Quasilinear and Hessian type equations with exponential reaction and measure data

Hung Nguyen Quoc, Laurent Veron

Quasilinear and Hessian type equations with exponential reaction and measure data

Nguyen Quoc Hung*
Laurent Véron†

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Contents

1 Introduction **2**

2 Estimates on potentials and Wolff integral equations **8**

3 Quasilinear Dirichlet Problems **19**

4 p-Superharmonic Funtions And Quasilinear Equations on $\mathbb{R}^N$ **22**

5 Hessian equations **24**

Abstract

We prove existence results concerning equations of the type $-\Delta_p u = P(u) + \mu$ for $p > 1$ and $F_k[-u] = P(u) + \mu$ with $1 \leq k < \frac{N}{2}$ in a bounded domain Ω, where μ is a positive Radon measure and $P(u) \sim e^{au^\beta}$ with $a > 0$ and $\beta \geq 1$. Sufficient conditions for existence are expressed in terms of the fractional maximal potential of μ. Two-sided estimates on the solutions are obtained in terms of some precise Wolff potentials of μ. Necessary conditions are obtained in terms of Orlicz capacities. We also establish existence results for a general Wolff potential equation under the form $u = \mathbf{W}^R_{\alpha,p}[P(u)] + f$.

2010 Mathematics Subject Classification. 31C15, 32 F10, 35J92, 35R06, 46E30.

Key words: quasilinear elliptic equations, Hessian equations, Wolff potential, maximal functions, Borel measures, Orlicz capacities.

*E-mail address: Hung.Nguyen-Quoc@lmpt.univ-tours.fr
†E-mail address: Laurent.Veron@lmpt.univ-tours.fr

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be either a bounded domain or the whole $\mathbb{R}^N$, $p > 1$ and $k \in \{1, 2, ..., N\}$. We denote by

$$\Delta_p u := div\left(|\nabla u|^{p-2}\nabla u\right)$$

the p-Laplace operator and by

$$F_k[u] = \sum_{1\le j_1<j_2<...<j_k\le N} \lambda_{j_1}\lambda_{j_2}...\lambda_{j_k}$$

the k-Hessian operator where $\lambda_1, ..., \lambda_N$ are the eigenvalues of the Hessian matrix D^2u. Let μ be a positive Radon measure in Ω; our aim is to study the existence of positive renormalized solutions to the quasilinear equation

$$\begin{array}{rl} -\Delta_p u = P(u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{1.1}$$

and the fully nonlinear equation

$$\begin{array}{rl} F_k[-u] = P(u) + \mu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{array} \tag{1.2}$$

where P is an exponential function. When $P(r) = r^q$ with $q > p - 1$, Phuc and Verbitsky published a seminal article [20] on the solvability of the corresponding problem (1.1). They obtained necessary and sufficient conditions involving Bessel capacities or Wolff potentials. For example, assume that Ω is bounded, they proved that if μ has compact support in Ω it is equivalent to solve (1.1) with $P(r) = r^q$ or to have

$$\mu(E) \le cC_{p,\frac{p}{q+1-p}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{1.3}$$

where c is a suitable positive constant and $C_{p,\frac{p}{q+1-p}}$ a Bessel capacity, or to have

$$\int_B \left[\mathbf{W}^{2R}_{1,p}[\mu_B(x)]\right]^q dx \le C\mu(B) \qquad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \ne \emptyset, \tag{1.4}$$

where $R = \text{diam}(\Omega)$. Other conditions are expressed in terms of Riesz potentials and maximal fractional potentials. Their construction is based upon sharp estimates of solutions of the non-homogeneous problem

$$\begin{array}{rl} -\Delta_p u = \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{1.5}$$

for positive measures ω. We refer to [4, 5, 6, 7, 9, 13, 23] for the previous studies of these and other related results. Concerning k-Hessian operator in a bounded $(k-1)$-convex domain Ω, they proved that if μ has compact support and $||\varphi||_{L^\infty(\partial\Omega)}$ is small enough, the corresponding problem (1.2) with $P(r) = r^q$ with $q > k$ admits a positive solution if and only if

$$\mu(E) \le cC_{2k,\frac{q}{q-k}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{1.6}$$

or equivalently

$$\int_B \left[\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}[\mu_B(x)]\right]^q dx \le C\mu(B) \qquad \text{for all ball } B \text{ s.t. } B\cap \text{supp}\mu \neq \emptyset. \tag{1.7}$$

In the linear case $p = 2$ and $k = 1$, can be found in [2, 3, 29]. The main tools in their proofs are derived from recent advances in potential theory for nonlinear elliptic equations obtained by Kilpelainen and Malý [15, 16], Trudinger and Wang [25, 26, 27], and Labutin [18] thanks to whom the authors first provide global pointwise estimates for solutions of the homogeneous Dirichlet problems in terms of Wolffs potentials of suitable order.

For $s > 1$, $0 \le \alpha < \frac{N}{s}$, $\eta \ge 0$ and $0 < T \le \infty$, we recall the expression of the *T-truncated Wolff potential* of a positive Radon measure μ defined in $\mathbb{R}^N$ by

$$\mathbf{W}^T_{\alpha,s}[\mu](x) = \int_0^T \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t}, \tag{1.8}$$

and *T-truncated Riesz potential* of a positive Radon measure μ by

$$\mathbf{I}^T_\alpha[\mu](x) = \int_0^T \frac{\mu(B_t(x))}{t^{N-\alpha}} \frac{dt}{t}, \tag{1.9}$$

and the *T-truncated η-fractional maximal potential* of μ by

$$\mathbf{M}^\eta_{\alpha,T}[\mu](x) = \sup\left\{\frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)} : 0 < t \le T\right\}, \tag{1.10}$$

where $h_\eta(t) = (-\ln t)^{-\eta}\chi_{(0,2^{-1}]}(t) + (\ln 2)^{-\eta}\chi_{[2^{-1},\infty)}(t)$. If $\eta = 0$, then $h_\eta = 1$ and we denote by $\mathbf{M}_{\alpha,T}[\mu]$ the corresponding *T-truncated fractional maximal potential* of μ. We also denote by $\mathbf{W}_{\alpha,s}[\mu]$ (resp $\mathbf{I}_\alpha[\mu]$, $\mathbf{M}_\alpha[\mu]$) the *∞-truncated Wolff potential (resp Riesz Potential, fractional maximal potential)* of μ. When the measures are only defined in an open subset $\Omega \subset \mathbb{R}^N$, they are naturally extended by 0 in Ω^c. For $l \in \mathbb{N}^*$, we define the *l-truncated exponential function*

$$H_l(r) = e^r - \sum_{j=0}^{l-1} \frac{r^j}{j!}, \tag{1.11}$$

and for $a > 0$ and $\beta \ge 1$, we set

$$P_{l,a,\beta}(r) = H_l(ar^\beta). \tag{1.12}$$

We put

$$Q_p(s) = \begin{cases} \sum\limits_{q=l}^{\infty} \frac{s^{\frac{\beta q}{p-1}}}{q^{\frac{\beta q}{p-1}} q!} & \text{if } p \neq 2, \\ H_l(s^\beta) & \text{if } p = 2, \end{cases} \tag{1.13}$$

$Q^*_p(r) = \max\{rs - Q_p(s) : s \ge 0\}$ is the complementary function to Q_p, and define

$$cap_{G_p,Q^*_p}(E) = \inf\left\{\int_{\mathbb{R}^N} Q^*_p(f)dx : G_p * f \ge \chi_E, f \ge 0, Q^*_p(f) \in L^1(\mathbb{R}^N)\right\}, \tag{1.14}$$

and

$$cap_{I_p,Q_p^*}(E) = \inf\left\{\int\limits_{\mathbb{R}^N} Q_p^*(f)dx : I_p * f \geq \chi_E, f \geq 0, Q_p^*(f) \in L^1(\mathbb{R}^N)\right\}, \tag{1.15}$$

where $G_p(x) = \mathcal{F}^{-1}\left((1+|.|^2)^{-\frac{p}{2}}\right)(x)$ is the Bessel kernel of order p and $I_p(x) = (N-p)^{-1}|x|^{-(N-p)}$.

We denote $a \wedge b$ and $a \vee b$ for $\min\{a,b\}$ and $\max\{a,b\}$ respectively. Our main results are the following theorems

Theorem 1.1 *Let $1 < p < N$, $a > 0$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p-1$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If μ is a nonnegative Radon measure in Ω, there exists $M > 0$ depending on N, p, l, a, β and $diam\,\Omega$ which is the diameter of Ω such that if*

$$||\mathbf{M}_{p,2diam\,\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M$$

the following Dirichlet problem

$$\begin{array}{rl} -\Delta_p u = P_{l,a,\beta}(u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{1.16}$$

admits a nonnegative renormalized solution u which satisfies $P_{l,a,\beta}\left(2c_pK_1\mathbf{W}_{1,p}^{2diam\,\Omega}[\omega]\right) \in L^1(\Omega)$ and

$$u(x) \leq 2c_pK_1\mathbf{W}_{1,p}^{2diam\,\Omega}[\omega](x) \quad \forall x \in \Omega, \tag{1.17}$$

where $\omega = M||\mathbf{M}_{p,2diam\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$. The role of $K_1 = K_1(N,p)$ will be made explicit in Theorem 3.5. Conversely, if (1.16) admits a nonnegative renormalized solution u and $P_{l,a,\beta}(u) \in L^1(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant C depending on N, p, l, a, β and $\mathrm{dist}\,(K, \partial\Omega)$ such that

$$\int\limits_E P_{l,\alpha,\beta}(u)dx + \mu(E) \leq Ccap_{G_p,Q_p^*}(E) \quad \text{for all Borel sets } E \subset K. \tag{1.18}$$

Furthermore, $u \in W_0^{1,p_1}(\Omega)$ for all $1 \leq p_1 < p$.

Theorem 1.2 *Let $1 < p < N$, $a > 0$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > \frac{N(p-1)}{N-p}$ and $R > 0$. If μ is a nonnegative Radon measure in $\mathbb{R}^N$ with $\mathrm{supp}\,(\mu) \subset B_R(0)$ there exists $M > 0$ depending on N, p, l, a, β and R such that if*

$$||\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M$$

the following problem

$$\begin{array}{rl} -\Delta_p u = P_{l,a,\beta}(u) + \mu & \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u = 0, & \end{array} \tag{1.19}$$

admits an p-superharmonic solution u which satisfies $P_{l,a,\beta}\left(2c_pK_1\mathbf{W}_{1,p}[\omega]\right) \in L^1(\mathbb{R}^N)$ *and*

$$u(x) \leq 2c_pK_1\mathbf{W}_{1,p}[\omega](x) \quad \forall x \in \mathbb{R}^N, \tag{1.20}$$

where $\omega = M||\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)} + \mu$ *and* $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$. *The role of* $K_1 = K_1(N,p)$ *will be made explicit in Theorem 3.5. Conversely, if* (1.19) *has a solution u and* $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ *then there exists a positive constant C depending on* N, p, l, a, β *such that*

$$\int_E P_{l,\alpha,\beta}(u)dx + \mu(E) \leq Ccap_{I_p,Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.} \tag{1.21}$$

Furthermore, $u \in W^{1,p_1}_{loc}(\mathbb{R}^N)$ *for all* $1 \leq p_1 < p$.

