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Hung Nguyen Quoc, Laurent Veron. Quasilinear and Hessian type equations with exponential reaction and measure data. 2013. hal-00823874v3

HAL Id: hal-00823874

<https://hal.science/hal-00823874v3>

Preprint submitted on 8 Jun 2013 (v3), last revised 9 May 2014 (v6)

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Quasilinear and Hessian type equations with exponential reaction and measure data

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Abstract

We prove existence results concerning equations of the type $-\Delta_p u = P(u) + \mu$ for $p > 1$ and $F_k[-u] = P(u) + \mu$ with $1 \leq k < \frac{N}{2}$ in a bounded domain Ω , where μ is a positive Radon measure and $P(u) \sim e^{au^\beta}$ with $a > 0$ and $\beta \geq 1$. Sufficient conditions for existence are expressed in terms of the fractional maximal potential of μ . Two-sided estimates on the solutions are obtained in terms of some precise Wolff potentials of μ . Necessary conditions are obtained in terms of Orlicz capacities. We also establish existence results for a general Wolff potential equation under the form $u = \mathbf{W}_{\alpha,p}^R[P(u)] + f$.

2010 Mathematics Subject Classification. 31C15, 32 F10, 35J92, 35R06, 46E30.

Key words: quasilinear elliptic equations, Hessian equations, Wolff potential, maximal functions, Borel measures, Orlicz capacities.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be either a bounded domain or the whole \mathbb{R}^N , $p > 1$ and $k \in \{1, 2, \dots, N\}$. We denote by

$$\Delta_p u := \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right)$$

the p-Laplace operator and by

$$F_k[u] = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq N} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k}$$

the k-Hessian operator where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of the Hessian matrix $D^2 u$. Let μ be a positive Radon measure in Ω ; our aim is to study the existence of positive renormalized solutions to the quasilinear equation

$$\begin{aligned} -\Delta_p u &= P(u) + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

and the fully nonlinear equation

$$\begin{aligned} F_k[-u] &= P(u) + \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (1.2)$$

where P is an exponential function. When $P(r) = r^q$ with $q > p - 1$, Phuc and Verbitsky published a seminal article [20] on the solvability of the corresponding problem (1.1). They obtained necessary and sufficient conditions involving Bessel capacities or Wolff potentials. For example, assume that Ω is bounded, they proved that if μ has compact support in Ω it is equivalent to solve (1.1) with $P(r) = r^q$ or to have

$$\mu(E) \leq cC_{p, \frac{p}{q+1-p}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.3)$$

where c is a suitable positive constant and $C_{p, \frac{p}{q+1-p}}$ a Bessel capacity, or to have

$$\int_B [\mathbf{W}_{1,p}^{2R}[\mu_B(x)]]^q dx \leq C\mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \operatorname{supp}\mu \neq \emptyset, \quad (1.4)$$

where $R = \operatorname{diam}(\Omega)$. Other conditions are expressed in terms of Riesz potentials and maximal fractional potentials. Their construction is based upon sharp estimates of solutions of the non-homogeneous problem

$$\begin{aligned} -\Delta_p u &= \omega && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.5)$$

for positive measures ω . We refer to [4, 5, 6, 7, 9, 13, 23] for the previous studies of these and other related results. Concerning k-Hessian operator in a bounded $(k - 1)$ -convex domain Ω , they proved that if μ has compact support and $\|\varphi\|_{L^\infty(\partial\Omega)}$ is small enough, the corresponding problem (1.2) with $P(r) = r^q$ with $q > k$ admits a positive solution if and only if

$$\mu(E) \leq cC_{2k, \frac{q}{q-k}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.6)$$

or equivalently

$$\int_B \left[\mathbf{W}_{\frac{2k}{k+1}, k+1}^{2R} [\mu_B(x)] \right]^q dx \leq C\mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \neq \emptyset. \quad (1.7)$$

In the linear case $p = 2$ and $k = 1$, can be found in [2, 3, 29]. The main tools in their proofs are derived from recent advances in potential theory for nonlinear elliptic equations obtained by Kilpelainen and Malý [15, 16], Trudinger and Wang [25, 26, 27], and Labutin [18] thanks to whom the authors first provide global pointwise estimates for solutions of the homogeneous Dirichlet problems in terms of Wolffs potentials of suitable order.

For $s > 1$, $0 \leq \alpha < \frac{N}{s}$, $\eta \geq 0$ and $0 < T \leq \infty$, we recall the expression of the T -truncated Wolff potential of a positive Radon measure μ defined in \mathbb{R}^N by

$$\mathbf{W}_{\alpha, s}^T[\mu](x) = \int_0^T \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t}, \quad (1.8)$$

and T -truncated Riesz potential of a positive Radon measure μ by

$$\mathbf{I}_\alpha^T[\mu](x) = \int_0^T \frac{\mu(B_t(x))}{t^{N-\alpha}} \frac{dt}{t}, \quad (1.9)$$

and the T -truncated η -fractional maximal potential of μ by

$$\mathbf{M}_{\alpha, T}^\eta[\mu](x) = \sup \left\{ \frac{\mu(B_t(x))}{t^{N-\alpha} h_\eta(t)} : 0 < t \leq T \right\}, \quad (1.10)$$

where $h_\eta(t) = (-\ln t)^{-\eta} \chi_{(0, 2^{-1})}(t) + (\ln 2)^{-\eta} \chi_{[2^{-1}, \infty)}(t)$. If $\eta = 0$, then $h_\eta = 1$ and we denote by $\mathbf{M}_{\alpha, T}[\mu]$ the corresponding T -truncated fractional maximal potential of μ . We also denote by $\mathbf{W}_{\alpha, s}[\mu]$ (resp $\mathbf{I}_\alpha[\mu]$, $\mathbf{M}_\alpha[\mu]$) the ∞ -truncated Wolff potential (resp Riesz Potential, fractional maximal potential) of μ . When the measures are only defined in an open subset $\Omega \subset \mathbb{R}^N$, they are naturally extended by 0 in Ω^c . For $l \in \mathbb{N}^*$, we define the l -truncated exponential function

$$H_l(r) = e^r - \sum_{j=0}^{l-1} \frac{r^j}{j!}, \quad (1.11)$$

and for $a > 0$ and $\beta \geq 1$, we set

$$P_{l, a, \beta}(r) = H_l(ar^\beta). \quad (1.12)$$

We put

$$Q_p(s) = \begin{cases} \sum_{q=l}^{\infty} \frac{s^{\frac{\beta q}{p-1}}}{q^{\frac{\beta q}{p-1}} q!} & \text{if } p \neq 2, \\ H_l(s^\beta) & \text{if } p = 2, \end{cases} \quad (1.13)$$

$Q_p^*(r) = \max \{rs - Q_p(s) : s \geq 0\}$ is the complementary function to Q_p , and define

$$\text{cap}_{G_p, Q_p^*}(E) = \inf \left\{ \int_{\mathbb{R}^N} Q_p^*(f) dx : G_p * f \geq \chi_E, f \geq 0, Q_p^*(f) \in L^1(\mathbb{R}^N) \right\}, \quad (1.14)$$

and

$$cap_{I_p, Q_p^*}(E) = \inf \left\{ \int_{\mathbb{R}^N} Q_p^*(f) dx : I_p * f \geq \chi_E, f \geq 0, Q_p^*(f) \in L^1(\mathbb{R}^N) \right\}, \quad (1.15)$$

where $G_p(x) = \mathcal{F}^{-1} \left((1 + |\cdot|^2)^{-\frac{p}{2}} \right) (x)$ is the Bessel kernel of order p and $I_p(x) = (N - p)^{-1} |x|^{-(N-p)}$.

We denote $a \wedge b$ and $a \vee b$ for $\min\{a, b\}$ and $\max\{a, b\}$ respectively. Our main results are the following theorems

Theorem 1.1 *Let $1 < p < N$, $a > 0$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p - 1$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If μ is a nonnegative Radon measure in Ω , there exists $M > 0$ depending on N, p, l, a, β and $diam \Omega$ which is the diameter of Ω such that if*

$$\| \mathbf{M}_{p, 2diam \Omega}^{\frac{(p-1)(\beta-1)}{\beta}} [\mu] \|_{L^\infty(\mathbb{R}^N)} \leq M$$

the following Dirichlet problem

$$\begin{aligned} -\Delta_p u &= P_{l, a, \beta}(u) + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.16)$$

admits a nonnegative renormalized solution u which satisfies $P_{l, a, \beta}(2c_p K_1 \mathbf{W}_{1, p}^{2diam \Omega}[\omega]) \in L^1(\Omega)$ and

$$u(x) \leq 2c_p K_1 \mathbf{W}_{1, p}^{2diam \Omega}[\omega](x) \quad \forall x \in \Omega, \quad (1.17)$$

where $\omega = M \| \mathbf{M}_{p, 2diam \Omega}^{\frac{(p-1)(\beta-1)}{\beta}} [1] \|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$. The role of $K_1 = K_1(N, p)$ will be made explicit in Theorem 3.5. Conversely, if (1.16) admits a nonnegative renormalized solution u and $P_{l, a, \beta}(u) \in L^1(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant C depending on N, p, l, a, β and $\text{dist}(K, \partial\Omega)$ such that

$$\int_E P_{l, a, \beta}(u) dx + \mu(E) \leq C cap_{G_p, Q_p^*}(E) \quad \text{for all Borel sets } E \subset K. \quad (1.18)$$

Furthermore, $u \in W_0^{1, p_1}(\Omega)$ for all $1 \leq p_1 < p$.

Theorem 1.2 *Let $1 < p < N$, $a > 0$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > \frac{N(p-1)}{N-p}$ and $R > 0$. If μ is a nonnegative Radon measure in \mathbb{R}^N with $\text{supp}(\mu) \subset B_R(0)$ there exists $M > 0$ depending on N, p, l, a, β and R such that if*

$$\| \mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}} [\mu] \|_{L^\infty(\mathbb{R}^N)} \leq M$$

the following problem

$$\begin{aligned} -\Delta_p u &= P_{l, a, \beta}(u) + \mu \text{ in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u &= 0, \end{aligned} \quad (1.19)$$

admits an p -superharmonic solution u which satisfies $P_{l,a,\beta}(2c_p K_1 \mathbf{W}_{1,p}[\omega]) \in L^1(\mathbb{R}^N)$ and

$$u(x) \leq 2c_p K_1 \mathbf{W}_{1,p}[\omega](x) \quad \forall x \in \mathbb{R}^N, \quad (1.20)$$

where $\omega = M \|\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}} [\chi_{B_R(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R(0)} + \mu$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$. The role of $K_1 = K_1(N, p)$ will be made explicit in Theorem 3.5. Conversely, if (1.19) has a solution u and $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ then there exists a positive constant C depending on N, p, l, a, β such that

$$\int_E P_{l,\alpha,\beta}(u) dx + \mu(E) \leq C \text{cap}_{I_p, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}. \quad (1.21)$$

Furthermore, $u \in W^{1,p_1}_{loc}(\mathbb{R}^N)$ for all $1 \leq p_1 < p$.

