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# Quasilinear and Hessian type equations with exponential reaction and measure data

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## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Estimates on potentials</b>	<b>6</b>
<b>3</b>	<b>Quasilinear Dirichlet Problems</b>	<b>15</b>
<b>4</b>	<b>Hessian equations</b>	<b>18</b>
<b>5</b>	<b>Wolff integral equations</b>	<b>20</b>

## Abstract

We prove existence results concerning equations of the type  $-\Delta_p u = F(u) + \mu$  for  $p > 1$  and  $F_k[-u]u = F(u) + \mu$  with  $1 \leq k < \frac{N}{2}$  in a bounded domain  $\Omega$ , where  $\mu$  is a positive Radon measure and  $F(u) \sim e^{au^\beta}$  with  $a > 0$  and  $\beta \geq 1$ . Sufficient conditions for existence are expressed in terms of the maximal fractional potential of  $\mu$ . Two-sided estimates on the solutions are obtained in terms of some precise Wolff potentials of  $\mu$ . Necessary conditions are obtained in terms of Orlicz capacities. We also establish existence results for a general Wolff potential equation under the form  $u = \mathbf{W}_{\alpha,p}[F(u)] + f$ .

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# 1 Introduction

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain,  $p > 1$  and  $k \in \{1, 2, \dots, N\}$ . We denote by

$$\Delta_p u := \operatorname{div} \left( |\nabla u|^{p-2} \nabla u \right)$$

the p-Laplace operator and by

$$F_k[u] = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq N} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_k}$$

the k-Hessian operator where  $\lambda_1, \dots, \lambda_N$  are the eigenvalues of the Hessian matrix  $D^2u$ . Let  $\mu$  be a positive Radon measure in  $\Omega$ ; our aim is to study the existence of positive renormalized solutions to the quasilinear equation

$$\begin{aligned} -\Delta_p u &= P(u) + \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \quad (1.1)$$

and the fully nonlinear equation

$$\begin{aligned} F_k[u] &= P(u) + \mu && \text{in } \Omega \\ u &= \varphi && \text{on } \partial\Omega \end{aligned} \quad (1.2)$$

where  $P$  an exponential function. When  $P(r) = r^q$  with  $q > p - 1$ , Phuc and Verbitsky published a seminal article [10] on the solvability of the corresponding problem (1.1). They obtained necessary and sufficient conditions involving Bessel capacities or Wolff potentials. For example, they proved that if  $\mu$  has compact support in  $\Omega$  it is equivalent to solve (1.1) with  $P(r) = r^q$  or to have

$$\mu(E) \leq cC_{p, \frac{p}{q+1-p}}(E) \quad \text{for all compact set } E \subset \Omega, \quad (1.3)$$

where  $c$  is a suitable positive constant and  $C_{p, \frac{p}{q+1-p}}$  a Bessel capacity, or to have

$$\int_B [\mathbf{W}_{1,p}^{2R}[\mu_B(x)]]^q dx \leq C[\mu(B)] \quad \text{for all ball } B \text{ s.t. } B \cap \operatorname{supp}\mu \neq \emptyset, \quad (1.4)$$

where  $R = \operatorname{supp}(\Omega)$ . Other conditions are expressed in terms of Riesz capacities and Riesz potentials. Their construction is based upon sharp estimates of solutions of the non-homogeneous problem

$$\begin{aligned} -\Delta_p u &= \omega && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \quad (1.5)$$

for positive measures  $\omega$ . Concerning k-Hessian operator in a bounded  $(k - 1)$ -convex domain  $\Omega$ , they proved that if  $\mu$  has compact support, the corresponding problem (1.2 with  $P(r) = r^q$  with  $q > k$  admits a positive solution if and only if

$$\mu(E) \leq cC_{2k, \frac{q}{q-k}}(E) \quad \text{for all compact set } E \subset \Omega \quad (1.6)$$

or equivalently

$$\int_B \left[ \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2R} [\mu_B(x)] \right]^q dx \leq C[\mu(B)] \quad \text{for all ball } B \text{ s.t. } B \cap \text{supp}\mu \neq \emptyset. \quad (1.7)$$

The main tools in their proofs are derived from recent advances in potential theory for nonlinear elliptic equations obtained by Kilpelainen and Maly [5], Trudinger and Wang [14, 15, 16], and Labutin [7] thanks to whom the authors first provide global pointwise estimates for solutions of the homogeneous Dirichlet problems in terms of Wolffs potentials of suitable order.

For  $s > 1$ ,  $0 \leq \alpha < \frac{N}{s}$ ,  $\eta \geq 0$  and  $0 < T \leq \infty$ , we recall the expression of the  $T$ -truncated Wolff potential of a positive Radon measure  $\mu$  defined in  $\mathbb{R}^N$  by

$$\mathbf{W}_{\alpha, s}^T[\mu](x) = \int_0^T \left( \frac{\mu(B_t(X))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t}, \quad (1.8)$$

and the  $T$ -truncated  $\eta$ -fractional maximal potential of  $\mu$  by

$$\mathbf{M}_{\alpha, T}^\eta[\mu](x) = \sup \left\{ \frac{\mu(B_t(X))}{t^{N-\alpha} h_\eta(t)} : 0 < t \leq T \right\}, \quad (1.9)$$

where  $h_\eta(t) = (-\ln t)^{-\eta} \chi_{(0, 2^{-1}]}(t) + (\ln 2)^{-\eta} \chi_{[2^{-1}, \infty)}(t)$ . If  $\eta = 0$ , then  $h_\eta = 1$  and we denote by  $\mathbf{M}_{\alpha, T}[\mu]$  the corresponding  $T$ -truncated fractional maximal potential of  $\mu$ . When the measures are only defined in an open subset  $\Omega \subset \mathbb{R}^N$ , they are naturally extended by 0 in  $\Omega^c$ . For  $l \in \mathbb{N}_*$ , we define the  $l$ -truncated exponential function

$$H_l(r) = e^r - \sum_{j=0}^{l-1} \frac{r^j}{j!}, \quad (1.10)$$

and for  $a > 0$  and  $\beta \geq 1$ , we set

$$P_{l, a, \beta}(r) = H_l(ar^\beta). \quad (1.11)$$

We put

$$Q_p(s) = \begin{cases} \sum_{q=l}^{\infty} \frac{s^{\frac{\beta q}{p-1}}}{q^{\frac{\beta q}{p-1}} q!} & \text{if } p \neq 2 \\ H_l(s^\beta) & \text{if } p = 2, \end{cases} \quad (1.12)$$

$Q_p^*(r) = \max \{rs - Q_p(s) : s \geq 0\}$  is the complementary function to  $Q_p$ , and define

$$\text{cap}_{G_p, Q_p^*}(E) = \inf \left\{ \int_{\mathbb{R}^N} Q_p^*(f) dx : G_p * f \geq \chi_E, f \geq 0, Q_p^*(f) \in L^1(\mathbb{R}^N) \right\} \quad (1.13)$$

where  $G_p(x) = \mathcal{F}^{-1} \left( (1 + |\cdot|^2)^{-\frac{p}{2}} \right) (x)$  is the Bessel kernel of order  $p$ .

We denote  $a \wedge b$  and  $a \vee b$  for  $\min\{a, b\}$  and  $\max\{a, b\}$  respectively. Our main results are the following theorems

**Theorem 1.1** *Let  $1 < p < N$ ,  $l \in \mathbb{N}$  and  $\beta \geq 1$  such that  $l\beta > p - 1$  and  $a > 0$ . Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain. If  $\mu$  is a nonnegative Radon measure in  $\Omega$ , there exist  $b_0 \in (0, 1]$  and  $M_0 > 0$  depending on  $N, p, \beta, a, l$  and  $\text{diam } \Omega$  which is the diameter of  $\Omega$  such that if*

$$\| \mathbf{M}_{p, 2\text{diam } \Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu] \|_{L^\infty(\mathbb{R}^N)} \leq M_0, \quad (1.14)$$

the following Dirichlet problem

$$\begin{aligned} -\Delta_p u &= P_{l,a,\beta}(u) + \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \quad (1.15)$$

admits a nonnegative renormalized solution  $u$ , which satisfies

$$u(x) \leq K_1 \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{1,p}^{2\text{diam } \Omega}[\mu](x) + 2b_0 \quad \forall x \in \Omega \quad (1.16)$$

and

$$\exp \left( K_1 \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{1,p}^{2\text{diam } \Omega}[\chi_\Omega \mu] + 2b_0 \right) \in L^1(\Omega). \quad (1.17)$$

The role of  $K_1 = K_1(N, p)$  will be made explicit in Theorem 3.5.

Conversely, if (1.15) admits a nonnegative renormalized solution  $u$ , then for any compact set  $K \subset \Omega$ , there exists a positive constant  $C$  depending on  $N, p, \beta$  and  $\text{dist}(K, \partial\Omega)$  such that

$$\int_E P_{l,\alpha,\beta}(u) dx + \mu(E) \leq C \text{cap}_{G_p, Q_p^*}(E) \quad \text{for all Borel sets } E \subset K. \quad (1.18)$$

Concerning the  $k$ -Hessian operator we recall some notions introduced by Trudinger and Wang [14, 15, 16], and we follow their notations. For  $k = 1, \dots, N$  and  $u \in C^2(\Omega)$  the  $k$ -hessian operator  $F_k$  is defined by

$$F_k[u] = S_k(\lambda(D^2u))$$

where  $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$  denotes the eigenvalues of the Hessian matrix of second partial derivatives  $D^2u$  and  $S_k$  is the  $k$ -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}$$

It is straightforward that

$$F_k[u] = [D^2u]_k,$$

where in general  $[A]_k$  denotes the sum of the  $k$ -th principal minors of a matrix  $A = (a_{ij})$ . In order that there exists a smooth  $k$ -admissible function which vanishes on  $\partial\Omega$ , the boundary  $\partial\Omega$  must satisfy a uniformly  $(k-1)$ -convex condition, that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega \quad (1.19)$$

for some positive constant  $c_0$ , where  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$  denote the principal curvatures of  $\partial\Omega$  with respect to its inner normal. We also denote by  $\Phi^k(\Omega)$  the class of upper-semicontinuous functions  $\Omega \mapsto [-\infty, \infty)$  which are  $k$ -convex, or subharmonic in the Perron sense (see Definition 4.1). In this paper we prove the following theorem