Concerning the k-Hessian operator we recall some notions notions introduced by Trudinger and Wang [25, 26, 27], and we follow their notations. For $k = 1, ..., N$ and $u \in C^2(\Omega)$ the k-hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < ... < i_k \leq N} \lambda_{i_1}...\lambda_{i_k}.$$

It is straightforward that

$$F_k[u] = \left[D^2u\right]_k,$$

where in general $[A]_k$ denotes the sum of the k-th principal minors of a matrix $A = (a_{ij})$. In order that there exists a smooth k-admissible function which vanishes on $\partial\Omega$, the boundary $\partial\Omega$ must satisfy a uniformly (k-1)-convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal. We also denotes by $\Phi^k(\Omega)$ the class of upper-semicontinuous functions $\Omega \mapsto [-\infty, \infty)$ which are k-convex, or subharmonic in the Perron sense (see Definition 5.1). In this paper we prove the following theorem

Theorem 1.3 *Let* $k \in \{1, 2, ..., N\}$ *such that* $2k < N$, $l \in \mathbb{N}^*$, $\beta \geq 1$ *such that* $l\beta > k$ *and* $a > 0$. *Let* Ω *be a bounded uniformly (k-1)-convex domain in* $\mathbb{R}^N$. *Let* φ *be a nonnegative continuous function on* $\partial\Omega$ *and* $\mu = \mu_1 + f$ *be a nonnegative Radon measure where* μ_1 *has compact support in* Ω *and* $f \in L^q(\Omega)$ *for some* $q > \frac{N}{2k}$. *Let* $K_2 = K_2(N,k)$ *be the constant* K_2 *which appears in Theorem 5.3. Then, there exist* $b > 0$ *and* $M_1, M_2 > 0$ *depending on* N, k, l, a, β *and* $diam(\Omega)$ *such that if* $\max_{\partial\Omega}\varphi \leq M_2$ *and*

$$||\mathbf{M}^{\frac{k(\beta-1)}{\beta}}_{2k,2diam(\Omega)}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M_1.$$

the following Dirichlet problem

$$\begin{array}{ll} F_k[-u] = P_{l,a,\beta}(u) + \mu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{array} \tag{1.22}$$

admits a nonnegative solution u, continuous near $\partial\Omega$, with $-u \in \Phi^k(\Omega)$ which satisfies $P_{l,a,\beta}\left(2K_2\mathbf{W}^{2diam\,\Omega}_{\frac{2k}{k+1},k+1}[\mu]+b\right) \in L^1(\Omega)$ and

$$u(x) \leq 2K_2\mathbf{W}^{2diam\,\Omega}_{\frac{2k}{k+1},k+1}[\mu](x) + b \qquad \forall x \in \Omega. \tag{1.23}$$

Conversely, if (1.22) *admits a nonnegative solution u, continuous near $\partial\Omega$ such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$, then for any a compact set $K \subset \Omega$, there exists a positive constant C depending on N, k, l, a, β and $dist(K, \partial\Omega)$ such that there holds*

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \leq C cap_{G_{2k},Q^*_{k+1}}(E) \qquad \forall E \subset K, E \text{ Borel}, \tag{1.24}$$

where $Q_{k+1}(s)$ is defined by (1.13) with $p = k+1$, Q^*_{k+1} is its complementary function and $cap_{G_{2k},Q^*_{k+1}}(E)$ is defined accordingly by (1.14).

Theorem 1.4 *Let $k \in \{1, 2, ..., N\}$ such that $2k < N$, $l \in \mathbb{N}^*$, $\beta \geq 1$ such that $l\beta > \frac{Nk}{N-2k}$ and $a > 0$, $R > 0$. Let $K_2 = K_2(N,k)$ be the constant K_2 which appears in Theorem 5.3. If μ is a nonnegative Radon measure in $\mathbb{R}^N$ with $\text{supp}\,(\mu) \subset B_R(0)$ there exists $M > 0$ depending on N, k, l, a, β and R such that if*

$$||\mathbf{M}_{2k}^{\frac{k(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M.$$

the following Dirichlet problem

$$\begin{array}{l} F_k[-u] = P_{l,a,\beta}(u) + \mu \quad \text{in } \mathbb{R}^N, \\ \inf_{\mathbb{R}^N} u = 0, \end{array} \tag{1.25}$$

admits a nonnegative solution u, $-u \in \Phi^k(\mathbb{R}^N)$ which satisfies $P_{l,a\beta}\left(2K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega]\right) \in L^1(\mathbb{R}^N)$ and

$$u(x) \leq 2K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\omega](x) \qquad \forall x \in \mathbb{R}^N, \tag{1.26}$$

where $\omega = M||\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)} + \mu$. Conversely, if (1.25) *admits a nonnegative solution u, $-u \in \Phi^k(\mathbb{R}^N)$ and $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$, then there exists a positive constant C depending on N, k, l, a, β and R such that there holds*

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \leq C cap_{I_{2k},Q^*_{k+1}}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}, \tag{1.27}$$

where $cap_{I_{2k},Q^*_{k+1}}(E)$ is defined accordingly by (1.15) with $p = k+1$.

The four previous theorems are connected to the following result which deals with a class of nonlinear *Wolff integral equations*.

Theorem 1.5 *Let $\alpha > 0$, $p > 1$, $a > 0$, $\varepsilon > 0$, $R_1, R_2 > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p-1$ and $0 < \alpha p < N$. Let f be a nonnegative measurable in $\mathbb{R}^N$ such that $\exp((a+\varepsilon)f^\beta) \in L^1_{loc}(\mathbb{R}^N)$. We set $d\mu_1 = P_{l,a+\varepsilon,\beta}(f)dx$. Let $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$.*
1. *There exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon$ and R_1 such that if*

$$||\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M$$

then there exists a nonnegative function u, such that $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ which satisfies

$$u = \mathbf{W}_{\alpha,p}^{R_1}[P_{l,a,\beta}(u) + \mu] + f \quad in \ \ \mathbb{R}^N, \tag{1.28}$$

and

$$u \leq F := 2c_p \mathbf{W}_{\alpha,p}^{R_1}[\omega_1] + 2c_p \mathbf{W}_{\alpha,p}^{R_1}[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1_{loc}(\mathbb{R}^N), \tag{1.29}$$

where $\omega_1 = M||\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$, $\omega_2 = M||\mathbf{M}_{\alpha p, R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu_1$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$.
Conversely, if (1.28) admits a nonnegative solution u and $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ then there exists a positive constant C depending on $N, \alpha, p, l, a, \beta$ and R_1 such that there holds

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) + \mu_1(E) \leq C cap_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, \ E \ Borel. \tag{1.30}$$

2. Assume that $l\beta > \frac{N(p-1)}{N-\alpha p}$. There exists $M > 0$ depending on $N, \alpha, p, l, a, \beta, \varepsilon$ and R_2 such that if $\operatorname{supp}(f), \operatorname{supp}(\mu) \subset B_{R_2}(0)$ and

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M \quad \text{and} \quad ||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

then there exists a nonnegative function u, such that $P_{l,a,\beta}(u) \in L^1(\mathbb{R}^N)$ which satisfies

$$u = \mathbf{W}_{\alpha,p}[P_{l,a,\beta}(u) + \mu] + f \quad in \ \ \mathbb{R}^N, \tag{1.31}$$

and

$$u \leq F := 2c_p \mathbf{W}_{\alpha,p}[\omega_1] + 2c_p \mathbf{W}_{\alpha,p}[\omega_2] + f, \quad P_{l,a,\beta}(F) \in L^1(\mathbb{R}^N), \tag{1.32}$$

where $\omega_1 = M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_{R_2}(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_{R_2}(0)} + \mu$, $\omega_2 = M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_{R_2}(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_{R_2}(0)} + \mu_1$. Conversely, if (1.31) admits a nonnegative solution u and $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ then there exists a positive constant C depending on $N, \alpha, p, l, a, \beta$ such that there holds

$$\int_E P_{l,a,\beta}(u)dx + \int_E P_{l,a,\beta}(f)dx + \mu(E) \leq C cap_{I_{\alpha_p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, \ E \ Borel \tag{1.33}$$

Remark 1.6 *If $p = 2$ equation (1.31) is equivalent to*

$$(-\Delta)^\alpha(u - f) = P_{l,a,\beta}(u) + \mu \qquad in \ \mathbb{R}^N,$$

since $W_{\alpha,2}[P_{l,a,\beta}(u) + \mu] = I_{2\alpha}[P_{l,a,\beta}(u) + \mu] = I_{2\alpha} * (P_{l,a,\beta}(u) + \mu)$.

2 Estimates on potentials and Wolff integral equations

We denote by $B_r(a)$ the ball of center a and radius $r>0$ and $B_r = B_r(0)$; χ_E the characteristic function of E for Borel set E. The next estimates are crucial in the sequel.

Theorem 2.1 ***1.*** *There exists a positive constant c_1, depending only on N, α, p such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > p-1$, $0 < R \le \infty$ we have*

$$(c_1 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} \left(\mathbf{I}^R_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx \le \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \le (c_1 q)^q \int_{\mathbb{R}^N} \left(\mathbf{I}^R_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx, \tag{2.1}$$

2. *Let $R>0$. There exists a positive constant c_2, depending only on N, α, p, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > p-1$ we have*

$$(c_2 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} \left(\mathbf{G}_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx \le \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \le (c_2 q)^q \int_{\mathbb{R}^N} \left(\mathbf{G}_{\alpha p}[\mu](x)\right)^{\frac{q}{p-1}} dx, \tag{2.2}$$

where $\mathbf{G}_{\alpha p}[\mu] := G_{\alpha p} * \mu$ denotes the Bessel potential of order αp of μ.

3. *There exists a positive constant c_3 depending only on N, α, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > 1$ we have*

$$c_3^{-q} \int_{\mathbb{R}^N} \left(\mathbf{G}_\alpha[\mu][\mu](x)\right)^q dx \le \int_{\mathbb{R}^N} \left(\mathbf{I}^R_\alpha[\mu](x)\right)^q dx \le c_3^q \int_{\mathbb{R}^N} \left(\mathbf{G}_\alpha[\mu](x)\right)^q dx. \tag{2.3}$$

Proof. Note that $W^R_{\alpha/2,2}[\mu] = I^R_\alpha[\mu]$. We can find proof of (2.3) in [8, Step 3, Theorem 2.3]. By [8, Step 2, Theorem 2.3], there is $c_4 > 0$ such that

$$\int\limits_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \ge c_4^q \int\limits_{\mathbb{R}^N} \left(\mathbf{M}_{\alpha p,R}[\mu](x)\right)^{\frac{q}{p-1}} dx \quad \forall q>0,\ 0<R\le\infty \text{ and } \ \mu \in \mathfrak{M}^+(\mathbb{R}^N). \tag{2.4}$$

We recall that $\mathbf{M}_{\alpha p,R}[\mu] = \mathbf{M}^0_{\alpha p,R}[\mu]$ by (1.10). Next we show that

$$\int\limits_{\mathbb{R}^N} \left(\mathbf{M}_{\alpha p,R}[\mu](x)\right)^{\frac{q}{p-1}} dx \ \ge (c_5 q)^{-q} \int\limits_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \ \ \forall q>0,\ 0<R\le\infty \text{ and } \ \mu \in \mathfrak{M}^+(\mathbb{R}^N), \tag{2.5}$$

for some a positive constant c_4. Indeed, we denote μ_n by $\chi_{B_n}\mu$ for $n \in \mathbb{N}^*$. By [17, Theorem 1.2] or [8, Proposition 2.2], there exist constants $c_6 = c_6(N,\alpha,p) > 0$, $a = a(\alpha,p) > 0$ and $\varepsilon_0 = \varepsilon(N,\alpha,p)$ such that for all $n \in \mathbb{N}^*$, $t>0$, $0<R\le\infty$ and $0<\varepsilon<\varepsilon_0$, there holds

$$\left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > 3t\right\}\right| \le c_6 \exp\left(-a\varepsilon^{-1}\right) \left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > t\right\}\right| + \left|\left\{\left(\mathbf{M}_{\alpha p,R}\mu_n\right)^{\frac{1}{p-1}} > \varepsilon t\right\}\right|.$$

