From dimension free concentration to Poincaré inequality

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FROM DIMENSION FREE CONCENTRATION TO POINCARÉ INEQUALITY

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Abstract. We prove that a probability measure on an abstract metric space satisfies a non trivial dimension free concentration inequality for the ℓ_2 metric if and only if it satisfies the Poincaré inequality.

1. Introduction

In all the paper $(\mathcal{X}, d)$ is a polish metric space and $\mathcal{P}(\mathcal{X})$ is the set of Borel probability measures on $\mathcal{X}$. On the product space $\mathcal{X}^n$, we consider the following ℓ_p product distance d_p defined by

$$d_p(x,y) = \left[\sum_{i=1}^n d^p(x_i,y_i)\right]^{1/p}, \qquad x,y \in \mathcal{X}^n.$$

If A is a Borel subset of $\mathcal{X}^n$, we define its enlargement $A_{r,p}$ (simply denoted by A_r when $n=1$), $r \geq 0$ as follows

$$A_{r,p} = \{x \in \mathcal{X}^n; d_p(x,A) \leq r\}.$$

In all what follows, $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ will always be a non increasing function. One will say that $\mu \in \mathcal{P}(\mathcal{X})$ satisfies the dimension free concentration property with the concentration profile α and with respect to the ℓ_p product structure if

$$\mu^n(A_{r,p}) \geq 1 - \alpha(r), \qquad \forall r \geq 0, \tag{1.1}$$

for all $A \subset \mathcal{X}^n$, with $\mu^n(A) \geq 1/2$. In this case, we will write that μ satisfies the dimension free concentration inequality $\mathbf{CI}_p^\infty(\alpha)$. If μ satisfies (1.1) only for $n=1$, we will write that μ satisfies $\mathbf{CI}(\alpha)$.

The general problem considered in this paper is to give a characterization of the class of probability measures satisfying $\mathbf{CI}_p^\infty(\alpha)$. The main result of the paper shows that the class of probability measures satisfying $\mathbf{CI}_2^\infty(\alpha)$, for some non trivial α, always contains the class of probability measures satisfying the Poincaré inequality. Moreover, these two classes coincide when α is exponential: $\alpha(r) = be^{-ar}$, for some $a, b > 0$.

Before stating this result, let us recall the definition of the Poincaré inequality: one says that $\mu \in \mathcal{P}(\mathcal{X})$ satisfies the Poincaré inequality with the constant $\lambda \in \mathbb{R}^+ \cup \{+\infty\}$, if

$$\lambda \mathrm{Var}_\mu(f) \leq \int |\nabla^- f|^2 \, d\mu, \tag{1.2}$$

for all Lipschitz function $f : \mathcal{X} \to \mathbb{R}$, where by definition

$$|\nabla^- f|(x) = \limsup_{y \to x} \frac{[f(y) - f(x)]_-}{d(y,x)},$$

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when x is not isolated in $\mathcal{X}$ (we set $|\nabla f|(x) = 0$, when x is isolated in $\mathcal{X}$). We take the convention $\infty \times 0 = 0$, so that $\lambda = +\infty$ if and only if μ is a Dirac measure.

1.1. **Main result.** The main result of this paper is the following theorem. In what follows, $\overline{\Phi}$ will denote the tail distribution function of the standard Gaussian measure $\gamma(dx) = (2\pi)^{-1/2} e^{-x^2/2}\, dx$ on $\mathbb{R}$ defined by

$$\overline{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-u^2/2}\, du, \qquad x \in \mathbb{R}.$$

Theorem 1.3. *If μ satisfies the dimension free concentration property $\mathbf{CI}_2^\infty(\alpha)$, then μ satisfies Poincaré inequality (1.2) with the constant λ defined by*

$$\sqrt{\lambda} = \sup\left\{ \frac{\overline{\Phi}^{-1}(\alpha(r))}{r}; r > 0 \ s.t\ \alpha(r) \leq 1/2 \right\}.$$

Moreover, if α is convex decreasing and such that $\alpha(0) = 1/2$, then $\lambda = (2\pi\alpha'_+(0)^2)$, where $\alpha'_+(0) \in [-\infty, 0)$ is the right derivative of α at 0.