Concerning the k -Hessian operator we recall some notions introduced by Trudinger and Wang [25, 26, 27], and we follow their notations. For $k = 1, \dots, N$ and $u \in C^2(\Omega)$ the k -hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives D^2u and S_k is the k -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}.$$

It is straightforward that

$$F_k[u] = [D^2u]_k,$$

where in general $[A]_k$ denotes the sum of the k -th principal minors of a matrix $A = (a_{ij})$. In order that there exists a smooth k -admissible function which vanishes on $\partial\Omega$, the boundary $\partial\Omega$ must satisfy a uniformly $(k-1)$ -convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega$$

for some positive constant c_0 , where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal. We also denote by $\Phi^k(\Omega)$ the class of upper-semicontinuous functions $\Omega \mapsto [-\infty, \infty)$ which are k -convex, or subharmonic in the Perron sense (see Definition 5.1). In this paper we prove the following theorem

Theorem 1.3 *Let $k \in \{1, 2, \dots, N\}$ such that $2k < N$, $l \in \mathbb{N}^*$, $\beta \geq 1$ such that $l\beta > k$ and $a > 0$. Let Ω be a bounded uniformly $(k-1)$ -convex domain in \mathbb{R}^N . Let φ be a nonnegative continuous function on $\partial\Omega$ and $\mu = \mu_1 + f$ be a nonnegative Radon measure where μ_1 has compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$. Let $K_2 = K_2(N, k)$ be the constant K_2 which appears in Theorem 5.3. Then, there exist $b > 0$ and $M_1, M_2 > 0$ depending on N, k, l, a, β and $\text{diam}(\Omega)$ such that if $\max_{\partial\Omega} \varphi \leq M_2$ and*

$$\|\mathbf{M}_{2k, 2\text{diam}(\Omega)}^{\frac{k(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M_1.$$

the following Dirichlet problem

$$\begin{aligned} F_k[-u] &= P_{l,a,\beta}(u) + \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (1.22)$$

admits a nonnegative solution u , continuous near $\partial\Omega$, with $-u \in \Phi^k(\Omega)$ which satisfies $P_{l,a,\beta} \left(2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2diam\Omega} [\mu] + b \right) \in L^1(\Omega)$ and

$$u(x) \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2diam\Omega} [\mu](x) + b \quad \forall x \in \Omega. \quad (1.23)$$

Conversely, if (1.22) admits a nonnegative solution u , continuous near $\partial\Omega$ such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$, then for any a compact set $K \subset \Omega$, there exists a positive constant C depending on N, k, l, a, β and $dist(K, \partial\Omega)$ such that there holds

$$\int_E P_{l,a,\beta}(u) dx + \mu(E) \leq C cap_{G_{2k}, Q_{k+1}^*}(E) \quad \forall E \subset K, E \text{ Borel}, \quad (1.24)$$

where $Q_{k+1}(s)$ is defined by (1.13) with $p = k + 1$, Q_{k+1}^* is its complementary function and $cap_{G_{2k}, Q_{k+1}^*}(E)$ is defined accordingly by (1.14).

Theorem 1.4 Let $k \in \{1, 2, \dots, N\}$ such that $2k < N$, $l \in \mathbb{N}^*$, $\beta \geq 1$ such that $l\beta > \frac{Nk}{N-2k}$ and $a > 0$, $R > 0$. Let $K_2 = K_2(N, k)$ be the constant K_2 which appears in Theorem 5.3. If μ is a nonnegative Radon measure in \mathbb{R}^N with $\text{supp}(\mu) \subset B_R(0)$ there exists $M > 0$ depending on N, k, l, a, β and R such that if

$$\| \mathbf{M}_{2k}^{\frac{k(\beta-1)}{\beta}} [\mu] \|_{L^\infty(\mathbb{R}^N)} \leq M.$$

the following Dirichlet problem

$$\begin{aligned} F_k[-u] &= P_{l,a,\beta}(u) + \mu && \text{in } \mathbb{R}^N, \\ \inf_{\mathbb{R}^N} u &= 0, \end{aligned} \quad (1.25)$$

admits a nonnegative solution u , $-u \in \Phi^k(\mathbb{R}^N)$ which satisfies $P_{l,a,\beta} \left(2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1} [\omega] \right) \in L^1(\mathbb{R}^N)$ and

$$u(x) \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1} [\omega](x) \quad \forall x \in \mathbb{R}^N, \quad (1.26)$$

where $\omega = M \| \mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}} [\chi_{B_R(0)}] \|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R(0)} + \mu$. Conversely, if (1.25) admits a nonnegative solution u , $-u \in \Phi^k(\mathbb{R}^N)$ and $P_{l,a,\beta}(u) \in L_{loc}^1(\mathbb{R}^N)$, then there exists a positive constant C depending on N, k, l, a, β and R such that there holds

$$\int_E P_{l,a,\beta}(u) dx + \mu(E) \leq C cap_{I_{2k}, Q_{k+1}^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}, \quad (1.27)$$

where $cap_{I_{2k}, Q_{k+1}^*}(E)$ is defined accordingly by (1.15) with $p = k + 1$.

The four previous theorems are connected to the following result which deals with a class of nonlinear *Wolff integral equations*.

Theorem 1.5 *Let $\alpha > 0$, $p > 1$, $a > 0$, $\varepsilon > 0$, $R_1, R_2 > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p-1$ and $0 < \alpha p < N$. Let f be a nonnegative measurable in \mathbb{R}^N such that $\exp((a+\varepsilon)f^\beta) \in L^1_{loc}(\mathbb{R}^N)$. We set $d\mu_1 = P_{l,a+\varepsilon,\beta}(f)dx$. Let $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$.*

1. *There exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon$ and R_1 such that if*

$$\|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M$$

then there exists a nonnegative function u , such that $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ which satisfies

$$u = \mathbf{W}_{\alpha,p}^{R_1}[P_{l,a,\beta}(u) + \mu] + f \quad \text{in } \mathbb{R}^N, \quad (1.28)$$

and

$$u \leq F := 2c_p \mathbf{W}_{\alpha,p}^{R_1}[\omega_1] + 2c_p \mathbf{W}_{\alpha,p}^{R_1}[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1_{loc}(\mathbb{R}^N), \quad (1.29)$$

where $\omega_1 = M \|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$, $\omega_2 = M \|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu_1$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$.

Conversely, if (1.28) admits a nonnegative solution u and $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ then there exists a positive constant C depending on $N, \alpha, p, l, a, \beta$ and R_1 such that there holds

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) + \mu_1(E) \leq C \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}. \quad (1.30)$$

2. *Assume that $l\beta > \frac{N(p-1)}{N-\alpha p}$. There exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon$ and R_2 such that if $\text{supp}(f), \text{supp}(\mu) \subset B_{R_2}(0)$ and*

$$\|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

then there exists a nonnegative function u , such that $P_{l,a,\beta}(u) \in L^1(\mathbb{R}^N)$ which satisfies

$$u = \mathbf{W}_{\alpha,p}[P_{l,a,\beta}(u) + \mu] + f \quad \text{in } \mathbb{R}^N, \quad (1.31)$$

and

$$u \leq F := 2c_p \mathbf{W}_{\alpha,p}[\omega_1] + 2c_p \mathbf{W}_{\alpha,p}[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1(\mathbb{R}^N), \quad (1.32)$$

where $\omega_1 = M \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_{R_2}(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$, $\omega_2 = M \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_{R_2}(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu_1$. Conversely, if (1.31) admits a nonnegative solution u and $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ then there exists a positive constant C depending on $N, \alpha, p, l, a, \beta$ such that there holds

$$\int_E P_{l,a,\beta}(u)dx + \int_E P_{l,a,\beta}(f)dx + \mu(E) \leq C \text{cap}_{I_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel} \quad (1.33)$$

Remark 1.6 *If $p = 2$ equation (1.31) is equivalent to*

$$(-\Delta)^\alpha(u - f) = P_{l,a,\beta}(u) + \mu \quad \text{in } \mathbb{R}^N,$$

*since $W_{\alpha,2}[P_{l,a,\beta}(u) + \mu] = I_{2\alpha}[P_{l,a,\beta}(u) + \mu] = I_{2\alpha} * (P_{l,a,\beta}(u) + \mu)$.*

2 Estimates on potentials and Wolff integral equations

We denote by $B_r(a)$ the ball of center a and radius $r > 0$ and $B_r = B_r(0)$; χ_E the characteristic function of E for Borel set E . The next estimates are crucial in the sequel.

Theorem 2.1 1. *There exists a positive constant c_1 , depending only on N, α, p such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > p - 1$, $0 < R \leq \infty$ we have*

$$(c_1 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} (\mathbf{I}_{\alpha p}^R[\mu](x))^{\frac{q}{p-1}} dx \leq \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha, p}^R[\mu](x))^q dx \leq (c_1 q)^q \int_{\mathbb{R}^N} (\mathbf{I}_{\alpha p}^R[\mu](x))^{\frac{q}{p-1}} dx, \quad (2.1)$$

2. *Let $R > 0$. There exists a positive constant c_2 , depending only on N, α, p, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > p - 1$ we have*

$$(c_2 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}} dx \leq \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha, p}^R[\mu](x))^q dx \leq (c_2 q)^q \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}} dx, \quad (2.2)$$

where $\mathbf{G}_{\alpha p}[\mu] := G_{\alpha p} * \mu$ denotes the Bessel potential of order αp of μ .