Multiplying by qt^{q-1} and integrating over $(0,\infty)$, we obtain

$$\begin{aligned}\int\limits_0^\infty qt^{q-1} \left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > 3t\right\}\right| dt \le c_6 \exp\left(-a\varepsilon^{-1}\right) &\int\limits_0^\infty qt^{q-1} \left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > t\right\}\right| dt \\ &+ \int\limits_0^\infty qt^{q-1} \left|\left\{\left(\mathbf{M}_{\alpha p,R}\mu_n\right)^{\frac{1}{p-1}} > \varepsilon t\right\}\right| dt,\end{aligned}$$

which implies

$$\varepsilon^q \left(3^{-q} - c_6 \exp\left(-a\varepsilon^{-1}\right)\right) \int\limits_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu_n](x)\right)^q dx \le \int\limits_{\mathbb{R}^N} \left(\mathbf{M}_{\alpha p,R}\mu_n\right)^{\frac{q}{p-1}} dx.$$

We see that $\sup\limits_{0<\varepsilon<\varepsilon_0} \varepsilon^q \left(3^{-q} - c_6 \exp\left(-a\varepsilon^{-1}\right)\right) \ge (c_7 q)^{-q}$ for some a constant c_7 which does not depend on q. Therefore, (2.5) follows by Fatou's lemma. Hence, it is easy to obtain (2.1) from (2.4) and (2.5). Then, we obtain (2.2) from (2.1) and (2.3). ∎

The next result is proved in [8].

Theorem 2.2 *Let $\alpha > 0$, $p > 1$, $0 \le \eta < p-1$, $0 < \alpha p < N$ and $L > 0$. Set $\delta = \frac{1}{2}\left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \log(2)$. Then there exists $C(L) > 0$, depending on N, α, p, η and L such that for any $R \in (0,\infty]$, $\mu \in \mathfrak{M} + (\mathbb{R}^N)$, any $a \in \mathbb{R}^N$ and $0 < r \le L$, there holds*

$$\frac{1}{|B_{2r}(a)|} \int\limits_{B_{2r}(a)} \exp\left(\delta \frac{\left(\mathbf{W}^R_{\alpha,p}[\mu_{B_r(a)}](x)\right)^{\frac{p-1}{p-1-\eta}}}{||\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_r}(a)]||^{\frac{1}{p-1-\eta}}_{L^\infty(B_r(a))}}\right) dx \le C(L), \tag{2.6}$$

where $\mu_{B_r(a)} = \chi_{B_r(a)}\mu$. Furthermore, if $\eta = 0$, C is independent of L.

Theorem 2.3 *Let $\alpha > 0$, $p > 1$ with $0 < \alpha p < N$, $\beta \ge 1$ and $R > 0$. Assume $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ satisfies*

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \le 1 \tag{2.7}$$

We set $\omega = ||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$. Then there exist positive constants C and δ_0 depending on μ such that

$$\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\beta\right) \in L^1_{loc}(\mathbb{R}^N) \quad \text{and} \quad \left\|\mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\beta\right)\right]\right\|_{L^\infty(\mathbb{R}^N)} \le C. \tag{2.8}$$

and

$$\mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\beta\right)\right] \le c\mathbf{W}^R_{\alpha,p}[\omega] \quad \text{in } \mathbb{R}^N \tag{2.9}$$

for some $c > 0$ independing on μ.

Proof. Let δ be as in Theorem 2.2. From (2.7), we have

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[\omega]||_{L^\infty(\mathbb{R}^N)} \le 2.$$

Let $x \in \mathbb{R}^N$. Since $\omega(B_t(y)) \le 2t^{N-\alpha p} h_{\frac{(p-1)(\beta-1)}{\beta}}(t)$, for all $r \in (0,R)$ and $y \in \mathbb{R}^N$ we have

$$\begin{aligned}
\mathbf{W}^R_{\alpha,p}[\omega](y) &= \mathbf{W}^r_{\alpha,p}[\omega](y) + \int\limits_r^R \left(\frac{\omega(B_t(y))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\le \mathbf{W}^r_{\alpha,p}[\omega](y) + 2^{\frac{1}{p-1}} \int\limits_{r\wedge 2^{-1}}^{2^{-1}} (-\ln t)^{-\frac{\beta-1}{\beta}} \frac{dt}{t} + 2^{\frac{1}{p-1}} \int\limits_{2^{-1}}^{R\vee 2^{-1}} (-\ln t)^{-\frac{\beta-1}{\beta}} \frac{dt}{t} \\
&\le \mathbf{W}^r_{\alpha,p}[\omega](y) + c_8(-\ln(r\wedge 2^{-1}))^{\frac{1}{\beta}} + c_8.
\end{aligned}$$

Thus,

$$\left(\mathbf{W}^R_{\alpha,p}[\omega](y)\right)^\beta \le 3^{\beta-1}\left(\mathbf{W}^r_{\alpha,p}[\omega](y)\right)^\beta + c_9 \ln\left(\frac{1}{r\wedge 2^{-1}}\right) + c_9 \tag{2.10}$$

Let $\theta\in(0,1]$, since $\exp\left(\frac{a+b}{2}\right)\le \exp(a)+\exp(b)$ for all $a,b\in\mathbb{R}$, we get from (2.10)

$$\begin{aligned}\exp\left(\theta\delta 3^{-\beta}\left(\mathbf{W}^R_{\alpha,p}[\omega](y)\right)^\beta\right) &\le \exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}[\omega](y)\right)^\beta\right) + c_{10}\exp\left(\theta c_{11}\ln\left(\frac{1}{r\wedge 2^{-1}}\right)\right)\\ &\le \exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}[\omega](y)\right)^\beta\right) + c_{10}\left(r\wedge 2^{-1}\right)^{-\theta c_{11}} \end{aligned}\tag{2.11}$$

For $r>0$, $0<t\le r$, $y\in B_r(x)$ we have $B_t(y)\subset B_{2r}(x)$. Thus, $\mathbf{W}^r_{\alpha,p}[\omega]=\mathbf{W}^r_{\alpha,p}[\omega_{B_{2r}(x)}]$ in $B_r(x)$. Then, using (2.6) from Theorem 2.2 with $L=2R$ we get

$$\int\limits_{B_r(x)} \exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}[\omega]\right)^\gamma\right) = \int\limits_{B_r(x)} \exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}\left[\omega_{B_{2r}(x)}\right]\right)^\gamma\right) \le c_{12}r^N.$$

So, take $\theta = 1\wedge\frac{\alpha p}{2c_{11}}$ we deduce from (2.11)

$$\begin{aligned}\mathbf{W}^R_{\alpha,p}\left[\exp\left(\theta\delta 3^{-\beta}\left(\mathbf{W}^R_{\alpha,p}[\omega]\right)^\gamma\right)\right](x) &\le \int\limits_0^R \left(c_{12}r^{\alpha p} + c_{13}\left(r\wedge 2^{-1}\right)^{-\theta c_{11}} r^{\alpha p}\right)^{\frac{1}{p-1}}\frac{dr}{r}\\ &\le \int\limits_0^R \left(c_{12}r^{\alpha p} + c_{13}\left(r\wedge 2^{-1}\right)^{-\frac{\alpha p}{2}} r^{\alpha p}\right)^{\frac{1}{p-1}}\frac{dr}{r}\\ &\le c_{14}\end{aligned}$$

Hence, we get (2.8) with $\delta_0 = \left(1\wedge\frac{\alpha p}{2c_{11}}\right)\delta 3^{-\beta}$ and we also get (2.9) since $\mathbf{W}^R_{\alpha,p}[\omega]\ge c_{15}$ for some a positive constant $c_{15}>0$. ∎

We recall that H_l and $P_{l,a,\beta}$ have been defined in (1.11) and (1.12).

Theorem 2.4 *Let $\alpha>0$, $p>1$, $l\in\mathbb{N}^*$ and $\beta\ge 1$ such that $0<\alpha p<N$, $l\beta>\frac{N(p-1)}{N-\alpha p}$ and $R>0$. Assume that $\mu\in\mathfrak{M}^+(\mathbb{R}^N)$ with $\operatorname{supp}\mu\subset B_R(0)$ satisfies*

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)}\le 1 \tag{2.12}$$

Set $\omega = ||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)}+\mu$. Then there exist $C=C(N,\alpha,p,l,\beta,R)>0$ and $\delta_1=\delta_1(N,\alpha,p,l,\beta,R)>0$ such that $H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right)\in L^1(\mathbb{R}^N)$ and

$$\mathbf{W}_{\alpha,p}\left[H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right)\right](x)\le C\mathbf{W}_{\alpha,p}[\omega](x)\quad\forall\ x\in\mathbb{R}^N. \tag{2.13}$$

Proof. We have from (2.12)

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]||_{L^\infty(\mathbb{R}^N)}\le 2. \tag{2.14}$$

Let $\delta_1 > 0$. Fixed $x \in \mathbb{R}^N$. We split the Wolff potential $\mathbf{W}_{\alpha,p}[\omega]$ into lower and upper parts defined by

$$\mathbf{L}^t_{\alpha,p}[\omega](x) = \int_t^{+\infty} \left(\frac{\omega(B_r(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r},$$

and

$$\mathbf{W}^t_{\alpha,p}[\omega](x) = \int_0^{t} \left(\frac{\omega(B_r(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Using the convexity property we have

$$H_l\left(\delta_1 \left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right) \leq H_l\left(\delta_1 2^\beta \left(\mathbf{L}^t_{\alpha,p}[\omega]\right)^\beta\right) + H_l\left(\delta_1 2^\beta \left(\mathbf{W}^t_{\alpha,p}[\omega]\right)^\beta\right).$$

Thus,

$$\mathbf{W}_{\alpha,p}\left[H_l\left(\delta_1 \left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right)\right](x) \leq c_{16}\int_0^{+\infty} \left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} + c_{16}\int_0^{+\infty} \left(\frac{\omega^2_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t},$$

where $d\omega^1_t = H_l\left(\delta_1 2^\beta \left(\mathbf{L}^t_{\alpha,p}[\omega]\right)^\beta\right) dx$ and $d\omega^2_t = H_l\left(\delta_1 2^\beta \left(\mathbf{W}^t_{\alpha,p}[\omega]\right)^\beta\right) dx$.
We will obtain (2.13) if we prove that

$$\int_0^{+\infty} \left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{17} W_{\alpha,p}[\omega](x), \tag{2.15}$$

and

$$\omega^2_t(B_t(x)) \leq c_{18}\omega(B_{4t}(x)). \tag{2.16}$$

To prove (2.15), we have $B_r(y) \subset B_{2r}(x)$ for $y \in B_t(x)$ and $r \geq t$, so

$$\mathbf{L}^t_{\alpha,p}[\omega](y) \leq \int_t^{+\infty} \left(\frac{\omega(B_{2r}(x))}{r^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} = 2^{\frac{N-\alpha p}{p-1}} \mathbf{L}^{t/2}_{\alpha,p}[\omega](x)$$

