function  $u : \mathbb{R}^N \rightarrow (-\infty, \infty]$  is called  $p$ -superharmonic if  $u$  is not identically infinite and if, for all open  $D \subset\subset \mathbb{R}^N$  and all  $v \in C(\overline{D})$ ,  $p$ -harmonic in  $D$ ,  $v \leq u$  on  $\partial D$  implies  $v \leq u$  in  $D$ .

Let  $u$  be a  $p$ -superharmonic in  $\mathbb{R}^N$ . It is well known that  $u \wedge k \in W_{loc}^{1,p}(\mathbb{R}^N)$  is a  $p$ -supersolution for all  $k > 0$  and  $u < \infty$  a.e in  $\mathbb{R}^N$ , thus,  $u$  has a gradient (see the previous section). We also have  $|\nabla u|^{p-1} \in L_{loc}^q(\mathbb{R}^N)$ ,  $\frac{|\nabla u|^p}{(|u|+1)^r} \in L_{loc}^1(\mathbb{R}^N)$  and  $u \in L_{loc}^s(\mathbb{R}^N)$  for  $1 \leq q < \frac{N}{N-1}$  and  $r > 1$ ,  $1 \leq s < \frac{N(p-1)}{N-p}$  (see [14, Theorem 7.46]). In particular, if  $e^{a|u|} \in L_{loc}^1(\mathbb{R}^N)$  for some  $a > 0$ , then  $u \in W_{loc}^{1,p_1}(\mathbb{R}^N)$  for all  $1 \leq p_1 < p$  by Hölder's inequality. Thus by the dominated convergence theorem,

$$\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla(u \wedge k)|^{p-2} \nabla(u \wedge k) \nabla \varphi \geq 0.$$

Hence, by the Riesz Representation Theorem we conclude that there is a nonnegative Radon measure denoted by  $\mu[u]$ , called Riesz measure, such that  $-\Delta_p u = \mu[u]$  in  $\mathcal{D}'(\mathbb{R}^N)$ .

The following weak convergence result for Riesz measures proved in [28] will be used to prove the existence of  $p$ -superharmonic solutions to quasilinear equations.

**Theorem 4.3** Suppose that  $\{u_n\}$  is a sequence of nonnegative  $p$ -superharmonic functions in  $\mathbb{R}^N$  that converges a.e to a  $p$ -superharmonic function  $u$ . Then the sequence of measures  $\{\mu[u_n]\}$  converges to  $\mu[u]$  in the weak sense of measures.

The next theorem is proved in [20]

**Theorem 4.4** Let  $\mu$  be a measure in  $\mathfrak{M}^+(\mathbb{R}^N)$ . Suppose that  $\mathbf{W}_{1,p}[\mu] < \infty$  a.e. Then there exists a nonnegative  $p$ -superharmonic function  $u$  in  $\mathbb{R}^N$  such that  $-\Delta_p u = \mu$  in  $\mathcal{D}'(\mathbb{R}^N)$ ,  $\inf_{\mathbb{R}^N} u = 0$  and

$$\frac{1}{K_1} \mathbf{W}_{1,p}[\mu](x) \leq u(x) \leq K_1 \mathbf{W}_{1,p}[\mu](x), \quad (4.1)$$

for all  $x$  in  $\mathbb{R}^N$ , where the constant  $K_1$  is as in Theorem 3.5. Furthermore any  $p$ -superharmonic function  $u$  in  $\mathbb{R}^N$ , such that  $\inf_{\mathbb{R}^N} u = 0$  satisfies (4.1) with  $\mu = -\Delta_p u$ .

**Proof of Theorem 1.2.** Let  $\{u_m\}_{m \in \mathbb{N}}$  be a sequence of  $p$ -superharmonic solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu & \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_0 &= 0, \end{aligned}$$



and, for  $m \in \mathbb{N}$ ,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_{m+1} &= 0. \end{aligned}$$

By Theorem 4.4 we have

$$\begin{aligned} u_0 &\leq K_1 \mathbf{W}_{1,p}[\mu], \\ u_{m+1} &\leq K_1 \mathbf{W}_{1,p}[P_{l,a,\beta}(u_m) + \mu] \quad \forall m \in \mathbb{N}. \end{aligned}$$