3. *There exists a positive constant c_3 depending only on N, α, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > 1$ we have*

$$c_3^{-q} \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha}[\mu][\mu](x))^q dx \leq \int_{\mathbb{R}^N} (\mathbf{I}_{\alpha}^R[\mu](x))^q dx \leq c_3^q \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha}[\mu](x))^q dx. \quad (2.3)$$

Proof. Note that $W_{\alpha/2, 2}^R[\mu] = I_{\alpha}^R[\mu]$. We can find proof of (2.3) in [8, Step 3, Theorem 2.3]. By [8, Step 2, Theorem 2.3], there is $c_4 > 0$ such that

$$\int_{\mathbb{R}^N} (\mathbf{W}_{\alpha, p}^R[\mu](x))^q dx \geq c_4^q \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p, R}[\mu](x))^{\frac{q}{p-1}} dx \quad \forall q > 0, 0 < R \leq \infty \text{ and } \mu \in \mathfrak{M}^+(\mathbb{R}^N). \quad (2.4)$$

We recall that $\mathbf{M}_{\alpha p, R}[\mu] = \mathbf{M}_{\alpha p, R}^0[\mu]$ by (1.10). Next we show that

$$\int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p, R}[\mu](x))^{\frac{q}{p-1}} dx \geq (c_5 q)^{-q} \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha, p}^R[\mu](x))^q dx \quad \forall q > 0, 0 < R \leq \infty \text{ and } \mu \in \mathfrak{M}^+(\mathbb{R}^N), \quad (2.5)$$

for some a positive constant c_4 . Indeed, we denote μ_n by $\chi_{B_n} \mu$ for $n \in \mathbb{N}^*$. By [17, Theorem 1.2] or [8, Proposition 2.2], there exist constants $c_6 = c_6(N, \alpha, p) > 0$, $a = a(\alpha, p) > 0$ and $\varepsilon_0 = \varepsilon(N, \alpha, p)$ such that for all $n \in \mathbb{N}^*$, $t > 0$, $0 < R \leq \infty$ and $0 < \varepsilon < \varepsilon_0$, there holds

$$|\{\mathbf{W}_{\alpha, p}^R \mu_n > 3t\}| \leq c_6 \exp(-a\varepsilon^{-1}) |\{\mathbf{W}_{\alpha, p}^R \mu_n > t\}| + \left| \left\{ (\mathbf{M}_{\alpha p, R} \mu_n)^{\frac{1}{p-1}} > \varepsilon t \right\} \right|.$$

Multiplying by qt^{q-1} and integrating over $(0, \infty)$, we obtain

$$\begin{aligned} \int_0^{\infty} qt^{q-1} |\{\mathbf{W}_{\alpha, p}^R \mu_n > 3t\}| dt &\leq c_6 \exp(-a\varepsilon^{-1}) \int_0^{\infty} qt^{q-1} |\{\mathbf{W}_{\alpha, p}^R \mu_n > t\}| dt \\ &\quad + \int_0^{\infty} qt^{q-1} \left| \left\{ (\mathbf{M}_{\alpha p, R} \mu_n)^{\frac{1}{p-1}} > \varepsilon t \right\} \right| dt, \end{aligned}$$

which implies

$$\varepsilon^q (3^{-q} - c_6 \exp(-a\varepsilon^{-1})) \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu_n](x))^q dx \leq \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R} \mu_n)^{\frac{q}{p-1}} dx.$$

We see that $\sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^q (3^{-q} - c_6 \exp(-a\varepsilon^{-1})) \geq (c_7 q)^{-q}$ for some a constant c_7 which does not depend on q . Therefore, (2.5) follows by Fatou's lemma. Hence, it is easy to obtain (2.1) from (2.4) and (2.5). Then, we obtain (2.2) from (2.1) and (2.3). \blacksquare

The next result is proved in [8].

Theorem 2.2 *Let $\alpha > 0$, $p > 1$, $0 \leq \eta < p - 1$, $0 < \alpha p < N$ and $L > 0$. Set $\delta = \frac{1}{2} \left(\frac{p-1-\eta}{12(p-1)} \right)^{\frac{p-1}{p-1-\eta}} \alpha p \log(2)$. Then there exists $C(L) > 0$, depending on N , α , p , η and L such that for any $R \in (0, \infty]$, $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$, any $a \in \mathbb{R}^N$ and $0 < r \leq L$, there holds*

$$\frac{1}{|B_{2r}(a)|} \int_{B_{2r}(a)} \exp \left(\delta \frac{(\mathbf{W}_{\alpha,p}^R[\mu_{B_r(a)}](x))^{\frac{p-1}{p-1-\eta}}}{\|\mathbf{M}_{\alpha p,R}^\eta[\mu_{B_r(a)}]\|_{L^\infty(B_r(a))}^{\frac{1}{p-1-\eta}}} \right) dx \leq C(L), \quad (2.6)$$

where $\mu_{B_r(a)} = \chi_{B_r(a)} \mu$. Furthermore, if $\eta = 0$, C is independent of L .

Theorem 2.3 *Let $\alpha > 0$, $p > 1$ with $0 < \alpha p < N$, $\beta \geq 1$ and $R > 0$. Assume $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ satisfies*

$$\|\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1 \quad (2.7)$$

We set $\omega = \|\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$. Then there exist positive constants C and δ_0 independent on μ such that

$$\exp \left(\delta_0 (\mathbf{W}_{\alpha,p}^R[\omega])^\beta \right) \in L^1_{loc}(\mathbb{R}^N) \quad \text{and} \quad \left\| \mathbf{W}_{\alpha,p}^R \left[\exp \left(\delta_0 (\mathbf{W}_{\alpha,p}^R[\omega])^\beta \right) \right] \right\|_{L^\infty(\mathbb{R}^N)} \leq C. \quad (2.8)$$

and

$$\mathbf{W}_{\alpha,p}^R \left[\exp \left(\delta_0 (\mathbf{W}_{\alpha,p}^R[\omega])^\beta \right) \right] \leq c \mathbf{W}_{\alpha,p}^R[\omega] \quad \text{in } \mathbb{R}^N \quad (2.9)$$

for some $c > 0$ independent on μ .

Proof. Let δ be as in Theorem 2.2. From (2.7), we have

$$\|\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq 2.$$

Let $x \in \mathbb{R}^N$. Since $\omega(B_t(y)) \leq 2t^{N-\alpha p} h_{\frac{(p-1)(\beta-1)}{\beta}}(t)$, for all $r \in (0, R)$ and $y \in \mathbb{R}^N$ we have

$$\begin{aligned} \mathbf{W}_{\alpha,p}^R[\omega](y) &= \mathbf{W}_{\alpha,p}^r[\omega](y) + \int_r^R \left(\frac{\omega(B_t(y))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha,p}^r[\omega](y) + 2^{\frac{1}{p-1}} \int_{r \wedge 2^{-1}}^{2^{-1}} (-\ln t)^{-\frac{\beta-1}{\beta}} \frac{dt}{t} + 2^{\frac{1}{p-1}} \int_{2^{-1}}^{R \vee 2^{-1}} (-\ln t)^{-\frac{\beta-1}{\beta}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha,p}^r[\omega](y) + c_8 (-\ln(r \wedge 2^{-1}))^{\frac{1}{\beta}} + c_8. \end{aligned}$$

Thus,

$$(\mathbf{W}_{\alpha,p}^R[\omega](y))^\beta \leq 3^{\beta-1}(\mathbf{W}_{\alpha,p}^r[\omega](y))^\beta + c_9 \ln\left(\frac{1}{r \wedge 2^{-1}}\right) + c_9 \quad (2.10)$$

Let $\theta \in (0, 1]$, since $\exp\left(\frac{a+b}{2}\right) \leq \exp(a) + \exp(b)$ for all $a, b \in \mathbb{R}$, we get from (2.10)

$$\begin{aligned} \exp\left(\theta\delta 3^{-\beta}(\mathbf{W}_{\alpha,p}^R[\omega](y))^\beta\right) &\leq \exp\left(\delta(\mathbf{W}_{\alpha,p}^r[\omega](y))^\beta\right) + c_{10} \exp\left(\theta c_{11} \ln\left(\frac{1}{r \wedge 2^{-1}}\right)\right) \\ &\leq \exp\left(\delta(\mathbf{W}_{\alpha,p}^r[\omega](y))^\beta\right) + c_{10} (r \wedge 2^{-1})^{-\theta c_{11}} \end{aligned} \quad (2.11)$$

For $r > 0$, $0 < t \leq r$, $y \in B_r(x)$ we have $B_t(y) \subset B_{2r}(x)$. Thus, $\mathbf{W}_{\alpha,p}^r[\omega] = \mathbf{W}_{\alpha,p}^r[\omega_{B_{2r}(x)}]$ in $B_r(x)$. Then, using (2.6) from Theorem 2.2 with $L = 2R$ we get

$$\int_{B_r(x)} \exp(\delta(\mathbf{W}_{\alpha,p}^r[\omega])^\gamma) = \int_{B_r(x)} \exp(\delta(\mathbf{W}_{\alpha,p}^r[\omega_{B_{2r}(x)}])^\gamma) \leq c_{12} r^N.$$

So, take $\theta = 1 \wedge \frac{\alpha p}{2c_{11}}$ we deduce from (2.11)

$$\begin{aligned} \mathbf{W}_{\alpha,p}^R \left[\exp\left(\theta\delta 3^{-\beta}(\mathbf{W}_{\alpha,p}^R[\omega])^\gamma\right) \right] (x) &\leq \int_0^R \left(c_{12} r^{\alpha p} + c_{13} (r \wedge 2^{-1})^{-\theta c_{11}} r^{\alpha p} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\leq \int_0^R \left(c_{12} r^{\alpha p} + c_{13} (r \wedge 2^{-1})^{-\frac{\alpha p}{2}} r^{\alpha p} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\leq c_{14} \end{aligned}$$

Hence, we get (2.8) with $\delta_0 = \left(1 \wedge \frac{\alpha p}{2c_{11}}\right) \delta 3^{-\beta}$ and we also get (2.9) since $\mathbf{W}_{\alpha,p}^R[\omega] \geq c_{15}$ for some a positive constant $c_{15} > 0$. \blacksquare

We recall that H_l and $P_{l,a,\beta}$ have been defined in (1.11) and (1.12).