It follows

$$\omega^1_t(B_t(x)) \leq |B_1(0)| t^N H_l\left(\delta_1 c_{19}\left(\mathbf{L}^{t/2}_{\alpha,p}[\omega](x)\right)^\beta\right)$$

Thus,

$$\int_0^{+\infty} \left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{20}\int_0^{\infty} A_t(x)dt \tag{2.17}$$

where $A_t(x) = \left(t^{\alpha p} H_l\left(\delta_1 c_{19}\left(\mathbf{L}^{t/2}_{\alpha,p}[\omega](x)\right)^\beta\right)\right)^{\frac{1}{p-1}} \frac{1}{t}$. Since $H_l(s) \leq s^l \exp(s)$ for all $s \geq 0$,

$$\begin{aligned} A_t(x) &\leq c_{21}\left(t^{\alpha p}\left(\mathbf{L}^{t/2}_{\alpha,p}[\omega](x)\right)^{l\beta} \exp\left(\delta_1 c_{19}\left(\mathbf{L}^{t/2}_{\alpha,p}[\omega](x)\right)^\beta\right)\right)^{\frac{1}{p-1}} \frac{1}{t} \\ &= c_{21} t^{\frac{\alpha p}{p-1}-1}\left(\mathbf{L}^{t/2}_{\alpha,p}[\omega](x)\right)^{\frac{l\beta-p+1}{p-1}} \exp\left(\delta_1 c_{22}\left(\mathbf{L}^{t/2}_{\alpha,p}[\omega](x)\right)^\beta\right) \mathbf{L}^{t/2}_{\alpha,p}[\omega](x) \end{aligned}$$

Now we estimate $L^{t/2}_{\alpha,p}[\omega]$.
Case 1: $t \in (0,1)$, from (2.14) we deduce

$$\begin{aligned} L^{t/2}_{\alpha,p}[\omega](x) &= \int_{t/2}^{1/2} \left(\frac{\omega(B_s(x))}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} + \int_{1/2}^{\infty} \left(\frac{\omega(B_s(x))}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\leq c_{23} \int_{t/2}^{1/2} (-ln(s))^{-1+\frac{1}{\beta}} \frac{ds}{s} + \int_{1/2}^{\infty} \left(\frac{\omega(B_R(0))}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &\leq c_{24} \left(-\ln(t/2)\right)^{\frac{1}{\beta}} \end{aligned}$$

Which implies

$$\begin{aligned} A_t(x) &\leq c_{25} t^{\frac{\alpha p}{p-1}-1} \left(-\ln(t/2)\right)^{\frac{l\beta-p+1}{\beta(p-1)}} \exp\left(\delta_1 c_{26}(-\ln(t/2))\right) \mathbf{L}^{t/2}_{\alpha,p}[\omega](x) \\ &= c_{27} t^{\frac{\alpha p}{p-1}-1} \left(-\ln(t/2)\right)^{\frac{l\beta-p+1}{\beta(p-1)}} t^{-\delta_1 c_{26}} \mathbf{L}^{t/2}_{\alpha,p}[\omega](x) \end{aligned}$$

We take $\delta_1 \leq \frac{1}{2c_{26}} \left(\frac{\alpha p}{p-1} - 1\right)$, we get

$$A_t(x) \leq c_{28} \mathbf{L}^{t/2}_{\alpha,p}[\omega](x) \quad \forall t \in (0,1). \tag{2.18}$$

Case 2: $t \geq 1$, we have

$$\begin{aligned} L^{t/2}_{\alpha,p}[\omega](x) &\leq \int_{t/2}^{\infty} \left(\frac{\omega(B_R(0))}{s^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} \\ &= c_{29} t^{-\frac{N-\alpha p}{p-1}} \end{aligned}$$

thus

$$\begin{aligned} A_t(x) &\leq c_{30} t^{\frac{\alpha p}{p-1}-1} t^{-\frac{(l\beta-p+1)(N-\alpha p)}{(p-1)^2}} \exp\left(\delta_1 c_{31} t^{-\frac{N-\alpha p}{p-1}}\right) \mathbf{L}^{t/2}_{\alpha,p}[\omega](x) \\ &\leq c_{32} t^{-1-\gamma} \mathbf{L}^{t/2}_{\alpha,p}[\omega](x) \end{aligned}$$

where $\gamma = \frac{1}{p-1} \left(\frac{l\beta(N-\alpha p)}{p-1} - N\right) > 0$.
Therefore, $A_t(x) \leq c_{33}(t \vee 1)^{-1-\gamma} \mathbf{L}^{t/2}_{\alpha,p}[\omega](x)$ for all $t > 0$. Therefore, from (2.17)

$$\int_0^{+\infty} \left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_{33} \int_0^{\infty} (t \vee 1)^{-1-\gamma} \mathbf{L}^{t/2}_{\alpha,p}[\omega](x) dt$$

Using Fubini Theorem we get

$$\begin{aligned} \int_0^{+\infty} \left(\frac{\omega^1_t(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} &\leq c_{34} \int_0^{\infty} \int_0^t (s \vee 1)^{-1-\gamma} ds \left(\frac{\omega(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq c_{35} \int_0^{\infty} \left(\frac{\omega(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= c_{35} \mathbf{W}_{\alpha,p}[\mu](x), \end{aligned}$$

which follows (2.15).
To prove (2.16), for $t>0$, $r\le t$ and $y\in B_t(x)$ we have $B_r(y)\subset B_{2t}(x)$. Therefore,

$$\omega_t^2(B_t(x))=\int_{B_t(x)} H_l\left(\delta_1 2^\beta\left(\mathbf{W}^t_{\alpha,p}[\omega_{B_{2t}(x)}](y)\right)^\beta\right)dy$$

By Theorem 2.2 there exists $c_{36}>0$ such that for $0<\delta_1\le c_{36}$, $0<t<2R$, $x\in\mathbb{R}^N$,

$$\int_{B_{4t}(x)}\exp\left(\delta_1 2^\beta\left(\mathbf{W}_{\alpha,p}[\omega_{B_{2t}(x)}](y)\right)^\beta\right)dy\le c_{37}t^N \tag{2.19}$$

We take $0<\delta_1\le c_{36}$.
Case 1: $x\in\overline{B}_R(0)$
If $0<t<2R$, from (2.19) we get

$$\omega_t^2(B_t(x))\le c_{37}t^N\le c_{38}\omega(B_{4t}(x)).$$

If $t\ge 2R$, since for any $|y|\ge 2R$,

$$\mathbf{W}_{\alpha,p}[\omega](y)=\int_{|y|/2}^\infty\left(\frac{\omega(B_t(y))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\le c_{39}\int_{|y|/2}^\infty t^{-1-\frac{N-\alpha p}{p-1}}dt\le c_{40}|y|^{-\frac{N-\alpha p}{p-1}},$$

and thanks to (2.19) we have

$$\begin{aligned}
\omega_t^2(B_t(x)) &\le \int_{B_{2R}(0)}\exp\left(\delta_1 2^\beta\left(\mathbf{W}_{\alpha,p}[\omega_{B_R(0)}](y)\right)^\beta\right)dy+\int_{B_{2R}(0)^c}H_l\left(\delta_1 2^\beta\left(\mathbf{W}_{\alpha,p}[\omega](y)\right)^\beta\right)dy\\
&\le c_{41}R^N+\int_{B_{2R}(0)^c}H_l\left(c_{42}|y|^{-\frac{\beta(N-\alpha p)}{p-1}}\right)dy\\
&\le c_{43}+c_{43}\int_{B_{2R}(0)^c}|y|^{-\frac{l\beta(N-\alpha p)}{p-1}}dy=c_{43}+c_{44}R^{N-\frac{l\beta(N-\alpha p)}{p-1}}\\
&\le c_{45}|B_{4t}(x)\cap B_R(0)|\le c_{46}\omega(B_{4t}(x)).
\end{aligned}$$

From this we also have $H_l\left(\delta_1\left(\mathbf{W}_{\alpha,p}[\omega]\right)^\beta\right)\in L^1(\mathbb{R}^N)$.
Case 2: $x\in B_R(0)^c$. If $|x|>R+t$ then $\omega_t^2(B_t(x))=0$. Now we consider case $R<|x|\le R+t$.
If $0<t<2R$, we have $B_t((R-\frac{t}{2})\frac{x}{|x|})\subset B_{4t}(x)\cap B_R(0)$ thus from (2.19) we get

$$\omega_t^2(B_t(x))\le c_{47}t^N=c_{48}\left|B_t\left((R-\frac{t}{2})\frac{x}{|x|}\right)\right|\le c_{48}\left|B_{4t}(x)\cap B_R(0)\right|\le c_{49}\omega(B_{4t}(x))$$

If $t\ge 2R$, as in Case 1 we also obtain $\omega_t^2(B_t(x))\le c_{50}\omega(B_{4t}(x))$.
Hence, we get (2.16). Therefore, the Theorem follows with $\delta_1=\frac{1}{2c_{26}}\left(\frac{\alpha p}{p-1}-1\right)\wedge c_{36}$. ∎

In the next result we obtain estimate on iterative solutions of Wolff integral inequalities.

Theorem 2.5 *Under the assumption as in Theorem 1.5. Let $K>0$.*
A. *Suppose that $\{u_m\}$ is a sequence of nonnegative functions in $\mathbb{R}^N$ that satisfies*

$$\begin{aligned}
u_{m+1}&\le K\mathbf{W}^{R_1}_{\alpha,p}[P_{l,a,\beta}(u_m)+\mu]+f\quad\forall m\in\mathbb{N},\\
u_0&\le K\mathbf{W}^{R_1}_{\alpha,p}[\mu]+f.
\end{aligned}\tag{2.20}$$

Then there exists $M>0$ depending on $N,\alpha,p,l,a,\beta,\varepsilon,K$ and R_1 such that if

$$||\mathbf{M}_{\alpha p,R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)}\le M \quad \textit{and} \quad ||\mathbf{M}_{\alpha p,R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)}\le M,$$

there holds

$$P_{l,a,\beta}\left(4c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_1]+4c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_2]+f\right)\in L^1_{loc}(\mathbb{R}^N), \tag{2.21}$$

and

$$u_m\le 2c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_1]+2c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_2]+f\quad\forall m\in\mathbb{N}, \tag{2.22}$$

where $\omega_1=M||\mathbf{M}_{\alpha p,R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)}+\mu$, $\omega_2=M||\mathbf{M}_{\alpha p,R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)}+\mu_1$ *and* $c_p=1\vee 4^{\frac{2-p}{p-1}}$.

B. *Assume that $l\beta>\frac{N(p-1)}{N-\alpha p}$ and $supp(f),supp(\mu)\subset B_{R_2}(0)$. Suppose that $\{u_m\}$ is a sequence of nonnegative functions in $\mathbb{R}^N$ that satisfies*

$$\begin{aligned} u_{m+1}&\le K\mathbf{W}_{\alpha,p}[P_{l,a,\beta}(u_m)+\mu]+f\quad\forall m\in\mathbb{N},\\ u_0&\le K\mathbf{W}_{\alpha,p}[\mu]+f. \end{aligned} \tag{2.23}$$

Then there exists $M>0$ depending on $N,\alpha,p,l,a,\beta,\varepsilon,K$ and R_2 such that if

$$||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)}\le M \quad \textit{and} \quad ||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)}\le M,$$

there holds

$$P_{l,a,\beta}\left(4c_pK\mathbf{W}_{\alpha,p}[\omega_3]+4c_pK\mathbf{W}_{\alpha,p}[\omega_4]+f\right)\in L^1(\mathbb{R}^N), \tag{2.24}$$

and

$$u_m\le 2c_pK\mathbf{W}_{\alpha,p}[\omega_3]+2c_pK\mathbf{W}_{\alpha,p}[\omega_4]+f\quad\forall m\in\mathbb{N}, \tag{2.25}$$

where $\omega_3=M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_{R_2}(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_{R_2}(0)}+\mu$, $\omega_4=M||\mathbf{M}_{\alpha p}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_{R_2}(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_{R_2}(0)}+\mu$.