Thus, by Theorem 2.5 with  $f \equiv 0$ , there exists  $M > 0$  depending on  $N, p, l, a, \beta, K_1$  and  $R$  such that  $P_{l,a,\beta}(4c_p K_1 \mathbf{W}_{1,p}[\omega]) \in L^1(\mathbb{R}^N)$  and

$$u_m \leq 2c_p K_1 \mathbf{W}_{1,p}[\omega] \quad \forall m \in \mathbb{N}, \quad (4.2)$$

provided that

$$\|M_p^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

where  $\omega = M \|M_p^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R(0)} + \mu$ . This implies that  $\{u_m\}$  is well defined; by [15, Theorem 1.17] it contains a subsequence that we still denote by  $\{u_m\}$  which converges a.e in  $\mathbb{R}^N$  to a  $p$ -superharmonic function  $u$  which satisfies (1.20) in  $\Omega$ . Furthermore, we deduce from (4.2) and the dominated convergence theorem that  $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$  in  $L^1(\mathbb{R}^N)$ . Finally, by Theorem 4.3 we conclude that  $u$  is a  $p$ -superharmonic solution of (1.19).

Conversely, assume that (1.19) admits a nonnegative renormalized solution  $u$ . By Theorem 4.4 there holds

$$u(x) \geq \frac{1}{K_1} \mathbf{W}_{1,p}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \mathbb{R}^N.$$

Hence, we obtain (1.21) from Theorem 2.7. ■

## 5 Hessian equations

In this section  $\Omega \subset \mathbb{R}^N$  is either a bounded domain with a  $C^2$  boundary or the whole  $\mathbb{R}^N$ . For  $k = 1, \dots, N$  and  $u \in C^2(\Omega)$  the  $k$ -hessian operator  $F_k$  is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where  $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$  denotes the eigenvalues of the Hessian matrix of second partial derivative  $D^2u$  and  $S_k$  is the  $k$ -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}.$$

We can see that

$$F_k[u] = [D^2u]_k,$$

where for a matrix  $A = (a_{ij})$ ,  $[A]_k$  denotes the sum of the  $k$ -th principal minors. We assume that  $\partial\Omega$  is uniformly  $(k-1)$ -convex, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \quad \text{on } \partial\Omega,$$

for some positive constant  $c_0$ , where  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$  denote the principal curvatures of  $\partial\Omega$  with respect to its inner normal.

**Definition 5.1** *An upper-semicontinuous function  $u : \Omega \rightarrow [-\infty, \infty)$  is  $k$ -convex ( $k$ -subharmonic) if, for every open set  $\Omega' \subset \overline{\Omega}' \subset \Omega$  and for every function  $v \in C^2(\Omega') \cap C(\overline{\Omega}')$  satisfying  $F_k[v] \leq 0$  in  $\Omega'$ , the following implication is true*

$$u \leq v \text{ on } \partial\Omega' \implies u \leq v \text{ in } \Omega'.$$

We denote by  $\Phi^k(\Omega)$  the class of all  $k$ -subharmonic functions in  $\Omega$  which are not identically equal to  $-\infty$ .

The following weak convergence result for  $k$ -Hessian operators proved in [26] is fundamental in our study.

**Theorem 5.2** *Let  $\Omega$  be either a bounded uniformly  $(k-1)$ -convex in  $\mathbb{R}^N$  or the whole  $\mathbb{R}^N$ . For each  $u \in \Phi^k(\Omega)$ , there exist a nonnegative Radon measure  $\mu_k[u]$  in  $\Omega$  such that*

**1**  $\mu_k[u] = F_k[u]$  for  $u \in C^2(\Omega)$ .

**2** If  $\{u_n\}$  is a sequence of  $k$ -convex functions which converges a.e to  $u$ , then  $\mu_k[u_n] \rightharpoonup \mu_k[u]$  in the weak sense of measures.

As in the case of quasilinear equations with measure data, precise estimates of solutions of  $k$ -Hessian equations with measures data are expressed in terms of Wolff potentials. The next results are proved in [26, 18, 20].