Conversely, it is well known since the work by Gromov and Milman [16] (see also [1], [6], [26] for related results) that a probability measure μ verifying Poincaré inequality satisfies a dimension free concentration property with a profile of the form $\alpha(r) = be^{-ar}$, for some $a, b > 0$. This is recalled in the following theorem (we refer to the appendix for a proof).

Theorem 1.4. *Suppose that μ satisfies Poincaré inequality (1.2) with a constant $\lambda > 0$, then it satisfies the dimension free concentration property with the profile*

$$\alpha(r) = b \exp(-a\sqrt{\lambda} r), \qquad r \geq 0,$$

where a, b are universal constants.

Theorem 1.3 and Theorem 1.4 thus give a full description of the set of probability distributions verifying a dimension free concentration property with a concentration profile α such that $\{r : \alpha(r) < 1/2\} \neq \emptyset$: this set coincides with the set of probability measures verifying the Poincaré inequality. An immediate corollary of Theorem 1.3 and Theorem 1.4 (see Corollary 4.1) is that any type of dimension free concentration inequality can always be improved into a dimension free concentration inequality with an exponential profile (up to universal constants). This was already noticed by Talagrand in [28]. See Section 4.3 for a further discussion.

Remark 1.5. *Let us make some comments on the constant λ appearing in Theorem 1.3.*

(1) Note that $\lambda > 0$ if and only if there is some $r_o > 0$ such that $\alpha(r_o) < 1/2$. In particular, Theorem 1.3 applies even in the case of a "minimal" profile $\alpha = \beta_{a_o, r_o}$, defined as follows

$$\beta_{a_o,r_o}(r) = 1/2, \quad \text{if } r < a_o \qquad \text{and} \qquad \beta_{a_o,r_o}(r) = a_o, \quad \text{if } r \geq r_o, \tag{1.6}$$

where $a_o \in [0, 1/2)$, $r_o > 0$. If a probability measure satisfies $\mathbf{CI}_2^\infty(\beta_{a_o,r_o})$, then it satisfies Poincaré with the constant

$$\sqrt{\lambda_{a_o,r_o}} := \frac{\overline{\Phi}^{-1}(a_o)}{r_o}$$

(2) Note that any non increasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, with $\alpha(0) = 1/2$ can be written as an infimum of minimal profiles:

$$\alpha = \inf_{r>0} \beta_{\alpha(r),r}.$$

The constant λ given in Theorem 1.3 is thus the supremum of the constants $\lambda_{\alpha(r),r}$ $r > 0$ defined above. This shows that the information contained in the concentration profile α is treated pointwise, and that the global behavior of α is not taken into account.

(3) *It is well known that the standard Gaussian measure γ satisfies the dimension free concentration property with the profile $\alpha = \overline{\Phi}$ (this follows from the isoperimetric theorem in Gauss space due to Sudakov-Cirelson [27] and Borell [7], see e.g [18]). So applying the preceding result, we conclude that γ satisfies Poincaré inequality with the constant $\lambda = 1$, which is well known to be optimal.*

(4) *If the concentration profile $\alpha(r)$ goes to zero too fast when $r \to \infty$, then $\lambda = +\infty$ and μ is a Dirac measure. This happens for instance when $\alpha(r) = be^{-ar^k}$, $r \geq 0$ with $k > 2$ and $a, b > 0$.*