Proof. We will using Theorem 2.3 and 2.4 to prove the Theorem. Here, we only prove A. Set $c_{a,\varepsilon}=2\left(1-\left(\frac{a}{a+\varepsilon}\right)^{1/\beta}\right)^{-1}$ and $\overline{a}=a\left(4c_{a,\varepsilon}c_pK\right)^\beta$. Take $0<M\le 1$ and put $\omega_1=M||\mathbf{M}_{\alpha p,R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)}+\mu$, $\omega_2=M||\mathbf{M}_{\alpha p,R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)}+\mu_1$. We now assume

$$||\mathbf{M}_{\alpha p,R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)}\le M \quad \text{and} \quad ||\mathbf{M}_{\alpha p,R_1}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu_1]||_{L^\infty(\mathbb{R}^N)}\le M.$$

First, we need to show that

$$\mathbf{W}_{\alpha,p}^{R_1}\left[H_l\left(\overline{a}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}]\right)^\beta\right)\right]\le\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}]. \tag{2.26}$$

By Theorem 2.3, then there exist positive constants c and δ_0 independing on μ such that $\exp\left(\delta_0\left(\mathbf{W}_{\alpha,p}^{R}\left[M^{-1}\omega_{1,2}\right]\right)^\beta\right)\in L^1_{loc}(\mathbb{R}^N)$ and

$$\mathbf{W}_{\alpha,p}^{R}\left[\exp\left(\delta_0\left(\mathbf{W}_{\alpha,p}^{R}\left[M^{-1}\omega_{1,2}\right]\right)^\beta\right)\right]\le c\mathbf{W}_{\alpha,p}^{R}[M^{-1}\omega_{1,2}]\quad\text{in }\mathbb{R}^N.$$

Since $\theta^{-l}H_l(s) \leq H_l(\theta^{-1}s)$ for all $s \geq 0$ and $0 < \theta \leq 1$, thus

$$\begin{aligned}
\mathbf{W}_{\alpha,p}^{R_1}\left[M^{-\frac{1}{2}\left(\frac{\beta l}{p-1}+1\right)}H_l\left(\delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}]\right)^{\beta}\right)\right] &\leq \mathbf{W}_{\alpha,p}^{R_1}\left[H_l\left(\delta_0 M^{-\frac{\beta}{p-1}}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}]\right)^{\beta}\right)\right] \\
&\leq \mathbf{W}_{\alpha,p}^{R_1}\left[\exp\left(\delta_0\left(\mathbf{W}_{\alpha,p}^{R_1}[M^{-1}\omega_{1,2}]\right)^{\beta}\right)\right] \\
&\leq cM^{-\frac{1}{p-1}}\mathbf{W}_{\alpha,p}^{R}[\omega_{1,2}].
\end{aligned}$$

So,

$$\mathbf{W}_{\alpha,p}^{R_1}\left[H_l\left(\delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}]\right)^{\beta}\right)\right] \leq cM^{\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)}\mathbf{W}_{\alpha,p}^{R}[\omega_{1,2}]$$

We see that (2.26) will be achieved if we prove

$$\overline{a} \leq \delta_0 M^{-\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)} \text{ and } cM^{\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)} \leq 1.$$

which is equivalent to

$$M \leq \left(\delta_0\overline{a}^{-1}\right)^{\left(\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)\right)^{-1}}.$$

Thus, one can choose $M = 1 \wedge \left(\delta_0\overline{a}^{-1}\right)^{\left(\frac{1}{2}\left(\frac{\beta}{p-1}-\frac{1}{l}\right)\right)^{-1}} \wedge c^{-\left(\frac{1}{2(p-1)}\left(\frac{\beta l}{p-1}-1\right)\right)^{-1}}$ we obtain (2.26) and $H_l\left(\overline{a}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_{1,2}]\right)^{\beta}\right) \in L^1_{loc}(\mathbb{R}^N)$.
Now, we prove (2.22). Clearly, (2.22) holds with $m = 0$. Now, assume that (2.22) holds with $m = n$, we need to prove that

$$u_{n+1} \leq 2c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_1] + 2c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_2] + f. \tag{2.27}$$

In fact, since (2.22) holds with $m = n$ and the convexity property of $P_{l,a,\beta}$, we have

$$\begin{aligned}
P_{l,a,\beta}\left(u_n\right) &\leq P_{l,a,\beta}\left(4c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_1] + 4c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_2] + f\right) \\
&\leq P_{l,a,\beta}\left(4c_{a,\varepsilon}c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_1]\right) + P_{l,\varepsilon,a}\left(4c_{a,\varepsilon}c_pK\mathbf{W}_{\alpha,p}^{R_1}[\omega_2]\right) + P_{l,a,\beta}\left(\left(\frac{a+\varepsilon}{a}\right)^{1/\beta} f\right) \\
&= H_l\left(\overline{a}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_1]\right)^{\beta}\right) + H_l\left(\overline{a}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_2]\right)^{\beta}\right) + P_{l,a+\varepsilon,\beta}(f).
\end{aligned}$$

From this we get (2.21). By the definition of u_{n+1} and sub-additive property of $\mathbf{W}_{\alpha,p}^{R_1}[.]$, we continue

$$\begin{aligned}
u_{n+1} &\leq K\mathbf{W}_{\alpha,p}^{R_1}\left[H_l\left(\overline{a}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_1]\right)^{\beta}\right) + H_l\left(\overline{a}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_2]\right)^{\beta}\right) + P_{l,a+\varepsilon,\beta}(f) + \mu\right] + f \\
&\leq c_pK\mathbf{W}_{\alpha,p}^{R_1}\left[H_l\left(\overline{a}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_1]\right)^{\beta}\right)\right] + c_pK\mathbf{W}_{\alpha,p}^{R_1}\left[H_l\left(\overline{a}\left(\mathbf{W}_{\alpha,p}^{R_1}[\omega_2]\right)^{\beta}\right)\right] \\
&\quad + c_pK\mathbf{W}_{\alpha,p}^{R_1}\left[P_{l,a+\varepsilon,\beta}(f)\right] + c_pK\mathbf{W}_{\alpha,p}^{R_1}\left[\mu\right] + f.
\end{aligned}$$

It follows (2.27) from (2.26). This completes the proof of the theorem. ∎

Let $P \in C(\mathbb{R}^+)$ be a decreasing positive function; the (α, P)-Orlicz-Bessel capacity of a Borel set $E \subset \mathbb{R}^N$ is defined by (see [1, Sect 2.6])

$$cap_{G_\alpha,P}(E) = \inf\left\{\int\limits_{\mathbb{R}^N} P(f) : G_\alpha * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N)\right\},$$

and the (α, P)-Orlicz-Riesz capacity

$$cap_{I_\alpha,P}(E) = \inf\left\{\int\limits_{\mathbb{R}^N} P(f) : I_\alpha * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N)\right\}.$$

Theorem 2.6 *Let $\alpha > 0$, $p > 1$, $a > 0$, $c > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p - 1$ and $0 < \alpha p < N$. Let $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$.*
1. Let $0 < R \leq \infty$. If u is a nonnegative Borel function in $\mathbb{R}^N$ such that $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ and

$$u(x) \geq cW^R_{\alpha,p}[P_{l,a,\beta}(u) + \mu](x) \ \forall x \in \mathbb{R}^N \tag{2.28}$$

then,
***Case 1**: $R < \infty$, there exists a positive constant C_1 depending on $N, \alpha, p, l, a, \beta, c$ and R such that*

$$\int\limits_E P_{l,a,\beta}(u)dx + \mu(E) \leq C_1 cap_{G_{\alpha p},Q^*_p}(E) \quad \forall E \subset \mathbb{R}^N, \ E \ Borel. \tag{2.29}$$

***Case 2**: $R = \infty$, there exists a positive constant C_2 depending on $N, \alpha, p, l, a, \beta, c$ such that*

$$\int\limits_E P_{l,a,\beta}(u)dx + \mu(E) \leq C_2 cap_{I_{\alpha p},Q^*_p}(E) \quad \forall E \subset \mathbb{R}^N, \ E \ Borel. \tag{2.30}$$

2. Let Ω be a bounded domain in $\mathbb{R}^N$ and $\delta \in (0, 1)$. If u is a nonnegative Borel function in Ω such that $P_{l,a,\beta}(u) \in L^1_{loc}(\Omega)$ and

$$u(x) \geq c\mathbf{W}^{\delta d(x,\partial\Omega)}_{\alpha,p}[P_{l,a,\beta}(u) + \mu](x) \qquad \forall x \in \Omega, \tag{2.31}$$

then for any a compact set $K \subset \Omega$, there exists a positive constant C_3 depending on $N, \alpha, p, l, a, \beta, c, \delta$ and $dist(K, \partial\Omega)$ such that

$$\int\limits_E P_{l,a,\beta}(u)dx + \mu(E) \leq C_3 cap_{G_{\alpha p},Q^*_p}(E) \quad \forall E \subset K, E \ Borel \tag{2.32}$$

where Q^*_p is the complementary function to Q_p.

Proof. Set $d\omega = P_{l,a,\beta}(u)dx + d\mu$.
1. We have

$$P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega]\right)dx \leq d\omega \quad \text{in } \mathbb{R}^N.$$

Let M_ω denote the centered Hardy-Littlewood maximal function which is defined for any $f \in L^1_{loc}(\mathbb{R}^N, d\omega)$ by

$$M_\omega f(x) = \sup_{t>0} \frac{\int\limits_{B_t(x)} |f| d\omega}{\omega(B_t(x))}.$$

Thus, if $E \subset \mathbb{R}^N$ is a Borel set, we have

$$\int\limits_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega]\right) dx \leq \int\limits_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

Since M_ω is bounded on $L^s(\mathbb{R}^N, d\omega)$, $s > 1$, we deduce from Fefferman's result [11] that

$$\int\limits_{\mathbb{R}^N} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega]\right) dx \leq c_{51}\omega(E),$$

for some constant c_{51} only depends on N and $\frac{l\beta}{p-1}$. Since $M_\omega \chi_E \leq 1$, we derive

$$\begin{aligned}(M_\omega \chi_E(x))^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega](x)\right) &\geq P_{l,a,\beta}\left(c\left(M_\omega \chi_E(x)\right)^{\frac{1}{p-1}} \mathbf{W}^R_{\alpha,p}[\omega](x)\right) \\ &\geq P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega_E](x)\right),\end{aligned}$$

where $\omega_E = \chi_E \omega$.
So,

$$\int\limits_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega_E]\right) dx \leq c_{51}\omega(E) \text{ for all Borel set } E \subset \mathbb{R}^N. \tag{2.33}$$

