Quasilinear and Hessian type equations with exponential reaction and measure data

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Abstract

We prove existence results concerning equations of the type $-\Delta_p u = P(u) + \mu$ for $p > 1$ and $F_k[-u] = P(u) + \mu$ with $1 \leq k < \frac{N}{2}$ in a bounded domain Ω, where μ is a positive bounded Radon measure and $P(u) \sim e^{au^\beta}$ with $a > 0$ and $\beta \geq 1$. Sufficient conditions for existence are expressed in terms of the fractional maximal potential of μ. Two-sided estimates on the solutions are obtained in terms of some precise Wolff potentials of μ. Necessary conditions are obtained in terms of Orlicz capacities. We also establish existence results for a general Wolff potential equation under the form $u = \mathbf{W}^R_{\alpha,p}[P(u)] + f$.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $p > 1$ and $k \in \{1, 2, ..., N\}$. We denote by

$$\Delta_p u := div\left(|\nabla u|^{p-2}\nabla u\right)$$

the p-Laplace operator and by

$$F_k[u] = \sum_{1\le j_1<j_2<...<j_k\le N} \lambda_{j_1}\lambda_{j_2}...\lambda_{j_k}$$

the k-Hessian operator where $\lambda_1, ..., \lambda_N$ are the eigenvalues of the Hessian matrix D^2u. Let μ be a positive bounded Radon measure in Ω; our aim is to study the existence of positive renormalized solutions to the quasilinear equation

$$\begin{array}{rl} -\Delta_p u = P(u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{1.1}$$

and the fully nonlinear equation

$$\begin{array}{rl} F_k[-u] = P(u) + \mu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{array} \tag{1.2}$$

where P is an exponential function. When $P(r) = r^q$ with $q > p - 1$, Phuc and Verbitsky published a seminal article [19] on the solvability of the corresponding problem (1.1). They obtained necessary and sufficient conditions involving Bessel capacities or Wolff potentials. For example, they proved that if μ has compact support in Ω it is equivalent to solve (1.1) with $P(r) = r^q$ or to have

$$\mu(E) \le cC_{p,\frac{q}{q+1-p}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{1.3}$$

where c is a suitable positive constant and $C_{p,\frac{q}{q+1-p}}$ a Bessel capacity, or to have

$$\int_B \left[\mathbf{W}^{2R}_{1,p}[\mu_B(x)]\right]^q dx \le C\mu(B) \qquad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \ne \emptyset, \tag{1.4}$$

where $R = \text{diam}(\Omega)$. Other conditions are expressed in terms of Riesz potentials and maximal fractional potentials. Their construction is based upon sharp estimates of solutions of the non-homogeneous problem

$$\begin{array}{rl} -\Delta_p u = \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{1.5}$$

for positive measures ω. We refer to [4, 5, 6, 7, 9, 13, 22] for the previous studies of these and other related results. Concerning k-Hessian operator in a bounded $(k-1)$-convex domain Ω, they proved that if μ has compact support and $||\varphi||_{L^\infty(\partial\Omega)}$ is small enough, the corresponding problem (1.2) with $P(r) = r^q$ with $q > k$ admits a positive solution if and only if

$$\mu(E) \le cC_{2k,\frac{q}{q-k}}(E) \qquad \text{for all compact set } E \subset \Omega, \tag{1.6}$$

or equivalently

$$\int_B \left[\mathbf{W}^{2R}_{\frac{2k}{k+1},k+1}[\mu_B(x)]\right]^q dx \leq C\mu(B) \qquad \text{for all ball } B \text{ s.t. } B\cap \text{supp}\mu \neq \emptyset. \tag{1.7}$$

In the linear case $p = 2$ and $k = 1$, can be found in [2, 3, 27]. The main tools in their proofs are derived from recent advances in potential theory for nonlinear elliptic equations obtained by Kilpelainen and Malý [14, 15], Trudinger and Wang [24, 25, 26], and Labutin [17] thanks to whom the authors first provide global pointwise estimates for solutions of the homogeneous Dirichlet problems in terms of Wolffs potentials of suitable order.

For $s > 1$, $0 \leq \alpha < \frac{N}{s}$, $\eta \geq 0$ and $0 < T \leq \infty$, we recall the expression of the *T-truncated Wolff potential* of a positive Radon measure μ defined in $\mathbb{R}^N$ by

$$\mathbf{W}^T_{\alpha,s}[\mu](x) = \int_0^T \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t}, \tag{1.8}$$

and the *T-truncated η-fractional maximal potential* of μ by

$$\mathbf{M}^\eta_{\alpha,T}[\mu](x) = \sup\left\{\frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)} : 0 < t \leq T\right\}, \tag{1.9}$$

where $h_\eta(t) = (-\ln t)^{-\eta}\chi_{(0,2^{-1}]}(t) + (\ln 2)^{-\eta}\chi_{[2^{-1},\infty)}(t)$. If $\eta = 0$, then $h_\eta = 1$ and we denote by $\mathbf{M}_{\alpha,T}[\mu]$ the corresponding *T-truncated fractional maximal potential* of μ. When the measures are only defined in an open subset $\Omega \subset \mathbb{R}^N$, they are naturally extended by 0 in Ω^c. For $l \in \mathbb{N}^*$, we define the *l-truncated exponential function*

$$H_l(r) = e^r - \sum_{j=0}^{l-1} \frac{r^j}{j!}, \tag{1.10}$$

and for $a > 0$ and $\beta \geq 1$, we set

$$P_{l,a,\beta}(r) = H_l(ar^\beta). \tag{1.11}$$

We put

$$Q_p(s) = \begin{cases} \sum\limits_{q=l}^{\infty} \frac{s^{\frac{\beta q}{p-1}}}{q^{\frac{\beta q}{p-1}} q!} & \text{if } p \neq 2, \\ H_l(s^\beta) & \text{if } p = 2, \end{cases} \tag{1.12}$$

$Q^*_p(r) = \max\{rs - Q_p(s) : s \geq 0\}$ is the complementary function to Q_p, and define

$$cap_{G_p,Q^*_p}(E) = \inf\left\{\int_{\mathbb{R}^N} Q^*_p(f)dx : G_p * f \geq \chi_E, f \geq 0, Q^*_p(f) \in L^1(\mathbb{R}^N)\right\}, \tag{1.13}$$

where $G_p(x) = \mathcal{F}^{-1}\left((1 + |.|^2)^{-\frac{p}{2}}\right)(x)$ is the Bessel kernel of order p.

We denote $a \wedge b$ and $a \vee b$ for $\min\{a, b\}$ and $\max\{a, b\}$ respectively. Our main results are the following theorems

Theorem 1.1 *Let $1 < p < N$, $l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p-1$ and $a > 0$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If μ is a nonnegative Radon measure in Ω, there exist $b_0 \in (0,1]$ and $M > 0$ depending on N, p, l, a, β and $diam\,\Omega$ which is the diameter of Ω such that if*

$$||\mathbf{M}_{p,2diam\,\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M, \tag{1.14}$$

the following Dirichlet problem

$$\begin{array}{ll} -\Delta_p u = P_{l,a,\beta}(u) + \mu & in\ \ \Omega, \\ u = 0 & on\ \ \partial\Omega, \end{array} \tag{1.15}$$

admits a nonnegative renormalized solution u, which satisfies

$$u(x) \leq K_1 \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{1,p}^{2diam\,\Omega}[\mu](x) + 2b_0 \quad \forall x \in \Omega, \tag{1.16}$$

and

$$P_{l,a,\beta}\left(K_1 \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{1,p}^{2diam\,\Omega}[\mu] + 2b_0\right) \in L^1(\Omega). \tag{1.17}$$

The role of $K_1 = K_1(N,p)$ will be made explicit in Theorem 3.5.
Conversely, if (1.15) admits a nonnegative renormalized solution u, then for any compact set $K \subset \Omega$, there exists a positive constant C depending on N, p, a, β and $\mathrm{dist}\,(K, \partial\Omega)$ such that

$$\int_E P_{l,\alpha,\beta}(u)dx + \mu(E) \leq C cap_{G_p, Q_p^*}(E) \quad \textit{for all Borel sets } E \subset K. \tag{1.18}$$

Concerning the k-Hessian operator we recall some notions notions introduced by Trudinger and Wang [24, 25, 26], and we follow their notations. For $k = 1, ..., N$ and $u \in C^2(\Omega)$ the k-hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < ... < i_k \leq N} \lambda_{i_1} ... \lambda_{i_k}.$$