Theorem 2.4 *Let $\alpha > 0$, $p > 1$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $0 < \alpha p < N$, $l\beta > \frac{N(p-1)}{N-\alpha p}$ and $R > 0$. Assume that $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ with $\text{supp } \mu \subset B_R(0)$ satisfies*

$$\|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq 1 \quad (2.12)$$

Set $\omega = \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R(0)} + \mu$. Then there exist $C = C(N, \alpha, p, l, \beta, R) > 0$ and $\delta_1 = \delta_1(N, \alpha, p, l, \beta, R) > 0$ such that $H_l\left(\delta_1(\mathbf{W}_{\alpha,p}[\omega])^\beta\right) \in L^1(\mathbb{R}^N)$ and

$$\mathbf{W}_{\alpha,p} \left[H_l\left(\delta_1(\mathbf{W}_{\alpha,p}[\omega])^\beta\right) \right] (x) \leq C \mathbf{W}_{\alpha,p}[\omega](x) \quad \forall x \in \mathbb{R}^N. \quad (2.13)$$

Proof. We have from (2.12)

$$\|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq 2. \quad (2.14)$$

Let $\delta_1 > 0$. Fixed $x \in \mathbb{R}^N$. We split the Wolff potential $\mathbf{W}_{\alpha,p}[\omega]$ into lower and upper parts defined by

$$\mathbf{L}_{\alpha,p}^t[\omega](x) = \int_t^{+\infty} \left(\frac{\omega(B_r(x))}{r^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r},$$

and

$$\mathbf{W}_{\alpha,p}^t[\omega](x) = \int_0^t \left(\frac{\omega(B_r(x))}{r^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Using the convexity property we have

$$H_l \left(\delta_1 (\mathbf{W}_{\alpha,p}[\omega])^\beta \right) \leq H_l \left(\delta_1 2^\beta (\mathbf{L}_{\alpha,p}^t[\omega])^\beta \right) + H_l \left(\delta_1 2^\beta (\mathbf{W}_{\alpha,p}^t[\omega])^\beta \right).$$

Thus,

$$\mathbf{W}_{\alpha,p} \left[H_l \left(\delta_1 (\mathbf{W}_{\alpha,p}[\omega])^\beta \right) \right] (x) \leq c_{16} \int_0^{+\infty} \left(\frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} + c_{16} \int_0^{+\infty} \left(\frac{\omega_t^2(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t},$$

where $d\omega_t^1 = H_l \left(\delta_1 2^\beta (\mathbf{L}_{\alpha,p}^t[\omega])^\beta \right) dx$ and $d\omega_t^2 = H_l \left(\delta_1 2^\beta (\mathbf{W}_{\alpha,p}^t[\omega])^\beta \right) dx$.

We will obtain (2.13) if we prove that

$$\int_0^{+\infty} \left(\frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{17} W_{\alpha,p}[\omega](x), \quad (2.15)$$

and

$$\omega_t^2(B_t(x)) \leq c_{18} \omega(B_{4t}(x)). \quad (2.16)$$

To prove (2.15), we have $B_r(y) \subset B_{2r}(x)$ for $y \in B_t(x)$ and $r \geq t$, so

$$\mathbf{L}_{\alpha,p}^t[\omega](y) \leq \int_t^{+\infty} \left(\frac{\omega(B_{2r}(x))}{r^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = 2^{\frac{N-\alpha p}{p-1}} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x)$$

It follows

$$\omega_t^1(B_t(x)) \leq |B_1(0)| t^N H_l \left(\delta_1 c_{19} \left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^\beta \right)$$

Thus,

$$\int_0^{+\infty} \left(\frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{20} \int_0^\infty A_t(x) dt \quad (2.17)$$

where $A_t(x) = \left(t^{\alpha p} H_l \left(\delta_1 c_{19} \left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^\beta \right) \right)^{\frac{1}{p-1}} \frac{1}{t}$. Since $H_l(s) \leq s^l \exp(s)$ for all $s \geq 0$,

$$\begin{aligned} A_t(x) &\leq c_{21} \left(t^{\alpha p} \left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^{l\beta} \exp \left(\delta_1 c_{19} \left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^\beta \right) \right)^{\frac{1}{p-1}} \frac{1}{t} \\ &= c_{21} t^{\frac{\alpha p}{p-1}-1} \left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^{\frac{l\beta-p+1}{p-1}} \exp \left(\delta_1 c_{22} \left(\mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \right)^\beta \right) \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \end{aligned}$$

Now we estimate $L_{\alpha,p}^{t/2}[\omega]$.

Case 1: $t \in (0, 1)$, from (2.14) we deduce

$$\begin{aligned} L_{\alpha,p}^{t/2}[\omega](x) &= \int_{t/2}^{1/2} \left(\frac{\omega(B_s(x))}{s^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} + \int_{1/2}^{\infty} \left(\frac{\omega(B_s(x))}{s^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\leq c_{23} \int_{t/2}^{1/2} (-\ln(s))^{-1+\frac{1}{\beta}} \frac{ds}{s} + \int_{1/2}^{\infty} \left(\frac{\omega(B_R(0))}{s^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\leq c_{24} (-\ln(t/2))^{\frac{1}{\beta}} \end{aligned}$$

Which implies

$$\begin{aligned} A_t(x) &\leq c_{25} t^{\frac{\alpha p}{p-1}-1} (-\ln(t/2))^{\frac{l\beta-p+1}{\beta(p-1)}} \exp(\delta_1 c_{26} (-\ln(t/2))) \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \\ &= c_{27} t^{\frac{\alpha p}{p-1}-1} (-\ln(t/2))^{\frac{l\beta-p+1}{\beta(p-1)}} t^{-\delta_1 c_{26}} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \end{aligned}$$

We take $\delta_1 \leq \frac{1}{2c_{26}} \left(\frac{\alpha p}{p-1} - 1 \right)$, we get

$$A_t(x) \leq c_{28} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \quad \forall t \in (0, 1). \quad (2.18)$$

Case 2: $t \geq 1$, we have

$$\begin{aligned} L_{\alpha,p}^{t/2}[\omega](x) &\leq \int_{t/2}^{\infty} \left(\frac{\omega(B_R(0))}{s^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &= c_{29} t^{-\frac{N-\alpha p}{p-1}} \end{aligned}$$

thus

$$\begin{aligned} A_t(x) &\leq c_{30} t^{\frac{\alpha p}{p-1}-1} t^{-\frac{(l\beta-p+1)(N-\alpha p)}{(p-1)^2}} \exp\left(\delta_1 c_{31} t^{-\frac{N-\alpha p}{p-1}}\right) \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \\ &\leq c_{32} t^{-1-\gamma} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) \end{aligned}$$

where $\gamma = \frac{1}{p-1} \left(\frac{l\beta(N-\alpha p)}{p-1} - N \right) > 0$.

Therefore, $A_t(x) \leq c_{33} (t \vee 1)^{-1-\gamma} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x)$ for all $t > 0$. Therefore, from (2.17)

$$\int_0^{+\infty} \left(\frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{33} \int_0^{\infty} (t \vee 1)^{-1-\gamma} \mathbf{L}_{\alpha,p}^{t/2}[\omega](x) dt$$

Using Fubini Theorem we get

$$\begin{aligned} \int_0^{+\infty} \left(\frac{\omega_t^1(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} &\leq c_{34} \int_0^{\infty} \int_0^t (s \vee 1)^{-1-\gamma} ds \left(\frac{\omega(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq c_{35} \int_0^{\infty} \left(\frac{\omega(B_t(x))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= c_{35} \mathbf{W}_{\alpha,p}[\mu](x), \end{aligned}$$

which follows (2.15).

To prove (2.16), for $t > 0$, $r \leq t$ and $y \in B_t(x)$ we have $B_r(y) \subset B_{2t}(x)$. Therefore,

$$\omega_t^2(B_t(x)) = \int_{B_t(x)} H_l \left(\delta_1 2^\beta (\mathbf{W}_{\alpha,p}^t[\omega_{B_{2t}(x)}](y))^\beta \right) dy$$

By Theorem 2.2 there exists $c_{36} > 0$ such that for $0 < \delta_1 \leq c_{36}$, $0 < t < 2R$, $x \in \mathbb{R}^N$,

$$\int_{B_{4t}(x)} \exp \left(\delta_1 2^\beta (\mathbf{W}_{\alpha,p}[\omega_{B_{2t}(x)}](y))^\beta \right) dy \leq c_{37} t^N \quad (2.19)$$

We take $0 < \delta_1 \leq c_{36}$.

Case 1: $x \in \overline{B_R(0)}$

If $0 < t < 2R$, from (2.19) we get

$$\omega_t^2(B_t(x)) \leq c_{37} t^N \leq c_{38} \omega(B_{4t}(x)).$$

If $t \geq 2R$, since for any $|y| \geq 2R$,

$$\mathbf{W}_{\alpha,p}[\omega](y) = \int_{|y|/2}^{\infty} \left(\frac{\omega(B_t(y))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{39} \int_{|y|/2}^{\infty} t^{-1-\frac{N-\alpha p}{p-1}} dt \leq c_{40} |y|^{-\frac{N-\alpha p}{p-1}},$$

and thanks to (2.19) we have

$$\begin{aligned} \omega_t^2(B_t(x)) &\leq \int_{B_{2R}(0)} \exp \left(\delta_1 2^\beta (\mathbf{W}_{\alpha,p}[\omega_{B_R(0)}](y))^\beta \right) dy + \int_{B_{2R}(0)^c} H_l \left(\delta_1 2^\beta (\mathbf{W}_{\alpha,p}[\omega](y))^\beta \right) dy \\ &\leq c_{41} R^N + \int_{B_{2R}(0)^c} H_l \left(c_{42} |y|^{-\frac{\beta(N-\alpha p)}{p-1}} \right) dy \\ &\leq c_{43} + c_{43} \int_{B_{2R}(0)^c} |y|^{-\frac{l\beta(N-\alpha p)}{p-1}} dy = c_{43} + c_{44} R^{N-\frac{l\beta(N-\alpha p)}{p-1}} \\ &\leq c_{45} |B_{4t}(x) \cap B_R(0)| \leq c_{46} \omega(B_{4t}(x)). \end{aligned}$$

From this we also have $H_l \left(\delta_1 (\mathbf{W}_{\alpha,p}[\omega])^\beta \right) \in L^1(\mathbb{R}^N)$.

Case 2: $x \in B_R(0)^c$. If $|x| > R + t$ then $\omega_t^2(B_t(x)) = 0$. Now we consider case $R < |x| \leq R + t$. If $0 < t < 2R$, we have $B_t((R - \frac{t}{2})\frac{x}{|x|}) \subset B_{4t}(x) \cap B_R(0)$ thus from (2.19) we get

$$\omega_t^2(B_t(x)) \leq c_{47} t^N = c_{48} \left| B_t \left((R - \frac{t}{2}) \frac{x}{|x|} \right) \right| \leq c_{48} |B_{4t}(x) \cap B_R(0)| \leq c_{49} \omega(B_{4t}(x))$$

If $t \geq 2R$, as in Case 1 we also obtain $\omega_t^2(B_t(x)) \leq c_{50} \omega(B_{4t}(x))$.

Hence, we get (2.16). Therefore, the Theorem follows with $\delta_1 = \frac{1}{2c_{26}} \left(\frac{\alpha p}{p-1} - 1 \right) \wedge c_{36}$. ■

In the next result we obtain estimate on iterative solutions of Wolff integral inequalities.