From (2.1), (2.2) and (2.3) we get

$$\int\limits_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega_E](x)\right) dx \geq \int\limits_{\mathbb{R}^N} Q_p\left(c_{52}\mathbf{G}_{\alpha p}[\omega_E](x)\right) dx \quad \text{if } R < \infty$$

and

$$\int\limits_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}^R_{\alpha,p}[\omega_E](x)\right) dx \geq \int\limits_{\mathbb{R}^N} Q_p\left(c_{53}\mathbf{I}_{\alpha p}[\omega_E](x)\right) dx \quad \text{if } R = \infty$$

where Q_p is defined by (1.13) and $c_{52} = (c_2\beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$ if $p \neq 2$, $c_{52} = c_3^{-1} a^{\frac{1}{\beta}} c$ if $p = 2$ (the constant c_2 defined in (2.2), depends on R, therefore $c_{52} = c_{52}(r_K)$) and $c_{53} = (c_1\beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$ if $p \neq 2$, $c_{53} = a^{\frac{1}{\beta}} c$ if $p = 2$. Thus, from (2.33) we obtain for all Borel set $E \subset \mathbb{R}^N$

$$\int\limits_{\mathbb{R}^N} Q_p\left(c_{52}\mathbf{G}_{\alpha p}[\omega_E](x)\right) dx \leq c_{51}\omega(E) \text{ if } R < \infty,$$

and

$$\int\limits_{\mathbb{R}^N} Q_p\left(c_{53}\mathbf{I}_{\alpha p}[\omega_E](x)\right) dx \leq c_{51}\omega(E) \text{ if } p = \infty.$$

We recall that $Q_p^*(s) = \sup_{t>0}\{st - Q_p(t)\}$ and we note that Q_p^* satisfies the sub-additivity Δ_2-condition (see Chapter 2 in [19]).
Case 1. $R < \infty$. For every $f \geq 0$, $Q_p^*(f) \in L^1(\Omega)$ such that $G_{\alpha p} * f \geq \chi_E$, we have

$$\begin{aligned}\omega(E) &\leq \int\limits_{\mathbb{R}^N} G_{\alpha p} * f d\omega_E = (2c_{51})^{-1} \int\limits_{\mathbb{R}^N} \left(c_{52}\mathbf{G}_{\alpha p}\left[\omega_E\right]\right)\left(2c_{51}c_{52}^{-1}f\right)dx \\ &\leq (2c_{51})^{-1} \int\limits_{\mathbb{R}^N} Q_p\left(c_{52}\mathbf{G}_{\alpha p}\left[\omega_E\right]\right)dx + (2c_{51})^{-1} \int\limits_{\mathbb{R}^N} Q_p^*\left(2c_{51}c_{52}^{-1}f\right)dx \\ &\leq 2^{-1}\omega(E) + c_{54}\int\limits_{\mathbb{R}^N} Q_p^*\left(f\right)dx,\end{aligned}$$

the last inequality following from the Δ_2-condition. Notice that c_{54}, as well as the next constant c_{55}, depends on r_K. Thus,

$$\omega(E) \leq 2c_{54}\int\limits_{\mathbb{R}^N} Q_p^*\left(f\right)dx.$$

Then, we get

$$\omega(E) \leq c_{55} cap_{G_{\alpha p}, Q_p^*}(E) \qquad \text{for all Borel set } E \subset \mathbb{R}^N.$$

Which implies (2.29).
Case 2. $R = \infty$. For every $f \geq 0$, $Q_p^*(f) \in L^1(\Omega)$ such that $I_{\alpha p} * f \geq \chi_E$, since $I_{\alpha p} * \omega_E = I_{\alpha p}[\omega_E]$, as above we have

$$\begin{aligned}\omega(E) &\leq \int\limits_{\mathbb{R}^N} I_{\alpha p} * f d\omega_E = \int\limits_{\mathbb{R}^N} \left(\mathbf{I}_{\alpha p} * \omega_E\right) f dx = \int\limits_{\mathbb{R}^N} \mathbf{I}_{\alpha p}\left[\omega_E\right] f dx \\ &\leq 2^{-1}\omega(E) + c_{56}\int\limits_{\mathbb{R}^N} Q_p^*\left(f\right)dx,\end{aligned}$$

Then, it follows (2.30).
2. Let $K \subset\subset \Omega$. Set $r_K = dist(K, \partial\Omega)$ and $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$. We have

$$P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega]\right)dx \leq d\omega \quad \text{in } \Omega.$$

Thus, for any Borel set $E \subset K$

$$\int\limits_{\Omega} \left(M_\omega \chi_E\right)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega]\right)dx \leq \int\limits_{\Omega} \left(M_\omega \chi_E\right)^{\frac{l\beta}{p-1}} d\omega.$$

As above we get

$$\int\limits_{\Omega} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x)\right)dx \leq c_{51}\omega(E) \quad \forall E \subset K, E \text{ Borel.} \tag{2.34}$$

Note that if $x \in \Omega$ and $d(x, \partial\Omega) \leq r_K/8$, then $B_t(x) \subset \Omega \backslash \Omega_K$ for all $t \in (0, \delta d(x, \partial\Omega))$; indeed, for all $y \in B_t(x)$

$$d(y, \partial\Omega) \leq d(x, \partial\Omega) + |x - y| < (1 + \delta) d(x, \partial\Omega) < \frac{1}{4} r_K,$$

thus

$$d(y, K) \geq d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4} r_K > \frac{1}{2} r_K,$$

which implies $y \notin \Omega_K$. We deduce that

$$\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \geq \mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K}[\omega_E](x) \qquad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K}[\omega_E](x) = 0 \qquad \forall x \in \Omega^c.$$

Hence we obtain from (2.34),

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c \mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K}[\omega_E](x)\right) dx \leq c_{51} \omega(E) \qquad \forall E \subset K,\ E \text{ Borel.} \tag{2.35}$$

As above we also obtain

$$\omega(E) \leq c_{57} cap_{G_{\alpha p}, Q_p^*}(E) \qquad \forall E \subset K,\ E \text{ Borel.}$$

where the positive constant c_{57} depends on r_K. It follows (2.32).
This completes the proof of the Theorem. ∎
Now we begin to prove Theorem 1.5.
Proof of Theorem 1.5. Here we only prove the part **A**. Consider the sequence $\{u_m\}_{m \geq 0}$ of nonnegative functions defined by $u_0 = f$ and

$$u_{m+1} = \mathbf{W}_{\alpha,p}^{R_1}[P_{l,a,\beta}(u_m)] + f \quad \text{in } \mathbb{R}^N \quad \forall m \geq 0.$$

By Theorem 2.5, $\{u_m\}_{m \geq 0}$ is well defined and (2.21) and (2.22) are satisfied. It is easy to see that u_m is nondecreasing. Hence, thanks to the dominated convergence theorem, we obtain that $u(x) = \lim_{m \to \infty} u_m(x)$ is a solution to equation (**??**) which satisfies (**??**).
Conversely, we get the converse directly from Theorem 2.6. ∎

3 Quasilinear Dirichlet Problems

Let Ω be a bounded domain in $\mathbb{R}^N$. If $\mu \in \mathfrak{M}_b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative part in the Jordan decomposition. We denote by $\mathfrak{M}_0(\Omega)$ the space of measures in Ω which are absolutely continuous with respect to the $c_{1,p}^{\Omega}$-capacity defined on a compact set $K \subset \Omega$ by

$$c_{1,p}^{\Omega}(K) = \inf\left\{\int_{\Omega} |\nabla\varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^{\infty}(\Omega)\right\}.$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c_{1,p}^{\Omega}$-capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. It is well known that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ can be written under the form $\mu_0 = f - div\ g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega)$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. If u is a measurable function defined and finite a.e. in Ω, such that $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u|\leq k} v$ a.e. in Ω and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [10].

Definition 3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu \quad in\ \Omega, \\ u &= 0 \quad on\ \partial\Omega, \end{aligned} \tag{3.1}$$

if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist λ_k^+ and λ_k^- belonging to $\mathfrak{M}_b^+ \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\mu_k^+ \rightharpoonup \mu_s^+$, $\mu_k^- \rightharpoonup \lambda_s^-$ in the narrow topology of measures and such that

$$\int_{\{|u|<k\}} |\nabla u|^{p-2} \nabla u . \nabla \varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_\Omega \varphi d\lambda_k^+ - \int_\Omega \varphi d\lambda_k^-,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2 *We recall that if u is a renormalized solution to problem (3.1) then $\frac{|\nabla u|^p}{(|u|+)^r} \in L^1(\Omega)$ for all $r > 1$ which deduce from Holder's inequality, $u \in W_0^{1,p_1}(\Omega)$ for all $1 \leq p_1 < p$ if $e^{a|u|} \in L^1(\Omega)$ for some $a > 0$. Furthermore, $u \geq 0$ a.e. in Ω if $\mu \in \mathfrak{M}_b^+(\Omega)$.*

The following general stability result has been proved in [10, Th 4.1, Sec 5.1].

Theorem 3.3 *If $\{\mu_n\}$ is a bounded sequence in $\mathfrak{M}_b(\Omega)$ and $\{u_n\}$ a sequence of renormalized solutions of problem*

$$\begin{aligned} -\Delta_p u_n &= \mu_n \quad &in\ \Omega, \\ u_n &= 0 \quad &on\ \partial\Omega, \end{aligned} \tag{3.2}$$

then, up to a subsequence, $\{u_n\}$ converges a.e. to a solution of equation $-\Delta_p u = \mu$ in $\mathcal{D}'(\Omega)$, and such that $\frac{1}{k}\int_\Omega |\nabla T_k(u)|^p dx \leq M$ for every $k > 0$.

The next result is a sharp extension of the stability Theorem 3.3.

Theorem 3.4 *Let $\mu = \mu_0 + \mu_s^+ - \mu_s^-$, with $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$ and μ_s^+, μ_s^- belonging to $\mathfrak{M}_s^+(\Omega)$. Let $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$ with $F_n \in L^1(\Omega)$, $g_n \in (L^{p'}(\Omega))^N$ and ρ_n, η_n belonging to $\mathfrak{M}_b^+(\Omega)$. Assume that $\{F_n\}$ converges to F weakly in $L^1(\Omega)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(\Omega))^N$ and $(\operatorname{div} g_n)$ is bounded in $\mathfrak{M}_b(\Omega)$; assume also that $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ to μ_s^- in the narrow topology. If $\{u_n\}$ is a sequence of renormalized solutions of (3.2), then, up to a subsequence, it converges a.e. in Ω to a renormalized solution u of problem (3.1). Furthermore $T_k(u_n)$ converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$ for any $k > 0$.*

We also recall the following estimate [20, Th 2.1].

Theorem 3.5 *Let Ω be a bounded domain of $\mathbb{R}^N$. Then there exists a constant $K_1 > 0$, depending on p and N such that if $\mu \in \mathfrak{M}_b^+(\Omega)$ and u is a nonnegative renormalized solution of problem (3.1) with data μ, there holds*

$$\frac{1}{K_1}\mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \le u(x) \le K_1\mathbf{W}_{1,p}^{2diam(\Omega)}[\mu](x) \quad \forall x \in \Omega, \tag{3.3}$$

where the positive constant K_1 only depends on N, p.

