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Quasilinear and Hessian type equations with exponential reaction and measure data

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Abstract

We prove existence results concerning equations of the type $-\Delta_p u = P(u) + \mu$ for $p > 1$ and $F_k[-u] = P(u) + \mu$ with $1 \leq k < \frac{N}{2}$ in a bounded domain Ω , where μ is a positive bounded Radon measure and $P(u) \sim e^{au^\beta}$ with $a > 0$ and $\beta \geq 1$. Sufficient conditions for existence are expressed in terms of the fractional maximal potential of μ . Two-sided estimates on the solutions are obtained in terms of some precise Wolff potentials of μ . Necessary conditions are obtained in terms of Orlicz capacities. We also establish existence results for a general Wolff potential equation under the form $u = \mathbf{W}_{\alpha,p}^R[P(u)] + f$.

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Key words: quasilinear elliptic equations, Hessian equations, Wolff potential, maximal functions, Borel measures, Orlicz capacities.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $p > 1$ and $k \in \{1, 2, \dots, N\}$. We denote by

$$\Delta_p u := \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right)$$

the p-Laplace operator and by

$$F_k[u] = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq N} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k}$$

the k-Hessian operator where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of the Hessian matrix D^2u . Let μ be a positive bounded Radon measure in Ω ; our aim is to study the existence of positive renormalized solutions to the quasilinear equation

$$\begin{aligned} -\Delta_p u &= P(u) + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

and the fully nonlinear equation

$$\begin{aligned} F_k[-u] &= P(u) + \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (1.2)$$

where P is an exponential function. When $P(r) = r^q$ with $q > p - 1$, Phuc and Verbitsky published a seminal article [19] on the solvability of the corresponding problem (1.1). They obtained necessary and sufficient conditions involving Bessel capacities or Wolff potentials. For example, they proved that if μ has compact support in Ω it is equivalent to solve (1.1) with $P(r) = r^q$ or to have

$$\mu(E) \leq cC_{p, \frac{q}{q+1-p}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.3)$$

where c is a suitable positive constant and $C_{p, \frac{q}{q+1-p}}$ a Bessel capacity, or to have

$$\int_B [\mathbf{W}_{1,p}^{2R}[\mu_B(x)]]^q dx \leq C\mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \operatorname{supp}\mu \neq \emptyset, \quad (1.4)$$

where $R = \operatorname{diam}(\Omega)$. Other conditions are expressed in terms of Riesz potentials and maximal fractional potentials. Their construction is based upon sharp estimates of solutions of the non-homogeneous problem

$$\begin{aligned} -\Delta_p u &= \omega && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.5)$$

for positive measures ω . We refer to [4, 5, 6, 7, 9, 13, 22] for the previous studies of these and other related results. Concerning k-Hessian operator in a bounded $(k - 1)$ -convex domain Ω , they proved that if μ has compact support and $\|\varphi\|_{L^\infty(\partial\Omega)}$ is small enough, the corresponding problem (1.2) with $P(r) = r^q$ with $q > k$ admits a positive solution if and only if

$$\mu(E) \leq cC_{2k, \frac{q}{q-k}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.6)$$

or equivalently

$$\int_B \left[\mathbf{W}_{\frac{2k}{k+1}, k+1}^{2R} [\mu_B(x)] \right]^q dx \leq C\mu(B) \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \neq \emptyset. \quad (1.7)$$

In the linear case $p = 2$ and $k = 1$, can be found in [2, 3, 27]. The main tools in their proofs are derived from recent advances in potential theory for nonlinear elliptic equations obtained by Kilpelainen and Malý [14, 15], Trudinger and Wang [24, 25, 26], and Labutin [17] thanks to whom the authors first provide global pointwise estimates for solutions of the homogeneous Dirichlet problems in terms of Wolffs potentials of suitable order.

For $s > 1$, $0 \leq \alpha < \frac{N}{s}$, $\eta \geq 0$ and $0 < T \leq \infty$, we recall the expression of the T -truncated Wolff potential of a positive Radon measure μ defined in \mathbb{R}^N by

$$\mathbf{W}_{\alpha, s}^T[\mu](x) = \int_0^T \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t}, \quad (1.8)$$

and the T -truncated η -fractional maximal potential of μ by

$$\mathbf{M}_{\alpha, T}^\eta[\mu](x) = \sup \left\{ \frac{\mu(B_t(x))}{t^{N-\alpha h_\eta(t)}} : 0 < t \leq T \right\}, \quad (1.9)$$

where $h_\eta(t) = (-\ln t)^{-\eta} \chi_{(0, 2^{-1}]}(t) + (\ln 2)^{-\eta} \chi_{[2^{-1}, \infty)}(t)$. If $\eta = 0$, then $h_\eta = 1$ and we denote by $\mathbf{M}_{\alpha, T}[\mu]$ the corresponding T -truncated fractional maximal potential of μ . When the measures are only defined in an open subset $\Omega \subset \mathbb{R}^N$, they are naturally extended by 0 in Ω^c . For $l \in \mathbb{N}^*$, we define the l -truncated exponential function

$$H_l(r) = e^r - \sum_{j=0}^{l-1} \frac{r^j}{j!}, \quad (1.10)$$

and for $a > 0$ and $\beta \geq 1$, we set

$$P_{l, a, \beta}(r) = H_l(ar^\beta). \quad (1.11)$$

We put

$$Q_p(s) = \begin{cases} \sum_{q=l}^{\infty} \frac{s^{\frac{\beta q}{p-1}}}{q^{\frac{\beta q}{p-1}} q!} & \text{if } p \neq 2, \\ H_l(s^\beta) & \text{if } p = 2, \end{cases} \quad (1.12)$$

$Q_p^*(r) = \max \{rs - Q_p(s) : s \geq 0\}$ is the complementary function to Q_p , and define

$$\text{cap}_{G_p, Q_p^*}(E) = \inf \left\{ \int_{\mathbb{R}^N} Q_p^*(f) dx : G_p * f \geq \chi_E, f \geq 0, Q_p^*(f) \in L^1(\mathbb{R}^N) \right\}, \quad (1.13)$$

where $G_p(x) = \mathcal{F}^{-1} \left((1 + |\cdot|^2)^{-\frac{p}{2}} \right) (x)$ is the Bessel kernel of order p .

We denote $a \wedge b$ and $a \vee b$ for $\min\{a, b\}$ and $\max\{a, b\}$ respectively. Our main results are the following theorems

Theorem 1.1 *Let $1 < p < N$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p - 1$ and $a > 0$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If μ is a nonnegative Radon measure in Ω , there exist $b_0 \in (0, 1]$ and $M > 0$ depending on N, p, l, a, β and $\text{diam } \Omega$ which is the diameter of Ω such that if*

$$\|\mathbf{M}_{p, 2\text{diam } \Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M, \quad (1.14)$$

the following Dirichlet problem

$$\begin{aligned} -\Delta_p u &= P_{l,a,\beta}(u) + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.15)$$

admits a nonnegative renormalized solution u , which satisfies

$$u(x) \leq K_1 \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{1,p}^{2\text{diam } \Omega}[\mu](x) + 2b_0 \quad \forall x \in \Omega, \quad (1.16)$$

and

$$P_{l,a,\beta} \left(K_1 \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{1,p}^{2\text{diam } \Omega}[\mu] + 2b_0 \right) \in L^1(\Omega). \quad (1.17)$$

The role of $K_1 = K_1(N, p)$ will be made explicit in Theorem 3.5.