**Theorem 1.2** Let  $k \in \{1, 2, \dots, N\}$  such that  $2k < N$ ,  $l \in \mathbb{N}^*$ ,  $\beta \geq 1$  such that  $l\beta > k$  and  $a > 0$ . Let  $\Omega$  be a bounded uniformly  $(k-1)$ -convex domain in  $\mathbb{R}^N$ . Let  $\varphi$  be a nonnegative continuous function on  $\partial\Omega$  and  $\mu = \mu_1 + f$  be a nonnegative Radon measure where  $\mu_1$  has compact support in  $\Omega$  and  $f \in L^q(\Omega)$  for some  $q > \frac{N}{2k}$ . Let  $K_2 = K_2(N, k)$  be the constant  $K_2$  which appears in Theorem 4.3. Then, there exist  $b_0 \in (0, 1]$  and  $M_0 > 0$  depending on  $N, k, \beta, a, l$  and  $\text{diam}(\Omega)$  such that if

$$\|\mathbf{M}_{2k, R}^{\frac{k(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M_0$$

and  $\max_{\partial\Omega} \varphi \leq \frac{b_0}{K_2}$ , the following Dirichlet problem

$$\begin{aligned} F_k[-u] &= P_{l, a, \beta}(u) + \mu && \text{in } \Omega \\ u &= \varphi && \text{on } \partial\Omega, \end{aligned} \quad (1.20)$$

admits a nonnegative solution  $u$ , continuous near  $\partial\Omega$ , with  $-u \in \Phi^k(\Omega)$  and  $P_{l, a, \beta}(u) \in L^1(\Omega)$  which satisfies

$$u(x) \leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2\text{diam}\Omega}[\chi_\Omega \mu](x) + 2b_0 \quad \forall x \in \Omega. \quad (1.21)$$

Conversely, if (1.20) admits a nonnegative solution  $u$ , continuous near  $\partial\Omega$ , such that  $-u \in \Phi^k(\Omega)$  and  $P_{l, a, \beta}(u) \in L^1(\Omega)$ , then for any a compact set  $K \subset \Omega$ , there exists a positive constant  $C$  depending on  $N, k, \beta$  and  $\text{dist}(K, \partial\Omega)$  such that there holds

$$\int_E P_{l, a, \beta}(u) dx + \mu(E) \leq C \text{cap}_{G_{2k}, Q_{k+1}^*}(E) \quad \forall E \subset K, E \text{ Borel}, \quad (1.22)$$

where  $Q_{k+1}(s)$  is defined by (1.12) with  $p = k + 1$ ,  $Q_{k+1}^*$  is its complementary function and  $\text{cap}_{G_{2k}, Q_{k+1}^*}(E)$  is defined accordingly by (1.13).

The two previous theorems are connected to the following result which deals with a class of nonlinear Wolff integral equations.

**Theorem 1.3** Let  $N \in \mathbb{N}$ ,  $p > 1$ ,  $\alpha > 0$ ,  $a > 0$ ,  $R > 0$ ,  $l \in \mathbb{N}$  and  $\beta \geq 1$  such that  $l\beta > p - 1$  and  $0 < \alpha p < N$ . Let  $f \in L_{loc}^1(\mathbb{R}^N)$ ,  $f \geq 0$ . For any  $\varepsilon > 0$  there exist constants  $b_0 \in (0, 1]$  and  $M_0 > 0$  depending on  $N, \alpha, p, \beta, a, l, R$  and  $\varepsilon$  such that if  $\exp((a + \varepsilon)f^\beta) \in L_{loc}^1(\mathbb{R}^N)$  and

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M_0$$

where  $d\mu = P_{l, a + \varepsilon, \beta}(f) dx$ , then there exists a nonnegative function  $u$ , such that  $P_{l, a, \beta}(u) \in L_{loc}^1(\mathbb{R}^N)$  which satisfies

$$u = \mathbf{W}_{\alpha, p}^R[P_{l, a, \beta}(u)] + f \quad \text{in } \mathbb{R}^N. \quad (1.23)$$

Furthermore

$$\mathbf{W}_{\alpha, p}^R[P_{l, a, \beta}(u)] + f \leq u \leq \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha, p}^R[\mu] + f + b_0 := F \quad \text{in } \mathbb{R}^N \quad (1.24)$$

Moreover  $\exp(aF^\beta) \in L_{loc}^1(\mathbb{R}^N)$  and  $\|\mathbf{W}_{\alpha, p}^R[P_{l, a, \beta}(F)]\|_{L^\infty(\mathbb{R}^N)} < +\infty$ .

Conversely, if (1.23) admits a nonnegative solution  $u$ , then there exists a positive constant  $C$

depending on  $N, \alpha, p, \beta$  and  $R$  such that there holds

$$\int_E P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)]) dx + \int_E P_{l,a,\beta}(f) dx \leq C \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel},$$

where  $Q_p, Q_p^*$  and  $\text{cap}_{G_{\alpha p}, Q_p^*}$  are defined in (1.12).

In the case  $\alpha = 1, R = \infty$ , (1.23) is equivalent to

$$-\Delta_p(u - f) = P_{l,a,\beta}(u) \quad \text{in } \mathbb{R}^N. \quad (1.25)$$

while when  $\alpha = \frac{2k}{k+1}$ , for some  $k \in \mathbb{N}_*$  and  $p = k + 1$ , it is equivalent to

$$F_k[-u + f] = P_{l,a,\beta}(u) \quad \text{in } \mathbb{R}^N. \quad (1.26)$$

Some other applications are also given to equations involving the fractional Laplacian.

## 2 Estimates on potentials

We denote by  $B_r(a)$  the ball of center  $a$  and radius  $r > 0$  and  $B_r = B_r(0)$ . The next estimates are crucial in the sequel.

**Theorem 2.1 1.** *There exists a positive constant  $c_1$ , depending only on  $N, \alpha, p, R$  such that for all  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  and  $q > p - 1$  we have*

$$(c_1 q)^{-\frac{q}{p-1}} \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}} dx \leq \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx \leq (c_1 q)^q \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p}[\mu](x))^{\frac{q}{p-1}} dx \quad (2.1)$$

where  $\mathbf{G}_{\alpha p}[\mu] := G_{\alpha p} * \mu$  denotes the Bessel potential of order  $\alpha p$  of  $\mu$ .

**2.** *There exists a positive constant  $c_2$  depending only on  $N, \alpha, p, R$  such that for all  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  and  $q > p - 1$  we have*

$$c_2^{-q} \int_{\mathbb{R}^N} (\mathbf{G}_{2\alpha}[\mu](x))^q dx \leq \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,2}^R[\mu](x))^q dx \leq c_2^q \int_{\mathbb{R}^N} (\mathbf{G}_{2\alpha}[\mu](x))^q dx. \quad (2.2)$$

**Proof.** We can find proof of (2.2) in [2, Step 3, Theorem 2.3]. By [2, Step 2, Theorem 2.3], there is  $c_3 > 0$  such that

$$\int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx \geq c_3^q \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}} dx \quad \forall q > 0, R > 0 \text{ and } \mu \in \mathfrak{M}^+(\mathbb{R}^N). \quad (2.3)$$

We recall that  $\mathbf{M}_{\alpha p,R}[\mu] = \mathbf{M}_{\alpha p,R}^0[\mu]$  by (1.9). Next we show that

$$\int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p,R}[\mu](x))^{\frac{q}{p-1}} dx \geq (c_4 q)^{-q} \int_{\mathbb{R}^N} (\mathbf{W}_{\alpha,p}^R[\mu](x))^q dx \quad \forall q > 0, R > 0 \text{ and } \mu \in \mathfrak{M}^+(\mathbb{R}^N), \quad (2.4)$$

for some a positive constant  $c_4$ . Indeed, we denote  $\mu_n$  by  $\chi_{B_n}\mu$  for  $n \in \mathbb{N}^*$ . By [2, Proposition 2.2], there exist constants  $c_5 = c_5(N, \alpha, p) > 0$ ,  $a = a(\alpha, p) > 0$  and  $\varepsilon_0 = \varepsilon(N, \alpha, p)$  such that for all  $n \in \mathbb{N}^*$ ,  $t > 0$ ,  $r > 0$ ,  $0 < R \leq \infty$  and  $0 < \varepsilon < \varepsilon_0$ , there holds

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R \mu_n > 3t, (\mathbf{M}_{\alpha p, R} \mu_n)^{\frac{1}{p-1}} \leq \varepsilon t \right\} \right| \leq c_5 \exp(-a\varepsilon^{-1}) \left| \left\{ \mathbf{W}_{\alpha,p}^R \mu_n > t \right\} \right|.$$

Thus,

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R \mu_n > 3t \right\} \right| \leq c_5 \exp(-a\varepsilon^{-1}) \left| \left\{ \mathbf{W}_{\alpha,p}^R \mu_n > t \right\} \right| + \left| \left\{ (\mathbf{M}_{\alpha p, R} \mu_n)^{\frac{1}{p-1}} > \varepsilon t \right\} \right|.$$

Multiplying by  $qt^{q-1}$  and integrating over  $(0, \infty)$ , we obtain

$$\begin{aligned} \int_0^\infty qt^{q-1} \left| \left\{ \mathbf{W}_{\alpha,p}^R \mu_n > 3t \right\} \right| dt &\leq c_5 \exp(-a\varepsilon^{-1}) \int_0^\infty qt^{q-1} \left| \left\{ \mathbf{W}_{\alpha,p}^R \mu_n > t \right\} \right| dt \\ &\quad + \int_0^\infty qt^{q-1} \left| \left\{ (\mathbf{M}_{\alpha p, R} \mu_n)^{\frac{1}{p-1}} > \varepsilon t \right\} \right| dt, \end{aligned}$$

which implies

$$3^{-q} \int_N (\mathbf{W}_{\alpha,p}^R [\mu_n](x))^q dx \leq c_5 \exp(-a\varepsilon^{-1}) \int_N (\mathbf{W}_{\alpha,p}^R [\mu_n](x))^q dx + \varepsilon^{-q} \int_N (\mathbf{M}_{\alpha p, R} \mu_n)^{\frac{q}{p-1}} dx.$$