Theorem 1.3 is in the same spirit as a previous result of the first author [10], where the Gaussian dimension free concentration level was characterized by a transport-entropy inequality. To state this result, let us recall that the Kantorovich-Rubinstein distance W_p, $p \geq 1$, between $\nu, \mu \in \mathcal{P}(\mathcal{X})$ is defined by

$$W_p^p(\nu, \mu) = \inf \mathbb{E}[d^p(X, Y)],$$

where the infimum runs over the set of couples of random variable (X, Y) such that $\mathrm{Law}(X) = \mu$, $\mathrm{Law}(Y) = \nu$. A probability measure satisfies the p-Talagrand transport-entropy inequality, for some $C > 0$ if

$$W_p^p(\nu, \mu) \leq CH(\nu|\mu), \qquad \forall \nu \in \mathcal{P}(\mathcal{X}), \tag{1.7}$$

where the relative entropy functional is defined by $H(\nu|\mu) = \int \log \frac{d\nu}{d\mu}\, d\nu$ if ν is absolutely continuous with respect to μ, and $H(\nu|\mu) = +\infty$ otherwise. Inequalities of this type were introduced by Marton and Talagrand in the nineties [20, 30]. We refer to the survey [11] for more informations on this topic.

Theorem 1.8. [10] *A probability measure μ satisfies the p-Talagrand transport inequality (1.7) for some $p \geq 2$ and $C > 0$ if and only if it satisfies the dimension free concentration inequality $\mathbf{CI}_p^\infty(\alpha)$, with a concentration profile of the form*

$$\alpha(r) = \exp\left(-\frac{1}{C}[r - r_o]_+^p\right), \qquad r \geq 0,$$

for some $r_o \geq 0$.

As we will see, the proofs of Theorem 1.3 and 1.8 are very different. Both makes use of probability limit theorems, but not at the same scale: Theorem 1.8 used Sanov's large deviations theorem, whereas Theorem 1.3 is an application of the central limit theorem. Moreover, contrary to what happens in Theorem 1.3 (see item (2) of Remark 1.5), the global behavior of the concentration profile is used in Theorem 1.8.

In view of Theorems 1.3 and 1.8, it is natural to formulate the following general question:

(**Q**) Which functional inequality is equivalent to $\mathbf{CI}_p^\infty(\alpha)$ for a concentration profile of the form

$$\alpha(r) = \exp(-a[r - r_o]_+^k), \qquad r \geq 0,$$

where $a > 0, r_o \geq 0$ and $k > 0$?

Remark 1.9. *It is easy to see, using the central limit theorem, that for $p \in [1, 2)$ the only probability measures verifying $\mathbf{CI}_p^\infty(\alpha)$, for some α such that $\alpha(r_o) < 1/2$ for at least one $r_o > 0$, are Dirac masses. Thus the question (**Q**) is interesting only for $p \geq 2$.*

To summarize, Theorem 1.8 shows that the answer to (**Q**) is the p-Talagrand inequality for $k = p$ and $p \geq 2$. Theorem 1.3 shows that the answer is the Poincaré inequality for

$p = 2$ and for $k \in (0, 1]$. Moreover the point (4) of Remark 1.5 above shows that for $p = 2$, the question is interesting only for $k \in [1; 2]$. The question for $k \in (1; 2)$ is still open.

Some partial results are known for $p = \infty$. In [5], Bobkov and Houdré characterized the set of probability measures on $\mathbb{R}$ satisfying $\mathbf{CI}_\infty^\infty(\beta_{a_o,r_o})$, with $a_o \in [0, 1/2)$, where β_{a_o,r_o} is the minimal concentration profile defined by (1.6). They showed that a probability measure μ belongs to this class if and only if the map U_μ defined by

$$U_\mu(x) = F_\mu^{-1}\left(\frac{1}{1+e^{-x}}\right), \qquad x \in \mathbb{R},$$

where $F_\mu(x) = \mu((-\infty, x])$ and $F_\mu^{-1}(p) = \inf\{x \in \mathbb{R}; F_\mu(x) \geq p\}$, $p \in (0, 1)$, satisfies the following inequality on the interval where it is defined:

$$|U_\mu(x) - U_\mu(y)| \leq a + b|x - y|,$$

for some $a, b \geq 0$.