Conversely, if (1.15) admits a nonnegative renormalized solution u , then for any compact set $K \subset \Omega$, there exists a positive constant C depending on N, p, a, β and $\text{dist}(K, \partial\Omega)$ such that

$$\int_E P_{l,\alpha,\beta}(u) dx + \mu(E) \leq C \text{cap}_{G_p, Q_p^*}(E) \quad \text{for all Borel sets } E \subset K. \quad (1.18)$$

Concerning the k -Hessian operator we recall some notions introduced by Trudinger and Wang [24, 25, 26], and we follow their notations. For $k = 1, \dots, N$ and $u \in C^2(\Omega)$ the k -hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives D^2u and S_k is the k -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}.$$

It is straightforward that

$$F_k[u] = [D^2u]_k,$$

where in general $[A]_k$ denotes the sum of the k -th principal minors of a matrix $A = (a_{ij})$. In order that there exists a smooth k -admissible function which vanishes on $\partial\Omega$, the boundary $\partial\Omega$ must satisfy a uniformly $(k-1)$ -convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \quad \text{on } \partial\Omega, \quad (1.19)$$

for some positive constant c_0 , where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal. We also denote by $\Phi^k(\Omega)$ the class of upper-semicontinuous functions $\Omega \mapsto [-\infty, \infty)$ which are k -convex, or subharmonic in the Perron sense (see Definition 4.1). In this paper we prove the following theorem

Theorem 1.2 Let $k \in \{1, 2, \dots, N\}$ such that $2k < N$, $l \in \mathbb{N}^*$, $\beta \geq 1$ such that $l\beta > k$ and $a > 0$. Let Ω be a bounded uniformly $(k-1)$ -convex domain in \mathbb{R}^N . Let φ be a nonnegative continuous function on $\partial\Omega$ and $\mu = \mu_1 + f$ be a nonnegative Radon measure where μ_1 has compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$. Let $K_2 = K_2(N, k)$ be the constant K_2 which appears in Theorem 4.3. Then, there exist $b_0 \in (0, 1]$ and $M > 0$ depending on N, k, l, a, β and $\text{diam}(\Omega)$ such that if

$$\| \mathbf{M}_{2k, 2\text{diam}(\Omega)}^{\frac{k(\beta-1)}{\beta}}[\mu] \|_{L^\infty(\mathbb{R}^N)} \leq M,$$

and $\max_{\partial\Omega} \varphi \leq \frac{b_0}{K_2}$, the following Dirichlet problem

$$\begin{aligned} F_k[-u] &= P_{l,a,\beta}(u) + \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (1.20)$$

admits a nonnegative solution u , continuous near $\partial\Omega$, with $-u \in \Phi^k(\Omega)$ which satisfies

$$u(x) \leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2\text{diam}\Omega}[\mu](x) + 2b_0 \quad \forall x \in \Omega, \quad (1.21)$$

and

$$P_{l,a,\beta} \left(K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2\text{diam}\Omega}[\mu] + 2b_0 \right) \in L^1(\Omega). \quad (1.22)$$

Conversely, if (1.20) admits a nonnegative solution u , continuous near $\partial\Omega$, such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$, then for any a compact set $K \subset \Omega$, there exists a positive constant C depending on N, k, a, β and $\text{dist}(K, \partial\Omega)$ such that there holds

$$\int_E P_{l,a,\beta}(u) dx + \mu(E) \leq C \text{cap}_{G_{2k}, Q_{k+1}^*}(E) \quad \forall E \subset K, E \text{ Borel}, \quad (1.23)$$

where $Q_{k+1}(s)$ is defined by (1.12) with $p = k + 1$, Q_{k+1}^* is its complementary function and $\text{cap}_{G_{2k}, Q_{k+1}^*}(E)$ is defined accordingly by (1.13).

The two previous theorems are connected to the following result which deals with a class of nonlinear Wolff integral equations.

Theorem 1.3 Let $p > 1$, $\alpha > 0$, $a > 0$, $R > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p - 1$ and $0 < \alpha p < N$. Let $f \in L_{loc}^1(\mathbb{R}^N)$, $f \geq 0$. For any $\varepsilon > 0$ there exist constants $b_0 \in (0, 1]$ and $M > 0$ depending on $N, \alpha, p, l, a, \beta, R$ and ε such that if $\exp((a + \varepsilon)f^\beta) \in L_{loc}^1(\mathbb{R}^N)$ and

$$\| \mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu] \|_{L^\infty(\mathbb{R}^N)} \leq M,$$

where $d\mu = P_{l,a+\varepsilon,\beta}(f)dx$, then there exists a nonnegative function u , such that $P_{l,a,\beta}(u) \in L_{loc}^1(\mathbb{R}^N)$ which satisfies

$$u = \mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)] + f \quad \text{in } \mathbb{R}^N. \quad (1.24)$$

Furthermore

$$\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(f)] + f \leq u \leq \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha,p}^R[\mu] + f + b_0 := F \quad \text{in } \mathbb{R}^N. \quad (1.25)$$

Moreover $\exp(aF^\beta) \in L^1_{loc}(\mathbb{R}^N)$ and $\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(F)] \in L^\infty(\mathbb{R}^N)$.

Conversely, if (1.24) admits a nonnegative solution u , then there exists a positive constant C depending on N, α, p, a, β and R such that there holds

$$\int_E P_{l,a,\beta}(u) dx + \int_E P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)]) dx + \int_E P_{l,a,\beta}(f) dx \leq C \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.}$$

2 Estimates on potentials

We denote by $B_r(a)$ the ball of center a and radius $r > 0$ and $B_r = B_r(0)$; χ_E the characteristic function of E for Borel set E . The next estimates are crucial in the sequel.

Theorem 2.1 1. *There exists a positive constant c_1 , depending only on N, α, p, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > p - 1$ we have*

$$(c_1 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}} dx \leq \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx \leq (c_1 q)^q \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}} dx, \quad (2.1)$$

where $\mathbf{G}_{\alpha p}[\mu] := G_{\alpha p} * \mu$ denotes the Bessel potential of order αp of μ .

2. *There exists a positive constant c_2 depending only on N, α, p, R such that for all $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ and $q > p - 1$ we have*

$$c_2^{-q} \int_{\mathbb{R}^N} (\mathbf{G}_{2\alpha}[\mu](x))^q dx \leq \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,2}^R[\mu](x))^q dx \leq c_2^q \int_{\mathbb{R}^N} (\mathbf{G}_{2\alpha}[\mu](x))^q dx. \quad (2.2)$$

Proof. We can find proof of (2.2) in [8, Step 3, Theorem 2.3]. By [8, Step 2, Theorem 2.3], there is $c_3 > 0$ such that

$$\int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx \geq c_3^q \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}} dx \quad \forall q > 0, R > 0 \text{ and } \mu \in \mathfrak{M}^+(\mathbb{R}^N). \quad (2.3)$$

We recall that $\mathbf{M}_{\alpha p,R}[\mu] = \mathbf{M}_{\alpha p,R}^0[\mu]$ by (1.9). Next we show that

$$\int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}} dx \geq (c_4 q)^{-q} \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx \quad \forall q > 0, R > 0 \text{ and } \mu \in \mathfrak{M}^+(\mathbb{R}^N), \quad (2.4)$$

for some a positive constant c_4 . Indeed, we denote μ_n by $\chi_{B_n} \mu$ for $n \in \mathbb{N}^*$. By [16, Theorem 1.2] or [8, Proposition 2.2], there exist constants $c_5 = c_5(N, \alpha, p) > 0$, $a = a(\alpha, p) > 0$ and $\varepsilon_0 = \varepsilon(N, \alpha, p)$ such that for all $n \in \mathbb{N}^*$, $t > 0$, $r > 0$, $0 < R \leq \infty$ and $0 < \varepsilon < \varepsilon_0$, there holds