Theorem 2.5 *Under the assumption as in Theorem 1.5. Let $K > 0$.*

A. *Suppose that $\{u_m\}$ is a sequence of nonnegative functions in \mathbb{R}^N that satisfies*

$$\begin{aligned} u_{m+1} &\leq K \mathbf{W}_{\alpha,p}^{R_1}[P_{l,\alpha,\beta}(u_m) + \mu] + f \quad \forall m \in \mathbb{N}, \\ u_0 &\leq K \mathbf{W}_{\alpha,p}^{R_1}[\mu] + f. \end{aligned} \quad (2.20)$$

Then there exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon, K$ and R_1 such that if

$$\|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$P_{l, a, \beta} (4c_p K \mathbf{W}_{\alpha, p}^{R_1}[\omega_1] + 4c_p K \mathbf{W}_{\alpha, p}^{R_1}[\omega_2] + f) \in L^1_{loc}(\mathbb{R}^N), \quad (2.21)$$

and

$$u_m \leq 2c_p K \mathbf{W}_{\alpha, p}^{R_1}[\omega_1] + 2c_p K \mathbf{W}_{\alpha, p}^{R_1}[\omega_2] + f \quad \forall m \in \mathbb{N}, \quad (2.22)$$

where $\omega_1 = M \|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$, $\omega_2 = M \|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu_1$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$.

B. Assume that $l\beta > \frac{N(p-1)}{N-\alpha p}$ and $\text{supp}(f), \text{supp}(\mu) \subset B_{R_2}(0)$. Suppose that $\{u_m\}$ is a sequence of nonnegative functions in \mathbb{R}^N that satisfies

$$\begin{aligned} u_{m+1} &\leq K \mathbf{W}_{\alpha, p}[P_{l, a, \beta}(u_m) + \mu] + f \quad \forall m \in \mathbb{N}, \\ u_0 &\leq K \mathbf{W}_{\alpha, p}[\mu] + f. \end{aligned} \quad (2.23)$$

Then there exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon, K$ and R_2 such that if

$$\|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$P_{l, a, \beta} (4c_p K \mathbf{W}_{\alpha, p}[\omega_3] + 4c_p K \mathbf{W}_{\alpha, p}[\omega_4] + f) \in L^1(\mathbb{R}^N), \quad (2.24)$$

and

$$u_m \leq 2c_p K \mathbf{W}_{\alpha, p}[\omega_3] + 2c_p K \mathbf{W}_{\alpha, p}[\omega_4] + f \quad \forall m \in \mathbb{N}, \quad (2.25)$$

where $\omega_3 = M \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_{R_2}(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_{R_2}(0)} + \mu$, $\omega_4 = M \|\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_{R_2}(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_{R_2}(0)} + \mu$.

Proof. We will use Theorem 2.3 and 2.4 to prove the Theorem. Here, we only prove A.

Set $c_{a, \varepsilon} = 2 \left(1 - \left(\frac{a}{a+\varepsilon}\right)^{1/\beta}\right)^{-1}$ and $\bar{a} = a(4c_{a, \varepsilon} c_p K)^\beta$. Take $0 < M \leq 1$ and put $\omega_1 = M \|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$, $\omega_2 = M \|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu_1$. We now assume

$$\|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad \|\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]\|_{L^\infty(\mathbb{R}^N)} \leq M.$$

First, we need to show that

$$\mathbf{W}_{\alpha, p}^{R_1} \left[H_l \left(\bar{a} (\mathbf{W}_{\alpha, p}^{R_1}[\omega_{1,2}])^\beta \right) \right] \leq \mathbf{W}_{\alpha, p}^{R_1}[\omega_{1,2}]. \quad (2.26)$$

By Theorem 2.3, then there exist positive constants c and δ_0 depending on μ such that $\exp\left(\delta_0 (\mathbf{W}_{\alpha, p}^R[M^{-1}\omega_{1,2}])^\beta\right) \in L^1_{loc}(\mathbb{R}^N)$ and

$$\mathbf{W}_{\alpha, p}^R \left[\exp\left(\delta_0 (\mathbf{W}_{\alpha, p}^R[M^{-1}\omega_{1,2}])^\beta\right) \right] \leq c \mathbf{W}_{\alpha, p}^R[M^{-1}\omega_{1,2}] \quad \text{in } \mathbb{R}^N.$$

Since $\theta^{-l}H_l(s) \leq H_l(\theta^{-1}s)$ for all $s \geq 0$ and $0 < \theta \leq 1$, thus

$$\begin{aligned} \mathbf{W}_{\alpha,p}^{R_1} \left[M^{-\frac{1}{2}(\frac{\beta l}{p-1}+1)} H_l \left(\delta_0 M^{-\frac{1}{2}(\frac{\beta}{p-1}-\frac{1}{l})} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}])^\beta \right) \right] &\leq \mathbf{W}_{\alpha,p}^{R_1} \left[H_l \left(\delta_0 M^{-\frac{\beta}{p-1}} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}])^\beta \right) \right] \\ &\leq \mathbf{W}_{\alpha,p}^{R_1} \left[\exp \left(\delta_0 (\mathbf{W}_{\alpha,p}^{R_1}[M^{-1}\omega_{1,2}])^\beta \right) \right] \\ &\leq cM^{-\frac{1}{p-1}} \mathbf{W}_{\alpha,p}^R[\omega_{1,2}]. \end{aligned}$$

So,

$$\mathbf{W}_{\alpha,p}^{R_1} \left[H_l \left(\delta_0 M^{-\frac{1}{2}(\frac{\beta}{p-1}-\frac{1}{l})} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}])^\beta \right) \right] \leq cM^{\frac{1}{2(p-1)}(\frac{\beta l}{p-1}-1)} \mathbf{W}_{\alpha,p}^R[\omega_{1,2}]$$

We see that (2.26) will be achieved if we prove

$$\bar{a} \leq \delta_0 M^{-\frac{1}{2}(\frac{\beta}{p-1}-\frac{1}{l})} \quad \text{and} \quad cM^{\frac{1}{2(p-1)}(\frac{\beta l}{p-1}-1)} \leq 1.$$

which is equivalent to

$$M \leq (\delta_0 \bar{a}^{-1})^{\left(\frac{1}{2}(\frac{\beta}{p-1}-\frac{1}{l})\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}(\frac{\beta l}{p-1}-1)\right)^{-1}}.$$

Thus, one can choose $M = 1 \wedge (\delta_0 \bar{a}^{-1})^{\left(\frac{1}{2}(\frac{\beta}{p-1}-\frac{1}{l})\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}(\frac{\beta l}{p-1}-1)\right)^{-1}}$ we obtain (2.26) and $H_l \left(\bar{a} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}])^\beta \right) \in L_{loc}^1(\mathbb{R}^N)$.

Now, we prove (2.22). Clearly, (2.22) holds with $m = 0$. Now, assume that (2.22) holds with $m = n$, we need to prove that

$$u_{n+1} \leq 2c_p K \mathbf{W}_{\alpha,p}^{R_1}[\omega_1] + 2c_p K \mathbf{W}_{\alpha,p}^{R_1}[\omega_2] + f. \quad (2.27)$$

In fact, since (2.22) holds with $m = n$ and the convexity property of $P_{l,a,\beta}$, we have

$$\begin{aligned} P_{l,a,\beta}(u_n) &\leq P_{l,a,\beta}(4c_p K \mathbf{W}_{\alpha,p}^{R_1}[\omega_1] + 4c_p K \mathbf{W}_{\alpha,p}^{R_1}[\omega_2] + f) \\ &\leq P_{l,a,\beta}(4c_{a,\varepsilon} c_p K \mathbf{W}_{\alpha,p}^{R_1}[\omega_1]) + P_{l,\varepsilon,a}(4c_{a,\varepsilon} c_p K \mathbf{W}_{\alpha,p}^{R_1}[\omega_2]) + P_{l,a,\beta} \left(\left(\frac{a+\varepsilon}{a} \right)^{1/\beta} f \right) \\ &= H_l \left(\bar{a} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_1])^\beta \right) + H_l \left(\bar{a} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_2])^\beta \right) + P_{l,a+\varepsilon,\beta}(f). \end{aligned}$$

From this we get (2.21). By the definition of u_{n+1} and sub-additive property of $\mathbf{W}_{\alpha,p}^{R_1}[\cdot]$, we continue

$$\begin{aligned} u_{n+1} &\leq K \mathbf{W}_{\alpha,p}^{R_1} \left[H_l \left(\bar{a} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_1])^\beta \right) + H_l \left(\bar{a} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_2])^\beta \right) + P_{l,a+\varepsilon,\beta}(f) + \mu \right] + f \\ &\leq c_p K \mathbf{W}_{\alpha,p}^{R_1} \left[H_l \left(\bar{a} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_1])^\beta \right) \right] + c_p K \mathbf{W}_{\alpha,p}^{R_1} \left[H_l \left(\bar{a} (\mathbf{W}_{\alpha,p}^{R_1}[\omega_2])^\beta \right) \right] \\ &\quad + c_p K \mathbf{W}_{\alpha,p}^{R_1} [P_{l,a+\varepsilon,\beta}(f)] + c_p K \mathbf{W}_{\alpha,p}^{R_1} [\mu] + f. \end{aligned}$$

It follows (2.27) from (2.26). This completes the proof of the theorem. \blacksquare

Let $P \in C(\mathbb{R}^+)$ be a decreasing positive function; the (α, P) -Orlicz-Bessel capacity of a Borel set $E \subset \mathbb{R}^N$ is defined by (see [1, Sect 2.6])

$$\text{cap}_{G_\alpha, P}(E) = \inf \left\{ \int_{\mathbb{R}^N} P(f) : G_\alpha * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N) \right\},$$

and the (α, P) -Orlicz-Riesz capacity

$$\text{cap}_{I_\alpha, P}(E) = \inf \left\{ \int_{\mathbb{R}^N} P(f) : I_\alpha * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N) \right\}.$$

Theorem 2.6 *Let $\alpha > 0$, $p > 1$, $a > 0$, $c > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p - 1$ and $0 < \alpha p < N$. Let $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$.*

1. *Let $0 < R \leq \infty$. If u is a nonnegative Borel function in \mathbb{R}^N such that $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ and*

$$u(x) \geq cW_{\alpha,p}^R[P_{l,a,\beta}(u) + \mu](x) \quad \forall x \in \mathbb{R}^N \quad (2.28)$$

then,

Case 1: *$R < \infty$, there exists a positive constant C_1 depending on $N, \alpha, p, l, a, \beta, c$ and R such that*

$$\int_E P_{l,a,\beta}(u) dx + \mu(E) \leq C_1 \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.} \quad (2.29)$$

Case 2: *$R = \infty$, there exists a positive constant C_2 depending on $N, \alpha, p, l, a, \beta, c$ such that*

$$\int_E P_{l,a,\beta}(u) dx + \mu(E) \leq C_2 \text{cap}_{I_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.} \quad (2.30)$$

2. *Let Ω be a bounded domain in \mathbb{R}^N and $\delta \in (0, 1)$. If u is a nonnegative Borel function in Ω such that $P_{l,a,\beta}(u) \in L^1_{loc}(\Omega)$ and*

$$u(x) \geq c\mathbf{W}_{\alpha,p}^{\delta d(x, \partial\Omega)}[P_{l,a,\beta}(u) + \mu](x) \quad \forall x \in \Omega, \quad (2.31)$$

then for any a compact set $K \subset \Omega$, there exists a positive constant C_3 depending on $N, \alpha, p, l, a, \beta, c, \delta$ and $\text{dist}(K, \partial\Omega)$ such that

$$\int_E P_{l,a,\beta}(u) dx + \mu(E) \leq C_3 \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel} \quad (2.32)$$

where Q_p^* is the complementary function to Q_p .