It is straightforward that

$$F_k[u] = \left[D^2u\right]_k,$$

where in general $[A]_k$ denotes the sum of the k-th principal minors of a matrix $A = (a_{ij})$. In order that there exists a smooth k-admissible function which vanishes on $\partial\Omega$, the boundary $\partial\Omega$ must satisfy a uniformly (k-1)-convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \ on\ \ \partial\Omega, \tag{1.19}$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal. We also denotes by $\Phi^k(\Omega)$ the class of upper-semicontinuous functions $\Omega \mapsto [-\infty, \infty)$ which are k-convex, or subharmonic in the Perron sense (see Definition 4.1). In this paper we prove the following theorem

Theorem 1.2 *Let $k \in \{1,2,...,N\}$ such that $2k < N$, $l \in \mathbb{N}^*$, $\beta \geq 1$ such that $l\beta > k$ and $a > 0$. Let Ω be a bounded uniformly (k-1)-convex domain in $\mathbb{R}^N$. Let φ be a nonnegative continuous function on $\partial\Omega$ and $\mu = \mu_1 + f$ be a nonnegative Radon measure where μ_1 has compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$. Let $K_2 = K_2(N,k)$ be the constant K_2 which appears in Theorem 4.3. Then, there exist $b_0 \in (0,1]$ and $M > 0$ depending on N, k, l, a, β and $diam(\Omega)$ such that if*

$$||\mathbf{M}_{2k,2diam(\Omega)}^{\frac{k(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

and $\max_{\partial\Omega}\varphi \leq \frac{b_0}{K_2}$*, the following Dirichlet problem*

$$\begin{array}{ll} F_k[-u] = P_{l,a,\beta}(u) + \mu & in\ \ \Omega, \\ u = \varphi & on\ \ \partial\Omega, \end{array} \tag{1.20}$$

admits a nonnegative solution u, continuous near $\partial\Omega$, with $-u \in \Phi^k(\Omega)$ which satisfies

$$u(x) \leq K_2 \mathbf{W}_{\frac{2k}{k+1},k+1}^{2diam\,\Omega}[\mu](x) + 2b_0 \qquad \forall x \in \Omega, \tag{1.21}$$

and

$$P_{l,a\beta}\left(K_2 \mathbf{W}_{\frac{2k}{k+1},k+1}^{2diam\,\Omega}[\mu] + 2b_0\right) \in L^1(\Omega). \tag{1.22}$$

Conversely, if (1.20) admits a nonnegative solution u, continuous near $\partial\Omega$, such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$, then for any a compact set $K \subset \Omega$, there exists a positive constant C depending on N, k, a, β and $dist(K, \partial\Omega)$ such that there holds

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \leq C cap_{G_{2k},Q_{k+1}^*}(E) \qquad \forall E \subset K, E\ Borel, \tag{1.23}$$

where $Q_{k+1}(s)$ is defined by (1.12) with $p = k+1$, Q_{k+1}^* is its complementary function and $cap_{G_{2k},Q_{k+1}^*}(E)$ is defined accordingly by (1.13).

The two previous theorems are connected to the following result which deals with a class of nonlinear *Wolff integral equations.*

Theorem 1.3 *Let $p > 1$, $\alpha > 0$, $a > 0$, $R > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p-1$ and $0 < \alpha p < N$. Let $f \in L^1_{loc}(\mathbb{R}^N)$, $f \geq 0$. For any $\varepsilon > 0$ there exist constants $b_0 \in (0,1]$ and $M > 0$ depending on $N, \alpha, p, l, a, \beta, R$ and ε such that if $\exp((a+\varepsilon)f^\beta) \in L^1_{loc}(\mathbb{R}^N)$ and*

$$||\mathbf{M}_{\alpha p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

where $d\mu = P_{l,a+\varepsilon,\beta}(f)dx$, then there exists a nonnegative function u, such that $P_{l,a,\beta}(u) \in L^1_{loc}(\mathbb{R}^N)$ which satisfies

$$u = \mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)] + f \quad in\ \ \mathbb{R}^N. \tag{1.24}$$

Furthermore

$$\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(f)] + f \leq u \leq \left(1 \vee 3^{\frac{2-p}{p-1}}\right)\mathbf{W}_{\alpha,p}^R[\mu] + f + b_0 := F \quad in\ \ \mathbb{R}^N. \tag{1.25}$$

Moreover $\exp(aF^\beta) \in L^1_{loc}(\mathbb{R}^N)$ *and* $\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(F)] \in L^\infty(\mathbb{R}^N)$.
Conversely, if (1.24) *admits a nonnegative solution* u*, then there exists a positive constant* C *depending on* N, α, p, a, β *and* R *such that there holds*

$$\int_E P_{l,a,\beta}(u)dx + \int_E P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u)]\right)dx + \int_E P_{l,a,\beta}(f)dx \le C cap_{G_{\alpha p},Q^*_p}(E) \quad \forall E \subset \mathbb{R}^N, \ E \text{ Borel.}$$

2 Estimates on potentials

We denote by $B_r(a)$ the ball of center a and radius $r > 0$ and $B_r = B_r(0)$; χ_E the characteristic function of E for Borel set E. The next estimates are crucial in the sequel.

Theorem 2.1 ***1.*** *There exists a positive constant* c_1*, depending only on* N, α, p, R *such that for all* $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ *and* $q > p-1$ *we have*

$$(c_1 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}}\,dx \le \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \le (c_1 q)^q \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}}\,dx, \tag{2.1}$$

where $\mathbf{G}_{\alpha p}[\mu] := G_{\alpha p} * \mu$ *denotes the Bessel potential of order* αp *of* μ.
2. *There exists a positive constant* c_2 *depending only on* N, α, p, R *such that for all* $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ *and* $q > p-1$ *we have*

$$c_2^{-q} \int_{\mathbb{R}^N} (\mathbf{G}_{2\alpha}[\mu][\mu](x))^q\,dx \le \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,2}[\mu](x)\right)^q dx \le c_2^q \int_{\mathbb{R}^N} (\mathbf{G}_{2\alpha}[\mu](x))^q\,dx. \tag{2.2}$$

Proof. We can find proof of (2.2) in [8, Step 3, Theorem 2.3]. By [8, Step 2, Theorem 2.3], there is $c_3 > 0$ such that

$$\int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \ge c_3^q \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}}\,dx \qquad \forall q > 0, \ R > 0 \text{ and } \ \mu \in \mathfrak{M}^+(\mathbb{R}^N). \tag{2.3}$$

We recall that $\mathbf{M}_{\alpha p,R}[\mu] = \mathbf{M}^0_{\alpha p,R}[\mu]$ by (1.9). Next we show that

$$\int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}}dx \ \ge (c_4 q)^{-q} \int_{\mathbb{R}^N} \left(\mathbf{W}^R_{\alpha,p}[\mu](x)\right)^q dx \qquad \forall q > 0, \ R > 0 \text{ and } \ \mu \in \mathfrak{M}^+(\mathbb{R}^N), \tag{2.4}$$

for some a positive constant c_4. Indeed, we denote μ_n by $\chi_{B_n}\mu$ for $n \in \mathbb{N}^*$. By [16, Theorem 1.2] or [8, Proposition 2.2], there exist constants $c_5 = c_5(N, \alpha, p) > 0$, $a = a(\alpha, p) > 0$ and $\varepsilon_0 = \varepsilon(N, \alpha, p)$ such that for all $n \in \mathbb{N}^*$, $t > 0$, $r > 0$, $0 < R \le \infty$ and $0 < \varepsilon < \varepsilon_0$, there holds

$$\left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > 3t\right\}\right| \le c_5 \exp\left(-a\varepsilon^{-1}\right)\left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > t\right\}\right| + \left|\left\{(\mathbf{M}_{\alpha p,R}\mu_n)^{\frac{1}{p-1}} > \varepsilon t\right\}\right|.$$

Multiplying by qt^{q-1} and integrating over $(0,\infty)$, we obtain

$$\int_0^\infty qt^{q-1}\left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > 3t\right\}\right|dt \le c_5\exp\left(-a\varepsilon^{-1}\right)\int_0^\infty qt^{q-1}\left|\left\{\mathbf{W}^R_{\alpha,p}\mu_n > t\right\}\right|dt + \int_0^\infty qt^{q-1}\left|\left\{(\mathbf{M}_{\alpha p,R}\mu_n)^{\frac{1}{p-1}} > \varepsilon t\right\}\right|dt,$$

which implies

$$3^{-q}\int_{\mathbb{R}^N}\left(\mathbf{W}^R_{\alpha,p}[\mu_n](x)\right)^q dx \le c_5\exp\left(-a\varepsilon^{-1}\right)\int_{\mathbb{R}^N}\left(\mathbf{W}^R_{\alpha,p}[\mu_n](x)\right)^q dx + \varepsilon^{-q}\int_N(\mathbf{M}_{\alpha p,R}\mu_n)^{\frac{q}{p-1}}dx.$$