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R \mu_n > 3t \right\} \right| \leq c_5 \exp(-a\varepsilon^{-1}) \left| \left\{ \mathbf{W}_{\alpha,p}^R \mu_n > t \right\} \right| + \left| \left\{ (\mathbf{M}_{\alpha p,R} \mu_n)^{\frac{1}{p-1}} > \varepsilon t \right\} \right|.$$

Multiplying by qt^{q-1} and integrating over $(0, \infty)$, we obtain

$$\begin{aligned} \int_0^\infty qt^{q-1} |\{\mathbf{W}_{\alpha,p}^R \mu_n > 3t\}| dt &\leq c_5 \exp(-a\varepsilon^{-1}) \int_0^\infty qt^{q-1} |\{\mathbf{W}_{\alpha,p}^R \mu_n > t\}| dt \\ &\quad + \int_0^\infty qt^{q-1} |\{\mathbf{M}_{\alpha p, R} \mu_n^{\frac{1}{p-1}} > \varepsilon t\}| dt, \end{aligned}$$

which implies

$$3^{-q} \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R [\mu_n](x))^q dx \leq c_5 \exp(-a\varepsilon^{-1}) \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R [\mu_n](x))^q dx + \varepsilon^{-q} \int_N (\mathbf{M}_{\alpha p, R} \mu_n)^{\frac{q}{p-1}} dx.$$

This leads to

$$\varepsilon^q (3^{-q} - c_5 \exp(-a\varepsilon^{-1})) \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R [\mu_n](x))^q dx \leq \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p, R} \mu_n)^{\frac{q}{p-1}} dx.$$

We see that $\sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^q (3^{-q} - c_5 \exp(-a\varepsilon^{-1})) \geq (c_6 q)^{-q}$ for some a constant c_6 which does not depend on q . Therefore, (2.4) follows by Fatou's lemma. From (2.3) and (2.4), we have

$$c_7^q \int_{\mathbb{R}^N} (\mathbf{W}_{\frac{\alpha p}{2}, 2}^R [\mu](x))^q dx \geq \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p, R} [\mu](x))^q dx \geq (c_7 q)^{-q} \int_N (\mathbf{W}_{\frac{\alpha p}{2}, 2}^R [\mu](x))^q dx,$$

for all $q > 0$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$, with $c_7 = c_7(N, \alpha, p) > 0$. Then, combining with (2.2) we get

$$c_8^q \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p} [\mu](x))^q dx \geq \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p, R} [\mu](x))^q dx \geq (c_8 q)^{-q} \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p} [\mu](x))^q dx,$$

with $c_8 = c_8(N, \alpha, p, R)$, for all $q > 0$ and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$. Therefore, jointly with (2.3) and (2.4) we derive (2.1). \blacksquare

The next result is proved in [8].

Theorem 2.2 *Let $\alpha > 0$, $p > 1$, $0 \leq \eta < p - 1$, $0 < \alpha p < N$ and $L > 0$. Set $\delta = \frac{1}{2} \left(\frac{p-1-\eta}{12(p-1)} \right)^{\frac{p-1}{p-1-\eta}} \alpha p \log(2)$. Then there exists $C(L) > 0$, depending on N , α , p , η and L such that for any $R \in (0, \infty]$, $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$, any $a \in \mathbb{R}^N$ and $0 < r \leq L$, there holds*

$$\frac{1}{|B_{2r}(a)|} \int_{B_{2r}(a)} \exp \left(\delta \frac{(\mathbf{W}_{\alpha,p}^R [\mu_{B_r(a)}](x))^{\frac{p-1}{p-1-\eta}}}{\|\mathbf{M}_{\alpha p, R}^\eta [\mu_{B_r(a)}]\|_{L^\infty(B_r(a))}^{\frac{1}{p-1-\eta}}} \right) dx \leq C(L), \quad (2.5)$$

where $\mu_{B_r(a)} = \chi_{B_r(a)} \mu$. Furthermore, if $\eta = 0$, C is independent of L .

Theorem 2.3 Assume α, p, η are as in Theorem 2.2 and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ satisfies for $R > 0$

$$\|\mathbf{M}_{\alpha,p,R}^\eta[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

for some $M > 0$. Then there exist $C = C(N, \alpha, p, \eta, R) > 0$ and $\delta_0 = \delta_0(N, \alpha, p, \eta) > 0$ such that

$$\left\| \mathbf{W}_{\alpha,p}^R \left[\exp \left(\delta_0 M^{-\frac{1}{p-1-\eta}} (\mathbf{W}_{\alpha,p}^R[\mu])^{\frac{p-1}{p-1-\eta}} \right) \right] \right\|_{L^\infty(\mathbb{R}^N)} \leq C. \quad (2.6)$$

Proof. Without loss of generality we may assume that $M = 1$. Let δ be as in Theorem 2.2 and $\gamma = \frac{p-1}{p-1-\eta}$. Let $x \in \mathbb{R}^N$. Since $\mu(B_t(y)) \leq Mt^{N-\alpha p} h_\eta(t)$, for all $r \in (0, R)$ and $y \in \mathbb{R}^N$ we have

$$\begin{aligned} \mathbf{W}_{\alpha,p}^R[\mu](y) &= \mathbf{W}_{\alpha,p}^r[\mu](y) + \int_r^R \left(\frac{\mu(B_t(y))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha,p}^r[\mu](y) + \int_{r \wedge 2^{-1}}^{2^{-1}} \left((-\ln t)^{-\eta} \right)^{\frac{1}{p-1}} \frac{dt}{t} + \int_{2^{-1}}^{R \vee 2^{-1}} \left((\ln 2)^{-\eta} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha,p}^r[\mu](y) + \gamma (-\ln(r \wedge 2^{-1}))^{\frac{1}{\gamma}} + (\ln 2)^{-\frac{\eta}{p-1}} \ln(2R \vee 1). \end{aligned}$$

Thus,

$$(\mathbf{W}_{\alpha,p}^R[\mu](y))^\gamma \leq 3^{\gamma-1} \left((\mathbf{W}_{\alpha,p}^r[\mu](y))^\gamma + \gamma^\gamma \ln \left(\frac{1}{r \wedge 2^{-1}} \right) + A_1(R) \right), \quad (2.7)$$

where $A_1(R) = (\ln 2)^{1-\gamma} (\ln(2R \vee 1))^\gamma$.

Let $\theta \in (0, 1]$, since $\exp\left(\frac{a+b}{2}\right) \leq \exp(a) + \exp(b)$ for all $a, b \in \mathbb{R}$, we get from (2.7)

$$\exp\left(\theta \delta 3^{-\gamma} (\mathbf{W}_{\alpha,p}^R[\mu](y))^\gamma\right) \leq \exp\left(\delta (\mathbf{W}_{\alpha,p}^r[\mu](y))^\gamma\right) + \exp\left(\theta \delta \gamma^\gamma \ln \left(\frac{1}{r \wedge 2^{-1}} \right) + \theta \delta A_1(R)\right). \quad (2.8)$$

For $r > 0, 0 < t \leq r, y \in B_r(x)$ we have $B_t(y) \subset B_{2r}(x)$. Thus, $\mathbf{W}_{\alpha,p}^r[\mu] = \mathbf{W}_{\alpha,p}^r[\mu_{B_{2r}(x)}]$ in $B_r(x)$. Then, using (2.5) with $L = 2R$ we get

$$\int_{B_r(x)} \exp\left(\delta (\mathbf{W}_{\alpha,p}^r[\mu])^\gamma\right) = \int_{B_r(x)} \exp\left(\delta (\mathbf{W}_{\alpha,p}^r[\mu_{B_{2r}(x)}])^\gamma\right) \leq C(2R) |B_4| r^N.$$