This leads to

$$\int_N (\mathbf{M}_{\alpha p, R} \mu_n)^{\frac{q}{p-1}} dx \varepsilon^q (3^{-q} - c_5 \exp(-a\varepsilon^{-1})) \int_N (\mathbf{W}_{\alpha,p}^R [\mu_n](x))^q dx.$$

We see that  $\sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^q (3^{-q} - c_5 \exp(-a\varepsilon^{-1})) \approx (c_6 q)^{-q}$  for some a constant  $c_6$  which does not depend on  $q$ . Therefore, (2.4) follows by Fatou's lemma. From (2.3) and (2.4), we have

$$c_7^q \int_{\mathbb{R}^N} \left( \mathbf{W}_{\frac{\alpha p}{2}, 2}^R [\mu](x) \right)^q dx \geq \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p, R} [\mu](x))^q dx \geq (c_7 q)^{-q} \int_N \left( \mathbf{W}_{\frac{\alpha p}{2}, 2}^R [\mu](x) \right)^q dx$$

for all  $q > 0$  and  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ , with  $c_7 = c_7(N, \alpha, p) > 0$ . Then, combining with (2.2) we get

$$c_8^q \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p} [\mu](x))^q dx \geq \int_{\mathbb{R}^N} (\mathbf{M}_{\alpha p, R} [\mu](x))^q dx \geq (c_8 q)^{-q} \int_{\mathbb{R}^N} (\mathbf{G}_{\alpha p} [\mu](x))^q dx$$

with  $c_8 = c_8(N, \alpha, p, R)$ , for all  $q > 0$  and  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ . Therefore, jointly with (2.3) and (2.4) we derive (2.1).  $\blacksquare$

The next result is proved in [2].



**Theorem 2.2** Let  $\alpha > 0$ ,  $p > 1$ ,  $0 \leq \eta < p - 1$ ,  $0 < \alpha p < N$  and  $L > 0$ . Set  $\delta = \frac{1}{2} \left( \frac{p-1-\eta}{12(p-1)} \right)^{\frac{p-1}{p-1-\eta}} \alpha p \log(2)$ . Then there exists  $C(L) > 0$ , depending on  $N$ ,  $\alpha$ ,  $p$ ,  $\eta$  and  $L$  such that for any  $R \in (0, \infty]$ ,  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ , any  $a \in \mathbb{R}^N$  and  $0 < r \leq L$ , there holds

$$\frac{1}{|B_{2r}(a)|} \int_{B_{2r}(a)} \exp \left( \frac{\delta}{\|\mathbf{M}_{\alpha p, R}^\eta[\mu_{B_r}(a)]\|_{L^\infty(B_r(a))}^{\frac{1}{p-1-\eta}}} (\mathbf{W}_{\alpha, p}^R[\mu_{B_r}(a)](x))^{\frac{p-1}{p-1-\eta}} \right) dx \leq C(L) \quad (2.5)$$

where  $\mu_{B_r}(a) = \chi_{B_r(a)} \mu$ . Furthermore, if  $\eta = 0$ ,  $C$  is independent of  $r$ .

**Theorem 2.3** Assume  $\alpha$ ,  $p$ ,  $\eta$  are as in Theorem 2.2 and  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$  satisfies for  $R > 0$

$$\|\mathbf{M}_{\alpha p, R}^\eta[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

for some  $M > 0$ . Then there exist  $C = C(N, \alpha, p, \eta, R) > 0$  and  $\delta_0 = \delta_0(N, \alpha, p, \eta) > 0$  such that

$$\left\| \mathbf{W}_{\alpha, p}^R \left[ \exp \left( \delta_0 M^{-\frac{1}{p-1-\eta}} (\mathbf{W}_{\alpha, p}^R[\mu])^{\frac{p-1}{p-1-\eta}} \right) \right] \right\|_{L^\infty(\mathbb{R}^N)} \leq C.$$

**Proof.** Let  $x \in \mathbb{R}^N$ . Since  $\mu(B_t(y)) \leq M t^{N-\alpha p} h_\eta(t)$ , for all  $r \in (0, R)$  and  $y \in \mathbb{R}^N$  we have

$$\begin{aligned} \mathbf{W}_{\alpha, p}^R[\mu](y) &= \mathbf{W}_{\alpha, p}^r[\mu](y) + \int_r^R \left( \frac{\mu(B_t(y))}{t^{N-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha, p}^r[\mu](y) + M^{\frac{1}{p-1}} \int_r^R (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha, p}^r[\mu](y) + M^{\frac{1}{p-1}} \int_{r \wedge 2^{-1}}^{2^{-1}} \left( (-\ln t)^{-\eta} \right)^{\frac{1}{p-1}} \frac{dt}{t} + M^{\frac{1}{p-1}} \int_{2^{-1}}^{R \vee 2^{-1}} \left( (\ln 2)^{-\eta} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha, p}^r[\mu](y) + \frac{(p-1)M^{\frac{1}{p-1}}}{p-1-\eta} (-\ln(r \wedge 2^{-1}))^{\frac{p-1-\eta}{p-1}} + M^{\frac{1}{p-1}} (\ln(2))^{-\frac{\eta}{p-1}} \ln(2R \vee 1). \end{aligned}$$

Thus,

$$(\mathbf{W}_{\alpha, p}^R[\mu](y))^{\frac{p-1}{p-1-\eta}} \leq A_1 (\mathbf{W}_{\alpha, p}^r[\mu](y))^{\frac{p-1}{p-1-\eta}} + A_2 M^{\frac{1}{p-1-\eta}} \ln \left( \frac{1}{r \wedge 0.5} \right) + A_3 M^{\frac{1}{p-1-\eta}}, \quad (2.6)$$

where

$$\begin{aligned} A_1 &= 3^{\frac{\eta}{p-1-\eta}}, \\ A_2 &= 3^{\frac{\eta}{p-1-\eta}} \left( \frac{p-1}{p-1-\eta} \right)^{\frac{p-1}{p-1-\eta}}, \end{aligned}$$

and

$$A_3(R) = 3^{\frac{\eta}{p-1-\eta}} (\ln(2))^{-\frac{\eta}{p-1-\eta}} (\ln(2R \vee 1))^{\frac{p-1}{p-1-\eta}}.$$

Let  $\theta \in (0, 1]$ , since  $\exp(a + b) \leq \frac{1}{2} \exp(2a) + \frac{1}{2} \exp(2b)$  for all  $a, b \in \mathbb{R}$ , we get from (2.6)

$$\begin{aligned} \exp\left(\theta \frac{\delta M^{-\frac{1}{p-1-\eta}}}{2A_1} (\mathbf{W}_{\alpha,p}^R[\mu])^{\frac{p-1}{p-1-\eta}}\right) &\leq \frac{1}{2} \exp\left(\delta M^{-\frac{1}{p-1-\eta}} (\mathbf{W}_{\alpha,p}^r[\mu])^{\frac{p-1}{p-1-\eta}}\right) \\ &\quad + \frac{1}{2} \exp\left(\theta \frac{A_2}{A_1} \ln\left(\frac{1}{r \wedge 2^{-1}}\right) + \theta \frac{A_3(R)}{A_1}\right) \end{aligned} \quad (2.7)$$

For  $r > 0$ ,  $0 < t \leq r$ ,  $y \in B_r(x)$  we have  $B_t(y) \subset B_{2r}(x)$ . Thus,  $\mathbf{W}_{\alpha,p}^r[\mu] = \mathbf{W}_{\alpha,p}^r[\mu_{B_{2r}}]$  in  $B_r(x)$ . Then, using (2.5) with  $L = 2R$  we get

$$\begin{aligned} \int_{B_r(x)} \exp\left(\delta M^{-\frac{1}{p-1-\eta}} (\mathbf{W}_{\alpha,p}^r[\mu])^{\frac{p-1}{p-1-\eta}}\right) &= \int_{B_r(x)} \exp\left(\delta M^{-\frac{1}{p-1-\eta}} (\mathbf{W}_{\alpha,p}^r[\mu_{B_{2r}(x)}])^{\frac{p-1}{p-1-\eta}}\right) \\ &\leq C(2R)|B_4|r^N. \end{aligned}$$

So, from (2.7) we deduce

$$\begin{aligned} \int_{B_r(x)} \exp\left(\theta \frac{\delta M^{-\frac{1}{p-1-\eta}}}{2A_1} (\mathbf{W}_{\alpha,p}^R[\mu])^{\frac{p-1}{p-1-\eta}}\right) &\leq \frac{1}{2} C(2R)|B_4|r^N + \frac{1}{2} \exp\left(\theta \frac{A_2}{A_1} \ln\left(\frac{1}{r \wedge 2^{-1}}\right) + \theta \frac{A_3(R)}{A_1}\right) |B_1|r^N \\ &\leq \frac{1}{2} C(2R)|B_4|r^N + \frac{1}{2} |B_1| \exp\left(\theta \frac{A_3(R)}{A_1}\right) (r \wedge 2^{-1})^{-\theta \frac{A_2}{A_1}} r^N. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{W}_{\alpha,p}^R \left[ \exp\left(\theta \frac{\delta M^{-\frac{1}{p-1-\eta}}}{2A_1} (\mathbf{W}_{\alpha,p}^R[\mu])^{\frac{p-1}{p-1-\eta}}\right) \right] (x) &\leq \frac{1}{2} \int_0^R \left( C(2R)|B_4|r^{\alpha p} + |B_1| \exp\left(\theta \frac{A_3(R)}{A_1}\right) (r \wedge 2^{-1})^{-\theta \frac{A_2}{A_1}} r^{\alpha p} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\leq \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \left(\frac{1}{2} C(2R)|B_4|\right)^{\frac{1}{p-1}} \int_0^R r^{\frac{\alpha p}{p-1}-1} dr \\ &\quad + \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \left(\frac{1}{2} |B_1| \exp\left(\theta \frac{A_3(R)}{A_1}\right)\right)^{\frac{1}{p-1}} \int_0^R (r \wedge 2^{-1})^{-\theta \frac{A_2}{(p-1)A_1}} r^{\frac{\alpha p}{p-1}-1} dr \\ &\leq A_4(R) + A_5(R) \int_0^R (r \wedge 2^{-1})^{-\theta \frac{A_2}{(p-1)A_1}} r^{\frac{\alpha p}{p-1}-1} dr, \end{aligned}$$

where

$$A_4(R) = \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \left(\frac{1}{2} C(2R)|B_4|\right)^{\frac{1}{p-1}} \frac{p-1}{\alpha p} R^{\frac{\alpha p}{p-1}}$$

and

$$A_5(R) = \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \left(\frac{1}{2}|B_1| \exp\left(\theta \frac{A_3(R)}{A_1}\right)\right)^{\frac{1}{p-1}}.$$