This leads to

$$\varepsilon^q\left(3^{-q}-c_5\exp\left(-a\varepsilon^{-1}\right)\right)\int_{\mathbb{R}^N}\left(\mathbf{W}^R_{\alpha,p}[\mu_n](x)\right)^q dx \le \int_{\mathbb{R}^N}(\mathbf{M}_{\alpha p,R}\mu_n)^{\frac{q}{p-1}}dx.$$

We see that $\sup_{0<\varepsilon<\varepsilon_0}\varepsilon^q\left(3^{-q}-c_5\exp\left(-a\varepsilon^{-1}\right)\right) \ge (c_6q)^{-q}$ for some a constant c_6 which does not depend on q. Therefore, (2.4) follows by Fatou's lemma. From (2.3) and (2.4), we have

$$c_7^q\int_{\mathbb{R}^N}\left(\mathbf{W}^R_{\frac{\alpha p}{2},2}[\mu](x)\right)^q dx \ge \int_{\mathbb{R}^N}(\mathbf{M}_{\alpha p,R}[\mu](x))^q dx \ \ge (c_7q)^{-q}\int_N\left(\mathbf{W}^R_{\frac{\alpha p}{2},2}[\mu](x)\right)^q dx,$$

for all $q>0$ and $\mu\in\mathfrak{M}^+(\mathbb{R}^N)$, with $c_7=c_7(N,\alpha,p)>0$. Then, combining with (2.2) we get

$$c_8^q\int_{\mathbb{R}^N}(\mathbf{G}_{\alpha p}[\mu](x))^q dx \ge \int_{\mathbb{R}^N}(\mathbf{M}_{\alpha p,R}[\mu](x))^q dx \ \ge (c_8q)^{-q}\int_{\mathbb{R}^N}(\mathbf{G}_{\alpha p}[\mu](x))^q dx,$$

with $c_8=c_8(N,\alpha,p,R)$, for all $q>0$ and $\mu\in\mathfrak{M}^+(\mathbb{R}^N)$. Therefore, jointly with (2.3) and (2.4) we derive (2.1). ∎

The next result is proved in [8].

Theorem 2.2 *Let $\alpha>0$, $p>1$, $0\le\eta<p-1$, $0<\alpha p<N$ and $L>0$. Set $\delta=\frac{1}{2}\left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\alpha p\log(2)$. Then there exists $C(L)>0$, depending on N, α, p, η and L such that for any $R\in(0,\infty]$, $\mu\in\mathfrak{M}+(\mathbb{R}^N)$, any $a\in\mathbb{R}^N$ and $0<r\le L$, there holds*

$$\frac{1}{|B_{2r}(a)|}\int_{B_{2r}(a)}\exp\left(\delta\frac{\left(\mathbf{W}^R_{\alpha,p}[\mu_{B_r(a)}](x)\right)^{\frac{p-1}{p-1-\eta}}}{||\mathbf{M}^\eta_{\alpha p,R}[\mu_{B_r}(a)]||^{\frac{1}{p-1-\eta}}_{L^\infty(B_r(a))}}\right)dx \le C(L), \tag{2.5}$$

where $\mu_{B_r(a)}=\chi_{B_r(a)}\mu$. Furthermore, if $\eta=0$, C is independent of L.

Theorem 2.3 *Assume α, p, η are as in Theorem 2.2 and $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ satisfies for $R > 0$*

$$||\mathbf{M}^{\eta}_{\alpha p,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

for some $M > 0$. Then there exist $C = C(N, \alpha, p, \eta, R) > 0$ and $\delta_0 = \delta_0(N, \alpha, p, \eta) > 0$ such that

$$\left\| \mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0 M^{-\frac{1}{p-1-\eta}}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^{\frac{p-1}{p-1-\eta}}\right)\right]\right\|_{L^\infty(\mathbb{R}^N)} \leq C. \tag{2.6}$$

Proof. Without loss of generality we may assume that $M = 1$. Let δ be as in Theorem 2.2 and $\gamma = \frac{p-1}{p-1-\eta}$. Let $x \in \mathbb{R}^N$. Since $\mu(B_t(y)) \leq Mt^{N-\alpha p}h_\eta(t)$, for all $r \in (0, R)$ and $y \in \mathbb{R}^N$ we have

$$\begin{aligned}
\mathbf{W}^R_{\alpha,p}[\mu](y) &= \mathbf{W}^r_{\alpha,p}[\mu](y) + \int_r^R \left(\frac{\mu(B_t(y))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \\
&\leq \mathbf{W}^r_{\alpha,p}[\mu](y) + \int_{r\wedge 2^{-1}}^{2^{-1}} \left((-\ln t)^{-\eta}\right)^{\frac{1}{p-1}}\frac{dt}{t} + \int_{2^{-1}}^{R\vee 2^{-1}} \left((\ln 2)^{-\eta}\right)^{\frac{1}{p-1}}\frac{dt}{t} \\
&\leq \mathbf{W}^r_{\alpha,p}[\mu](y) + \gamma(-\ln(r\wedge 2^{-1}))^{\frac{1}{\gamma}} + (\ln(2))^{-\frac{\eta}{p-1}}\ln(2R\vee 1).
\end{aligned}$$

Thus,

$$\left(\mathbf{W}^R_{\alpha,p}[\mu](y)\right)^\gamma \leq 3^{\gamma-1}\left(\left(\mathbf{W}^r_{\alpha,p}[\mu](y)\right)^\gamma + \gamma^\gamma \ln\left(\frac{1}{r\wedge 2^{-1}}\right) + A_1(R)\right), \tag{2.7}$$

where $A_1(R) = (\ln(2))^{1-\gamma}(\ln(2R\vee 1))^\gamma$.
Let $\theta \in (0, 1]$, since $\exp\left(\frac{a+b}{2}\right) \leq \exp(a) + \exp(b)$ for all $a, b \in \mathbb{R}$, we get from (2.7)

$$\exp\left(\theta\delta 3^{-\gamma}\left(\mathbf{W}^R_{\alpha,p}[\mu](y)\right)^\gamma\right) \leq \exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}[\mu](y)\right)^\gamma\right) + \exp\left(\theta\delta\gamma^\gamma \ln\left(\frac{1}{r\wedge 2^{-1}}\right) + \theta\delta A_1(R)\right). \tag{2.8}$$

For $r > 0$, $0 < t \leq r$, $y \in B_r(x)$ we have $B_t(y) \subset B_{2r}(x)$. Thus, $\mathbf{W}^r_{\alpha,p}[\mu] = \mathbf{W}^r_{\alpha,p}[\mu_{B_{2r}(x)}]$ in $B_r(x)$. Then, using (2.5) with $L = 2R$ we get

$$\int_{B_r(x)} \exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}[\mu]\right)^\gamma\right) = \int_{B_r(x)} \exp\left(\delta\left(\mathbf{W}^r_{\alpha,p}\left[\mu_{B_{2r}(x)}\right]\right)^\gamma\right) \leq C(2R)|B_4|r^N.$$

So, from (2.8) we deduce

$$\begin{aligned}
&\mathbf{W}^R_{\alpha,p}\left[\exp\left(\theta\delta 3^{-\gamma}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\gamma\right)\right](x) \\
&\quad\leq \int_0^R \left(C(2R)|B_4|r^{\alpha p} + |B_1|\exp(\theta\delta A_1(R))\left(r\wedge 2^{-1}\right)^{-\theta\delta\gamma^\gamma} r^{\alpha p}\right)^{\frac{1}{p-1}}\frac{dr}{r} \\
&\quad\leq \left(1\vee 2^{\frac{2-p}{p-1}}\right)(C(2R)|B_4|)^{\frac{1}{p-1}}\int_0^R r^{\frac{\alpha p}{p-1}-1}dr \\
&\qquad + \left(1\vee 2^{\frac{2-p}{p-1}}\right)(|B_1|\exp(\theta\delta A_1(R)))^{\frac{1}{p-1}}\int_0^R \left(r\wedge 2^{-1}\right)^{-\frac{\theta\delta\gamma^\gamma}{p-1}} r^{\frac{\alpha p}{p-1}-1}dr.
\end{aligned}$$

Now we choose $\theta = 1 \wedge \frac{\alpha p}{2\delta\gamma^\gamma}$, we have

$$\begin{aligned}\int_0^R \left(r \wedge 2^{-1}\right)^{-\frac{\theta\delta\gamma^\gamma}{p-1}} r^{\frac{\alpha p}{p-1}-1} dr &\leq \int_0^R \left(\frac{r}{r \wedge 2^{-1}}\right)^{\frac{\alpha p}{2(p-1)}} r^{\frac{\alpha p}{2(p-1)}-1} dr \\ &\leq \frac{2(p-1)}{\alpha p}(2R+1)^{\frac{\alpha p}{2(p-1)}} R^{\frac{\alpha p}{2(p-1)}}.\end{aligned}$$

Hence

$$\mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0\big(\mathbf{W}^R_{\alpha,p}[\mu]\big)^\gamma\right)\right](x) \leq A_2(R),$$

where $\delta_0 = \theta\delta 3^{-\gamma} = \left(1 \wedge \frac{\alpha p}{2\delta\gamma^\gamma}\right)\delta 3^{-\gamma}$ and

$$\begin{aligned}A_2(R) &= \left(1 \vee 2^{\frac{2-p}{p-1}}\right)(C(2R)|B_4|)^{\frac{1}{p-1}} \frac{p-1}{\alpha p} R^{\frac{\alpha p}{p-1}} \\ &\quad + \left(1 \vee 2^{\frac{2-p}{p-1}}\right)(|B_1| \exp(\theta\delta A_1(R)))^{\frac{1}{p-1}} \frac{2(p-1)}{\alpha p}(2R+1)^{\frac{\alpha p}{2(p-1)}} R^{\frac{\alpha p}{2(p-1)}}.\end{aligned}$$

Which yields (2.6). ∎

In the next result we obtain estimate on iterative solutions of Wolff integral inequalities. We recall that H_l and $P_{l,a,\beta}$ have been defined in (1.10) and (1.11).