So, from (2.8) we deduce

$$\begin{aligned} &\mathbf{W}_{\alpha,p}^R \left[\exp\left(\theta \delta 3^{-\gamma} (\mathbf{W}_{\alpha,p}^R[\mu])^\gamma\right) \right](x) \\ &\leq \int_0^R \left(C(2R) |B_4| r^{\alpha p} + |B_1| \exp(\theta \delta A_1(R)) (r \wedge 2^{-1})^{-\theta \delta \gamma^\gamma} r^{\alpha p} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\leq \left(1 \vee 2^{\frac{2-p}{p-1}} \right) (C(2R) |B_4|)^{\frac{1}{p-1}} \int_0^R r^{\frac{\alpha p}{p-1}-1} dr \\ &\quad + \left(1 \vee 2^{\frac{2-p}{p-1}} \right) (|B_1| \exp(\theta \delta A_1(R)))^{\frac{1}{p-1}} \int_0^R (r \wedge 2^{-1})^{-\frac{\theta \delta \gamma^\gamma}{p-1}} r^{\frac{\alpha p}{p-1}-1} dr. \end{aligned}$$

Now we choose $\theta = 1 \wedge \frac{\alpha p}{2\delta\gamma^\gamma}$, we have

$$\begin{aligned} \int_0^R (r \wedge 2^{-1})^{-\frac{\theta\delta\gamma^\gamma}{p-1}} r^{\frac{\alpha p}{p-1}-1} dr &\leq \int_0^R \left(\frac{r}{r \wedge 2^{-1}}\right)^{\frac{\alpha p}{2(p-1)}} r^{\frac{\alpha p}{2(p-1)}-1} dr \\ &\leq \frac{2(p-1)}{\alpha p} (2R+1)^{\frac{\alpha p}{2(p-1)}} R^{\frac{\alpha p}{2(p-1)}}. \end{aligned}$$

Hence

$$\mathbf{W}_{\alpha,p}^R \left[\exp \left(\delta_0 (\mathbf{W}_{\alpha,p}^R [\mu])^\gamma \right) \right] (x) \leq A_2(R),$$

where $\delta_0 = \theta\delta 3^{-\gamma} = \left(1 \wedge \frac{\alpha p}{2\delta\gamma^\gamma}\right) \delta 3^{-\gamma}$ and

$$\begin{aligned} A_2(R) &= \left(1 \vee 2^{\frac{2-p}{p-1}}\right) (C(2R)|B_4|)^{\frac{1}{p-1}} \frac{p-1}{\alpha p} R^{\frac{\alpha p}{p-1}} \\ &\quad + \left(1 \vee 2^{\frac{2-p}{p-1}}\right) (|B_1| \exp(\theta\delta A_1(R)))^{\frac{1}{p-1}} \frac{2(p-1)}{\alpha p} (2R+1)^{\frac{\alpha p}{2(p-1)}} R^{\frac{\alpha p}{2(p-1)}}. \end{aligned}$$

Which yields (2.6). ■

In the next result we obtain estimate on iterative solutions of Wolff integral inequalities. We recall that H_l and $P_{l,a,\beta}$ have been defined in (1.10) and (1.11).

Theorem 2.4 *Let $\alpha > 0$, $p > 1$, $a > 0$, $b \geq 0$, $K > 0$, $R > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p-1$ and $0 < \alpha p < N$. Suppose that $\{u_m\}$ is a sequence of nonnegative functions in \mathbb{R}^N that satisfies*

$$\begin{aligned} u_{m+1} &\leq K \mathbf{W}_{\alpha,p}^R [P_{l,a,\beta}(u_m) + \mu] + b \quad \forall m \in \mathbb{N}, \\ u_0 &\leq K \mathbf{W}_{\alpha,p}^R [\mu] + b, \end{aligned} \tag{2.9}$$

where $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$. Then there exist $b_0 \in (0, 1]$ and $M > 0$ depending on $N, \alpha, p, l, a, \beta, K$ and R such that if $b \leq b_0$ and

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$\exp \left(a (c_p K \mathbf{W}_{1,p}^R [\mu] + 2b_0)^\beta \right) \in L_{loc}^1(\mathbb{R}^N), \tag{2.10}$$

and

$$\mathbf{W}_{\alpha,p}^R [P_{l,a,\beta}(c_p K \mathbf{W}_{\alpha,p}^R [\mu] + 2b_0)] \in L^\infty(\mathbb{R}^N), \tag{2.11}$$

and finally

$$u_m \leq c_p K \mathbf{W}_{\alpha,p}^R [\mu] + 2b_0 \quad \forall m \in \mathbb{N}, \tag{2.12}$$

where $c_p = 1 \vee 2^{\frac{2-p}{p-1}}$.

Proof. By Theorem 2.3 with $\eta = \frac{(p-1)(\beta-1)}{\beta}$, there exist $C = C(N, \alpha, p, \beta, R) > 0$ and $\delta_0 = \delta_0(N, \alpha, p, \beta) > 0$ such that if

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

then

$$\mathbf{W}_{\alpha, p}^R \left[\exp \left(\delta_0 M^{-\frac{\beta}{p-1}} (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) \right] (x) \leq C \quad \forall x \in \mathbb{R}^N. \quad (2.13)$$

Take $b_0 \in [b, 1]$. We will choose b_0 and M later on. Clearly, (2.12) holds with $m = 0$. Now, assume that (2.12) holds with $m = n$, we need to prove that

$$u_{n+1} \leq c_p K \mathbf{W}_{\alpha, p}^R[\mu] + 2b_0.$$

In fact, by the definition of u_{n+1} and the sub-additive property of $\mathbf{W}_{\alpha, p}^R[\cdot]$, we have

$$u_{n+1} \leq c_p K \mathbf{W}_{\alpha, p}^R[\mu] + c_p K \mathbf{W}_{\alpha, p}^R[P_{l, a, \beta}(u_n)] + b.$$

So, it is sufficient to prove that

$$c_p K \mathbf{W}_{\alpha, p}^R[P_{l, a, \beta}(u_n)] \leq b_0. \quad (2.14)$$

Since (2.12) holds with $m = n$ and the convexity property of H_l in $[0, \infty)$ and $b_0 \leq 1$, we have

$$\begin{aligned} P_{l, a, \beta}(u_n) &\leq H_l \left(a (c_p K \mathbf{W}_{\alpha, p}^R[\mu] + 2b_0)^\beta \right) \\ &\leq H_l \left(a c_p^\beta K^\beta 2^{\beta-1} (\mathbf{W}_{\alpha, p}^R[\mu])^\beta + a 2^{2\beta-1} b_0^\beta \right) \\ &\leq H_l \left(a c_p^\beta K^\beta 2^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) + H_l(a 2^{2\beta} b_0^\beta) \\ &\leq H_l \left(a c_p^\beta K^\beta 2^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) + b_0^{l\beta} H_l(a 2^{2\beta}), \end{aligned}$$

which implies

$$\begin{aligned} c_p K \mathbf{W}_{\alpha, p}^R[P_{l, a, \beta}(u_n)] &\leq c_p^2 K \mathbf{W}_{\alpha, p}^R \left[H_l \left(a c_p^\beta K^\beta 2^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) \right] + c_p^2 K \mathbf{W}_{\alpha, p}^R \left[b_0^{l\beta} H_l(a 2^{2\beta}) \right] \\ &= c_p^2 K \mathbf{W}_{\alpha, p}^R \left[H_l \left(a c_p^\beta K^\beta 2^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) \right] + A(R) b_0^{\frac{l\beta}{p-1}}, \end{aligned} \quad (2.15)$$

where $A(R) = \frac{p-1}{\alpha p} c_p^2 K (H_l(a 2^{2\beta}))^{\frac{1}{p-1}} |B_1|^{\frac{1}{p-1}} R^{\frac{\alpha p}{p-1}}$.