Now we choose  $\theta = 1 \wedge \frac{\alpha p A_1}{2A_2}$ , we have

$$\begin{aligned} \mathbf{W}_{\alpha,p}^R \left[ \exp\left(\theta \frac{\delta M^{-\frac{1}{p-1-\eta}}}{2A_1} (\mathbf{W}_{\alpha,p}^R[\mu])^{\frac{p-1}{p-1-\eta}}\right) \right] (x) &\leq A_4 + A_5 \int_0^R \left(\frac{r}{r \wedge 2^{-1}}\right)^{\frac{\alpha p}{2(p-1)}} r^{\frac{\alpha p}{2(p-1)}-1} dr \\ &\leq A_4 + 2^{\frac{\alpha p}{2(p-1)}} A_5 \int_0^R r^{\frac{\alpha p}{2(p-1)}-1} dr. \end{aligned}$$

In other words,

$$\mathbf{W}_{\alpha,p}^R \left[ \exp\left(\delta_0 M^{-\frac{1}{p-1-\eta}} (\mathbf{W}_{\alpha,p}^R[\mu])^{\frac{p-1}{p-1-\eta}}\right) \right] (x) \leq A_6(R),$$

where

$$\begin{aligned} A_6(R) &= A_4(R) + 2^{\frac{\alpha p}{2(p-1)}} \frac{2(p-1)}{\alpha p} A_5(R) R^{\frac{\alpha p}{2(p-1)}}, \\ \delta_0 &= \frac{\theta \delta}{2A_1} = \frac{1}{4} 3^{-\frac{\eta}{p-1-\eta}} \left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \left(1 \wedge \left(\frac{\alpha p}{2} \left(\frac{p-1}{p-1-\eta}\right)^{-\frac{p-1}{p-1-\eta}}\right)\right) \alpha p \log(2). \end{aligned}$$

Which completes the proof of the Theorem.  $\blacksquare$

In the next result we obtain estimate on iterative solutions of Wolff integral inequalities. We recall that  $H_l$  and  $P_{l,a,\beta}$  have been defined in (1.14) and (1.11).

**Theorem 2.4** *Let  $N \in \mathbb{N}$ ,  $p > 1, \alpha > 0, a > 0, b \geq 0, K > 0, R > 0, l \in \mathbb{N}^*$  and  $\beta \geq 1$  such that  $l\beta > p-1$  and  $0 < \alpha p < N$ . Suppose that  $\{u_m\}$  is a sequence of nonnegative functions in  $\mathbb{R}^N$  that satisfies*

$$\begin{aligned} u_{m+1} &\leq K \mathbf{W}_{\alpha,p}^R [P_{l,a,\beta}(u_m) + \mu] + b \quad \forall m \in \mathbb{N}, \\ u_0 &\leq K \mathbf{W}_{\alpha,p}^R [\mu] + b, \end{aligned} \tag{2.8}$$

where  $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ . Then there exist  $b_0 \in (0, 1]$  and  $M_0 > 0$  depending on  $N, \alpha, p, \beta, a, l, K$  and  $R$  such that if  $b \leq b_0$  and

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M_0,$$

there holds

$$\exp\left(a \left(K \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha,p}^R[\mu] + 2b_0\right)^\beta\right) \in L_{loc}^1(\mathbb{R}^N)$$

and

$$\left\| \mathbf{W}_{\alpha,p}^R \left[ P_{l,a,\beta} \left( K \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{\alpha,p}^R[\mu] + 2b_0 \right) \right] \right\|_{L^\infty(\mathbb{R}^N)} < +\infty$$

and finally

$$u_m \leq K \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha,p}^R[\mu] + 2b_0 \quad \forall m \in \mathbb{N}. \tag{2.9}$$

**Proof.** By Theorem 2.3 with  $\eta = \frac{(p-1)(\beta-1)}{\beta}$ , there exist  $C = C(N, \alpha, p, \beta, R) > 0$  and  $\delta_0 = \delta_0(N, \alpha, p, \beta) > 0$  such that if

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

then

$$\mathbf{W}_{\alpha, p}^R \left[ \exp \left( \delta_0 \mathbf{M}^{-\frac{\beta}{p-1}} (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) \right] (x) \leq C \quad x \in \mathbb{R}^N. \quad (2.10)$$

Take  $b_0 \in [b, 1]$ . We will choose  $b_0$  and  $M_0$  later on. Clearly, (2.9) holds with  $m = 0$ . Now, assume that (2.9) holds with  $m = n$ , we need to prove that

$$u_{n+1} \leq K \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{\alpha, p}^R[\mu] + 2b_0.$$

In fact, by the definition of  $u_{n+1}$  and the sub-additive property of  $\mathbf{W}_{\alpha, p}^R[\cdot]$ , we have

$$u_{n+1} \leq K \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{\alpha, p}^R[\mu] + K \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{\alpha, p}^R[P_{l, a, \beta}(u_n)] + b.$$

So, it is sufficient to prove that

$$P_1 := K \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{\alpha, p}^R[P_{l, a, \beta}(u_n)] \leq b_0. \quad (2.11)$$

Since (2.9) holds with  $m = n$ ,

$$u_n^\beta \leq K^\beta 2^{\beta-1} \left( 1 \vee 2^{\frac{2-p}{p-1}} \right)^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta + 2^{2\beta-1} b_0^\beta,$$

which implies

$$\begin{aligned} P_{l, a, \beta}(u_n) &\leq H_l \left( K^\beta 2^{\beta-1} \left( 1 \vee 2^{\frac{2-p}{p-1}} \right)^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta + 2^{2\beta-1} b_0^\beta \right) \\ &\leq \frac{1}{2} H_l \left( K^\beta 2^\beta \left( 1 \vee 2^{\frac{2-p}{p-1}} \right)^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) + \frac{1}{2} H_l(2^{2\beta} b_0^\beta) \\ &\leq \frac{1}{2} H_l \left( K^\beta 2^\beta \left( 1 \vee 2^{\frac{2-p}{p-1}} \right)^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) + \frac{1}{2} b_0^{l\beta} H_l(2^{2\beta}), \end{aligned}$$

since  $b_0 \leq 1$ . Thus,

$$\begin{aligned} \mathbf{W}_{\alpha, p}^R[P_{l, a, \beta}(u_n)] &\leq \mathbf{W}_{\alpha, p}^R \left[ \frac{1}{2} H_l \left( K^\beta 2^\beta \left( 1 \vee 2^{\frac{2-p}{p-1}} \right)^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) + \frac{1}{2} b_0^{l\beta} H_l(2^{2\beta}) \right] \\ &\leq 2^{-\frac{1}{p-1}} \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{\alpha, p}^R \left[ H_l \left( K^\beta 2^\beta \left( 1 \vee 2^{\frac{2-p}{p-1}} \right)^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) \right] \\ &\quad + 2^{-\frac{1}{p-1}} \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{\alpha, p}^R \left[ b_0^{l\beta} H_l(2^{2\beta}) \right] \\ &= 2^{-\frac{1}{p-1}} \left( 1 \vee 2^{\frac{2-p}{p-1}} \right) \mathbf{W}_{\alpha, p}^R \left[ H_l \left( K^\beta 2^\beta \left( 1 \vee 2^{\frac{2-p}{p-1}} \right)^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta \right) \right] \\ &\quad + C_1(R) b_0^{\frac{l\beta}{p-1}}, \end{aligned}$$

where

$$C_1(R) = \frac{p-1}{2^{\frac{1}{p-1}} \alpha p} \left(1 \vee 2^{\frac{2-p}{p-1}}\right) (H_l(2^{2\beta}))^{\frac{1}{p-1}} |B_1|^{\frac{1}{p-1}} R^{\frac{\alpha p}{p-1}}.$$

Thus

$$\begin{aligned} P_1 \leq K 2^{-\frac{1}{p-1}} \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^2 \mathbf{W}_{\alpha,p}^R \left[ H_l \left( K^\beta 2^\beta \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^\beta (\mathbf{W}_{\alpha,p}^R[\mu])^\beta \right) \right] \\ + C_1(R) K \left(1 \vee 2^{\frac{2-p}{p-1}}\right) b_0^{\frac{1\beta}{p-1}}. \end{aligned} \quad (2.12)$$

Note that

$$C_1(R) K \left(1 \vee 2^{\frac{2-p}{p-1}}\right) b_0^{\frac{1\beta}{p-1}} \leq \frac{b_0}{2} \Leftrightarrow b_0 \leq \left(2C_1(R) K \left(1 \vee 2^{\frac{2-p}{p-1}}\right)\right)^{\frac{p-1}{1\beta-p+1}}.$$

Now we choose

$$b_0 = b_0(N, \alpha, p, \beta, K, R) := 1 \wedge \left(2C_1(R) K \left(1 \vee 2^{\frac{2-p}{p-1}}\right)\right)^{\frac{p-1}{1\beta-p+1}},$$

and we derive

$$C_1(R) K \left(1 \vee 2^{\frac{2-p}{p-1}}\right) b_0^{\frac{1\beta}{p-1}} \leq \frac{b_0}{2}. \quad (2.13)$$

Next, set  $\gamma \in (0, 1]$ ; since  $H_l(s) \leq \gamma H_l(\gamma^{-1}s)$  for all  $s \geq 0$ ,

$$\begin{aligned} P_2 &:= K 2^{-\frac{1}{p-1}} \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^2 \mathbf{W}_{\alpha,p}^R \left[ H_l \left( K^\beta 2^\beta \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^\beta (\mathbf{W}_{\alpha,p}^R[\mu])^\beta \right) \right] \\ &\leq K 2^{-\frac{1}{p-1}} \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^2 \gamma^{\frac{1}{p-1}} \mathbf{W}_{\alpha,p}^R \left[ H_l \left( \gamma^{-1} K^\beta 2^\beta \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^\beta (\mathbf{W}_{\alpha,p}^R[\mu])^\beta \right) \right]. \end{aligned}$$