Theorem 2.4 *Let $\alpha > 0$, $p > 1$, $a > 0$, $b \geq 0$, $K > 0$, $R > 0$, $l \in \mathbb{N}^*$ and $\beta \geq 1$ such that $l\beta > p-1$ and $0 < \alpha p < N$. Suppose that $\{u_m\}$ is a sequence of nonnegative functions in $\mathbb{R}^N$ that satisfies*

$$\begin{aligned}u_{m+1} &\leq K\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u_m) + \mu] + b \quad \forall m \in \mathbb{N}, \\ u_0 &\leq K\mathbf{W}^R_{\alpha,p}[\mu] + b,\end{aligned} \tag{2.9}$$

where $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$. Then there exist $b_0 \in (0,1]$ and $M > 0$ depending on $N, \alpha, p, l, a, \beta, K$ and R such that if $b \leq b_0$ and

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$\exp\left(a\big(c_pK\mathbf{W}^R_{1,p}[\mu] + 2b_0\big)^\beta\right) \in L^1_{loc}(\mathbb{R}^N), \tag{2.10}$$

and

$$\mathbf{W}^R_{\alpha,p}\left[P_{l,a,\beta}\left(c_pK\mathbf{W}^R_{\alpha,p}[\mu] + 2b_0\right)\right] \in L^\infty(\mathbb{R}^N), \tag{2.11}$$

and finally

$$u_m \leq c_pK\mathbf{W}^R_{\alpha,p}[\mu] + 2b_0 \quad \forall m \in \mathbb{N}, \tag{2.12}$$

where $c_p = 1 \vee 2^{\frac{2-p}{p-1}}$.

Proof. By Theorem 2.3 with $\eta = \frac{(p-1)(\beta-1)}{\beta}$, there exist $C = C(N, \alpha, p, \beta, R) > 0$ and $\delta_0 = \delta_0(N, \alpha, p, \beta) > 0$ such that if

$$||\mathbf{M}_{\alpha p, R^{\beta}}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^{\infty}(\mathbb{R}^N)} \leq M,$$

then

$$\mathbf{W}_{\alpha,p}^{R}\left[\exp\left(\delta_0 M^{-\frac{\beta}{p-1}}\left(\mathbf{W}_{\alpha,p}^{R}[\mu]\right)^{\beta}\right)\right](x) \leq C \qquad \forall x \in \mathbb{R}^N. \tag{2.13}$$

Take $b_0 \in [b, 1]$. We will choose b_0 and M later on. Clearly, (2.12) holds with $m = 0$. Now, assume that (2.12) holds with $m = n$, we need to prove that

$$u_{n+1} \leq c_p K \mathbf{W}_{\alpha,p}^{R}[\mu] + 2b_0.$$

In fact, by the definition of u_{n+1} and the sub-additive property of $\mathbf{W}_{\alpha,p}^{R}[.]$, we have

$$u_{n+1} \leq c_p K \mathbf{W}_{\alpha,p}^{R}[\mu] + c_p K \mathbf{W}_{\alpha,p}^{R}[P_{l,a,\beta}(u_n)] \ + b.$$

So, it is sufficient to prove that

$$c_p K \mathbf{W}_{\alpha,p}^{R}[P_{l,a,\beta}(u_n)] \leq b_0. \tag{2.14}$$

Since (2.12) holds with $m = n$ and the convexity property of H_l in $[0, \infty)$ and $b_0 \leq 1$, we have

$$\begin{aligned}
P_{l,a,\beta}(u_n) &\leq H_l\left(a\left(c_p K \mathbf{W}_{\alpha,p}^{R}[\mu] + 2b_0\right)^{\beta}\right) \\
&\leq H_l\left(a c_p^{\beta} K^{\beta} 2^{\beta-1}\left(\mathbf{W}_{\alpha,p}^{R}[\mu]\right)^{\beta} + a 2^{2\beta-1} b_0^{\beta}\right) \\
&\leq H_l\left(a c_p^{\beta} K^{\beta} 2^{\beta}\left(\mathbf{W}_{\alpha,p}^{R}[\mu]\right)^{\beta}\right) + H_l(a 2^{2\beta} b_0^{\beta}) \\
&\leq H_l\left(a c_p^{\beta} K^{\beta} 2^{\beta}\left(\mathbf{W}_{\alpha,p}^{R}[\mu]\right)^{\beta}\right) + b_0^{l\beta} H_l(a 2^{2\beta}),
\end{aligned}$$

which implies

$$\begin{aligned}
c_p K \mathbf{W}_{\alpha,p}^{R}\left[P_{l,a,\beta}(u_n)\right] &\leq c_p^2 K \mathbf{W}_{\alpha,p}^{R}\left[H_l\left(a c_p^{\beta} K^{\beta} 2^{\beta}\left(\mathbf{W}_{\alpha,p}^{R}[\mu]\right)^{\beta}\right)\right] + c_p^2 K \mathbf{W}_{\alpha,p}^{R}\left[b_0^{l\beta} H_l(a 2^{2\beta})\right] \\
&= c_p^2 K \mathbf{W}_{\alpha,p}^{R}\left[H_l\left(a c_p^{\beta} K^{\beta} 2^{\beta}\left(\mathbf{W}_{\alpha,p}^{R}[\mu]\right)^{\beta}\right)\right] + A(R) b_0^{\frac{l\beta}{p-1}},
\end{aligned} \tag{2.15}$$

where $A(R) = \frac{p-1}{\alpha p} c_p^2 K \left(H_l(a 2^{2\beta})\right)^{\frac{1}{p-1}} |B_1|^{\frac{1}{p-1}} R^{\frac{\alpha p}{p-1}}$.
Now we choose $b_0 = 1 \wedge \left((2A(R))^{-\frac{p-1}{l\beta-p+1}}\right)$ and we derive

$$A(R) b_0^{\frac{l\beta}{p-1}} \leq \frac{b_0}{2}. \tag{2.16}$$

Next, set $\theta \in (0, 1]$; since $H_l(s) \leq \theta H_l(\theta^{-1} s)$ for all $s \geq 0$,

$$c_p^2 K \mathbf{W}_{\alpha,p}^{R}\left[H_l\left(a c_p^{\beta} K^{\beta} 2^{\beta}\left(\mathbf{W}_{\alpha,p}^{R}[\mu]\right)^{\beta}\right)\right] \leq \theta^{\frac{1}{p-1}} c_p^2 K \mathbf{W}_{\alpha,p}^{R}\left[H_l\left(\theta^{-1} a c_p^{\beta} K^{\beta} 2^{\beta}\left(\mathbf{W}_{\alpha,p}^{R}[\mu]\right)^{\beta}\right)\right].$$

From (2.13), we assert that if

$$\theta^{-1}ac_p^\beta K^\beta 2^\beta \le \delta_0 M^{-\frac{\beta}{p-1}} \Longleftrightarrow M \le \left(\frac{\delta_0}{ac_p^\beta K^\beta 2^\beta}\right)^{\frac{p-1}{\beta}} \theta^{\frac{p-1}{\beta}},$$

then

$$c_p^2 K \mathbf{W}_{\alpha,p}^R \left[H_l\left(ac_p^\beta K^\beta 2^\beta \left(\mathbf{W}_{\alpha,p}^R[\mu]\right)^\beta\right)\right] \le \theta^{\frac{1}{p-1}} c_p^2 KC.$$