Now we choose $b_0 = 1 \wedge \left((2A(R))^{-\frac{p-1}{l\beta-p+1}} \right)$ and we derive

$$A(R) b_0^{\frac{l\beta}{p-1}} \leq \frac{b_0}{2}. \quad (2.16)$$

Next, set $\theta \in (0, 1]$; since $H_l(s) \leq \theta H_l(\theta^{-1}s)$ for all $s \geq 0$,

$$c_p^2 K \mathbf{W}_{\alpha, p}^R \left[H_l \left(a c_p^\beta K^\beta 2^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) \right] \leq \theta^{\frac{1}{p-1}} c_p^2 K \mathbf{W}_{\alpha, p}^R \left[H_l \left(\theta^{-1} a c_p^\beta K^\beta 2^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) \right].$$

From (2.13), we assert that if

$$\theta^{-1}ac_p^\beta K^\beta 2^\beta \leq \delta_0 M^{-\frac{\beta}{p-1}} \iff M \leq \left(\frac{\delta_0}{ac_p^\beta K^\beta 2^\beta} \right)^{\frac{p-1}{\beta}} \theta^{\frac{p-1}{\beta}},$$

then

$$c_p^2 K \mathbf{W}_{\alpha,p}^R \left[H_l \left(ac_p^\beta K^\beta 2^\beta (\mathbf{W}_{\alpha,p}^R[\mu])^\beta \right) \right] \leq \theta^{\frac{1}{p-1}} c_p^2 K C.$$

This will be achieved if we prove that

$$\theta^{\frac{1}{p-1}} c_p^2 K C \leq \frac{b_0}{2},$$

which is equivalent to

$$\theta \leq \left(\frac{b_0}{2c_p^2 K C} \right)^{p-1}.$$

Thus, we can choose $\theta = 1 \wedge \left(\frac{b_0}{2c_p^2 K C} \right)^{p-1}$ and $M = \left(\frac{\delta_0}{ac_p^\beta K^\beta 2^\beta} \right)^{\frac{p-1}{\beta}} \theta^{\frac{p-1}{\beta}}$ and we obtain that

$$c_p^2 K \mathbf{W}_{\alpha,p}^R \left[H_l \left(ac_p^\beta K^\beta 2^\beta (\mathbf{W}_{\alpha,p}^R[\mu])^\beta \right) \right] \leq \frac{b_0}{2}.$$

Combining this with (2.16) and (2.15) we obtain (2.14) and (2.10), (2.11). This completes the proof of the theorem. \blacksquare

Let $P \in C(\mathbb{R}^+)$ be a decreasing positive function; the (α, P) -Orlicz-Bessel capacity of a Borel set $E \subset \mathbb{R}^N$ is defined by (see [1, Sect 2.6])

$$\text{cap}_{G_{\alpha,P}}(E) = \inf \left\{ \int_{\mathbb{R}^N} P(f) : G_\alpha * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N) \right\}.$$

Theorem 2.5 *Let $\alpha > 0, p > 1, a > 0, c > 0, \delta \in (0, 1], l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p - 1$ and $0 < \alpha p < N$. Let Ω be a bounded domain in \mathbb{R}^N and $\mu \in \mathfrak{M}_b^+(\Omega)$. If u is a nonnegative Borel function in Ω such that $P_{l,\alpha,\beta}(u) \in L^1(\Omega)$ and*

$$u(x) \geq c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega](x) \quad \forall x \in \Omega, \quad (2.17)$$

where $d\omega = P_{l,\alpha,\beta}(u)dx + d\mu$, then, for any a compact set $K \subset \Omega$, there exists a positive constant C depending on $N, \alpha, p, a, c, \beta, \delta$ and $\text{dist}(K, \partial\Omega)$ such that

$$\int_E P_{l,\alpha,\beta}(u)dx + \mu(E) \leq C \text{cap}_{G_{\alpha,p}, Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel},$$

where Q_p^* is the complementary function to Q_p .

Proof. Let $K \subset \Omega$. Set $r_K = \text{dist}(K, \partial\Omega)$ and $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$. We have

$$P_{l,a,\beta} \left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega] \right) dx \leq d\omega \quad \text{in } \Omega.$$

Let M_ω denote the centered Hardy-Littlewood maximal function which is defined for any $f \in L^1_{loc}(\mathbb{R}^N, d\omega)$ by

$$M_\omega f(x) = \sup_{t>0} \frac{\int_{B_t(x)} |f| d\omega}{\omega(B_t(x))}.$$

Thus, for any Borel set $E \subset K$

$$\int_{\Omega} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta} \left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega] \right) dx \leq \int_{\Omega} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

Since M_ω is bounded on $L^s(\mathbb{R}^N, d\omega)$, $s > 1$, we deduce from Fefferman's result [11] that

$$\int_{\Omega} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta} \left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega] \right) dx \leq c_9 \omega(E),$$

for some constant c_9 only depends on N and $\frac{l\beta}{p-1}$. Since $M_\omega \chi_E \leq 1$, we derive

$$\begin{aligned} (M_\omega \chi_E(x))^{\frac{l\beta}{p-1}} P_{l,a,\beta} \left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega](x) \right) &\geq P_{l,a,\beta} \left(c(M_\omega \chi_E(x))^{\frac{1}{p-1}} \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega](x) \right) \\ &\geq P_{l,a,\beta} \left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \right), \end{aligned}$$

where $\omega_E = \chi_E \omega$. This implies

$$\int_{\Omega} P_{l,a,\beta} \left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \right) dx \leq c_9 \omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.18)$$

Note that if $x \in \Omega$ and $d(x, \partial\Omega) \leq r_K/8$, then $B_t(x) \subset \Omega \setminus \Omega_K$ for all $t \in (0, \delta d(x, \partial\Omega))$; indeed, for all $y \in B_t(x)$

$$d(y, \partial\Omega) \leq d(x, \partial\Omega) + |x - y| < (1 + \delta)d(x, \partial\Omega) < \frac{1}{4}r_K,$$

thus

$$d(y, K) \geq d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4}r_K > \frac{1}{2}r_K,$$

which implies $y \notin \Omega_K$. We deduce that

$$\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \geq \mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) \quad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) = 0 \quad \forall x \in \Omega^c.$$

Hence we obtain from (2.18),

$$\int_{\mathbb{R}^N} P_{l,a,\beta} \left(c \mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K} [\omega_E](x) \right) dx \leq c_9 \omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.19)$$

From (2.1) and (2.2) we get

$$\int_{\mathbb{R}^N} P_{l,a,\beta} \left(c \mathbf{W}_{\alpha,p}^{\frac{\delta}{8} r_K} [\omega_E](x) \right) dx \geq \int_{\mathbb{R}^N} Q_p (c_{10} \mathbf{G}_{\alpha p} [\omega_E](x)) dx,$$

where Q_p is defined by (1.12) and $c_{10} = (c_1 \beta)^{-1} a^{\frac{p-1}{\beta}} c^{p-1}$ if $p \neq 2$, $c_{10} = c_1^{-1} a^{\frac{1}{\beta}} c$ if $p = 2$ (the constant c_1 defined in (2.1), depends on R , therefore $c_{10} = c_{10}(r_K)$). Thus, from (2.19) we obtain

$$\int_{\mathbb{R}^N} Q_p (c_{10} \mathbf{G}_{\alpha p} [\omega_E](x)) dx \leq c_9 \omega(E) \quad \forall E \subset K, E \text{ Borel.}$$