Proof of Theorem 1.1. Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of nonnegative renormalized solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu && \text{in } \Omega,\\ u_0 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu && \text{in } \Omega,\\ u_{m+1} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By Theorem 3.5 we have

$$\begin{aligned} \chi_\Omega u_0 &\le K_1\mathbf{W}_{1,p}^R[\mu],\\ \chi_\Omega u_{m+1} &\le K_1\mathbf{W}_{1,p}^R[P_{l,a,\beta}(u_m) + \mu] \quad \forall m \in \mathbb{N}, \end{aligned}$$

where $R = 2diam(\Omega)$. Thus, by Theorem 2.5 with $f \equiv 0$, there exists $M > 0$ depending on N, p, l, a, β, K_1 and R such that $P_{l,a,\beta}(4c_pK_1\mathbf{W}_{1,p}^R[\omega]) \in L^1(\Omega)$ and

$$u_m(x) \le 2c_pK_1\mathbf{W}_{1,p}^R[\omega](x) \quad \forall x \in \Omega, m \in \mathbb{N}, \tag{3.4}$$

provided that

$$||M_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M,$$

where $\omega = M||\mathbf{M}_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||_{L^\infty(\mathbb{R}^N)}^{-1} + \mu$ and $c_p = 1 \vee 4^{\frac{2-p}{p-1}}$. This implies that $\{u_m\}$ is well defined; by Theorem 3.3 it contains a subsequence that we still denote by $\{u_m\}$ which converges a.e in Ω to function u for which (1.17) is satisfied in Ω. Furthermore, we deduce from (3.4) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 3.4 we obtain that u is a renormalized solution of (1.16).
Conversely, assume that (1.16) admits a nonnegative renormalized solution u. By Theorem 3.5 there holds

$$u(x) \ge \frac{1}{K_1}\mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \Omega.$$

Hence, we achieve (1.18) from Theorem 2.6. ∎

Applications. We consider the case $p = 2$, $\beta = 1$. Then $l = 2$ and

$$P_{l,a,\beta}(r) = e^{ar} - 1 - ar.$$

If Ω is a bounded domain in $\mathbb{R}^N$, there exists $M > 0$ such that if μ is a positive Radon measure in Ω which satisfies

$$\mu(B_t(x)) \leq Mt^{N-2} \qquad \forall t > 0 \text{ and almost all } x \in \Omega,$$

there exists a positive solution u to the following problem

$$\begin{array}{ll} -\Delta u = e^{au} - 1 - au + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array}$$

Furthermore

$$u(x) \leq K(N) \int_0^{2\,diam\,\Omega} \frac{\omega(B_t(x))}{t^{N-1}} dt = K(N) \int_0^{2\,diam\,\Omega} \frac{\mu(B_t(x))}{t^{N-1}} dt + b \qquad \forall x \in \Omega.$$

where $b = 2K(N)M||\mathbf{M}_{p,2diam\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||_{L^\infty(\mathbb{R}^N)}^{-1}(diam\,\Omega)^2$.
In the case $N = 2$ this result has already been proved by Richard and Véron [22, Prop 2.4].

4 p-Superharmonic Funtions And Quasilinear Equations on $\mathbb{R}^N$

We recall some definitions and properties of p-superharmonic functions.
If $u \in W_{loc}^{1,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $-\Delta_p u = 0$ in $\mathcal{D}'(\mathbb{R}^N)$ then u is said to be p-harmonic in $\mathbb{R}^N$. A function u is called an p-supersolution in $\mathbb{R}^N$ if $u \in W_{loc}^{1,p}(\mathbb{R}^N)$ and $-\Delta_p u \geq 0$ in $\mathcal{D}'(\mathbb{R}^N)$.

Definition 4.1 *A Lower semicontinuous (l.s.c) function $u : \mathbb{R}^N \to (-\infty, \infty]$ is called p-superharmonic if u is not identically infinite and if for all open $D \subset\subset \mathbb{R}^N$ and all $v \in C(\overline{D})$, p-harmonic in D, $v \leq u$ on ∂D implies $v \leq u$ in D.*

Let u be an p-superharmonic in $\mathbb{R}^N$. It is well known that $u \wedge k \in W_{loc}^{1,p}(\mathbb{R}^N)$ is a p-supersolution for all $k > 0$ and $u < \infty$ a.e in $\mathbb{R}^N$. Thus, u has gradient, see previous section. We also have $|\nabla u|^{p-1} \in L_{loc}^q(\mathbb{R}^N)$, $\frac{|\nabla u|^p}{(|u|+1)^r} \in L_{loc}^1(\mathbb{R}^N)$ and $u \in L_{loc}^s(\mathbb{R}^N)$ for $1 \leq q < \frac{N}{N-1}$ and $r > 1$, $1 \leq s < \frac{N(p-1)}{N-p}$ see [14, Theorem 7.46]. In particular, thanks to Holder's inequality we have if $e^{a|u|} \in L_{loc}^1(\mathbb{R}^N)$ for some $a > 0$, then $u \in W_{loc}^{1,p_1}(\mathbb{R}^N)$ for all $1 \leq p_1 < p$. Thus by the dominated convergence theorem,

$$\langle -\Delta_p u, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \lim_{k \to \infty} \int_{\mathbb{R}^N} |\nabla (u \wedge k)|^{p-2} \nabla (u \wedge k) \nabla \varphi \geq 0$$

Hence, by the Riesz Representation Theorem we conclude that there is a nonnegative Radon measure denoted by $\mu[u]$, is called Riesz measure such that $-\Delta_p u = \mu[u]$ in $\mathcal{D}'(\mathbb{R}^N)$.
The following weak convergence result for Riesz measures proved in [28] will be used to prove the existence of p-superharmonic solutions to quasilinear equations.

Theorem 4.2 *Suppose that $\{u_n\}$ is a sequence of nonnegative p-superharmonic function in $\mathbb{R}^N$ that converges a.e to an p-supersolution function u. Then the sequence of measures $\{\mu[u_n]\}$ converges to $\mu[u]$ in the weak sense of measure.*

The next theorem is proved in [20]

Theorem 4.3 *Let μ be a measure in $\mathcal{M}^+(\mathbb{R}^N)$. Suppose that $\mathbf{W}_{1,p}[\mu] < \infty$ a.e. Then there is a nonnegative p-superharmonic function u in $\mathbb{R}^N$ such that $-\Delta_p u = \mu$ in $\mathcal{D}'(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} u = 0$ and*

$$\frac{1}{K_1}\mathbf{W}_{1,p}[\mu](x) \le u(x) \le K_1 \mathbf{W}_{1,p}[\mu](x) \tag{4.1}$$

for all x in $\mathbb{R}^N$, where the constant K_1 is as in Theorem 3.5. Furthermore, any p-superharmonic function u in $\mathbb{R}^N$, $\inf_{\mathbb{R}^N} u = 0$ then (4.1) holds with $\mu = -\Delta_p u$.

Proof of Theorem (1.2)**.** Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of p-superharmonic solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu \ \text{ in } \ \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_0 &= 0, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu \ \text{ in } \ \mathcal{D}'(\mathbb{R}^N), \\ \inf_{\mathbb{R}^N} u_{m+1} &= 0. \end{aligned}$$

By Theorem 4.3 we have

$$\begin{aligned} u_0 &\le K_1 \mathbf{W}_{1,p}[\mu], \\ u_{m+1} &\le K_1 \mathbf{W}_{1,p}[P_{l,a,\beta}(u_m) + \mu] \ \ \forall m \in \mathbb{N}. \end{aligned}$$

Thus, by Theorem 2.5 with $f \equiv 0$. there exists $M > 0$ depending on N, p, l, a, β, K_1 and R such that $P_{l,a,\beta}(4c_p K_1 \mathbf{W}_{1,p}[\omega]) \in L^1(\mathbb{R}^N)$ and

$$u_m \le 2c_p K_1 \mathbf{W}_{1,p}[\omega] \quad \forall m \in \mathbb{N}, \tag{4.2}$$

provided that

$$||M_p^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M.$$

where $\omega = M||\mathbf{M}_p^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)} + \mu$. This implies that $\{u_m\}$ is well defined; by [15, Theorem 1.17] it contains a subsequence that we still denote by $\{u_m\}$ which converges a.e in $\mathbb{R}^N$ to function u for which (1.20) is satisfied in Ω. Furthermore, we deduce from (4.2) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\mathbb{R}^N)$. Finally, by Theorem 4.2 we obtain that u is an p-superharmonic solution of (1.19).
Conversely, assume that (1.19) admits a nonnegative renormalized solution u. By Theorem 4.3 there holds

$$u(x) \ge \frac{1}{K_1}\mathbf{W}_{1,p}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \mathbb{R}^N.$$

Hence, we obtain (1.21) from Theorem 2.6. ∎

5 Hessian equations

In this section $\Omega \subset \mathbb{R}^N$ is either a bounded domain with a C^2 boundary or the whole $\mathbb{R}^N$. For $k = 1, .., N$ and $u \in C^2(\Omega)$ the k-hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivative D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \le i_1 < ... < i_k \le N} \lambda_{i_1} ... \lambda_{i_k}$$

We can see that

$$F_k[u] = \left[D^2u\right]_k,$$

where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the k -th principal minors. We assume that $\partial\Omega$ is uniformly (k-1)-convex that is

$$S_{k-1}(\kappa) \ge c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal.

Definition 5.1 *An upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is k-convex (k-subharmonic) if, for every open set $\Omega' \subset \overline{\Omega}' \subset \Omega$ and for every function $v \in C^2(\Omega') \cap C(\overline{\Omega})$ satisfying $F_k[v] \le 0$ in Ω', the following implication is true*

$$u \le v \text{ on } \partial\Omega' \quad \Longrightarrow \quad u \le v \text{ in } \Omega'.$$

We denote by $\Phi^k(\Omega)$ the class of all k-subharmonic functions in Ω which are not identically equal to $-\infty$.

The following weak convergence result for k-Hessian operators proved in [26] is fundamental in our study.

Theorem 5.2 *Let Ω be either a bounded uniformly (k-1)-convex in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$. For each $u \in \Phi^k(\Omega)$, there exist a nonnegative Radon measure $\mu_k[u]$ in Ω such that*

a. *$\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$.*

b. *If $\{u_n\}$ is a sequence of k-convex functions which converges to u a.e then $\mu_k[u_n] \rightharpoonup \mu_k[u]$ in the weak sense of measures.*

As in the case of quasilinear equations with measure data, precise estimates of solutions of k-Hessian equations with measures data are expressed in terms of Wolff potentials. The next results are proved in [26, 18, 20].

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2, uniformly $(k-1)$-convex domain. Let φ be a nonnegative continuous function on $\partial\Omega$ and μ be a nonnegative Radon measure. Suppose that μ can be decomposed as*

$$\mu = \mu_1 + f$$

such that μ_1 is a measure with compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$ if $k \leq \frac{N}{2}$ or $p = 1$ if $k > \frac{N}{2}$. Then there exists a nonnegative function u in Ω such that $-u \in \Phi^k(\Omega)$, continuous near $\partial\Omega$ and u is a solution to the problem

$$\begin{aligned} F_k[-u] &= \mu \qquad \text{in } \Omega, \\ u &= \varphi \qquad \text{on } \partial\Omega. \end{aligned}$$

Furthermore, for any nonnegative function u such that $-u \in \Phi^k(\Omega)$, which is continuous near $\partial\Omega$ and is a solution of above equation, there holds

$$\frac{1}{K_2}\mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[\mu] \leq u(x) \leq K_2\left(\mathbf{W}^{2diam\,\Omega}_{\frac{2k}{k+1},k+1}[\mu](x) + \max_{\partial\Omega}\varphi\right), \tag{5.1}$$

where K_2 is a positive constant independent of x, u and Ω.