This will be achieved if we prove that

$$\theta^{\frac{1}{p-1}} c_p^2 KC \le \frac{b_0}{2},$$

which is equivalent to

$$\theta \le \left(\frac{b_0}{2c_p^2 KC}\right)^{p-1}.$$

Thus, we can choose $\theta = 1 \wedge \left(\frac{b_0}{2c_p^2 KC}\right)^{p-1}$ and $M = \left(\frac{\delta_0}{ac_p^\beta K^\beta 2^\beta}\right)^{\frac{p-1}{\beta}} \theta^{\frac{p-1}{\beta}}$ and we obtain that

$$c_p^2 K \mathbf{W}_{\alpha,p}^R \left[H_l\left(ac_p^\beta K^\beta 2^\beta \left(\mathbf{W}_{\alpha,p}^R[\mu]\right)^\beta\right)\right] \le \frac{b_0}{2}.$$

Combining this with (2.16) and (2.15) we obtain (2.14) and (2.10), (2.11). This completes the proof of the theorem. ∎

Let $P \in C(\mathbb{R}^+)$ be a decreasing positive function; the (α, P)-Orlicz-Bessel capacity of a Borel set $E \subset \mathbb{R}^N$ is defined by (see [1, Sect 2.6])

$$cap_{G_\alpha,P}(E) = \inf\left\{\int_{\mathbb{R}^N} P(f) : G_\alpha * f \ge \chi_E, f \ge 0, P(f) \in L^1(\mathbb{R}^N)\right\}.$$

Theorem 2.5 *Let $\alpha > 0$,$p > 1$,$a > 0$,$c > 0$,$\delta \in (0,1]$, $l \in \mathbb{N}^*$ and $\beta \ge 1$ such that $l\beta > p-1$ and $0 < \alpha p < N$. Let Ω be a bounded domain in $\mathbb{R}^N$ and $\mu \in \mathfrak{M}_b^+(\Omega)$. If u is a nonnegative Borel function in Ω such that $P_{l,a,\beta}(u) \in L^1(\Omega)$ and*

$$u(x) \ge c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega](x) \qquad \forall x \in \Omega, \tag{2.17}$$

where $d\omega = P_{l,a,\beta}(u)dx + d\mu$, then, for any a compact set $K \subset \Omega$, there exists a positive constant C depending on $N, \alpha, p, a, c, \beta, \delta$ and $dist(K, \partial\Omega)$ such that

$$\int_E P_{l,a,\beta}(u)dx + \mu(E) \le C cap_{G_{\alpha p},Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel},$$

where Q_p^* is the complementary function to Q_p.

Proof. Let $K \subset \Omega$. Set $r_K = dist(K, \partial\Omega)$ and $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$. We have

$$P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega]\right)dx \leq d\omega \quad \text{in } \Omega.$$

Let M_ω denote the centered Hardy-Littlewood maximal function which is defined for any $f \in L^1_{loc}(\mathbb{R}^N, d\omega)$ by

$$M_\omega f(x) = \sup_{t>0} \frac{\int\limits_{B_t(x)} |f| d\omega}{\omega(B_t(x))}.$$

Thus, for any Borel set $E \subset K$

$$\int\limits_\Omega (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega]\right)dx \leq \int\limits_\Omega (M_\omega \chi_E)^{\frac{l\beta}{p-1}} d\omega.$$

Since M_ω is bounded on $L^s(\mathbb{R}^N, d\omega)$, $s > 1$, we deduce from Fefferman's result [11] that

$$\int\limits_\Omega (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega]\right)dx \leq c_9\omega(E),$$

for some constant c_9 only depends on N and $\frac{l\beta}{p-1}$. Since $M_\omega \chi_E \leq 1$, we derive

$$\begin{aligned}(M_\omega \chi_E(x))^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega](x)\right) &\geq P_{l,a,\beta}\left(c\left(M_\omega \chi_E(x)\right)^{\frac{1}{p-1}} \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega](x)\right)\\ &\geq P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x)\right),\end{aligned}$$

where $\omega_E = \chi_E \omega$. This implies

$$\int\limits_\Omega P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x)\right)dx \leq c_9\omega(E) \quad \forall E \subset K, E \text{ Borel.} \tag{2.18}$$

Note that if $x \in \Omega$ and $d(x, \partial\Omega) \leq r_K/8$, then $B_t(x) \subset \Omega \backslash \Omega_K$ for all $t \in (0, \delta d(x, \partial\Omega))$; indeed, for all $y \in B_t(x)$

$$d(y, \partial\Omega) \leq d(x, \partial\Omega) + |x - y| < (1 + \delta)d(x, \partial\Omega) < \frac{1}{4}r_K,$$

thus

$$d(y, K) \geq d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4}r_K > \frac{1}{2}r_K,$$

which implies $y \notin \Omega_K$. We deduce that

$$\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \geq \mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) \qquad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) = 0 \qquad \forall x \in \Omega^c.$$

Hence we obtain from (2.18),

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\frac{\delta}{\delta}r_K}[\omega_E](x)\right)dx \le c_9\omega(E) \qquad \forall E\subset K,\ E \text{ Borel.} \tag{2.19}$$

From (2.1) and (2.2) we get

$$\int_{\mathbb{R}^N} P_{l,a,\beta}\left(c\mathbf{W}_{\alpha,p}^{\frac{\delta}{\delta}r_K}[\omega_E](x)\right)dx \ge \int_{\mathbb{R}^N} Q_p\left(c_{10}\mathbf{G}_{\alpha p}[\omega_E](x)\right)dx,$$

where Q_p is defined by (1.12) and $c_{10}=(c_1\beta)^{-1}a^{\frac{p-1}{\beta}}c^{p-1}$ if $p\neq 2$, $c_{10}=c_1^{-1}a^{\frac{1}{\beta}}c$ if $p=2$ (the constant c_1 defined in (2.1), depends on R, therefore $c_{10}=c_{10}(r_K)$). Thus, from (2.19) we obtain

$$\int_{\mathbb{R}^N} Q_p\left(c_{10}\mathbf{G}_{\alpha p}[\omega_E](x)\right)dx \le c_9\omega(E) \qquad \forall E\subset K,\ E \text{ Borel.}$$

We recall that $Q_p^*(s)=\sup_{t>0}\{st-Q_p(t)\}$ and we note that Q_p^* satisfies the sub-additivity Δ_2-condition (see Chapter 2 in [18]). Let $E\subset K$ be a Borel set. For every $f\ge 0$, $Q_p^*(f)\in L^1(\Omega)$ such that $G_{\alpha p}*f\ge\chi_E$, we have

$$\begin{aligned}
\omega(E) &\le \int_{\mathbb{R}^N} G_{\alpha p}*f d\omega_E = \int_{\mathbb{R}^N}\mathbf{G}_{\alpha p}[\omega_E]\,f dx\\
&= (2c_9)^{-1}\int_{\mathbb{R}^N}\left(c_{10}\mathbf{G}_{\alpha p}[\omega_E]\right)\left(2c_9c_{10}^{-1}f\right)dx\\
&\le (2c_9)^{-1}\int_{\mathbb{R}^N} Q_p\left(c_{10}\mathbf{G}_{\alpha p}[\omega_E]\right)dx + (2c_9)^{-1}\int_{\mathbb{R}^N} Q_p^*\left(2c_9c_{10}^{-1}f\right)dx\\
&\le 2^{-1}\omega(E) + c_{11}\int_{\mathbb{R}^N} Q_p^*(f)\,dx,
\end{aligned}$$

the last inequality following from the Δ_2-condition. Notice that c_{11}, as well as the next constant c_{12}, depends on r_K. Thus,

$$\omega(E)\le 2c_{11}\int_{\mathbb{R}^N} Q_p^*(f)\,dx.$$

Then, we get

$$\omega(E)\le c_{12}cap_{G_{\alpha p},Q_p^*}(E) \qquad \forall E\subset K,\ E \text{ Borel.}$$

This completes the proof of the Theorem. ∎

3 Quasilinear Dirichlet Problems

Let Ω be a bounded domain in $\mathbb{R}^N$. If $\mu\in\mathfrak{M}_b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative part in the Jordan decomposition. We denote by $\mathfrak{M}_0(\Omega)$ the space of

measures in Ω which are absolutely continuous with respect to the $c_{1,p}^{\Omega}$-capacity defined on a compact set $K \subset \Omega$ by

$$c_{1,p}^{\Omega}(K) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^{\infty}(\Omega) \right\}.$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c_{1,p}^{\Omega}$-capacity. Classically, any $\mu \in \mathfrak{M}_b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. It is well known that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$ can be written under the form $\mu_0 = f - div\ g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega)$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s, k\}, -k\}$. If u is a measurable function defined and finite a.e. in Ω, such that $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u| \leq k} v$ a.e. in Ω and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [10].