We recall that $Q_p^*(s) = \sup_{t>0} \{st - Q_p(t)\}$ and we note that Q_p^* satisfies the sub-additivity Δ_2 -condition (see Chapter 2 in [18]). Let $E \subset K$ be a Borel set. For every $f \geq 0$, $Q_p^*(f) \in L^1(\Omega)$ such that $G_{\alpha p} * f \geq \chi_E$, we have

$$\begin{aligned} \omega(E) &\leq \int_{\mathbb{R}^N} G_{\alpha p} * f d\omega_E = \int_{\mathbb{R}^N} \mathbf{G}_{\alpha p} [\omega_E] f dx \\ &= (2c_9)^{-1} \int_{\mathbb{R}^N} (c_{10} \mathbf{G}_{\alpha p} [\omega_E]) (2c_9 c_{10}^{-1} f) dx \\ &\leq (2c_9)^{-1} \int_{\mathbb{R}^N} Q_p (c_{10} \mathbf{G}_{\alpha p} [\omega_E]) dx + (2c_9)^{-1} \int_{\mathbb{R}^N} Q_p^* (2c_9 c_{10}^{-1} f) dx \\ &\leq 2^{-1} \omega(E) + c_{11} \int_{\mathbb{R}^N} Q_p^* (f) dx, \end{aligned}$$

the last inequality following from the Δ_2 -condition. Notice that c_{11} , as well as the next constant c_{12} , depends on r_K . Thus,

$$\omega(E) \leq 2c_{11} \int_{\mathbb{R}^N} Q_p^* (f) dx.$$

Then, we get

$$\omega(E) \leq c_{12} \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel.}$$

This completes the proof of the Theorem. ■

3 Quasilinear Dirichlet Problems

Let Ω be a bounded domain in \mathbb{R}^N . If $\mu \in \mathfrak{M}_b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative part in the Jordan decomposition. We denote by $\mathfrak{M}_0(\Omega)$ the space of

measures in Ω which are absolutely continuous with respect to the $c_{1,p}^\Omega$ -capacity defined on a compact set $K \subset \Omega$ by

$$c_{1,p}^\Omega(K) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega) \right\}.$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c_{1,p}^\Omega$ -capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. It is well known that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ can be written under the form $\mu_0 = f - \operatorname{div} g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega)$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. If u is a measurable function defined and finite a.e. in Ω , such that $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \rightarrow \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u| \leq k} v$ a.e. in Ω and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [10].

Definition 3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist λ_k^+ and λ_k^- belonging to $\mathfrak{M}_b^+ \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\mu_k^+ \rightharpoonup \mu_s^+$, $\mu_k^- \rightharpoonup \lambda_s^-$ in the narrow topology of measures and such that

$$\int_{\{|u| < k\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\{|u| < k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda_k^+ - \int_{\Omega} \varphi d\lambda_k^-,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2 *We recall that if u is a renormalized solution to problem (3.1) and $\mu \in \mathfrak{M}_b^+(\Omega)$, then $u \geq 0$ a.e. in Ω .*

The following general stability result has been proved in [10, Th 4.1, Sec 5.1].

Theorem 3.3 *If $\{\mu_n\}$ is a bounded sequence in $\mathfrak{M}_b(\Omega)$ and $\{u_n\}$ a sequence of renormalized solutions of problem*

$$\begin{aligned} -\Delta_p u_n &= \mu_n & \text{in } \Omega, \\ u_n &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

then, up to a subsequence, $\{u_n\}$ converges a.e. to a solution of equation $-\Delta_p u = \mu$ in $\mathcal{D}'(\Omega)$, and such that $\frac{1}{k} \int_{\Omega} |\nabla T_k(u)|^p dx \leq M$ for every $k > 0$.

The next result is a sharp extension of the stability Theorem 3.3.

Theorem 3.4 *Let $\mu = \mu_0 + \mu_s^+ - \mu_s^-$, with $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$ and μ_s^+, μ_s^- belonging to $\mathfrak{M}_s^+(\Omega)$. Let $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$ with $F_n \in L^1(\Omega)$, $g_n \in (L^{p'}(\Omega))^N$ and ρ_n, η_n belonging to $\mathfrak{M}_b^+(\Omega)$. Assume that $\{F_n\}$ converges to F weakly in $L^1(\Omega)$, $\{g_n\}$ converges to g strongly*

in $(L^{p'}(\Omega))^N$ and $(\operatorname{div} g_n)$ is bounded in $\mathfrak{M}_b(\Omega)$; assume also that $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ to μ_s^- in the narrow topology. If $\{u_n\}$ is a sequence of renormalized solutions of (3.2), then, up to a subsequence, it converges a.e. in Ω to a renormalized solution u of problem (3.1). Furthermore $T_k(u_n)$ converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$ for any $k > 0$.

We also recall the following estimate [19, Th 2.1].

Theorem 3.5 *Let Ω be a bounded domain of \mathbb{R}^N . Then there exists a constant $K_1 > 0$, depending on p and N such that if $\mu \in \mathfrak{M}_b^+(\Omega)$ and u is a nonnegative renormalized solution of problem (3.1) with data μ , there holds*

$$\frac{1}{K_1} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \leq u(x) \leq K_1 \mathbf{W}_{1,p}^{2\operatorname{diam}(\Omega)}[\mu](x) \quad \forall x \in \Omega, \quad (3.3)$$

where K_1 is a constant independent of x, u and Ω .

Proof of Theorem 1.1. Let $\{u_m\}_{m \in \mathbb{N}}$ be a sequence of nonnegative renormalized solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu && \text{in } \Omega, \\ u_0 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu && \text{in } \Omega, \\ u_{m+1} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By Theorem 3.5 we have for all $x \in \mathbb{R}^N$

$$\begin{aligned} \chi_\Omega u_0(x) &\leq K_1 \mathbf{W}_{1,p}^R[\mu](x), \\ \chi_\Omega u_{m+1}(x) &\leq K_1 \mathbf{W}_{1,p}^R[P_{l,a,\beta}(u_m) + \mu](x), \end{aligned}$$

where $R = 2\operatorname{diam}(\Omega)$. Thus, by Theorem 2.4 there exist $b_0 \in (0, 1]$ and $M > 0$ depending on N, p, l, a, β, K_1 and R such that

$$u_m(x) \leq K_1 \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{1,p}^R[\mu](x) + 2b_0 \quad \forall x \in \Omega, m \in \mathbb{N}, \quad (3.4)$$

and

$$\exp\left(a \left(K \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{1,p}^R[\mu] + 2b_0\right)^\beta\right) \in L_{loc}^1(\mathbb{R}^N), \quad (3.5)$$

provided that

$$\|M_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M.$$

This implies that $\{u_m\}$ is well defined; by Theorem 3.3 it contains a subsequence that we still denote by $\{u_m\}$ which converges a.e in Ω to function u for which (1.16) is satisfied in Ω . Furthermore, we deduce from (3.4)-(3.5) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 3.4 we obtain that u is a renormalized solution of (1.15).

Conversely, assume that (1.15) admits a nonnegative renormalized solution u . By Theorem 3.5 there holds

$$u(x) \geq \frac{1}{K_1} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\omega](x) \quad \text{for all } x \in \Omega,$$

where $d\omega = P_{l,a,\beta}(u)dx + d\mu$. By Theorem 2.5, we obtain (1.18). ■

Applications. We consider the case $p = 2$, $\beta = 1$. Then $l = 2$ and

$$P_{l,a,\beta}(r) = e^{ar} - 1 - ar.$$

If Ω is a bounded domain in \mathbb{R}^N , there exists $M > 0$ and $b_0 \in (0, 1)$ such that if μ is a positive Radon measure in Ω which satisfies

$$\mu(B_t(x)) \leq Mt^{N-2} \quad \forall t > 0 \text{ and almost all } x \in \Omega,$$

there exists a positive solution u to the following problem

$$\begin{aligned} -\Delta u &= e^{au} - 1 - au + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Furthermore

$$u(x) \leq K(N) \int_0^{2 \operatorname{diam} \Omega} \frac{\mu(B_t(x))}{t^{N-1}} dt + 2b_0 \quad \forall x \in \Omega.$$

In the case $N = 2$ this result has already been proved by Richard and Véron [21, Prop 2.4].