**Theorem 5.3** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded  $C^2$ , uniformly  $(k-1)$ -convex domain. Let  $\varphi$  be a nonnegative continuous function on  $\partial\Omega$  and  $\mu$  be a nonnegative Radon measure. Suppose that  $\mu$  can be decomposed under the form*

$$\mu = \mu_1 + f$$

where  $\mu_1$  is a measure with compact support in  $\Omega$  and  $f \in L^q(\Omega)$  for some  $q > \frac{N}{2k}$  if  $k \leq \frac{N}{2}$ , or  $p = 1$  if  $k > \frac{N}{2}$ . Then there exists a nonnegative function  $u$  in  $\Omega$  such that  $-u \in \Phi^k(\Omega)$ , continuous near  $\partial\Omega$  and  $u$  is a solution of the problem

$$\begin{aligned} F_k[-u] &= \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega. \end{aligned}$$

Furthermore, any nonnegative function  $u$  such that  $-u \in \Phi^k(\Omega)$  which is continuous near  $\partial\Omega$  and is a solution of above equation, satisfies

$$\frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{s}}[\mu] \leq u(x) \leq K_2 \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2 \text{diam } \Omega}[\mu](x) + \max_{\partial\Omega} \varphi \right), \quad (5.1)$$

where  $K_2$  is a positive constant independent of  $x, u$  and  $\Omega$ .

**Theorem 5.4** *Let  $\mu$  be a measure in  $\mathfrak{M}^+(\mathbb{R}^N)$  and  $2k < N$ . Suppose that  $\mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu] < \infty$  a.e. Then there exists  $u, -u \in \Phi^k(\mathbb{R}^N)$  such that  $\inf_{\mathbb{R}^N} u = 0$  and  $F_k[-u] = \mu$  in  $\mathbb{R}^N$  and*

$$\frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x) \leq u(x) \leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x) \quad (5.2)$$

for all  $x$  in  $\mathbb{R}^N$ , where the constant  $K_2$  is the one of the previous Theorem. Furthermore, if  $u$  is a nonnegative function such that  $-u \in \Phi^k(\mathbb{R}^N)$ , then (5.2) holds with  $\mu = F_k[-u]$ .

**Proof of Theorem 1.3.** We defined a sequence of nonnegative functions  $u_m$ , continuous near  $\partial\Omega$  and such that  $-u_m \in \Phi^k(\Omega)$ , by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu & \text{in } \Omega, \\ u_0 &= \varphi & \text{on } \partial\Omega, \end{aligned} \quad (5.3)$$

and, for  $m \geq 0$ ,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l,a,\beta}(u_m) + \mu & \text{in } \Omega, \\ u_{m+1} &= \varphi & \text{on } \partial\Omega. \end{aligned} \quad (5.4)$$

By Theorem 5.3 we have

$$\begin{aligned} \chi_\Omega u_0 &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\mu] + b_0, \\ \chi_\Omega u_{m+1} &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [P_{l,a,\beta}(u_m) + \mu] + b_0, \end{aligned} \quad (5.5)$$

where  $b_0 = K_2 \max_{\partial\Omega} \varphi$  and  $R = 2 \operatorname{diam}(\Omega)$ . We set  $\omega_1 = M \|\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}} [1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$ ,  $\omega_2 = M \|\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}} [1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + P_{l,2a,\beta}(b_0)$  and  $g = 4c_p K \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\omega_2] + b_0$ . Since  $\omega_2$  is constant,  $g$  has the same property and actually  $g = 2c_p K (c_N \omega_2)^{\frac{1}{k}} R^2 + b_0$ . Then, by Theorem 2.5 with  $f = b_0$  and  $\varepsilon = a$ , there exists  $M_1 > 0$  depending on  $N, k, l, a, \beta$  and  $R$  such that  $P_{l,a,\beta} \left( 4K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\omega_1] + g \right) \in L^1(\Omega)$  and

$$u_m(x) \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\omega_1](x) + g \quad \forall x \in \Omega, \forall m \geq 0, \quad (5.6)$$

provided that

$$\|M_{2k,R}^{\frac{k(\beta-1)}{\beta}} [\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \quad \text{and} \quad \|M_{2k,R}^{\frac{k(\beta-1)}{\beta}} [P_{l,2a,\beta}(b_0)]\|_{L^\infty(\mathbb{R}^N)} \leq M_1.$$