Definition 3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{array}{rl} -\Delta_p u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{3.1}$$

if $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist λ_k^+ and λ_k^- belonging to $\mathfrak{M}_b^+ \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\mu_k^+ \rightharpoonup \mu_s^+$, $\mu_k^- \rightharpoonup \lambda_s^-$ in the narrow topology of measures and such that

$$\int_{\{|u|<k\}} |\nabla u|^{p-2} \nabla u . \nabla \varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda_k^+ - \int_{\Omega} \varphi d\lambda_k^-,$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$.

Remark 3.2 *We recall that if u is a renormalized solution to problem (3.1) and $\mu \in \mathfrak{M}_b^+(\Omega)$, then $u \geqq 0$ a.e. in Ω.*

The following general stability result has been proved in [10, Th 4.1, Sec 5.1].

Theorem 3.3 *If $\{\mu_n\}$ is a bounded sequence in $\mathfrak{M}_b(\Omega)$ and $\{u_n\}$ a sequence of renormalized solutions of problem*

$$\begin{array}{ll} -\Delta_p u_n = \mu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{array} \tag{3.2}$$

then, up to a subsequence, $\{u_n\}$ converges a.e. to a solution of equation $-\Delta_p u = \mu$ in $\mathcal{D}'(\Omega)$, and such that $\frac{1}{k} \int_{\Omega} |\nabla T_k(u)|^p dx \leq M$ for every $k > 0$.

The next result is a sharp extension of the stability Theorem 3.3.

Theorem 3.4 *Let $\mu = \mu_0 + \mu_s^+ - \mu_s^-$, with $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$ and μ_s^+, μ_s^- belonging to $\mathfrak{M}_s^+(\Omega)$. Let $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$ with $F_n \in L^1(\Omega)$, $g_n \in (L^{p'}(\Omega))^N$ and ρ_n, η_n belonging to $\mathfrak{M}_b^+(\Omega)$. Assume that $\{F_n\}$ converges to F weakly in $L^1(\Omega)$, $\{g_n\}$ converges to g strongly*

in $(L^{p'}(\Omega))^N$ *and* $(\operatorname{div} g_n)$ *is bounded in* $\mathfrak{M}_b(\Omega)$*; assume also that* $\{\rho_n\}$ *converges to* μ_s^+ *and* $\{\eta_n\}$ *to* μ_s^- *in the narrow topology. If* $\{u_n\}$ *is a sequence of renormalized solutions of (3.2), then, up to a subsequence, it converges a.e. in* Ω *to a renormalized solution* u *of problem (3.1). Furthermore* $T_k(u_n)$ *converges to* $T_k(u)$ *in* $W_0^{1,p}(\Omega)$ *for any* $k > 0$*.*

We also recall the following estimate [19, Th 2.1].

Theorem 3.5 *Let* Ω *be a bounded domain of* $\mathbb{R}^N$*. Then there exists a constant* $K_1 > 0$*, depending on* p *and* N *such that if* $\mu \in \mathfrak{M}_b^+(\Omega)$ *and* u *is a nonnegative renormalized solution of problem (3.1) with data* μ*, there holds*

$$\frac{1}{K_1}\mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \le u(x) \le K_1\mathbf{W}_{1,p}^{2diam(\Omega)}[\mu](x) \quad \forall x \in \Omega, \tag{3.3}$$

where K_1 *is a constant independent of* x, u *and* Ω*.*

Proof of Theorem 1.1. Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of nonnegative renormalized solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu && \text{in } \Omega,\\ u_0 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and, for $m \in \mathbb{N}$,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu && \text{in } \Omega,\\ u_{m+1} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By Theorem 3.5 we have for all $x \in \mathbb{R}^N$

$$\chi_\Omega u_0(x) \le K_1\mathbf{W}_{1,p}^R[\mu](x),$$
$$\chi_\Omega u_{m+1}(x) \le K_1\mathbf{W}_{1,p}^R[P_{l,a,\beta}(u_m) + \mu](x),$$

where $R = 2diam(\Omega)$. Thus, by Theorem 2.4 there exist $b_0 \in (0,1]$ and $M > 0$ depending on N, p, l, a, β, K_1 and R such that

$$u_m(x) \le K_1\left(1 \vee 2^{\frac{2-p}{p-1}}\right)\mathbf{W}_{1,p}^R[\mu](x) + 2b_0 \quad \forall x \in \Omega, m \in \mathbb{N}, \tag{3.4}$$

and

$$\exp\left(a\Big(K\left(1 \vee 2^{\frac{2-p}{p-1}}\right)\mathbf{W}_{1,p}^R[\mu] + 2b_0\Big)^\beta\right) \in L^1_{loc}(\mathbb{R}^N), \tag{3.5}$$

provided that

$$||M_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M.$$

This implies that $\{u_m\}$ is well defined; by Theorem 3.3 it contains a subsequence that we still denote by $\{u_m\}$ which converges a.e in Ω to function u for which (1.16) is satisfied in Ω. Furthermore, we deduce from (3.4)-(3.5) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 3.4 we obtain that u is a renormalized solution of (1.15).
Conversely, assume that (1.15) admits a nonnegative renormalized solution u. By Theorem 3.5 there holds

$$u(x) \ge \frac{1}{K_1}\mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\omega](x) \quad \text{for all } x \in \Omega,$$

where $d\omega = P_{l,a,\beta}(u)dx + d\mu$. By Theorem 2.5, we obtain (1.18). ∎

Applications. We consider the case $p = 2$, $\beta = 1$. Then $l = 2$ and

$$P_{l,a,\beta}(r) = e^{ar} - 1 - ar.$$

If Ω is a bounded domain in $\mathbb{R}^N$, there exists $M > 0$ and $b_0 \in (0, 1)$ such that if μ is a positive Radon measure in Ω which satisfies

$$\mu(B_t(x)) \leq Mt^{N-2} \qquad \forall t > 0 \text{ and almost all } x \in \Omega,$$

there exists a positive solution u to the following problem

$$\begin{aligned} -\Delta u &= e^{au} - 1 - au + \mu && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Furthermore

$$u(x) \leq K(N) \int_0^{2\,diam\,\Omega} \frac{\mu(B_t(x))}{t^{N-1}} dt + 2b_0 \qquad \forall x \in \Omega.$$

In the case $N = 2$ this result has already been proved by Richard and Véron [21, Prop 2.4].

4 Hessian equations

In this section $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2 boundary. For $k = 1, .., N$ and $u \in C^2(\Omega)$ the k-hessian operator F_k is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ denotes the eigenvalues of the Hessian matrix of second partial derivative D^2u and S_k is the k-th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < ... < i_k \leq N} \lambda_{i_1} ... \lambda_{i_k}$$

We can see that

$$F_k[u] = \left[D^2u\right]_k,$$

where for a matrix $A = (a_{ij})$, $[A]_k$ denotes the sum of the k -th principal minors. We assume that $\partial\Omega$ is uniformly (k-1)-convex that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega,$$

for some positive constant c_0, where $\kappa = (\kappa_1, \kappa_2, ..., \kappa_{n-1})$ denote the principal curvatures of $\partial\Omega$ with respect to its inner normal.

Definition 4.1 *An upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is k-convex (k-subharmonic) if, for every open set $\Omega' \subset \overline{\Omega}' \subset \Omega$ and for every function $v \in C^2(\Omega') \cap C(\overline{\Omega})$ satisfying $F_k[v] \leq 0$ in Ω', the following implication is true*

$$u \leq v \text{ on } \partial\Omega' \quad \Longrightarrow \quad u \leq v \text{ in } \Omega'.$$

We denote by $\Phi^k(\Omega)$ the class of all k-subharmonic functions in Ω which are not identically equal to $-\infty$.

The following weak convergence result for k-Hessian operators proved in [25] is fundamental in our study.

Theorem 4.2 *Let Ω be a bounded uniformly (k-1)-convex in $\mathbb{R}^N$. For each $u \in \Phi^k(\Omega)$, there exist a nonnegative Radon measure $\mu_k[u]$ in Ω such that*

a. *$\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$.*

b. *If $\{u_n\}$ is a sequence of k-convex functions which converges to u a.e then $\mu_k[u_n] \rightharpoonup \mu_k[u]$ in the weak sense of measures.*

As in the case of quasilinear equations with measure data, precise estimates of solutions of k-Hessian equations with measures data are expressed in terms of Wolff potentials. The next result is proved in [25, 17, 19].