From (2.10), we assert that if

$$\gamma^{-1} K^\beta 2^\beta \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^\beta \leq \delta_0 \mathbf{M}^{-\frac{\beta}{p-1}} \Leftrightarrow M \leq \left(\delta_0 K^{-\beta} 2^{-\beta} \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^{-\beta}\right)^{\frac{p-1}{\beta}} \gamma^{\frac{p-1}{\beta}},$$

then

$$P_2 \leq K 2^{-\frac{1}{p-1}} \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^2 \gamma^{\frac{1}{p-1}} C.$$

This will be achieved if we prove that

$$K 2^{-\frac{1}{p-1}} \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^2 \gamma^{\frac{1}{p-1}} C \leq \frac{b_0}{2},$$

which is equivalent to

$$\gamma \leq \left(K^{-1} 2^{\frac{2-p}{p-1}} \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^{-2} C^{-1} b_0\right)^{p-1}.$$

Thus, we can choose

$$\gamma = \gamma(N, \alpha, p, \beta, K, R) = 1 \wedge \left(K^{-1} 2^{\frac{2-p}{p-1}} \left(1 \vee 2^{\frac{2-p}{p-1}}\right)^{-2} C^{-1} b_0\right)^{p-1}$$

and

$$M_0 = M_0(N, \alpha, p, \beta, K, R) = \left( \delta_0 K^{-\beta} 2^{-\beta} \left( 1 \vee 2^{\frac{2-p}{p-1}} \right)^{-\beta} \right)^{\frac{p-1}{\beta}} \gamma^{\frac{p-1}{\beta}}$$

and we obtain that if  $M \leq M_0$  there holds

$$P_2 \leq \frac{b_0}{2}.$$

Combining this with (2.13) gives (2.11). This completes the proof of the theorem.  $\blacksquare$

Let  $P \in C(\mathbb{R}^+)$  be a decreasing positive function; the  $(\alpha, P)$ -Orlicz-Bessel capacity of a Borel set  $E \subset \mathbb{R}^N$  is defined by (see [1, Sect 2.6])

$$\text{cap}_{G_\alpha, P}(E) = \inf \left\{ \int_{\mathbb{R}^N} P(f) : G_\alpha * f \geq \chi_E, f \geq 0, P(f) \in L^1(\mathbb{R}^N) \right\}.$$

**Theorem 2.5** *Let  $N \in \mathbb{N}$ ,  $p > 1$ ,  $\alpha > 0$ ,  $a > 0$ ,  $c > 0$ ,  $\delta \in (0, 1]$ ,  $l \in \mathbb{N}^*$  and  $\beta \geq 1$  such that  $l\beta > p - 1$  and  $0 < \alpha p < N$ . Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  and  $\mu \in \mathfrak{M}_b^+(\Omega)$ . If  $u$  is a nonnegative Borel function in  $\Omega$  such that  $P_{l, a, \beta}(u) \in L^1(\Omega)$  and*

$$u(x) \geq c \mathbf{W}_{\alpha, p}^{\delta d(x, \partial\Omega)}[\omega](x) \quad \forall x \in \Omega \quad (2.14)$$

where  $d\omega = P_{l, a, \beta}(u)dx + d\mu$ , then, for any a compact set  $K \subset \Omega$ , there exists a positive constant  $C$  depending on  $N, \alpha, p, c, \beta$  and  $\text{dist}(K, \partial\Omega)$  such that

$$\int_E P_{l, a, \beta}(u)dx + \mu(E) \leq C \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel},$$

where  $Q_p^*$  is the complementary function to  $Q_p$ .

**Proof.** Let  $K \subset \Omega$ . Set  $r_K = \text{dist}(K, \partial\Omega)$  and  $\Omega_K = \{x \in \Omega : d(x, K) < r_K/2\}$ . We have

$$P_{l, a, \beta} \left( c \mathbf{W}_{\alpha, p}^{\delta d(x, \partial\Omega)}[\omega] \right) dx \leq d\omega \quad \text{in } \Omega$$

Let  $M_\omega$  denote the centered Hardy-Littlewood maximal function which is defined for any  $f \in L_{loc}(\mathbb{R}^N, d\omega)$  by

$$M_\omega f(x) = \sup_{t>0} \frac{\int_{B_t(x)} |f| d\omega}{\omega(B_t(x))}.$$

Thus, for any Borel set  $E \subset K$

$$\int_\Omega (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l, a, \beta} \left( c \mathbf{W}_{\alpha, p}^{\delta d(x, \partial\Omega)}[\omega] \right) dx \leq \int_\Omega (M_\omega \chi_E)^{\frac{m\beta}{p-1}} d\omega.$$

Since  $M_\omega$  is bounded on  $L^s(\mathbb{R}^N, d\omega)$ ,  $s > 1$ , we deduce from Fefferman's result [9] that

$$\int_{\Omega} (M_\omega \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega] \right) dx \leq c_4 \omega(E),$$

for some constant  $c_4$  only depends on  $N$  and  $\frac{l\beta}{p-1}$ . Since  $M_\omega \chi_E \leq 1$ , we derive

$$\begin{aligned} (M_\omega \chi_E(x))^{\frac{l\beta}{p-1}} P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega](x) \right) &\geq P_{l,a,\beta} \left( c (M_\omega \chi_E(x))^{\frac{1}{p-1}} \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega](x) \right) \\ &\geq P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \right). \end{aligned}$$

This implies

$$\int_{\Omega} P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \right) dx \leq c_4 \omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.15)$$

Note that if  $x \in \Omega$  and  $d(x, \partial\Omega) \leq r_K/8$ , then  $B_t(x) \subset \Omega \setminus \Omega_K$  for all  $t \in (0, \delta d(x, \partial\Omega))$ ; indeed, for all  $y \in B_t(x)$

$$d(y, \partial\Omega) \leq d(x, \partial\Omega) + |x - y| < (1 + \delta)d(x, \partial\Omega) < \frac{1}{4}r_K$$

thus

$$d(y, K) \geq d(K, \partial\Omega) - d(y, \partial\Omega) > \frac{3}{4}r_K > \frac{1}{2}r_K$$

which implies  $y \notin \Omega_K$ . We deduce that

$$\mathbf{W}_{\alpha,p}^{\delta d(x,\partial\Omega)}[\omega_E](x) \geq \mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) \quad \forall x \in \Omega,$$

and

$$\mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) = 0 \quad \forall x \in \Omega^c.$$

Hence we obtain from (2.15),

$$\int_{\mathbb{R}^N} P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) \right) dx \leq c_4 \omega(E) \quad \forall E \subset K, E \text{ Borel.} \quad (2.16)$$

From (2.1) and (2.2) we get

$$\int_{\mathbb{R}^N} P_{l,a,\beta} \left( c \mathbf{W}_{\alpha,p}^{\frac{\delta}{8}r_K}[\omega_E](x) \right) dx \geq \int_{\mathbb{R}^N} Q_p (c_5 \mathbf{G}_{\alpha p}[\omega_E](x)) dx,$$

where  $Q_p$  is defined by (1.12) and  $c_5 = (c_1\beta)^{-1}a^{\frac{p-1}{\beta}}c^{p-1}$  if  $p \neq 2$ ,  $c_5 = c_1^{-1}a^{\frac{1}{\beta}}c$  if  $p = 2$  (the constant  $c_1$  defined in (2.1), depends on  $R$ , therefore  $c_5 = c_5(r_K)$ ). Thus, from (2.16) we obtain

$$\int_{\mathbb{R}^N} Q_p (c_5 \mathbf{G}_{\alpha p}[\omega_E](x)) dx \leq c_4 \omega(E) \quad \forall E \subset K, E \text{ Borel.}$$

We recall that  $Q_p^*(s) = \sup_{t>0} \{st - Q_p(t)\}$  and we note that  $Q_p^*$  satisfies the sub-additivity  $\Delta_2$ -condition (see Chapter 2 in [8]). Let  $E \subset K$  be a Borel set. For every  $f \geq 0$ ,  $Q_p^*(f) \in L^1(\Omega)$  such that  $G_{\alpha p} * f \geq \chi_E$ , we have

$$\begin{aligned}
\omega(E) &\leq \int_{\mathbb{R}^N} G_{\alpha p} * f d\omega_E = \int_{\mathbb{R}^N} \mathbf{G}_{\alpha p}[\omega_E] f dx \\
&= (2c_4)^{-1} \int_{\mathbb{R}^N} (c_5 \mathbf{G}_{\alpha p}[\omega_E]) (2c_4 c_5^{-1} f) dx \\
&\leq (2c_4)^{-1} \int_{\mathbb{R}^N} Q_p(c_5 \mathbf{G}_{\alpha p}[\omega_E]) dx + (2c_4)^{-1} \int_{\mathbb{R}^N} Q_p^*(2c_4 c_5^{-1} f) dx \\
&\leq 2^{-1} \omega(E) + c_6 \int_{\mathbb{R}^N} Q_p^*(f) dx,
\end{aligned}$$

the last inequality following from the  $\Delta_2$ -condition. Notice that  $c_6$ , as well as the next constant  $c_7$ , depends on  $r_K$ . Thus,

$$\omega(E) \leq 2^{-1} c_6 \int_R Q_p^*(f) dx.$$

Then, we get

$$\omega(E) \leq c_7 \text{cap}_{G_{\alpha p}, Q_p^*}(E) \quad \forall E \subset K, E \text{ Borel.}$$

This completes the proof of the Theorem. ■

### 3 Quasilinear Dirichlet Problems

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ . If  $\mu \in \mathfrak{M}_b(\Omega)$ , we denote by  $\mu^+$  and  $\mu^-$  respectively its positive and negative part in the Jordan decomposition. We denote by  $\mathfrak{M}_0(\Omega)$  the space of measures in  $\Omega$  which are absolutely continuous with respect to the  $c_{1,p}^\Omega$ -capacity defined on a compact set  $K \subset \Omega$  by

$$c_{1,p}^\Omega(K) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega) \right\}.$$

We also denote  $\mathfrak{M}_s(\Omega)$  the space of measures in  $\Omega$  with support on a set of zero  $c_{1,p}^\Omega$ -capacity. Classically, any  $\mu \in \mathfrak{M}_b(\Omega)$  can be written in a unique way under the form  $\mu = \mu_0 + \mu_s$  where  $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$  and  $\mu_s \in \mathfrak{M}_s(\Omega)$ . It is well known that any  $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_b(\Omega)$  can be written under the form  $\mu_0 = f - \text{div } g$  where  $f \in L^1(\Omega)$  and  $g \in L^{p'}(\Omega)$ .