Theorem 5.4 *Let μ be a measure in $\mathcal{M}^+(\mathbb{R}^N)$ and $2k < N$. Suppose that $\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu] < \infty$ a.e. Then there exists u, $-u \in \Phi^k(\mathbb{R}^N)$ such that $\inf_{\mathbb{R}^N} u = 0$ and $F_k[-u] = \mu$ in $\mathbb{R}^N$ and*

$$\frac{1}{K_2}\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu](x) \leq u(x) \leq K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu](x) \tag{5.2}$$

for all x in $\mathbb{R}^N$, where the constant K_2 is in above Theorem.
Furthermore, for any nonnegative function u such that $-u \in \Phi^k(\mathbb{R}^N)$ then (5.2) holds with $\mu = F_k[-u]$.

Proof of Theorem 1.3. We defined a sequence of nonnegative functions u_m, continuous near $\partial\Omega$ and such that $-u_m \in \Phi^k(\Omega)$, by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu \qquad \text{in } \Omega, \\ u_0 &= \varphi \qquad \text{on } \partial\Omega, \end{aligned} \tag{5.3}$$

and, for $m \geq 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l,a,\beta}(u_m) + \mu \qquad \text{in } \Omega, \\ u_{m+1} &= \varphi \qquad \text{on } \partial\Omega. \end{aligned} \tag{5.4}$$

By Theorem 5.3 we have

$$\begin{aligned} &\chi_\Omega u_0 \leq K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu] + b_0, \\ &\chi_\Omega u_{m+1} \leq K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u_m) + \mu] + b_0, \end{aligned} \tag{5.5}$$

where $b_0 = K_2 \max_{\partial\Omega}\varphi$ and $R = 2diam\Omega$. Thus, by Theorem 2.5 with $f = b_0$ and $\varepsilon = a$, there exists $M_1 > 0$ depending on N, k, l, a, β and R such that $P_{l,a,\beta}\left(4K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega_1] + g\right) \in L^1(\Omega)$ and

$$u_m(x) \leq 2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega_1](x) + g(x) \qquad \forall x \in \Omega,\ \forall m \geq 0, \tag{5.6}$$

provided that

$$||M^{\frac{k(\beta-1)}{\beta}}_{2k,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M_1 \ \text{ and } ||M^{\frac{k(\beta-1)}{\beta}}_{2k,R}[P_{l,2a,\beta}(b_0)]||_{L^\infty(\mathbb{R}^N)} \leq M_1.$$

where $\omega_1 = M||\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + \mu$, $\omega_2 = M||\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[1]||^{-1}_{L^\infty(\mathbb{R}^N)} + P_{l,2a,\beta}(b_0)$
and $g(x) = 4c_pK\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega_2](x) + b_0$.
We see that one can find constants M_2 depending on N, k, l, a, β, R and M_1 such that if $\max_{\partial\Omega}\varphi \le M_2$ then $||M_{2k,R}^{\frac{k(\beta-1)}{\beta}}[P_{l,2a,\beta}(b_0)]||_{L^\infty(\mathbb{R}^N)} \le M_1$. Beside, g is constant. Hence, we deduce from (5.6) that $P_{l,a,\beta}\left(2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu] + b_2\right) \in L^1(\Omega)$ and

$$u_m(x) \le 2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu](x) + b_2 \qquad \forall x \in \Omega,\ \forall m \ge 0, \tag{5.7}$$

for some the constant b_2 depending on N, k, l, a, β, R and M_1.
Note that because we can write

$$\omega = P_{l,a,\beta}(u_m) + \mu = (\mu_1 + \chi_{\Omega_\delta}P_{l,a,\beta}(u_m)) + ((1-\chi_{\Omega_\delta})P_{l,a,\beta}(u_m) + f),$$

where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$ and $\delta > 0$ is small enough, then ω satisfies the assumptions of the data in Theorem 5.3 since u_m is continuous near $\partial\Omega$. Therefore the sequence $\{u_m\}$ is well defined. Since $-u_m$ is k-subharmonic and the sequence $\{u_m\}$ is bounded in $L^1(\Omega)$, the sequence $\{u_m\}$ is also bounded in $W^{1,1}_{loc}(\Omega)$ (see e.g. [26]); we can find a subsequence, still denoted by the index m, such that u_m converges a.e in Ω to function u for which (1.23) is satisfied in Ω with the constant $b_2 = g$. Furthermore, we deduce from (5.6) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 5.2, we obtain that u satisfies (1.22) and (1.23).
Conversely, assume that (1.22) admits nonnegative solution u, continuous near $\partial\Omega$, such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$. Then by Theorem 5.3 we have

$$u(x) \ge \frac{1}{K_2}\mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u) + \mu](x) \quad \text{for all } x \in \Omega.$$

Using Theorem 2.6, we conclude that (1.24) holds. ∎

Proof of Theorem 1.4. we defined a sequence of nonnegative functions u_m, continuous near $\partial\Omega$ and such that $-u_m \in \Phi^k(\Omega)$, by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu \qquad \text{in } \mathbb{R}^N, \\ \inf_{\mathbb{R}^N} u_0 &= 0 \end{aligned} \tag{5.8}$$

and, for $m \ge 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l,a,\beta}(u_m) + \mu \qquad \text{in } \mathbb{R}^N \\ \inf_{\mathbb{R}^N} u_{m+1} &= 0. \end{aligned} \tag{5.9}$$

By Theorem 5.4, we have

$$\begin{aligned} &u_0 \le K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[\mu], \\ &u_{m+1} \le K_2\mathbf{W}_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u_m) + \mu]. \end{aligned} \tag{5.10}$$

Thus, by Theorem 2.5 with $f \equiv 0$ there exists $M > 0$ depending on N, k, l, a, β and R such that $P_{l,a,\beta}\left(2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega]\right) \in L^1(\mathbb{R}^N)$

$$u_m \le 2K_2\mathbf{W}^R_{\frac{2k}{k+1},k+1}[\omega] \qquad \forall m \ge 0, \tag{5.11}$$

provided that $||M_{2k,R}^{\frac{k(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M$, where $\omega = M||\mathbf{M}_{2k}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_{B_R(0)}]||^{-1}_{L^\infty(\mathbb{R}^N)}\chi_{B_R(0)} + \mu$. Therefore the sequence $\{u_m\}$ is well defined. By arguing as in the proof of theorem 1.3 we obtain that u satisfies (1.25) and (1.26).
Conversely, assume that (1.25) admits nonnegative solution u and $-u \in \Phi^k(\mathbb{R}^N)$, $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$. Then by Theorem 5.4 we have

$$u \geq \frac{1}{K_2}\mathbf{W}_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u) + \mu]$$

Using Theorem 2.6, we conclude that (1.27) holds. ∎

References

[1] D. R. Adams, L.I. Heberg: *Function Spaces and Potential Theory*, Grundlehrender mathematischen Wisenschaften 31, Springer-Verlag (1999).

[2] D.R. Adams and M. Pierre,*Capacitary strong type estimates in semilinear problems*, **Ann. Inst. Fourier (Grenoble)** 41 (1991) 117-135.

[3] P. Baras, M. Pierre, *Critre d'existence des solutions positives pour des quations semi-linaires non monotones*, *Ann. Inst. H. Poincar* 3 (1985) 185-212.

[4] M. F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type* , **Arch. Ration. Mech. Anal.** 107 (1989) 293-324.

[5] M. F. Bidaut-Véron, *Necessary conditions of existence for an elliptic equation with source term and measure data involving p-Laplacian*, in: Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Electron. J. Differ. Equ. Conf. 8 (2002) 23-34.

[6] M. F. Bidaut-Véron, *Removable singularities and existence for a quasilinear equation with absorption or source term and measure data*, **Adv. Nonlinear Stud.** 3 (2003) 25-63.

[7] M. F. Bidaut-Véron and S. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, **J. Anal. Math.** 84 (2001) 1-49.

[8] M. F. Bidaut-Véron, H. Nguyen Quoc, L. Véron: *Quasilinear Lane-Emden equations with absorption and measure data*, **arXiv:1212.6314**, preprint.

[9] I. Birindelli and F. Demengel, *Some Liouville theorems for the p-Laplacian*, in: Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, Electron. J. Differ. Equ. Conf. 8 (2002) 35-46.

[10] G. Dal Maso, F. Murat, L. Orsina, A. Prignet: *Renormalized solutions of elliptic equations with general measure data*, **Ann. Sc. Norm. Sup. Pisa, 28** (1999), 741-808.

[11] R. Fefferman, *Strong differentiation with respect to measure*: **Amer. J. Math 103**, 33-40 (1981).

[12] F. Ferrari, B. Franchi, I. Verbitsky: *Hessian inequality and the fractional Laplacian*, **J. Reine Angew. Math. 667**, 133-148 (2012).

[13] N. Grenon. *Existence results for semilinear elliptic equations with small measure data.*, **Ann. Inst. H. Poincaré Anal. Non Linéaire** 19 (2002), 1-11.

[14] Heinonen, Juha; Kilpelinen, Tero; Martio, Olli: *Nonlinear potential theory of degenerate elliptic equations*. Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006. xii+404 pp.

[15] T. Kilpelainen, J. Malý, *Degenerate elliptic equation with measure data and nonlinear potentials*, **Ann. Sc. Norm. Super. Pisa, Cl. Sci.** 19(1992) 591-613.

[16] T. Kilpelainen, J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. 172(1994) 137-161.

[17] P. Honzik and B. Jaye: *On the good-λ inequality for nonlinear potentials*, **Proc. Amer. Math. Soc. 140**, 4167-4180 (2012).

[18] D. Labutin *Potential estimates for a class of fully nonlinear elliptic equations*: **Duke Math. J. 111**, 1-49 (2002).

[19] M.M. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, Textbooks in Pure and Applied Mathematics (1991).

[20] N. C. Phuc, I. E. Verbitsky: *Quasilinear and Hessian equations of Lane-Emden type*, **Ann. Math. 168**, 859-914 (2008).

[21] N. C. Phuc, I. E. Verbitsky: *Singular quasilinear and Hessian equation and inequalities*, **J. Functional Analysis 256**, 1875-1906 (2009).

[22] Y. Richard, L. Véron: *Isotropic singularities of solutions of nonlinear elliptic inequalities*, **Ann. I.H.P. Analyse Non Linéaire 6**, 37-72 (1989).

[23] J. Serrin, H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities* **Acta Math.** 189(2002) 79-142.

[24] B. O. Turesson: *Nonlinear Potential Theory and weighted Sobolev Spaces*, Lecture Notes in Mathematics, Vol. 1736, Springer-Verlag (2000).

[25] N.S. Trudinger and X.J. Wang: *Hessian measures*, **Topological Methods in Nonlinear Analysis 10**, 225239 (1997).

[26] N.S. Trudinger and X.J. Wang: *Hessian measures II*, **Annals of Mathematics 150**, 579-604 (1999).

[27] N.S. Trudinger and Wang: *Hessian measures III*, **Journal of Functional Analysis 193**, 123 (2002).

[28] N.S. Trudinger and X.J. Wang: *On the weak continuity of elliptic operators and applications to potential theory.* **Amer. J. Math.** 124 (2002), no. 2, 369410.

[29] L. Véron: *Elliptic equations involving measures*,in: Stationary Partial Differential Equations, vol. I, Handbook of Equations, North-Holland, Amsterdam, 2004, pp. 593-712.