Proof. Set $d\omega = P_{l,a,\beta}(u) dx + d\mu$.

1. We have

$$P_{l,a,\beta}(c\mathbf{W}_{\alpha,p}^R[\omega]) dx \leq d\omega \quad \text{in } \mathbb{R}^N.$$

Let M_ω denote the centered Hardy-Littlewood maximal function which is defined for any $f \in L^1_{loc}(\mathbb{R}^N, d\omega)$ by

$$M_\omega f(x) = \sup_{t>0} \frac{\int_{B_t(x)} |f| d\omega}{\omega(B_t(x))}.$$

Thus, if $E \subset \mathbb{R}^N$ is a Borel set, we have

$$\int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{1\beta}{p-1}} P_{l,a,\beta} (c\mathbf{W}_{\alpha,p}^R[\omega]) dx \leq \int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{1\beta}{p-1}} d\omega.$$

Since M_ω is bounded on $L^s(\mathbb{R}^N, d\omega)$, $s > 1$, we deduce from Fefferman's result [11] that

$$\int_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{1\beta}{p-1}} P_{l,a,\beta} (c\mathbf{W}_{\alpha,p}^R[\omega]) dx \leq c_{51}\omega(E),$$

for some constant c_{51} only depends on N and $\frac{1\beta}{p-1}$. Since $M_\omega \chi_E \leq 1$, we derive

$$\begin{aligned} (M_\omega \chi_E(x))^{\frac{1\beta}{p-1}} P_{l,a,\beta} (c\mathbf{W}_{\alpha,p}^R[\omega](x)) &\geq P_{l,a,\beta} \left(c (M_\omega \chi_E(x))^{\frac{1}{p-1}} \mathbf{W}_{\alpha,p}^R[\omega](x) \right) \\ &\geq P_{l,a,\beta} (c\mathbf{W}_{\alpha,p}^R[\omega_E](x)), \end{aligned}$$

where $\omega_E = \chi_E \omega$.

So,

$$\int_{\mathbb{R}^N} P_{l,a,\beta} (c\mathbf{W}_{\alpha,p}^R[\omega_E]) dx \leq c_{51}\omega(E) \quad \text{for all Borel set } E \subset \mathbb{R}^N. \quad (2.33)$$

From (2.1), (2.2) and (2.3) we get

$$\int_{\mathbb{R}^N} P_{l,a,\beta} (c\mathbf{W}_{\alpha,p}^R[\omega_E](x)) dx \geq \int_{\mathbb{R}^N} Q_p (c_{52}\mathbf{G}_{\alpha p}[\omega_E](x)) dx \quad \text{if } R < \infty$$

and

$$\int_{\mathbb{R}^N} P_{l,a,\beta} (c\mathbf{W}_{\alpha,p}^R[\omega_E](x)) dx \geq \int_{\mathbb{R}^N} Q_p (c_{53}\mathbf{I}_{\alpha p}[\omega_E](x)) dx \quad \text{if } R = \infty$$

where Q_p is defined by (1.13) and $c_{52} = (c_2\beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$ if $p \neq 2$, $c_{52} = c_3^{-1} a^{\frac{1}{\beta}} c$ if $p = 2$ (the constant c_2 defined in (2.2), depends on R , therefore $c_{52} = c_{52}(r_K)$) and $c_{53} = (c_1\beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$ if $p \neq 2$, $c_{53} = a^{\frac{1}{\beta}} c$ if $p = 2$. Thus, from (2.33) we obtain for all Borel set $E \subset \mathbb{R}^N$

$$\int_{\mathbb{R}^N} Q_p (c_{52}\mathbf{G}_{\alpha p}[\omega_E](x)) dx \leq c_{51}\omega(E) \quad \text{if } R < \infty,$$

and

$$\int_{\mathbb{R}^N} Q_p (c_{53}\mathbf{I}_{\alpha p}[\omega_E](x)) dx \leq c_{51}\omega(E) \quad \text{if } p = \infty.$$

We recall that $Q_p^*(s) = \sup_{t>0} \{st - Q_p(t)\}$ and we note that Q_p^* satisfies the sub-additivity Δ_2 -condition (see Chapter 2 in [19]).

Case 1. $R < \infty$. For every $f \geq 0$, $Q_p^*(f) \in L^1(\Omega)$ such that $G_{\alpha p} * f \geq \chi_E$, we have

$$\begin{aligned} \omega(E) &\leq \int_{\mathbb{R}^N} G_{\alpha p} * f d\omega_E = (2c_{51})^{-1} \int_{\mathbb{R}^N} (c_{52} \mathbf{G}_{\alpha p} [\omega_E]) (2c_{51} c_{52}^{-1} f) dx \\ &\leq (2c_{51})^{-1} \int_{\mathbb{R}^N} Q_p (c_{52} \mathbf{G}_{\alpha p} [\omega_E]) dx + (2c_{51})^{-1} \int_{\mathbb{R}^N} Q_p^* (2c_{51} c_{52}^{-1} f) dx \\ &\leq 2^{-1} \omega(E) + c_{54} \int_{\mathbb{R}^N} Q_p^* (f) dx, \end{aligned}$$

the last inequality following from the Δ_2 -condition. Notice that c_{54} , as well as the next constant c_{55} , depends on r_K . Thus,

$$\omega(E) \leq 2c_{54} \int_{\mathbb{R}^N} Q_p^* (f) dx.$$

Then, we get

$$\omega(E) \leq c_{55} \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \text{for all Borel set } E \subset \mathbb{R}^N.$$

Which implies (2.29).

Case 2. $R = \infty$. For every $f \geq 0$, $Q_p^*(f) \in L^1(\Omega)$ such that $I_{\alpha p} * f \geq \chi_E$, since $I_{\alpha p} * \omega_E = I_{\alpha p}[\omega_E]$, as above we have

$$\begin{aligned} \omega(E) &\leq \int_{\mathbb{R}^N} I_{\alpha p} * f d\omega_E = \int_{\mathbb{R}^N} (\mathbf{I}_{\alpha p} * \omega_E) f dx = \int_{\mathbb{R}^N} \mathbf{I}_{\alpha p} [\omega_E] f dx \\ &\leq 2^{-1} \omega(E) + c_{56} \int_{\mathbb{R}^N} Q_p^* (f) dx, \end{aligned}$$

Then, it follows (2.30).

2. Let $K \subset \subset \Omega$. Set $r_K = \text{dist}(K, \partial\Omega)$ and $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$. We have

$$P_{l,a,\beta} \left(c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega] \right) dx \leq d\omega \quad \text{in } \Omega.$$

Thus, for any Borel set $E \subset K$

$$\int_{\Omega} (M_{\omega} \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta} \left(c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega] \right) dx \leq \int_{\Omega} (M_{\omega} \chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

As above we get

$$\int_{\Omega} P_{l,a,\beta} \left(c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \right) dx \leq c_{51} \omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.34)$$

Note that if $x \in \Omega$ and $d(x, \partial\Omega) \leq r_K/8$, then $B_t(x) \subset \Omega \setminus \Omega_K$ for all $t \in (0, \delta d(x, \partial\Omega))$; indeed, for all $y \in B_t(x)$

$$d(y, \partial\Omega) \leq d(x, \partial\Omega) + |x - y| < (1 + \delta)d(x, \partial\Omega) < \frac{1}{4}r_K,$$

thus

$$d(y, K) \geq d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4}r_K > \frac{1}{2}r_K,$$

which implies $y \notin \Omega_K$. We deduce that

$$\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \geq \mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) \quad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) = 0 \quad \forall x \in \Omega^c.$$

Hence we obtain from (2.34),

$$\int_{\mathbb{R}^N} P_{l,a,\beta} \left(c \mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) \right) dx \leq c_{51} \omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.35)$$

As above we also obtain

$$\omega(E) \leq c_{57} \text{cap}_{G_{\alpha,p}, Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel.}$$

where the positive constant c_{57} depends on r_K . It follows (2.32). ■

This completes the proof of the Theorem. ■

Now we begin to prove Theorem 1.5.

Proof of Theorem 1.5. Here we only prove the part **A**. Consider the sequence $\{u_m\}_{m \geq 0}$ of nonnegative functions defined by $u_0 = f$ and

$$u_{m+1} = \mathbf{W}_{\alpha,p}^{R_1}[P_{l,a,\beta}(u_m)] + f \quad \text{in } \mathbb{R}^N \quad \forall m \geq 0.$$

By Theorem 2.5, $\{u_m\}_{m \geq 0}$ is well defined and (2.21) and (2.22) are satisfied. It is easy to see that u_m is nondecreasing. Hence, thanks to the dominated convergence theorem, we obtain that $u(x) = \lim_{m \rightarrow \infty} u_m(x)$ is a solution to equation (??) which satisfies (??). ■

Conversely, we get the converse directly from Theorem 2.6. ■

3 Quasilinear Dirichlet Problems

Let Ω be a bounded domain in \mathbb{R}^N . If $\mu \in \mathfrak{M}_b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative part in the Jordan decomposition. We denote by $\mathfrak{M}_0(\Omega)$ the space of measures in Ω which are absolutely continuous with respect to the $c_{1,p}^\Omega$ -capacity defined on a compact set $K \subset \Omega$ by

$$c_{1,p}^\Omega(K) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega) \right\}.$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c_{1,p}^\Omega$ -capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. It is well known that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ can be written under the form $\mu_0 = f - \operatorname{div} g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega)$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. If u is a measurable function defined and finite a.e. in Ω , such that $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \rightarrow \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u| \leq k} v$ a.e. in Ω and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [10].