Theorem 4.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2, uniformly $(k-1)$-convex domain. Let φ be a nonnegative continuous function on $\partial\Omega$ and μ be a nonnegative Radon measure. Suppose that μ can be decomposed as*

$$\mu = \mu_1 + f$$

such that μ_1 is a measure with compact support in Ω and $f \in L^q(\Omega)$ for some $q > \frac{N}{2k}$ if $k \leq \frac{N}{2}$ or $p = 1$ if $k > \frac{N}{2}$. Then there exists a nonnegative function u in Ω such that $-u \in \Phi^k(\Omega)$, continuous near $\partial\Omega$ and u is a solution to the problem

$$\begin{aligned} F_k[-u] &= \mu && \text{in } \Omega, \\ u &= \varphi && \text{on } \partial\Omega. \end{aligned}$$

Furthermore, for any nonnegative function u such that $-u \in \Phi^k(\Omega)$, which is continuous near $\partial\Omega$ and is a solution of above equation, there holds

$$\frac{1}{K_2}\mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[\mu] \leq u(x) \leq K_2\left(\mathbf{W}^{2diam\,\Omega}_{\frac{2k}{k+1},k+1}[\mu](x) + \max_{\partial\Omega}\varphi\right), \tag{4.1}$$

where K_2 is a positive constant independent of x, u and Ω.

Proof of Theorem 1.2. We defined a sequence of nonnegative functions u_m, continuous near $\partial\Omega$ and such that $-u_m \in \Phi^k(\Omega)$, by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu && \text{in } \Omega, \\ u_0 &= \varphi && \text{on } \partial\Omega, \end{aligned} \tag{4.2}$$

and, for $m \geq 0$,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l,a,\beta}(u_m) + \mu && \text{in } \Omega, \\ u_{m+1} &= \varphi && \text{on } \partial\Omega. \end{aligned} \tag{4.3}$$

By Theorem 4.3 we have for all $x \in \mathbb{R}^N$,

$$\begin{aligned} \chi_\Omega u_0(x) &\le K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu](x) + b, \\ \chi_\Omega u_{m+1}(x) &\le K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[P_{l,a,\beta}(u_m) + \mu](x) + b, \end{aligned} \tag{4.4}$$

where $b = K_2 \max_{\partial\Omega} \varphi$. Thus, by Theorem 2.4 there exist $b_0 \in (0,1]$ and $M > 0$ depending on N, k, l, a, β and R such that

$$u_m(x) \le K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu](x) + 2b_0 \qquad \forall x \in \Omega,\ \forall m \ge 0, \tag{4.5}$$

and

$$\exp\left(a\left(K_2 \mathbf{W}^R_{\frac{2k}{k+1},k+1}[\mu] + 2b_0\right)^\beta\right) \in L^1_{loc}(\mathbb{R}^N), \tag{4.6}$$

provided that $b \le b_0$ and

$$||M^{\frac{k(\beta-1)}{\beta}}_{2k,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M.$$

Note that because we can write

$$\omega = P_{l,a,\beta}(u_m) + \mu = (\mu_1 + \chi_{\Omega_\delta} P_{l,a,\beta}(u_m)) + ((1 - \chi_{\Omega_\delta}) P_{l,a,\beta}(u_m) + f),$$

where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$ and $\delta > 0$ is small enough, then ω satisfies the assumptions of the data in Theorem 4.3 since u_m is continuous near $\partial\Omega$. Therefore the sequence $\{u_m\}$ is well defined. Since $-u_m$ is k-subharmonic and the sequence $\{u_m\}$ is bounded in $L^1(\Omega)$, the sequence $\{u_m\}$ is also bounded in $W^{1,1}_{loc}(\Omega)$ (see e.g. [25]); we can find a subsequence, still denoted by the index m, such that u_m converges a.e in Ω to function u for which (1.21) is satisfied in Ω. Furthermore, we deduce from (4.5)-(4.6) and the dominated convergence theorem that $P_{l,a,\beta}(u_m) \to P_{l,a,\beta}(u)$ in $L^1(\Omega)$. Finally, by Theorem 4.2, we obtain that u satisfies (1.20) and (1.21).
Conversely, assume that (1.20) admits nonnegative solution u, continuous near $\partial\Omega$, such that $-u \in \Phi^k(\Omega)$ and $P_{l,a,\beta}(u) \in L^1(\Omega)$. Then by Theorem 4.3 we have

$$u(x) \ge \frac{1}{K_2} \mathbf{W}^{\frac{d(x,\partial\Omega)}{8}}_{\frac{2k}{k+1},k+1}[\omega](x) \quad \text{for all } x \in \Omega,$$

where $d\omega = P_{l,a,\beta}(u)dx + d\mu$. Using Theorem 2.5, we conclude that (1.23) holds. ∎

5 Wolff integral equations

Proof of Theorem 1.3. Let $\varepsilon > 0$, assume that $\exp((a+\varepsilon)f^\beta) \in L^1_{loc}(\mathbb{R}^N)$ and consider the measure $d\mu = P_{l,a+\varepsilon,\beta}(f)\,dx$. By Theorem 2.3 with $\eta = \frac{(p-1)(\beta-1)}{\beta}$, there exist $C = C(N,\alpha,p,\beta,R) > 0$ and $\delta_0 = \delta_0(N,\alpha,p,\beta) > 0$ such that if

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M,$$

there holds

$$\mathbf{W}^R_{\alpha,p}\left[\exp\left(\delta_0 M^{-\frac{\beta}{p-1}}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\beta\right)\right](x) \le C \qquad \forall x \in \mathbb{R}^N. \tag{5.1}$$

Consider the sequence $\{u_m\}_{m\ge 0}$ of nonnegative functions defined by $u_0 = f$ and

$$u_{m+1} = \mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u_m)] + f \quad \text{in } \mathbb{R}^N \quad \forall m \ge 0.$$

Step 1. We claim that there exist $b_0 \in (0,1]$ and $M > 0$ depending on $N, \alpha, p, l, a, \beta, R$ and ε such that if

$$||\mathbf{M}^{\frac{(p-1)(\beta-1)}{\beta}}_{\alpha p,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \le M,$$

then the sequence $\{u_m\}_{m\ge 0}$ is well defined and there holds

$$u_m \le F := \tilde{c}_p \mathbf{W}^R_{\alpha,p}[\mu] + f + b_0 \qquad \forall m \ge 0, \text{ with } \tilde{c}_p = 1 \vee 3^{\frac{2-p}{p-1}}, \tag{5.2}$$

and

$$\exp(aF^\beta) \in L^1_{loc}(\mathbb{R}^N) \text{ and } \mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(F)] \in L^\infty(\mathbb{R}^N). \tag{5.3}$$

Clearly, (5.2) holds with $m = 0$, thus we assume that (5.2) holds with $m = n$, then it follows

$$u_n{}^\beta \le F^\beta \le c_{13}\tilde{c}_p^\beta \left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\beta + c_{13} b_0{}^\beta + \frac{a + 2^{-1}\varepsilon}{a} f^\beta,$$

for some constant c_{13} only depends on β and $\frac{a+2^{-1}\varepsilon}{a}$.
Thanks to the convexity property of H_l in $[0,\infty)$ and $b_0 \le 1$, we have

$$\begin{aligned} H_l(aF^\beta) &\le \frac{\varepsilon}{4(a+\varepsilon)} H_l\left(c_{14}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\beta\right) + \frac{\varepsilon}{4(a+\varepsilon)} H_l\left(c_{15} b_0{}^\beta\right) + \frac{2a+\varepsilon}{2(a+\varepsilon)} H_l\left((a+\varepsilon) f^\beta\right) \\ &\le H_l\left(c_{14}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\beta\right) + b_0^{l\beta} H_l\left(c_{15}\right) + P_{l,a+\varepsilon,\beta}(f), \end{aligned}$$

where $c_{14} = \frac{4a(a+\varepsilon)}{\varepsilon} c_{13}\tilde{c}_p^\beta$ and $c_{15} = \frac{4a(a+\varepsilon)}{\varepsilon} c_{13}$.
Thus,

$$\begin{aligned} u_{n+1} &= \mathbf{W}^R_{\alpha,p}\left[H_l(a u_n{}^\beta)\right] + f \le \mathbf{W}^R_{\alpha,p}\left[H_l(aF^\beta)\right] + f \\ &\le \tilde{c}_p \mathbf{W}^R_{\alpha,p}\left[H_l\left(c_{14}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\beta\right)\right] + \tilde{c}_p b_0{}^{\frac{l\beta}{p-1}} \mathbf{W}^R_{\alpha,p}\left[H_l(c_{15})\right] + \tilde{c}_p \mathbf{W}^R_{\alpha,p}\left[P_{l,a+\varepsilon,\beta}(f)\right] + f \\ &= \tilde{c}_p \mathbf{W}^R_{\alpha,p}[\mu] + f + \tilde{c}_p \mathbf{W}^R_{\alpha,p}\left[H_l\left(c_{14}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\beta\right)\right] + c_{16} b_0{}^{\frac{l\beta}{p-1}}, \end{aligned} \tag{5.4}$$

where $c_{16} = \tilde{c}_p (H_l(c_{15}))^{\frac{1}{p-1}} |B_1|^{\frac{1}{p-1}} \frac{p-1}{\alpha p} R^{\frac{\alpha p}{p-1}}$.