It follows that one can find constants  $M_2$  depending on  $N, k, l, a, \beta, R$  and  $M_1$  such that if  $\max_{\partial\Omega} \varphi \leq M_2$ , then  $\|M_{2k,R}^{\frac{k(\beta-1)}{\beta}} [P_{l,2a,\beta}(b_0)]\|_{L^\infty(\mathbb{R}^N)} \leq M_1$ . Hence, we deduce from (5.6) that  $P_{l,a,\beta} \left( 2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\mu] + b_2 \right) \in L^1(\Omega)$  and

$$u_m(x) \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\mu](x) + b_2 \quad \forall x \in \Omega, \forall m \geq 0, \quad (5.7)$$

for some constant  $b_2 (= g)$  depending on  $N, k, l, a, \beta, R$  and  $M_1$ . Note that because we can write

$$\omega = P_{l,a,\beta}(u_m) + \mu = (\mu_1 + \chi_{\Omega_\delta} P_{l,a,\beta}(u_m)) + ((1 - \chi_{\Omega_\delta}) P_{l,a,\beta}(u_m) + f),$$

where  $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$  and  $\delta > 0$  is small enough and since  $u_m$  is continuous near  $\partial\Omega$ , then  $\omega$  satisfies the assumptions of the data in Theorem 5.3. Therefore the sequence  $\{\omega_m\}$  is well defined. Since  $-u_m$  is  $k$ -subharmonic and the sequence  $\{u_m\}$  is bounded in  $L^1(\Omega)$ ,

the sequence  $\{u_m\}$  is also bounded in  $W_{loc}^{1,1}(\Omega)$  (see e.g. [26]); we can find a subsequence, still denoted by the index  $m$ , such that  $u_m$  converges a.e in  $\Omega$  to a function  $u$  for which (1.23) is satisfied in  $\Omega$  with the constant  $b_2 = g$ . Furthermore, we deduce from (5.6) and the dominated convergence theorem that  $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$  in  $L^1(\Omega)$ . Finally, by Theorem 5.2, we obtain that  $u$  satisfies (1.22) and (1.23).

Conversely, assume that (1.22) admits nonnegative solution  $u$ , continuous near  $\partial\Omega$ , such that  $-u \in \Phi^k(\Omega)$  and  $P_{l,a,\beta}(u) \in L^1(\Omega)$ . Then by Theorem 5.3 we have

$$u(x) \geq \frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}} [P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \Omega.$$

Using Theorem 2.7, we conclude that (1.24) holds.  $\blacksquare$

**Proof of Theorem 1.4.** We define a sequence of nonnegative functions  $u_m$ , continuous near  $\partial\Omega$  and such that  $-u_m \in \Phi^k(\Omega)$ , by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu & \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_0 &= 0, \end{aligned} \tag{5.8}$$

and, for  $m \geq 0$ ,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l,a,\beta}(u_m) + \mu & \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_{m+1} &= 0. \end{aligned} \tag{5.9}$$

By Theorem 5.4, we have

$$\begin{aligned} u_0 &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu], \\ u_{m+1} &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}[P_{l,a,\beta}(u_m) + \mu]. \end{aligned} \tag{5.10}$$

Thus, by Theorem 2.5 with  $f \equiv 0$ , there exists  $M > 0$  depending on  $N, k, l, a, \beta$  and  $R$  such that  $P_{l,a,\beta}(2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega]) \in L^1(\mathbb{R}^N)$ ,

$$u_m \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega] \quad \forall m \geq 0, \tag{5.11}$$

provided that  $\|M_{2k,R}^{\frac{k(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M$ , where  $\omega = M \|\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R} + \mu$ .

Therefore the sequence  $\{u_m\}$  is well defined. By arguing as in the proof of theorem 1.3 we obtain that  $u$  satisfies (1.25) and (1.26).

Conversely, assume that (1.25) admits a nonnegative solution  $u$  and  $-u \in \Phi^k(\mathbb{R}^N)$  such that  $P_{l,a,\beta}(u) \in L_{loc}^1(\mathbb{R}^N)$ , then by Theorem 5.4 we have

$$u \geq \frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}[P_{l,a,\beta}(u) + \mu].$$

Using Theorem 2.7, we conclude that (1.27) holds.  $\blacksquare$

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