1.2. **Alternative formulation in terms of observable diameters.** It is possible to give an alternative formulation of Theorem 1.3 and Theorem 1.4 using the notion of observable diameter introduced by Gromov ([15, Chapter 3.1/2]). Recall that if $(\mathcal{X}, d, \mu)$ is a metric space equipped with a probability measure and $t \in [0, 1]$, the partial diameter of $(\mathcal{X}, d)$ is defined as the infimum of the diameters of subsets $A \subset \mathcal{X}$ such that $\mu(A) \geq 1 - t$. It is denoted by $\operatorname{Part}\operatorname{Diam}(\mathcal{X}, d, \mu, t)$. If $f : \mathcal{X} \to \mathbb{R}$ is some 1-Lipschitz function, let us denote by $\mu_f \in \mathcal{P}(\mathbb{R})$ the push forward of μ under f. The observable diameter of $(\mathcal{X}, d, \mu)$ is defined as follows

$$\operatorname{Obs}\operatorname{Diam}(\mathcal{X}, d, \mu, t) = \sup_{f \ 1-\mathrm{Lip}} \operatorname{Part}\operatorname{Diam}(\mathbb{R}, |\cdot|, \mu_f, t) \in \mathbb{R}^+ \cup \{+\infty\}$$

We define accordingly the observable diameters of $(\mathcal{X}^n, d_2, \mu^n)$ for all $n \in \mathbb{N}^*$.

The observable diameters are related to concentration profiles by the following lemma (see e.g [9, Lemma 2.22]).

Lemma 1.10. *If μ satisfies $\mathbf{CI}(\alpha)$, then*

$$\operatorname{Obs}\operatorname{Diam}(\mathcal{X}, d, \mu, 2\alpha(r)) \leq 2r,$$

for all $r \geq 0$ such that $\alpha(r) \leq 1/2$.
Conversely, for all $t \in [0, 1/2]$, for all $A \subset \mathcal{X}$, with $\mu(A) \geq 1/2$, it holds

$$\mu(A_{r(t)}) \geq 1 - t$$

with $r(t) = \operatorname{Obs}\operatorname{Diam}(\mathcal{X}, d, \mu, t)$.

The following corollary gives an interpretation of Poincaré inequality in terms of the boundedness of the observable diameters of the sequence of metric probability spaces $(\mathcal{X}^n, d_2, \mu^n)_{n\in\mathbb{N}^*}$.

Corollary 1.11. *A probability measure μ on $(\mathcal{X}, d)$ satisfies the Poincaré inequality (1.2) with the optimal constant λ if and only if for some $t \in (0, 1/2)$*

$$r_\infty(t) := \sup_{n\in\mathbb{N}^*} \operatorname{Obs}\operatorname{Diam}(\mathcal{X}^n, d_2, \mu^n, t) < \infty.$$

Moreover,

$$\overline{\Phi}^{-1}(t) \leq r_\infty(t)\sqrt{\lambda} \leq a \log\left(\frac{b}{t}\right), \qquad \forall t \in (0, 1/2)$$

where $a > 0$ and $b \geq 1$ are some universal constants.

1.3. **Tools.** The main tool in the proof of Theorem 1.3 is a new alternative formulation of concentration of measure in terms of deviation inequalities for inf-convolution operators that was first obtained in [12]. Recall that for all $t > 0$, the infimum convolution operator $f \mapsto Q_t f$ is defined for all $f : \mathcal{X}^n \to \mathbb{R} \cup \{+\infty\}$ bounded from below as follows

$$Q_t f(x) = \inf_{y \in \mathcal{X}^n} \left\{ f(y) + \frac{1}{t} d_2^2(x,y) \right\}, \qquad x \in \mathcal{X}^n \tag{1.12}$$

(we should write $Q_t^{(n)}$, but we will omit the dimension n in the notation).

We recall below a result from [12] giving a new way to express concentration of measure.

Proposition 1.13. *Let $\mu \in \mathcal{P}(\mathcal{X})$; μ satisfies $\mathbf{CI}_2^\infty(\alpha)$ if and only if for all $n \in \mathbb{N}^*$ and for all measurable function $f : \mathcal{X}^n \to \mathbb{R} \cup \{+\infty\}$ bounded from below and such that $\mu^n(f = +\infty) < 1/2$, it holds*

$$\mu^n(Q_t f > m(f) + r) \leq \alpha(\sqrt{tr}), \quad \forall r, t > 0, \tag{1.14}$$

where $m(f)$ is any number such that $\mu^n(f \leq m(f)) \geq 1/2$.