Definition 3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.1)$$

if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist λ_k^+ and λ_k^- belonging to $\mathfrak{M}_b^+ \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\mu_k^+ \rightarrow \mu_s^+$, $\mu_k^- \rightarrow \lambda_s^-$ in the narrow topology of measures and such that

$$\int_{\{|u| < k\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\{|u| < k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda_k^+ - \int_{\Omega} \varphi d\lambda_k^-,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2 *We recall that if u is a renormalized solution to problem (3.1) then $\frac{|\nabla u|^p}{(|u|^+)^r} \in L^1(\Omega)$ for all $r > 1$ which deduce from Holder's inequality, $u \in W_0^{1,p_1}(\Omega)$ for all $1 \leq p_1 < p$ if $e^{a|u|} \in L^1(\Omega)$ for some $a > 0$. Furthermore, $u \geq 0$ a.e. in Ω if $\mu \in \mathfrak{M}_b^+(\Omega)$.*

The following general stability result has been proved in [10, Th 4.1, Sec 5.1].

Theorem 3.3 *If $\{\mu_n\}$ is a bounded sequence in $\mathfrak{M}_b(\Omega)$ and $\{u_n\}$ a sequence of renormalized solutions of problem*

$$\begin{aligned} -\Delta_p u_n &= \mu_n && \text{in } \Omega, \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.2)$$

then, up to a subsequence, $\{u_n\}$ converges a.e. to a solution of equation $-\Delta_p u = \mu$ in $\mathcal{D}'(\Omega)$, and such that $\frac{1}{k} \int_{\Omega} |\nabla T_k(u)|^p dx \leq M$ for every $k > 0$.

The next result is a sharp extension of the stability Theorem 3.3.

Theorem 3.4 *Let $\mu = \mu_0 + \mu_s^+ - \mu_s^-$, with $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$ and μ_s^+, μ_s^- belonging to $\mathfrak{M}_s^+(\Omega)$. Let $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$ with $F_n \in L^1(\Omega)$, $g_n \in (L^{p'}(\Omega))^N$ and ρ_n, η_n belonging to $\mathfrak{M}_b^+(\Omega)$. Assume that $\{F_n\}$ converges to F weakly in $L^1(\Omega)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(\Omega))^N$ and $(\operatorname{div} g_n)$ is bounded in $\mathfrak{M}_b(\Omega)$; assume also that $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ to μ_s^- in the narrow topology. If $\{u_n\}$ is a sequence of renormalized solutions of (3.2), then, up to a subsequence, it converges a.e. in Ω to a renormalized solution u of problem (3.1). Furthermore $T_k(u_n)$ converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$ for any $k > 0$.*

We also recall the following estimate [20, Th 2.1].

Theorem 3.5 *Let Ω be a bounded domain of \mathbb{R}^N . Then there exists a constant $K_1 > 0$, depending on p and N such that if $\mu \in \mathfrak{M}_b^+(\Omega)$ and u is a nonnegative renormalized solution of problem (3.1) with data μ , there holds*

$$\frac{1}{K_1} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \leq u(x) \leq K_1 \mathbf{W}_{1,p}^{2\text{diam}(\Omega)}[\mu](x) \quad \forall x \in \Omega, \quad (3.3)$$

where the positive constant K_1 only depends on N, p .

Proof of Theorem 1.1. Let $\{u_m\}_{m \in \mathbb{N}}$ be a sequence of nonnegative renormalized solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu && \text{in } \Omega, \\ u_0 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu && \text{in } \Omega, \\ u_{m+1} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By Theorem 3.5 we have

$$\begin{aligned} \chi_\Omega u_0 &\leq K_1 \mathbf{W}_{1,p}^R[\mu], \\ \chi_\Omega u_{m+1} &\leq K_1 \mathbf{W}_{1,p}^R[P_{l,a,\beta}(u_m) + \mu] \quad \forall m \in \mathbb{N}, \end{aligned}$$

where $R = 2\text{diam}(\Omega)$. Thus, by Theorem 2.5 with $f \equiv 0$, there exists $M > 0$ depending on N, p, l, a, β, K_1 and R such that $P_{l,a,\beta}(4c_p K_1 \mathbf{W}_{1,p}^R[\omega]) \in L^1(\Omega)$ and

$$u_m(x) \leq 2c_p K_1 \mathbf{W}_{1,p}^R[\omega](x) \quad \forall x \in \Omega, m \in \mathbb{N}, \quad (3.4)$$

provided that

$$\|M_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

where $\omega = M \|M_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$. This implies that $\{u_m\}$ is well defined; by Theorem 3.3 it contains a subsequence that we still denote by $\{u_m\}$ which converges a.e in Ω to function u for which (1.17) is satisfied in Ω . Furthermore, we deduce from (3.4) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 3.4 we obtain that u is a renormalized solution of (1.16).

Conversely, assume that (1.16) admits a nonnegative renormalized solution u . By Theorem 3.5 there holds

$$u(x) \geq \frac{1}{K_1} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \Omega.$$

Hence, we achieve (1.18) from Theorem 2.6. ■

Applications. We consider the case $p = 2, \beta = 1$. Then $l = 2$ and

$$P_{l,a,\beta}(r) = e^{ar} - 1 - ar.$$

If Ω is a bounded domain in \mathbb{R}^N , there exists $M > 0$ such that if μ is a positive Radon measure in Ω which satisfies

$$\mu(B_t(x)) \leq Mt^{N-2} \quad \forall t > 0 \text{ and almost all } x \in \Omega,$$

there exists a positive solution u to the following problem

$$\begin{aligned} -\Delta u &= e^{au} - 1 - au + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Furthermore

$$u(x) \leq K(N) \int_0^{2 \operatorname{diam} \Omega} \frac{\omega(B_t(x))}{t^{N-1}} dt = K(N) \int_0^{2 \operatorname{diam} \Omega} \frac{\mu(B_t(x))}{t^{N-1}} dt + b \quad \forall x \in \Omega.$$

where $b = 2K(N)M \|\mathbf{M}_{p, 2 \operatorname{diam} \Omega}^{(\frac{p-1}{\beta})(\beta-1)}[1]\|_{L^\infty(\mathbb{R}^N)}^{-1} (\operatorname{diam} \Omega)^2$.

In the case $N = 2$ this result has already been proved by Richard and Véron [22, Prop 2.4].

4 p-Superharmonic Functions And Quasilinear Equations on \mathbb{R}^N

We recall some definitions and properties of p -superharmonic functions.

If $u \in W_{loc}^{1,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $-\Delta_p u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$ then u is said to be p -harmonic in \mathbb{R}^N . A function u is called an p -supersolution in \mathbb{R}^N if $u \in W_{loc}^{1,p}(\mathbb{R}^N)$ and $-\Delta_p u \geq 0$ in $\mathcal{D}'(\mathbb{R}^N)$.

Definition 4.1 *A Lower semicontinuous (l.s.c) function $u : \mathbb{R}^N \rightarrow (-\infty, \infty]$ is called p -superharmonic if u is not identically infinite and if for all open $D \subset \subset \mathbb{R}^N$ and all $v \in C(\overline{D})$, p -harmonic in D , $v \leq u$ on ∂D implies $v \leq u$ in D .*

Let u be an p -superharmonic in \mathbb{R}^N . It is well known that $u \wedge k \in W_{loc}^{1,p}(\mathbb{R}^N)$ is a p -supersolution for all $k > 0$ and $u < \infty$ a.e in \mathbb{R}^N . Thus, u has gradient, see previous section. We also have $|\nabla u|^{p-1} \in L_{loc}^q(\mathbb{R}^N)$, $\frac{|\nabla u|^p}{(|u|+1)^r} \in L_{loc}^1(\mathbb{R}^N)$ and $u \in L_{loc}^s(\mathbb{R}^N)$ for $1 \leq q < \frac{N}{N-1}$ and $r > 1$, $1 \leq s < \frac{N(p-1)}{N-p}$ see [14, Theorem 7.46]. In particular, thanks to Holder's inequality we have if $e^{a|u|} \in L_{loc}^1(\mathbb{R}^N)$ for some $a > 0$, then $u \in W_{loc}^{1,p_1}(\mathbb{R}^N)$ for all $1 \leq p_1 < p$. Thus by the dominated convergence theorem,

$$\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla(u \wedge k)|^{p-2} \nabla(u \wedge k) \nabla \varphi \geq 0$$

Hence, by the Riesz Representation Theorem we conclude that there is a nonnegative Radon measure denoted by $\mu[u]$, is called Riesz measure such that $-\Delta_p u = \mu[u]$ in $\mathcal{D}'(\mathbb{R}^N)$.

The following weak convergence result for Riesz measures proved in [28] will be used to prove the existence of p -superharmonic solutions to quasilinear equations.

Theorem 4.2 *Suppose that $\{u_n\}$ is a sequence of nonnegative p -superharmonic function in \mathbb{R}^N that converges a.e to an p -supersolution function u . Then the sequence of measures $\{\mu[u_n]\}$ converges to $\mu[u]$ in the weak sense of measure.*

The next theorem is proved in [20]

Theorem 4.3 *Let μ be a measure in $\mathcal{M}^+(\mathbb{R}^N)$. Suppose that $\mathbf{W}_{1,p}[\mu] < \infty$ a.e. Then there is a nonnegative p -superharmonic function u in \mathbb{R}^N such that $-\Delta_p u = \mu$ in $\mathcal{D}'(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} u = 0$ and*

$$\frac{1}{K_1} \mathbf{W}_{1,p}[\mu](x) \leq u(x) \leq K_1 \mathbf{W}_{1,p}[\mu](x) \quad (4.1)$$

for all x in \mathbb{R}^N , where the constant K_1 is as in Theorem 3.5. Furthermore, any p -superharmonic function u in \mathbb{R}^N , $\inf_{\mathbb{R}^N} u = 0$ then (4.1) holds with $\mu = -\Delta_p u$.