For  $k > 0$  and  $s \in \mathbb{R}^N$  we set  $T_k(s) = \max\{\min\{s, k\}, -k\}$ . If  $u$  is a measurable function defined and finite a.e. in  $\Omega$ , such that  $T_k(u) \in W_0^{1,p}(\Omega)$  for any  $k > 0$ , there exists a measurable function  $v : \Omega \rightarrow \mathbb{R}^N$  such that  $\nabla T_k(u) = \chi_{|u| \leq k} v$  a.e. in  $\Omega$  and for all  $k > 0$ . We define the gradient  $\nabla u$  of  $u$  by  $v = \nabla u$ . We recall the definition of a renormalized solution given in [3].



**Definition 3.1** Let  $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$ . A measurable function  $u$  defined in  $\Omega$  and finite a.e. is called a renormalized solution of

$$\begin{aligned} -\Delta_p u &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.1)$$

if  $T_k(u) \in W_0^{1,p}(\Omega)$  for any  $k > 0$ ,  $|\nabla u|^{p-1} \in L^r(\Omega)$  for any  $0 < r < \frac{N}{N-1}$ , and  $u$  has the property that for any  $k > 0$  there exist  $\lambda_k^+$  and  $\lambda_k^-$  belonging to  $\mathfrak{M}_b^+ \cap \mathfrak{M}_0(\Omega)$ , respectively concentrated on the sets  $u = k$  and  $u = -k$ , with the property that  $\mu_k^+ \rightharpoonup \mu_s^+$ ,  $\mu_k^- \rightharpoonup \lambda_s^-$  in the narrow topology of measures and such that

$$\int_{\{|u|<k\}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda_k^+ - \int_{\Omega} \varphi d\lambda_k^-$$

for every  $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ .

**Remark 3.2** We recall that if  $u$  is a renormalized solution to problem (3.1) and  $\mu \in \mathfrak{M}_b^+(\Omega)$ , then  $u \geq 0$  a.e. in  $\Omega$ .

The following general stability result has been proved in [3, Th 4.1, Sec 5.1].

**Theorem 3.3** 1- Let  $\{u_n\}$  be a sequence of measurable functions, finite a.e. in  $\Omega$ . We assume that  $T_k(u_n) \in W_0^{1,p}(\Omega)$  for any  $k > 0$ ,  $|\nabla u_n|^{p-1} \in L^r(\Omega) \forall r < \frac{N}{N-1}$ , and for any  $n \in \mathbb{N}$   $\frac{1}{k} \int_{\Omega} |\nabla T_k(u_n)|^p dx \leq M$  for some constant  $M$ . Then, up to a subsequence,  $\{u_n\}$  and  $\{\nabla u_n\}$  are Cauchy sequences in measure;  $\{u_n\}$  converges a.e. to a function  $u$  finite a.e. in  $\Omega$  such that  $T_k(u) \in W_0^{1,p}(\Omega)$  for any  $k > 0$ ,  $\{T_k(u_n)\}$  converges to  $T_k(u)$  weakly in  $W_0^{1,p}(\Omega)$  and  $\nabla u_n$  converges a.e. to  $\nabla u$ ; finally  $|\nabla u|^{p-1} \in L^r(\Omega)$  for all  $r \in [1, \frac{N}{N-1})$  and  $\{|\nabla u_n|^{p-2} \nabla u_n\}$  converges to  $|\nabla u|^{p-2} \nabla u$  strongly in  $L^r(\Omega)$ .

2- As a consequence, if  $\{\mu_n\}$  is a bounded sequence in  $\mathfrak{M}_b(\Omega)$  and  $\{u_n\}$  a sequence of renormalized solutions of problem

$$\begin{aligned} -\Delta_p u_n &= \mu_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (3.2)$$

then, up to a subsequence,  $\{u_n\}$  converges a.e. to a solution of equation  $-\Delta_p u = \mu$  in  $\mathcal{D}'(\Omega)$ , and such that  $\frac{1}{k} \int_{\Omega} |\nabla T_k(u)|^p dx \leq M$  for every  $k > 0$ .

The next result is a sharp extension of the stability Theorem 3.3.

**Theorem 3.4** Let  $\mu = \mu_0 + \mu_s^+ - \mu_s^-$ , with  $\mu_0 = F - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$  and  $\mu_s^+$ ,  $\mu_s^-$  belonging to  $\mathfrak{M}_s^+(\Omega)$ . Let  $\mu_n = F_n - \operatorname{div} g_n + \rho_n - \eta_n$  with  $F_n \in L^1(\Omega)$ ,  $g_n \in (L^{p'}(\Omega))^N$  and  $\rho_n, \eta_n$  belonging to  $\mathfrak{M}_b^+(\Omega)$ . Assume that  $\{F_n\}$  converges to  $F$  weakly in  $L^1(\Omega)$ ,  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(\Omega))^N$  and  $(\operatorname{div} g_n)$  is bounded in  $\mathfrak{M}_b(\Omega)$ ; assume also that  $\{\rho_n\}$  converges to  $\mu_s^+$  and  $\{\eta_n\}$  to  $\mu_s^-$  in the narrow topology. If  $\{u_n\}$  is a sequence of renormalized solutions of (3.2), then, up to a subsequence, it converges a.e. in  $\Omega$  to a renormalized solution  $U$  of problem (3.1). Furthermore  $\{T_k(u_n)\}$  converges to  $T_k(u)$  in  $W_0^{1,p}(\Omega)$ .

We also recall the following estimate [10, Th 2.1].

**Theorem 3.5** *Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ . Then there exists a constant  $K_1 > 0$ , depending on  $p$  and  $N$  such that if  $\mu \in \mathfrak{M}_b^+(\Omega)$  and  $u$  is a nonnegative renormalized solution of problem (3.1) with data  $\mu$ , there holds*

$$\frac{1}{K_1} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\mu](x) \leq u(x) \leq K_1 \mathbf{W}_{1,p}^R[\mu](x) \quad \forall x \in \Omega \quad (3.3)$$

where  $R = \frac{\text{diam}(\Omega)}{2}$  and  $K_1$  is a constant independent of  $x, u$  and  $\Omega$ .

**Proof of Theorem 1.1.** Let  $\{u_m\}_{m \in \mathbb{N}}$  be a sequence of nonnegative renormalized solutions of the following problems

$$\begin{aligned} -\Delta_p u_0 &= \mu & \text{in } \Omega \\ u_0 &= 0 & \text{on } \partial\Omega, \end{aligned}$$

and, for  $m \in \mathbb{N}$ ,

$$\begin{aligned} -\Delta_p u_{m+1} &= P_{l,a,\beta}(u_m) + \mu & \text{in } \Omega \\ u_{m+1} &= 0 & \text{on } \partial\Omega. \end{aligned}$$

By Theorem 3.5 we have for all  $x \in \Omega$

$$\begin{aligned} \chi_\Omega u_0(x) &\leq K_1 \mathbf{W}_{1,p}^R[\chi_\Omega \mu](x) \\ \chi_\Omega u_{m+1}(x) &\leq K_1 \mathbf{W}_{1,p}^R[\chi_\Omega P_{l,a,\beta}(u_m) + \chi_\Omega \mu]. \end{aligned}$$

Thus, by Theorem 2.4 there exist  $b_0 \in (0, 1]$  and  $M_0 > 0$  depending on  $N, p, \beta, a, l$  and  $R$  such that

$$u_m(x) \leq K_1 \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{1,p}^R[\chi_\Omega \mu](x) + 2b_0 \quad \forall x \in \Omega, m \in \mathbb{N} \quad (3.4)$$

and

$$\exp\left(a \left(K \left(1 \vee 2^{\frac{2-p}{p-1}}\right) \mathbf{W}_{1,p}^R[\mu] + 2b_0\right)^\beta\right) \in L_{loc}^1(\mathbb{R}^N), \quad (3.5)$$

provided that

$$\|M_{p,R}^{\frac{(p-1)(\beta-1)}{\beta}}[\chi_\Omega \mu]\|_{L^\infty(\mathbb{R}^N)} \leq M_0.$$

This implies that  $\{u_m\}$  is well defined; by Theorem 3.3 it contains a subsequence that we still denote by  $\{u_m\}$  which converges a.e in  $\Omega$  to function  $u$  for which (1.16) is satisfied in  $\Omega$ . Furthermore, we deduce from (3.4)-(3.5) and the dominated convergence theorem that  $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$  in  $L^1(\Omega)$ . Finally, by Theorem 3.4 we obtain that  $u$  is a renormalized solution of (1.15).

Conversely, assume that (1.15) admits a nonnegative renormalized solution  $u$ . By Theorem 3.5 there holds

$$u(x) \geq \frac{1}{K_1} \mathbf{W}_{1,p}^{\frac{d(x,\partial\Omega)}{3}}[\omega](x) \quad \text{for all } x \in \Omega,$$

where  $d\omega = (P_{l,a,\beta}(u)dx + d\mu)$ . By Theorem 2.5, we obtain (1.18). ■

**Applications 1.** We consider the case  $p = 2, \beta = 1$ . Then  $l = 2$  and

$$P_{l,a,\beta}(r) = e^{ar} - 1 - ar.$$

If  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ , there exists  $M_0 > 0$  and  $b_0 \in (0, 1)$  such that if  $\mu$  is a positive Radon measure in  $\Omega$  which satisfies

$$\mu(B_t(x)) \leq M_0 t^{N-2} \quad \forall t > 0 \text{ and almost all } x \in \Omega, \quad (3.6)$$

there exists a positive solution  $u$  to the following problem

$$\begin{aligned} -\Delta u &= e^{au} - 1 - au && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \quad (3.7)$$

Furthermore

$$u(x) \leq K(N) \int_0^{2 \operatorname{diam} \Omega} \frac{\mu(B_t(x))}{t^{N-1}} dt + 2b_0 \quad \forall x \in \Omega. \quad (3.8)$$

In the case  $N = 2$  this result has already been proved by Richard and Véron [12, Prop 2.4].