The second main tool is the well known fact that the function $u : (t,x) \mapsto Q_t f(x)$ is, in some weak sense, solution of Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} = -\frac{1}{4} |\nabla_x u|^2.$$

This result is very classical on $\mathbb{R}^k$ (see e.g [8]) ; extensions to metric spaces were proposed in [19], [3], [2] or [13]. This will be discussed in the next section.

The third tool is the celebrated Berry-Esseen Inequality.

Theorem 1.15 (Berry-Esseen)**.** *Let $(X_i)_{i \in \mathbb{N}^*}$ be an i.i.d. sequence of real random variables such that $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = \sigma^2$ and $\mathbb{E}[|X_i|^3] = \rho < \infty$. There exists a universal constant $\kappa > 0$ such that, for all $n \in \mathbb{N}^*$,*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}\sigma} > x \right) - \overline{\Phi}(x) \right| \leq \kappa \frac{\rho}{\sigma^3 \sqrt{n}},$$

where $\overline{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-u^2/2}\, du$, $x \in \mathbb{R}$.

The paper is organized as follows. Section 2 puts Theorem 1.3 in perspective. We compare it to a result by E. Milman on Poincaré inequalities in non-negative curvature. We show in particular that an immediate consequence of Theorem 1.3 as well as E. Milman's result is a reduction of the KLS conjecture for isotropic log-concave probability measures. In Section 3, we recall some properties of the infimum convolutions operators that will be used in the proofs. Section 4 contains the proof of Theorem 1.3.

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2. Comparison with other results

2.1. **Dimension free concentration v.s non negative curvature.** Theorem 1.3 is reminiscent of the following recent result by E. Milman showing that under non-negative curvature the Poincaré constant of a probability measure can be expressed through very weak concentration properties of the measure [22, 23].

We recall that the Minkowski content of a set $A \subset \mathcal{X}$ is defined as follows

$$\mu^+(A) = \liminf_{r \to 0} \frac{\mu(A_r) - \mu(A)}{r}.$$

Theorem 2.1 (Milman [23]). *Let $\mu(dx) = e^{-V(x)}\,dx$ be an absolutely continuous probability measure on a smooth complete separable Riemannian manifold M equipped with its geodesic distance d. Suppose that $V : M \to \mathbb{R}$ is a function of class $\mathcal{C}^2$ such that*

$$\mathrm{Ric} + \mathrm{Hess}\, V \geq 0,$$

and that μ satisfies the following concentration of measure inequality

$$\mu(A_r) \geq 1 - \alpha(r), \qquad \forall r \geq 0,$$

for a function $\alpha : [0,\infty) \to [0,1/2]$ such that $\alpha(r_o) < 1/2$, for some $r_o > 0$, then μ satisfies Cheeger's inequality

$$\mu^+(A) \geq D \min(\mu(A); 1-\mu(A)), \qquad \forall A \subset M,$$

with

$$D = \sup\left\{ \frac{\Psi(\alpha(r))}{r}; r > 0 \ s.t\ \alpha(r) < 1/2 \right\},$$

where $\Psi : [0,1/2)$ is some universal function.

We recall that Cheeger's inequality with the constant D implies Poincaré inequality (1.2) with the constant $\lambda = D^2/4$. In our result the non-negative curvature assumption of Milman's result is replaced by the assumption that the concentration is dimension free.

Remark 2.2. *If M has non-negative Ricci curvature and $\mu(dx) = \frac{1}{|K|}\mathbf{1}_K(x)\,dx$ is the normalized restriction of the Riemmanian volume to a geodesically convex set K, then E. Milman obtains in [24] that*

$$D = \sup\left\{ \frac{1-2\alpha(r)}{r}; r > 0 \right\}.$$

This bound is optimal (see [24]).