Now, we take $b_0 = 1 \wedge (2c_{16})^{-\frac{p-1}{l\beta - p + 1}}$ which yields $c_{16} b_0^{\frac{l\beta}{p-1}} \le \frac{b_0}{2}$. Hence, we will obtain (5.2) with $m = n+1$ and (5.3) if we show that

$$\tilde{c}_p \mathbf{W}^R_{\alpha,p}\left[H_l\left(c_{14}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\beta\right)\right] \le \frac{b_0}{2}. \tag{5.5}$$

As in the proof of Theorem 2.4, we have

$$\tilde{c}_p \mathbf{W}^R_{\alpha,p}\left[H_l\left(c_{14}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\beta\right)\right] \le \theta^{\frac{1}{p-1}}\tilde{c}_p \mathbf{W}^R_{\alpha,p}\left[H_l\left(\theta^{-1}c_{14}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right)^\beta\right)\right] \quad \text{with } \theta \in (0,1],$$

and thanks to (5.1) we will achieve (5.5) if $\theta^{-1}c_{14} \le \delta_0 M^{-\frac{\beta}{p-1}}$ and $\theta^{\frac{1}{p-1}}\tilde{c}_p C \le \frac{b_0}{2}$ are satisfied. Thus, we can choose $\theta = 1 \wedge \left(\frac{b_0}{2\tilde{c}_p C}\right)^{p-1}$ and $M = \left(\frac{\delta_0}{c_{14}}\right)^{\frac{p-1}{\beta}} \theta^{\frac{p-1}{\beta}}$.
Therefore $\{u_m\}_{m\ge 0}$ is well defined and (5.2) and (5.3) are satisfied. It is easy to see that u_m is nondecreasing. Hence, from (5.2) and (5.3), we obtain that $u(x) = \lim_{m\to\infty} u_m(x)$ is a solution to equation (1.24) which satisfies (1.25).
Step 2. Conversely, we assume that (1.24) admits a nonnegative solution u. There holds

$$u^\beta \ge \left(\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u)]\right)^\beta + f^\beta,$$

and thus,

$$P_{l,a,\beta}(u) \ge P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u)]\right) + P_{l,a,\beta}(f).$$

This implies

$$d\mu \ge P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right) dx,$$

where

$$d\mu = P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u)]\right) dx + P_{l,a,\beta}\left(f\right) dx.$$

If $E \subset \mathbb{R}^N$ is a Borel set, we deduce

$$\int\limits_{\mathbb{R}^N} (M_\mu \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right) dx \le \int\limits_{\mathbb{R}^N} (M_\mu \chi_E)^{\frac{l\beta}{p-1}} d\mu.$$

Since M_μ is bounded in $L^s(\mathbb{R}^N, d\mu)$ for $s > 1$ by Fefferman's differentiation theorem [11], we infer that

$$\int\limits_{\mathbb{R}^N} (M_\mu \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right) dx \le c_{17}\mu(E),$$

for some constant c_{17} only depends on N and $\frac{l\beta}{p-1}$. Because

$$(M_\mu \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[\mu]\right) \ge P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[\mu_E]\right),$$

where $\mu_E = \chi_E \mu$. We finally deduce

$$\int\limits_{\mathbb{R}^N} P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[\mu_E]\right) dx \le c_{17}\mu(E) \qquad \forall E \subset \mathbb{R}^N,\ E \text{ Borel}.$$

As in the proof of Theorem 2.5, it is easy to see that

$$\mu(E) \le c_{18} cap_{G_{\alpha p}, Q_p^*}(E) \qquad \forall E \subset \mathbb{R}^N,\ E \text{ Borel},$$

where the positive constant c_{18} depends on N, α, p, a, β and R. Finally since $d\mu \geq P_{l,a,\beta}(u)dx$ we obtain that for any Borel subset $E \subset \mathbb{R}^N$ there holds

$$\int\limits_E P_{l,a,\beta}(u)dx + \int\limits_E P_{l,a,\beta}\left(\mathbf{W}^R_{\alpha,p}[P_{l,a,\beta}(u)]\right)dx + \int\limits_E P_{l,a,\beta}(f)dx \leq 2c_{12}cap_{G_{\alpha p},Q_p^*}(E),$$

which completes the proof of the Theorem. ∎

References

[1] D. R. Adams, L.I. Heberg: *Function Spaces and Potential Theory*, Grundlehrender mathematischen Wisenschaften 31, Springer-Verlag (1999).

[2] D.R. Adams and M. Pierre,*Capacitary strong type estimates in semilinear problems*, **Ann. Inst. Fourier (Grenoble)** 41 (1991), 117-135.

[3] P. Baras, M. Pierre, *Critères d'existence des solutions positives pour des équations semilinéaires non monotones*, **Ann. Inst. H. Poincaré** 3 (1985), 185-212.

[4] M. F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type* , **Arch. Ration. Mech. Anal.** 107 (1989), 293-324.

[5] M. F. Bidaut-Véron, *Necessary conditions of existence for an elliptic equation with source term and measure data involving p-Laplacian*, in: Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, **Electron. J. Differ. Equ. Conf.** 8 (2002), 23-34.

[6] M. F. Bidaut-Véron, *Removable singularities and existence for a quasilinear equation with absorption or source term and measure data*, **Adv. Nonlinear Stud.** 3 (2003), 25-63.

[7] M. F. Bidaut-Véron and S. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, **J. Anal. Math.** 84 (2001), 1-49.

[8] M. F. Bidaut-Véron, H. Nguyen Quoc, L. Véron: *Quasilinear Lane-Emden equations with absorption and measure data*, **arXiv:1212.6314**, preprint.

[9] I. Birindelli and F. Demengel, *Some Liouville theorems for the p-Laplacian*, in: Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, **Electron. J. Differ. Equ. Conf.** 8 (2002), 35-46.

[10] G. Dal Maso, F. Murat, L. Orsina, A. Prignet: *Renormalized solutions of elliptic equations with general measure data*, **Ann. Sc. Norm. Sup. Pisa** 28 (1999), 741-808.

[11] R. Fefferman, *Strong differentiation with respect to measure*: **Amer. J. Math.** 103 (1981), 33-40.

[12] F. Ferrari, B. Franchi, I. Verbitsky: *Hessian inequality and the fractional Laplacian*, **J. Reine Angew. Math.** 667 (2012), 133-148.

[13] N. Grenon. *Existence results for semilinear elliptic equations with small measure data.*, **Ann. Inst. H. Poincaré Anal. Non Linéaire** 19 (2002), 1-11.

[14] T. Kilpelainen, J. Malý, *Degenerate elliptic equation with measure data and nonlinear potentials*, **Ann. Sc. Norm. Super. Pisa, Cl. Sci.** 19 (1992), 591-613.

[15] T. Kilpelainen, J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, **Acta Math.** 172 (1994), 137-161.

[16] P. Honzik and B. Jaye: *On the good-λ inequality for nonlinear potentials*, **Proc. Amer. Math. Soc.** 140 (2012), 4167-4180.

[17] D. Labutin *Potential estimates for a class of fully nonlinear elliptic equations*: **Duke Math. J.** 111 (2002), 1-49.

[18] M.M. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, Textbooks in Pure and Applied Mathematics (1991).

[19] N. C. Phuc, I. E. Verbitsky: *Quasilinear and Hessian equations of Lane-Emden type*, **Ann. Math.** 168 (2008), 859-914.

[20] N. C. Phuc, I. E. Verbitsky: *Singular quasilinear and Hessian equation and inequalities*, **J. Functional Analysis** 256 (2009), 1875-1906.

[21] Y. Richard, L. Véron: *Isotropic singularities of solutions of nonlinear elliptic inequalities*, **Ann. I.H.P. Analyse Non Linéaire** 6 (1989), 37-72.

[22] J. Serrin, H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities* **Acta Math.** 189 (2002), 79-142.

[23] B. O. Tureson: *Nonlinear Potential Theory and weighted Sobolev Spaces*, Lecture Notes in Mathematics, Vol. 1736, Springer-Verlag (2000).

[24] N.S. Trudinger and Wang: *Hessian measures*, **Topological Methods in Nonlinear Analysis** 10 (1997), 225-239.

[25] N.S. Trudinger and Wang: *Hessian measures II*, **Annals of Mathematics** 150 (1999), 579-604.

[26] N.S. Trudinger and Wang: *Hessian measures III*, **Journal of Functional Analysis** 193 (2002), 1-23.

[27] L. Véron: *Elliptic equations involving measures*, in Stationary Partial Differential Equations, vol. I, Handbook of Equations, North-Holland, Amsterdam (2004), pp. 593-712.