## 4 Hessian equations

In this section  $\Omega \subset \mathbb{R}^N$  is a bounded domain with a  $C^2$  boundary. For  $k = 1, \dots, N$  and  $u \in C^2(\Omega)$  the  $k$ -hessian operator  $F_k$  is defined by

$$F_k[u] = S_k(\lambda(D^2u))$$

where  $\lambda(D^2u) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$  denotes the eigenvalues of the Hessian matrix of second partial derivative  $D^2u$  and  $S_k$  is the  $k$ -th elementary symmetric polynomial that is

$$S_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq N} \lambda_{i_1} \dots \lambda_{i_k}$$

We can see that

$$F_k[u] = [D^2u]_k$$

where for a matrix  $A = (a_{ij})$ ,  $[A]_k$  denotes the sum of the  $k$ -th principal minors. We assume that  $\partial\Omega$  is uniformly  $(k-1)$ -convex that is

$$S_{k-1}(\kappa) \geq c_0 > 0 \text{ on } \partial\Omega$$

for some positive constant  $c_0$ , where  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$  denote the principal curvatures of  $\partial\Omega$  with respect to its inner normal.

**Definition 4.1** *An upper-semicontinuous function  $u : \Omega \rightarrow [-\infty, \infty)$  is  $k$ -convex ( $k$ -subharmonic) if, for every open set  $\Omega' \subset \overline{\Omega}' \subset \Omega$  and for every function  $v \in C^2(\Omega') \cap C(\overline{\Omega})$  satisfying  $F_k[v] \leq 0$  in  $\Omega'$ , the following implication is true*

$$u \leq v \text{ on } \partial\Omega' \implies u \leq v \text{ in } \Omega'.$$

We denote by  $\Phi^k(\Omega)$  the class of all  $k$ -subharmonic functions in  $\Omega$  which are not identically equal to  $-\infty$ .

The following weak convergence result for  $k$ -Hessian operators proved in [TW2] is fundamental in our study.

**Theorem 4.2** *Let  $\Omega$  be a bounded uniformly  $(k-1)$ -convex in  $\mathbb{R}^N$ . For each  $u \in \Phi^k(\Omega)$ , there exist a nonnegative Radon measure  $\mu_k[u]$  in  $\Omega$  such that*

- a.  $\mu_k[u] = F_k[u]$  for  $u \in C^2(\Omega)$ .
- b. If  $\{u_n\}$  is a sequence of  $k$ -convex functions which converges to  $u$  a.e then  $\mu_k[u_n] \rightharpoonup \mu_k[u]$  in the weak sense of measures.

As in the case of quasilinear equations with measure data, precise estimates of solutions of  $k$ -Hessian equations with measures data are expressed in terms of Wolff potentials. The next result is proved in [7].

**Theorem 4.3** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded  $C^2$ , uniformly  $(k-1)$ -convex domain. Let  $\varphi$  be a nonnegative continuous function on  $\partial\Omega$  and  $\mu$  be a nonnegative Radon measure. Suppose that  $\mu$  can be decomposed as*

$$\mu = \mu_1 + f$$

such that  $\mu_1$  is a measure with compact support in  $\Omega$  and  $f \in L^q(\Omega)$  for some  $q > \frac{N}{2k}$  if  $k \leq \frac{N}{2}$  or  $p = 1$  if  $k > \frac{N}{2}$ . Then there exists a nonnegative function  $u$  in  $\Omega$  such that  $-u \in \Phi^k(\Omega)$ , continuous near  $\partial\Omega$  and  $u$  is a solution to the problem

$$\begin{aligned} F_k[-u] &= \mu && \text{in } \Omega \\ u &= \varphi && \text{on } \partial\Omega. \end{aligned}$$

Furthermore, for any nonnegative function  $u$  such that  $-u \in \Phi^k(\Omega)$ , which is continuous near  $\partial\Omega$  and is a solution of above equation, there holds

$$\frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}}[\mu] \leq u(x) \leq K_2 \left( \mathbf{W}_{\frac{2k}{k+1}, k+1}^{2\text{diam } \Omega}[\mu](x) + \max_{\partial\Omega} \varphi \right). \quad (4.1)$$

where  $K_2$  is a positive constant independent of  $x, u$  and  $\Omega$ .

**Proof of Theorem 1.2.** We defined a sequence of nonnegative functions  $u_m$ , continuous near  $\partial\Omega$  and such that  $-u_m \in \Phi^k(\Omega)$ , by the following iterative scheme

$$\begin{aligned} F_k[-u_0] &= \mu && \text{in } \Omega \\ u_0 &= \varphi && \text{on } \partial\Omega \end{aligned} \quad (4.2)$$

and, for  $m \geq 0$ ,

$$\begin{aligned} F_k[-u_{m+1}] &= P_{l, a, \beta} u_m + \mu && \text{in } \Omega \\ u_{m+1} &= \varphi && \text{on } \partial\Omega. \end{aligned} \quad (4.3)$$

By Theorem 3.5 we have for all  $x \in \mathbb{R}^N$ ,

$$\begin{aligned} \chi_\Omega u_0(x) &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\chi_\Omega \mu](x) + b \\ \chi_\Omega u_{m+1}(x) &\leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R[\chi_\Omega H_l(a u_m^\beta) + \chi_\Omega \mu](x) + b, \end{aligned} \quad (4.4)$$

where  $b = K_2 \max_{\partial\Omega} \varphi$ . Thus, by Theorem 2.4 there exist  $b_0 \in (0, 1]$  and  $M_0 > 0$  depending on  $N, p, \beta, a, m$  and  $R$  such that

$$u_m(x) \leq K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\chi_\Omega \mu](x) + 2b_0 \quad \forall x \in \Omega, \forall m \geq 0, \quad (4.5)$$

and

$$P_{l,a,\beta} \left( K_2 \mathbf{W}_{\frac{2k}{k+1}, k+1}^R [\mu] + 2b_0 \right) \in L_{loc}^1(\mathbb{R}^N), \quad (4.6)$$

provided that

$$\|M_{2k,R}^{\frac{k(\beta-1)}{\beta}} [\chi_\Omega \mu]\|_{L^\infty(\mathbb{R}^N)} \leq M_0.$$

Note that because we can write

$$\omega = P_{l,a,\beta}(u_m) + \mu = (\mu_1 + \chi_{\Omega_\delta} P_{l,a,\beta}(u_m)) + ((1 - \chi_{\Omega_\delta}) P_{l,a,\beta}(u_m) + f),$$

where  $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}$  and  $\delta > 0$  is small enough, then  $\omega$  satisfies the assumptions of the data in Theorem 4.3 since  $u_m$  is continuous near  $\partial\Omega$ . Therefore the sequence  $\{u_m\}$  is well defined. Since  $u_m$  is  $k$ -superharmonic and (4.5) holds, the sequence  $\{u_m\}$  is relatively compact in  $L_{loc}^1$  (see e.g. [4]); we can find a subsequence, still denoted by the index  $m$ , such that  $u_m$  converges a.e in  $\Omega$  to function  $u$  for which (1.21) is satisfied in  $\Omega$ . Furthermore, we deduce from (4.5)-(4.6) and the dominated convergence theorem that  $P_{l,a,\beta}(u_m) \rightarrow P_{l,a,\beta}(u)$  in  $L^1(\Omega)$ . Finally, by Theorem 4.2, we obtain that  $u$  satisfies (1.20) and (1.21).

Conversely, assume that (1.20) admits nonnegative solution  $u$ , continuous near  $\partial\Omega$ , such that  $-u \in \Phi^k(\Omega)$  and  $P_{l,a,\beta}(u) \in L^1(\Omega)$ . Then by Theorem 4.3 we have

$$u(x) \geq \frac{1}{K_2} \mathbf{W}_{\frac{2k}{k+1}, k+1}^{\frac{d(x, \partial\Omega)}{8}} [\omega](x) \quad \text{for all } x \in \Omega,$$

where  $d\omega = P_{l,a,\beta}(u)dx + d\mu$ . Using Theorem 2.5, we conclude that (1.22) holds.  $\blacksquare$

## 5 Wolff integral equations

**Proof of Theorem 1.3.** Let  $\varepsilon > 0$ , assume that  $\exp((a + \varepsilon)f^\beta) \in L_{loc}^1(\mathbb{R}^N)$  and consider the measure  $d\mu = P_{l,a,\beta}(a + \varepsilon)f dx$ . By Theorem 2.3 with  $\eta = \frac{(p-1)(\beta-1)}{\beta}$ , there exist  $C = C(N, \alpha, p, \beta, R) > 0$  and  $\delta_0 = \delta_0(N, \alpha, p, \beta) > 0$  such that if

$$\|M_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}} [\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M,$$

there holds

$$\mathbf{W}_{\alpha, p}^R \left[ \exp \left( \delta_0 M^{-\frac{\beta}{p-1}} (\mathbf{W}_{\alpha, p}^R [\mu])^\beta \right) \right] (x) \leq C \quad \forall x \in \mathbb{R}^N. \quad (5.1)$$

Consider the sequence  $\{u_m\}_{m \geq 0}$  of nonnegative functions defined by  $u_0 = f$  and

$$u_{m+1} = \mathbf{W}_{\alpha, p}^R [P_{l,a,\beta}(u_m)] + f \quad \text{in } \mathbb{R}^N \quad \forall m \geq 0.$$