2.2. **A remark on the KLS conjecture.** In this section, $\mathbb{R}^k$ is always equipped with its standard Euclidean norm $|\cdot|$.

Let us recall the celebrated conjecture by Kannan, Lovász and Simonovits [17]. Recall that a probability measure μ on $\mathbb{R}^k$ is isotropic if $\int x\,\mu(dx) = 0$ and $\int x_i x_j\,\mu(dx) = \delta_{ij}$ for all $1 \leq i,j \leq n$. It is log-concave if it has a density of the form e^{-V}, where $V : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ is a convex function.

Conjecture 2.3 (Kannan-Lovász-Simonovits [17]). *There is a universal constant $D > 0$ such that for all $k \in \mathbb{N}^*$, any log-concave and isotropic probability measure μ on $\mathbb{R}^k$ satisfies the following Cheeger inequality*

$$\mu^+(A) \geq D \min(\mu(A); 1-\mu(A)), \qquad \forall A \subset \mathbb{R}^k.$$

Equivalently, there is a universal constant $\lambda > 0$ such that for all $k \in \mathbb{N}^*$, any log-concave and isotropic probability measure μ on $\mathbb{R}^k$ satisfies the following Poincaré inequality

$$\lambda \mathrm{Var}_\mu(f) \leq \int |\nabla f|^2\,d\mu,$$

for all $f : \mathbb{R}^k \to \mathbb{R}$ Lipschitz.

According to E. Milman's Theorem 2.1, the above conjecture can be reduced to a statement about universal concentration inequalities for log-concave isotropic probabilities.

Corollary 2.4. *The KLS conjecture is equivalent to the following statement. There exists $r_o > 0, a_o \in [0,1/2)$ such that for any $m \in \mathbb{N}^*$, any log-concave and isotropic probability ν on $\mathbb{R}^m$ satisfies*

$$\nu(A + r_o B_2) \geq 1 - a_o, \qquad \forall A \subset \mathbb{R}^m \ s.t.\ \nu(A) \geq 1/2, \tag{2.5}$$

where B_2 is the Euclidean unit ball of $\mathbb{R}^m$.

This corollary follows immediately from Theorem 2.1. Below, we propose an alternative proof based on our main result Theorem 1.3.

Proof of Corollary 2.4. According to Theorem 1.4, it is clear that the KLS conjecture implies uniform exponential concentration estimates for isotropic log-concave probability measures.

Conversely, let μ be isotropic and log-concave on $\mathbb{R}^k$. For all $n \in \mathbb{N}^*$, the probability μ^n is still isotropic and log-concave on $\left(\mathbb{R}^k\right)^n$. So applying (2.5) to $\nu = \mu^n$ on $\left(\mathbb{R}^k\right)^n$, for all $n \in \mathbb{N}^*$, we conclude that μ satisfies $\mathbf{CI}_2^\infty(\beta_{a_o,r_o})$, where the concentration profile β_{a_o,r_o} is defined by (1.6). According to Theorem 1.3, we conclude that μ satisfies Poincaré inequality with the constant $\lambda = \left(\overline{\Phi}^{-1}(a_o)/r_o\right)^2$. Since this holds for any isotropic log-concave probability measure in any dimension, this ends the proof. □

2.3. **Euclidean v.s Talagrand type enlargements.** Theorem 1.3 improves a preceding result by the first author [10] where a stronger form of exponential dimension free concentration, introduced by Talagrand [28, 29], was shown to be equivalent to a transport-entropy inequality in turn equivalent to Poincaré inequality.

In what follows, if $A \subset \mathcal{X}^n$, for some $n \in \mathbb{N}^*$, we will consider the following family of enlargements of A:

$$\widetilde{A}_{a,r} = \left\{ x \in \mathcal{X}^n ; \exists y \in A \text{ s.t } \sum_{i=1}^n \theta(a d(x_i, y_i)) \leq r \right\}, \qquad \forall a > 0, \quad \forall r \geq 0$$

where $\theta(t) = t^2$, if $t \in [0,1]$ and $\theta(t) = 2t - 1$, if $t \geq 1$.