4 Hessian equations

In this section $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2 boundary. For $k = 1, \dots, N$ and $u \in C^2(\Omega)$ the k-hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivative D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}$$

We can see that

$$F_k[u] = [D^2u]_k,$$

where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the k -th principal minors. We assume that $\partial\Omega$ is uniformly (k-1)-convex that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant c_0 , where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal.

Definition 4.1 An upper-semicontinuous function $u : \Omega \rightarrow [-\infty, \infty)$ is k -convex (k -subharmonic) if, for every open set $\Omega' \subset \overline{\Omega'} \subset \Omega$ and for every function $v \in C^2(\Omega') \cap C(\overline{\Omega'})$ satisfying $F_k[v] \leq 0$ in Ω' , the following implication is true

$$u \leq v \text{ on } \partial\Omega' \implies u \leq v \text{ in } \Omega'.$$

We denote by $\Phi^k(\Omega)$ the class of all k -subharmonic functions in Ω which are not identically equal to $-\infty$.

The following weak convergence result for k -Hessian operators proved in [25] is fundamental in our study.

Theorem 4.2 Let Ω be a bounded uniformly $(k-1)$ -convex in \mathbb{R}^N . For each $u \in \Phi^k(\Omega)$, there exist a nonnegative Radon measure $\mu_k[u]$ in Ω such that

- a. $\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$.
- b. If $\{u_n\}$ is a sequence of k -convex functions which converges to u a.e then $\mu_k[u_n] \rightharpoonup \mu_k[u]$ in the weak sense of measures.

As in the case of quasilinear equations with measure data, precise estimates of solutions of k -Hessian equations with measures data are expressed in terms of Wolff potentials. The next result is proved in [25, 17, 19].

Theorem 4.3 Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 , uniformly $(k-1)$ -convex domain. Let φ be a nonnegative continuous function on $\partial\Omega$ and μ be a nonnegative Radon measure. Suppose that μ can be decomposed as

$$\mu = \mu_1 + f$$

such that μ_1 is a measure with compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$ if $k \leq \frac{N}{2}$ or $p = 1$ if $k > \frac{N}{2}$. Then there exists a nonnegative function u in Ω such that $-u \in \Phi^k(\Omega)$, continuous near $\partial\Omega$ and u is a solution to the problem

$$\begin{aligned} F_k[-u] &= \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega. \end{aligned}$$

Furthermore, for any nonnegative function u such that $-u \in \Phi^k(\Omega)$, which is continuous near $\partial\Omega$ and is a solution of above equation, there holds

$$\frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}}[\mu] \leq u(x) \leq K_2 \left(\mathbf{W}_{\frac{2k}{k+1}, k+1}^{2 \text{diam } \Omega}[\mu](x) + \max_{\partial\Omega} \varphi \right), \quad (4.1)$$

where K_2 is a positive constant independent of x, u and Ω .

Proof of Theorem 1.2. We defined a sequence of nonnegative functions u_m , continuous near $\partial\Omega$ and such that $-u_m \in \Phi^k(\Omega)$, by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu && \text{in } \Omega, \\ u_0 &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (4.2)$$

and, for $m \geq 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l, \alpha, \beta}(u_m) + \mu && \text{in } \Omega, \\ u_{m+1} &= \varphi && \text{on } \partial\Omega. \end{aligned} \quad (4.3)$$

By Theorem 4.3 we have for all $x \in \mathbb{R}^N$,

$$\begin{aligned}\chi_\Omega u_0(x) &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu](x) + b, \\ \chi_\Omega u_{m+1}(x) &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[P_{l,a,\beta}(u_m) + \mu](x) + b,\end{aligned}\tag{4.4}$$

where $b = K_2 \max_{\partial\Omega} \varphi$. Thus, by Theorem 2.4 there exist $b_0 \in (0, 1]$ and $M > 0$ depending on N, k, l, a, β and R such that

$$u_m(x) \leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu](x) + 2b_0 \quad \forall x \in \Omega, \forall m \geq 0,\tag{4.5}$$

and

$$\exp\left(a \left(K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\mu] + 2b_0\right)^\beta\right) \in L_{loc}^1(\mathbb{R}^N),\tag{4.6}$$

provided that $b \leq b_0$ and

$$\|M_{2k, R}^{\frac{k(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M.$$

Note that because we can write

$$\omega = P_{l,a,\beta}(u_m) + \mu = (\mu_1 + \chi_{\Omega_\delta} P_{l,a,\beta}(u_m)) + ((1 - \chi_{\Omega_\delta}) P_{l,a,\beta}(u_m) + f),$$

where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$ and $\delta > 0$ is small enough, then ω satisfies the assumptions of the data in Theorem 4.3 since u_m is continuous near $\partial\Omega$. Therefore the sequence $\{u_m\}$ is well defined. Since $-u_m$ is k -subharmonic and the sequence $\{u_m\}$ is bounded in $L^1(\Omega)$, the sequence $\{u_m\}$ is also bounded in $W_{loc}^{1,1}(\Omega)$ (see e.g. [25]); we can find a subsequence, still denoted by the index m , such that u_m converges a.e in Ω to function u for which (1.21) is satisfied in Ω . Furthermore, we deduce from (4.5)-(4.6) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 4.2, we obtain that u satisfies (1.20) and (1.21).

Conversely, assume that (1.20) admits nonnegative solution u , continuous near $\partial\Omega$, such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$. Then by Theorem 4.3 we have

$$u(x) \geq \frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}}[\omega](x) \quad \text{for all } x \in \Omega,$$

where $d\omega = P_{l,a,\beta}(u)dx + d\mu$. Using Theorem 2.5, we conclude that (1.23) holds. \blacksquare

5 Wolff integral equations

Proof of Theorem 1.3. Let $\varepsilon > 0$, assume that $\exp((a + \varepsilon)f^\beta) \in L_{loc}^1(\mathbb{R}^N)$ and consider the measure $d\mu = P_{l,a+\varepsilon,\beta}(f) dx$. By Theorem 2.3 with $\eta = \frac{(p-1)(\beta-1)}{\beta}$, there exist $C = C(N, \alpha, p, \beta, R) > 0$ and $\delta_0 = \delta_0(N, \alpha, p, \beta) > 0$ such that if

$$\|M_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$\mathbf{W}_{\alpha,p}^R \left[\exp \left(\delta_0 M^{-\frac{\beta}{p-1}} (\mathbf{W}_{\alpha,p}^R [\mu])^\beta \right) \right] (x) \leq C \quad \forall x \in \mathbb{R}^N. \quad (5.1)$$

Consider the sequence $\{u_m\}_{m \geq 0}$ of nonnegative functions defined by $u_0 = f$ and

$$u_{m+1} = \mathbf{W}_{\alpha,p}^R [P_{l,a,\beta}(u_m)] + f \quad \text{in } \mathbb{R}^N \quad \forall m \geq 0.$$