Proof of Theorem (1.2). Let $\{u_m\}_{m \in \mathbb{N}}$ be a sequence of p -superharmonic solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu \text{ in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_0 &= 0, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu \text{ in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_{m+1} &= 0. \end{aligned}$$

By Theorem 4.3 we have

$$\begin{aligned} u_0 &\leq K_1 \mathbf{W}_{1,p}[\mu], \\ u_{m+1} &\leq K_1 \mathbf{W}_{1,p}[P_{l,a,\beta}(u_m) + \mu] \quad \forall m \in \mathbb{N}. \end{aligned}$$

Thus, by Theorem 2.5 with $f \equiv 0$, there exists $M > 0$ depending on N, p, l, a, β, K_1 and R such that $P_{l,a,\beta}(4c_p K_1 \mathbf{W}_{1,p}[\omega]) \in L^1(\mathbb{R}^N)$ and

$$u_m \leq 2c_p K_1 \mathbf{W}_{1,p}[\omega] \quad \forall m \in \mathbb{N}, \quad (4.2)$$

provided that

$$\|M_p^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M.$$

where $\omega = M \|M_p^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1} \chi_{B_R(0)} + \mu$. This implies that $\{u_m\}$ is well defined; by [15, Theorem 1.17] it contains a subsequence that we still denote by $\{u_m\}$ which converges a.e in \mathbb{R}^N to function u for which (1.20) is satisfied in Ω . Furthermore, we deduce from (4.2) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$ in $L^1(\mathbb{R}^N)$. Finally, by Theorem 4.2 we obtain that u is an p -superharmonic solution of (1.19).

Conversely, assume that (1.19) admits a nonnegative renormalized solution u . By Theorem 4.3 there holds

$$u(x) \geq \frac{1}{K_1} \mathbf{W}_{1,p}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \mathbb{R}^N.$$

Hence, we obtain (1.21) from Theorem 2.6. ■

5 Hessian equations

In this section $\Omega \subset \mathbb{R}^N$ is either a bounded domain with a C^2 boundary or the whole \mathbb{R}^N . For $k = 1, \dots, N$ and $u \in C^2(\Omega)$ the k -hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivative D^2u and S_k is the k -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}$$

We can see that

$$F_k[u] = [D^2u]_k,$$

where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the k -th principal minors. We assume that $\partial\Omega$ is uniformly $(k-1)$ -convex that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant c_0 , where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal.

Definition 5.1 *An upper-semicontinuous function $u : \Omega \rightarrow [-\infty, \infty)$ is k -convex (k -subharmonic) if, for every open set $\Omega' \subset \overline{\Omega}' \subset \Omega$ and for every function $v \in C^2(\Omega') \cap C(\overline{\Omega})$ satisfying $F_k[v] \leq 0$ in Ω' , the following implication is true*

$$u \leq v \text{ on } \partial\Omega' \implies u \leq v \text{ in } \Omega'.$$

We denote by $\Phi^k(\Omega)$ the class of all k -subharmonic functions in Ω which are not identically equal to $-\infty$.

The following weak convergence result for k -Hessian operators proved in [26] is fundamental in our study.

Theorem 5.2 *Let Ω be either a bounded uniformly $(k-1)$ -convex in \mathbb{R}^N or the whole \mathbb{R}^N . For each $u \in \Phi^k(\Omega)$, there exist a nonnegative Radon measure $\mu_k[u]$ in Ω such that*

- a. $\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$.
- b. If $\{u_n\}$ is a sequence of k -convex functions which converges to u a.e then $\mu_k[u_n] \rightharpoonup \mu_k[u]$ in the weak sense of measures.

As in the case of quasilinear equations with measure data, precise estimates of solutions of k -Hessian equations with measures data are expressed in terms of Wolff potentials. The next results are proved in [26, 18, 20].

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 , uniformly $(k-1)$ -convex domain. Let φ be a nonnegative continuous function on $\partial\Omega$ and μ be a nonnegative Radon measure. Suppose that μ can be decomposed as*

$$\mu = \mu_1 + f$$

such that μ_1 is a measure with compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$ if $k \leq \frac{N}{2}$ or $p = 1$ if $k > \frac{N}{2}$. Then there exists a nonnegative function u in Ω such that $-u \in \Phi^k(\Omega)$, continuous near $\partial\Omega$ and u is a solution to the problem

$$\begin{aligned} F_k[-u] &= \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega. \end{aligned}$$

Furthermore, for any nonnegative function u such that $-u \in \Phi^k(\Omega)$, which is continuous near $\partial\Omega$ and is a solution of above equation, there holds

$$\frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}}[\mu] \leq u(x) \leq K_2 \left(\mathbf{W}_{\frac{2k}{k+1}, k+1}^{2diam\Omega}[\mu](x) + \max_{\partial\Omega} \varphi \right), \quad (5.1)$$

where K_2 is a positive constant independent of x, u and Ω .

Theorem 5.4 Let μ be a measure in $\mathcal{M}^+(\mathbb{R}^N)$ and $2k < N$. Suppose that $\mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu] < \infty$ a.e. Then there exists $u, -u \in \Phi^k(\mathbb{R}^N)$ such that $\inf_{\mathbb{R}^N} u = 0$ and $F_k[-u] = \mu$ in \mathbb{R}^N and

$$\frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x) \leq u(x) \leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}[\mu](x) \quad (5.2)$$

for all x in \mathbb{R}^N , where the constant K_2 is in above Theorem.

Furthermore, for any nonnegative function u such that $-u \in \Phi^k(\mathbb{R}^N)$ then (5.2) holds with $\mu = F_k[-u]$.

Proof of Theorem 1.3. We defined a sequence of nonnegative functions u_m , continuous near $\partial\Omega$ and such that $-u_m \in \Phi^k(\Omega)$, by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu && \text{in } \Omega, \\ u_0 &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (5.3)$$

and, for $m \geq 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l,a,\beta}(u_m) + \mu && \text{in } \Omega, \\ u_{m+1} &= \varphi && \text{on } \partial\Omega. \end{aligned} \quad (5.4)$$

By Theorem 5.3 we have

$$\begin{aligned} \chi_\Omega u_0 &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu] + b_0, \\ \chi_\Omega u_{m+1} &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[P_{l,a,\beta}(u_m) + \mu] + b_0, \end{aligned} \quad (5.5)$$

where $b_0 = K_2 \max_{\partial\Omega} \varphi$ and $R = 2diam\Omega$. Thus, by Theorem 2.5 with $f = b_0$ and $\varepsilon = a$, there exists $M_1 > 0$ depending on N, k, l, a, β and R such that $P_{l,a,\beta} \left(4K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega_1] + g \right) \in L^1(\Omega)$ and

$$u_m(x) \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\omega_1](x) + g(x) \quad \forall x \in \Omega, \forall m \geq 0, \quad (5.6)$$

provided that

$$\|M_{2k,R}^{\frac{k(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \quad \text{and} \quad \|M_{2k,R}^{\frac{k(\beta-1)}{\beta}}[P_{l,2a,\beta}(b_0)]\|_{L^\infty(\mathbb{R}^N)} \leq M_1.$$

where $\omega_1 = M \|\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}} [1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$, $\omega_2 = M \|\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}} [1]\|_{L^\infty(\mathbb{R}^N)}^{-1} + P_{l,2a,\beta}(b_0)$

and $g(x) = 4c_p K \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\omega_2](x) + b_0$.

We see that one can find constants M_2 depending on N, k, l, a, β, R and M_1 such that if $\max_{\partial\Omega} \varphi \leq M_2$ then $\|M_{2k,R}^{\frac{k(\beta-1)}{\beta}} [P_{l,2a,\beta}(b_0)]\|_{L^\infty(\mathbb{R}^N)} \leq M_1$. Beside, g is constant. Hence, we deduce from (5.6) that $P_{l,a,\beta} \left(2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\mu] + b_2 \right) \in L^1(\Omega)$ and

$$u_m(x) \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\mu](x) + b_2 \quad \forall x \in \Omega, \forall m \geq 0, \quad (5.7)$$

for some the constant b_2 depending on N, k, l, a, β, R and M_1 .

Note that because we can write

$$\omega = P_{l,a,\beta}(u_m) + \mu = (\mu_1 + \chi_{\Omega_\delta} P_{l,a,\beta}(u_m)) + ((1 - \chi_{\Omega_\delta}) P_{l,a,\beta}(u_m) + f),$$

where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$ and $\delta > 0$ is small enough, then ω satisfies the assumptions of the data in Theorem 5.3 since u_m is continuous near $\partial\Omega$. Therefore the sequence $\{u_m\}$ is well defined. Since $-u_m$ is k -subharmonic and the sequence $\{u_m\}$ is bounded in $L^1(\Omega)$, the sequence $\{u_m\}$ is also bounded in $W_{loc}^{1,1}(\Omega)$ (see e.g. [26]); we can find a subsequence, still denoted by the index m , such that u_m converges a.e in Ω to function u for which (1.23) is satisfied in Ω with the constant $b_2 = g$. Furthermore, we deduce from (5.6) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 5.2, we obtain that u satisfies (1.22) and (1.23).

Conversely, assume that (1.22) admits nonnegative solution u , continuous near $\partial\Omega$, such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$. Then by Theorem 5.3 we have

$$u(x) \geq \frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}} [P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \Omega.$$

Using Theorem 2.6, we conclude that (1.24) holds. ■

Proof of Theorem 1.4. we defined a sequence of nonnegative functions u_m , continuous near $\partial\Omega$ and such that $-u_m \in \Phi^k(\Omega)$, by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu & \text{in } \mathbb{R}^N, \\ \inf_{\mathbb{R}^N} u_0 &= 0 \end{aligned} \quad (5.8)$$

and, for $m \geq 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l,a,\beta}(u_m) + \mu & \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_{m+1} &= 0. \end{aligned} \quad (5.9)$$

By Theorem 5.4, we have

$$\begin{aligned} u_0 &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\mu], \\ u_{m+1} &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [P_{l,a,\beta}(u_m) + \mu]. \end{aligned} \quad (5.10)$$

Thus, by Theorem 2.5 with $f \equiv 0$ there exists $M > 0$ depending on N, k, l, a, β and R such that $P_{l,a,\beta} \left(2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\omega] \right) \in L^1(\mathbb{R}^N)$

$$u_m \leq 2K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\omega] \quad \forall m \geq 0, \quad (5.11)$$

provided that $\|M_{2k,R}^{\frac{k(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M$, where $\omega = M\|\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]\|_{L^\infty(\mathbb{R}^N)}^{-1}\chi_{B_R(0)} + \mu$. Therefore the sequence $\{u_m\}$ is well defined. By arguing as in the proof of theorem 1.3 we obtain that u satisfies (1.25) and (1.26). Conversely, assume that (1.25) admits nonnegative solution u and $-u \in \Phi^k(\mathbb{R}^N)$, $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$. Then by Theorem 5.4 we have

$$u \geq \frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}[P_{l,a,\beta}(u) + \mu]$$

Using Theorem 2.6, we conclude that (1.27) holds. ■

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