*Step 1.* We claim that there exist  $b_0 \in (0, 1]$  and  $M_0 > 0$  depending on  $N, \alpha, p, \beta, a, m$  and  $R$  such that if

$$\|\mathbf{M}_{\alpha p, R}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]\|_{L^\infty(\mathbb{R}^N)} \leq M_0,$$

then the sequence  $\{u_m\}_{m \geq 0}$  is well defined and there holds

$$u_m \leq F := \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha, p}^R[\mu] + f + b_0 \quad \forall m \geq 0, \quad (5.2)$$

and

$$\exp(aF^\beta) \in L_{loc}^1(\mathbb{R}^N) \quad \text{and} \quad \|\mathbf{W}_{\alpha, p}^R[Pl_{a, \beta}(F)]\|_{L^\infty(\mathbb{R}^N)} < +\infty. \quad (5.3)$$

Clearly, (5.2) holds with  $m = 0$ , thus we assume that (5.2) holds with  $m = n$ , then it follows

$$u_n^\beta \leq C_1 \left(1 \vee 3^{\frac{2-p}{p-1}}\right)^\beta (\mathbf{W}_{\alpha, p}^R[\mu])^\beta + C_2 b_0^\beta + \frac{a + 2^{-1}\varepsilon}{a} f^\beta.$$

Here, we have used the inequality

$$(X + Y + Z)^\beta \leq C_1 X^\beta + C_2 Y^\beta + \frac{2a + \varepsilon}{2a} Z^\beta \quad \forall X, Y, Z \geq 0$$

where the constants  $C_1, C_2$  depend on  $a, \varepsilon$  and  $\beta$ . Since  $s \mapsto H_l(s)$  is convex in  $[0, +\infty)$ , there holds

$$\begin{aligned} H_l(au_n^\beta) &\leq \frac{\varepsilon}{4(a + \varepsilon)} H_l\left(C_3 (\mathbf{W}_{\alpha, p}^R[\mu])^\beta\right) + \frac{\varepsilon}{4(a + \varepsilon)} H_l\left(C_4 b_0^\beta\right) + \frac{2a + \varepsilon}{2(a + \varepsilon)} H_l\left((a + \varepsilon)f^\beta\right) \\ &\leq H_l\left(C_3 (\mathbf{W}_{\alpha, p}^R[\mu])^\beta\right) + b_0^{m\beta} H_l(C_4) + H_l\left((a + \varepsilon)f^\beta\right), \end{aligned}$$

where

$$C_3 = aC_1 \left(1 \vee 3^{\frac{2-p}{p-1}}\right)^\beta \frac{4(a + \varepsilon)}{\varepsilon} \quad \text{and} \quad C_4 = aC_2 \frac{4(a + \varepsilon)}{\varepsilon}.$$

Thus,

$$\begin{aligned} u_{n+1} &= \mathbf{W}_{\alpha, p}^R [H_l(au_n^\beta)] + f \\ &\leq \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha, p}^R \left[ H_l\left(C_3 (\mathbf{W}_{\alpha, p}^R[\mu])^\beta\right) \right] + \left(1 \vee 3^{\frac{2-p}{p-1}}\right) b_0^{\frac{l\beta}{p-1}} \mathbf{W}_{\alpha, p}^R [H_l(C_4)] \\ &\quad + \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha, p}^R [H_l((a + \varepsilon)f^\beta)] + f \\ &\leq \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha, p}^R [H_l((a + \varepsilon)f^\beta)] + f \\ &\quad + \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha, p}^R \left[ H_l\left(C_3 (\mathbf{W}_{\alpha, p}^R[\mu])^\beta\right) \right] + C_5 b_0^{\frac{l\beta}{p-1}}, \end{aligned} \quad (5.4)$$

where

$$C_5 = \left(1 \vee 3^{\frac{2-p}{p-1}}\right) (H_l(C_4))^{\frac{1}{p-1}} |B_1|^{\frac{1}{p-1}} \frac{p-1}{\alpha p} R^{\frac{\alpha p}{p-1}}.$$

Note that

$$C_5 b_0^{\frac{l\beta}{p-1}} \leq \frac{b_0}{2} \Leftrightarrow b_0 \leq (2C_5)^{-\frac{p-1}{l\beta - p + 1}}.$$

Next, if we choose  $b_0 = b_0(N, \alpha, p, \beta, K, R) = 1 \wedge (2C_5)^{-\frac{p-1}{m\beta-p+1}}$  we have

$$C_5 b_0^{\frac{l\beta}{p-1}} \leq \frac{b_0}{2}. \quad (5.5)$$

Furthermore, since  $H_l(s) \leq \gamma H_l(\gamma^{-1}s)$  for all  $\gamma \in (0, 1]$  and  $s \geq 0$ , we have

$$P := \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha,p}^R \left[ H_l \left( C_3 \left( \mathbf{W}_{\alpha,p}^R[\mu] \right)^\beta \right) \right] \leq \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \gamma^{\frac{1}{p-1}} \mathbf{W}_{\alpha,p}^R \left[ H_l \left( \gamma^{-1} C_3 \left( \mathbf{W}_{\alpha,p}^R[\mu] \right)^\beta \right) \right].$$

If  $[\gamma^{-1} C_3 \leq \delta_0 \mathbf{M}^{-\frac{\beta}{p-1}}$ , which is equivalent to  $M \leq (\delta_0 C_3^{-1})^{\frac{p-1}{\beta}} \gamma^{\frac{p-1}{\beta}}$ , we deduce

$$P \leq \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \gamma^{\frac{1}{p-1}} C$$

from (5.1). In order to insure

$$\left(1 \vee 3^{\frac{2-p}{p-1}}\right) \gamma^{\frac{1}{p-1}} C \leq \frac{b_0}{2},$$

which is equivalent to

$$\gamma \leq b_0^{p-1} \left( 2C \left( 1 \vee 3^{\frac{2-p}{p-1}} \right) \right)^{-p+1},$$

we can choose

$$\gamma = \gamma(N, \alpha, p, \beta, K, R) = 1 \wedge \left( b_0^{p-1} \left( 2C \left( 1 \vee 3^{\frac{2-p}{p-1}} \right) \right)^{-p+1} \right)$$

and

$$M_0 = M_0(N, \alpha, p, \beta, K, R) = (\delta_0 C_3^{-1})^{\frac{p-1}{\beta}} \gamma^{\frac{p-1}{\beta}}.$$

Having made those choices, we obtain

$$P \leq \frac{b_0}{2} \quad (5.6)$$

provided  $M \leq M_0$ . Jointly with (5.5) and (5.4) it yields to

$$u_{n+1} \leq \left(1 \vee 3^{\frac{2-p}{p-1}}\right) \mathbf{W}_{\alpha,p}^R[\mu] + f + b_0. \quad (5.7)$$

This implies that  $\{u_m\}_{m \geq 0}$  is well defined and (5.2) and (5.3) are satisfied. It is easy to see that  $u_n$  is nondecreasing. Hence, from (5.2) and (5.3), we obtain that  $u(x) = \lim_{n \rightarrow \infty} u_n(x)$  is a solution to equation (1.23) which satisfies (1.24).

*Step 2.* Conversely, we assume that (1.23) admits a nonnegative solution  $u$ . There holds

$$u^\beta \geq \left( \mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)] \right)^\beta + f^\beta,$$

and thus,

$$P_{l,a,\beta}(u) \geq P_{l,a,\beta} \left( \mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)] \right) + P_{l,a,\beta}(f).$$

This implies

$$d\mu \geq P_{l,a,\beta} \left( \mathbf{W}_{\alpha,p}^R[\mu] \right) dx,$$

where

$$d\mu = P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)]) dx + P_{l,a,\beta}(f) dx.$$

If  $E \subset \mathbb{R}^N$  is a Borel set, we deduce

$$\int_{\mathbb{R}^N} (M_\mu \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu]) dx \leq \int_{\mathbb{R}^N} (M_\mu \chi_E)^{\frac{l\beta}{p-1}} d\mu.$$

Since  $M_\omega$  is bounded in  $L^s(\mathbb{R}^N, d\omega)$  for  $s > 1$  by Fefferman's differentiation theorem [9], we derive

$$\int_{\mathbb{R}^N} (M_\mu \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu]) dx \leq c_8 \mu(E),$$

for some constant  $c_4$  only depends on  $N$  and  $\frac{l\beta}{p-1}$ . Because

$$(M_\mu \chi_E)^{\frac{l\beta}{p-1}} P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu]) \geq P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu E]),$$

we finally deduce

$$\int_E P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[\mu]) dx \leq c_8 \mu(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.}$$

As in the proof of Theorem 2.5, it is easy to see that we get

$$\mu(E) \leq c_9 \text{cap}_{G_{\alpha,p}, Q_p^*}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel,}$$

where the positive constant  $c_9$  depends on  $N, \alpha, p, \beta$  and  $R$ . Finally we obtain that for any Borel subset  $E \subset \mathbb{R}^N$  there holds

$$\int_E P_{l,a,\beta}(\mathbf{W}_{\alpha,p}^R[P_{l,a,\beta}(u)]) dx + \int_E P_{l,a,\beta}(f) dx \leq c_9 \text{cap}_{G_{\alpha,p}, Q_p^*}(E),$$

which completes the proof of the Theorem. ■

**Applications 2.** When  $\alpha = 1$ , the equation (1.20) with  $R = \infty$  is equivalent to

$$-\Delta_p(u - f) = P_{l,a,\beta}(u) \quad \text{in } \mathbb{R}^N.$$

and when  $\alpha = \frac{2k}{k+1}$  and  $p = k + 1$ , it is equivalent to

$$F_k(-u + f) = P_{l,a,\beta}(u) \quad \text{in } \mathbb{R}^N.$$

If  $p = 2$  equation (1.20) becomes linear. If we set  $\gamma = 2\alpha$ , then

$$\begin{aligned} \mathbf{W}_{\gamma,2}[\mu](x) &= \int_0^\infty \mu(B_t(x)) \frac{dt}{t^{N-\gamma-1}} \\ &= \int_{\mathbb{R}^N} \left( \int_{|x-y|}^\infty \frac{dt}{t^{N-\gamma-1}} \right) d\mu(y) \\ &= \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-\gamma}} \\ &= I_\gamma * \mu \end{aligned}$$



where  $I_\gamma$  is the Riesz kernel of order  $\gamma$ . Thus (1.20) with  $R = \infty$  is equivalent to

$$(-\Delta)^\alpha(u - f) = P_{l,a,\beta}(u) \quad \text{in } \mathbb{R}^N.$$

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