Step 1. We claim that there exist $b_0 \in (0, 1]$ and $M > 0$ depending on $N, \alpha, p, l, a, \beta, R$ and ε such that if

$$\|\mathbf{M}_{\alpha,p,R}^{\frac{(p-1)(\beta-1)}{\beta}} [\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

then the sequence $\{u_m\}_{m \geq 0}$ is well defined and there holds

$$u_m \leq F := \tilde{c}_p \mathbf{W}_{\alpha,p}^R [\mu] + f + b_0 \quad \forall m \geq 0, \quad \text{with } \tilde{c}_p = 1 \vee 3^{\frac{2-p}{p-1}}, \quad (5.2)$$

and

$$\exp(aF^\beta) \in L_{loc}^1(\mathbb{R}^N) \quad \text{and} \quad \mathbf{W}_{\alpha,p}^R [P_{l,a,\beta}(F)] \in L^\infty(\mathbb{R}^N). \quad (5.3)$$

Clearly, (5.2) holds with $m = 0$, thus we assume that (5.2) holds with $m = n$, then it follows

$$u_n^\beta \leq F^\beta \leq c_{13} \tilde{c}_p^\beta (\mathbf{W}_{\alpha,p}^R [\mu])^\beta + c_{13} b_0^\beta + \frac{a + 2^{-1}\varepsilon}{a} f^\beta,$$

for some constant c_{13} only depends on β and $\frac{a+2^{-1}\varepsilon}{a}$.

Thanks to the convexity property of H_l in $[0, \infty)^a$ and $b_0 \leq 1$, we have

$$\begin{aligned} H_l(aF^\beta) &\leq \frac{\varepsilon}{4(a+\varepsilon)} H_l \left(c_{14} (\mathbf{W}_{\alpha,p}^R [\mu])^\beta \right) + \frac{\varepsilon}{4(a+\varepsilon)} H_l \left(c_{15} b_0^\beta \right) + \frac{2a+\varepsilon}{2(a+\varepsilon)} H_l \left((a+\varepsilon) f^\beta \right) \\ &\leq H_l \left(c_{14} (\mathbf{W}_{\alpha,p}^R [\mu])^\beta \right) + b_0^\beta H_l(c_{15}) + P_{l,a+\varepsilon,\beta}(f), \end{aligned}$$

where $c_{14} = \frac{4a(a+\varepsilon)}{\varepsilon} c_{13} \tilde{c}_p^\beta$ and $c_{15} = \frac{4a(a+\varepsilon)}{\varepsilon} c_{13}$.

Thus,

$$\begin{aligned} u_{n+1} &= \mathbf{W}_{\alpha,p}^R [H_l(a u_n^\beta)] + f \leq \mathbf{W}_{\alpha,p}^R [H_l(a F^\beta)] + f \\ &\leq \tilde{c}_p \mathbf{W}_{\alpha,p}^R \left[H_l \left(c_{14} (\mathbf{W}_{\alpha,p}^R [\mu])^\beta \right) \right] + \tilde{c}_p b_0^{\frac{l\beta}{p-1}} \mathbf{W}_{\alpha,p}^R [H_l(c_{15})] + \tilde{c}_p \mathbf{W}_{\alpha,p}^R [P_{l,a+\varepsilon,\beta}(f)] + f \\ &= \tilde{c}_p \mathbf{W}_{\alpha,p}^R [\mu] + f + \tilde{c}_p \mathbf{W}_{\alpha,p}^R \left[H_l \left(c_{14} (\mathbf{W}_{\alpha,p}^R [\mu])^\beta \right) \right] + c_{16} b_0^{\frac{l\beta}{p-1}}, \end{aligned} \quad (5.4)$$

where $c_{16} = \tilde{c}_p (H_l(c_{15}))^{\frac{1}{p-1}} |B_1|^{\frac{1}{p-1}} \frac{p-1}{\alpha p} R^{\frac{\alpha p}{p-1}}$.

Now, we take $b_0 = 1 \wedge (2c_{16})^{-\frac{p-1}{l\beta-p+1}}$ which yields $c_{16} b_0^{\frac{l\beta}{p-1}} \leq \frac{b_0}{2}$. Hence, we will obtain (5.2) with $m = n + 1$ and (5.3) if we show that

$$\tilde{c}_p \mathbf{W}_{\alpha,p}^R \left[H_l \left(c_{14} (\mathbf{W}_{\alpha,p}^R [\mu])^\beta \right) \right] \leq \frac{b_0}{2}. \quad (5.5)$$

As in the proof of Theorem 2.4, we have

$$\tilde{c}_p \mathbf{W}_{\alpha,p}^R \left[H_l \left(c_{14} (\mathbf{W}_{\alpha,p}^R[\mu])^\beta \right) \right] \leq \theta^{\frac{1}{p-1}} \tilde{c}_p \mathbf{W}_{\alpha,p}^R \left[H_l \left(\theta^{-1} c_{14} (\mathbf{W}_{\alpha,p}^R[\mu])^\beta \right) \right] \quad \text{with } \theta \in (0, 1],$$

and thanks to (5.1) we will achieve (5.5) if $\theta^{-1} c_{14} \leq \delta_0 M^{-\frac{\beta}{p-1}}$ and $\theta^{\frac{1}{p-1}} \tilde{c}_p C \leq \frac{b_0}{2}$ are satisfied.

Thus, we can choose $\theta = 1 \wedge \left(\frac{b_0}{2\tilde{c}_p C} \right)^{p-1}$ and $M = \left(\frac{\delta_0}{c_{14}} \right)^{\frac{p-1}{\beta}} \theta^{\frac{p-1}{\beta}}$.

Therefore $\{u_m\}_{m \geq 0}$ is well defined and (5.2) and (5.3) are satisfied. It is easy to see that u_m is nondecreasing. Hence, from (5.2) and (5.3), we obtain that $u(x) = \lim_{m \rightarrow \infty} u_m(x)$ is a solution to equation (1.24) which satisfies (1.25).

Step 2. Conversely, we assume that (1.24) admits a nonnegative solution u . There holds

$$u^\beta \geq (\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)])^\beta + f^\beta,$$

and thus,

$$P_{l,a,\beta}(u) \geq P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)]) + P_{l,a,\beta}(f).$$

This implies

$$d\mu \geq P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu]) dx,$$

where

$$d\mu = P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)]) dx + P_{l,a,\beta}(f) dx.$$

If $E \subset \mathbb{R}^N$ is a Borel set, we deduce

$$\int_{\mathbb{R}^N} (M_\mu \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu]) dx \leq \int_{\mathbb{R}^N} (M_\mu \chi_E)^{\frac{l\beta}{p-1}} d\mu.$$

Since M_μ is bounded in $L^s(\mathbb{R}^N, d\mu)$ for $s > 1$ by Fefferman's differentiation theorem [11], we infer that

$$\int_{\mathbb{R}^N} (M_\mu \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu]) dx \leq c_{17} \mu(E),$$

for some constant c_{17} only depends on N and $\frac{l\beta}{p-1}$. Because

$$(M_\mu \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu]) \geq P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu_E]),$$

where $\mu_E = \chi_E \mu$. We finally deduce

$$\int_{\mathbb{R}^N} P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu_E]) dx \leq c_{17} \mu(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.}$$

As in the proof of Theorem 2.5, it is easy to see that

$$\mu(E) \leq c_{18} \text{cap}_{G_{\alpha,p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel,}$$

where the positive constant c_{18} depends on N, α, p, a, β and R . Finally since $d\mu \geq P_{l,a,\beta}(u)dx$ we obtain that for any Borel subset $E \subset \mathbb{R}^N$ there holds

$$\int_E P_{l,a,\beta}(u)dx + \int_E P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)]) dx + \int_E P_{l,a,\beta}(f)dx \leq 2c_{12}cap_{G_{\alpha p}, Q_p^*}(E),$$

which completes the proof of the Theorem. ■

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