Definition 2.6. *A probability μ on $\mathcal{X}$ satisfies the Talagrand exponential type dimension free concentration inequality with constants $a, b \geq 0$ if for all $n \in \mathbb{N}^*$, for all $A \subset \mathcal{X}^n$ with $\mu^n(A) \geq 1/2$, it holds*

$$\mu^n(\widetilde{A}_{a,r}) \geq 1 - be^{-r}, \qquad \forall r \geq 0. \tag{2.7}$$

Since $t \mapsto \theta(\sqrt{t})$ is concave and vanishes at 0, it is thus sub-additive and we have the following inequality

$$\sum_{i=1}^n \theta(a d(x_i, y_i)) \geq \theta\left(\sqrt{\sum_{i=1}^n a^2 d^2(x_i, y_i)} \right) = \theta(a d_2(x, y)), \qquad \forall x, y \in \mathcal{X}^n.$$

Therefore,

$$\widetilde{A}_{a,\theta(ar)} \subset A_{r,2},$$

and so if μ satisfies the Talagrand concentration inequality (2.7), then it obviously verify the dimension free concentration inequality with the profile $\alpha(u) = be^{-\theta(au)} \leq ebe^{-2au}$, $u \geq 0$.

The following theorem summarizes the known links between Talagrand exponential type dimension free concentration and Poincaré inequality.

Theorem 2.8. *Let μ be a probability measure on $\mathcal{X}$. The following statements are equivalent*

1. *The probability μ satisfies Poincaré inequality (1.2) with a constant $\lambda > 0$.*
2. *The probability μ satisfies the following dimension free concentration inequality: for all $n \in \mathbb{N}^*$, for all $A \subset \mathcal{X}^n$ such that $\mu^n(A) \geq 1/2$,*

$$\mu^n\left(\widetilde{A}_{a,r}\right) \geq 1 - be^{-r}, \qquad \forall r \geq 0,$$

for some constants $a, b > 0$.

(3) *The probability measure μ satisfies the following transport-entropy inequality*

$$\inf_{\pi\in\Pi(\mu,\nu)} \iint \theta(Cd(x,y))\,\pi(dxdy) \le H(\nu|\mu), \qquad \forall \nu \in \mathcal{P}(\mathcal{X}),$$

for some constant $C>0$, where $\Pi(\mu,\nu)$ is the set of couplings π between μ and ν and the relative entropy is defined as follows $H(\nu|\mu) = \int \log \frac{d\nu}{d\mu}\, d\nu$, when $\nu \ll \mu$ and $+\infty$ otherwise.

Moreover the constants above are related as follows :
(1) $\Rightarrow$ (2) *with $a = \kappa\sqrt{\lambda}$ and $b=1$, for some universal constant κ.*
(2) $\Rightarrow$ (3) *with $C = a$.*
(3) $\Rightarrow$ (1) *with $\lambda = 2C^2$.*

Let us make some comments about the different implications in Theorem 2.9. The implication (1) $\Rightarrow$ (2) is due to Bobkov and Ledoux [6], the implication (2) $\Rightarrow$ (3) is due to the first author [10, Theorem 5.1], and the implication (3) $\Rightarrow$ (1) is due to Maurey [21] or Otto-Villani [25]. The equivalence between (1) and (3) was first proved by Bobkov, Gentil and Ledoux in [4].

Remark 2.9. *It is worth noting that the implication (2) $\Rightarrow$ (3) follows from Theorem 1.8 for $p=2$ by a change of metric argument. Namely, suppose that μ satisfies the concentration property (2) of Theorem , for some $a>0$, and define $\tilde{d}(x,y) = \sqrt{\theta(ad(x,y))}$ for all $x,y\in\mathcal{X}$. It is not difficult to check that the function $\theta^{1/2}$ is subadditive, and therefore $\tilde{d}$ defines a new distance on $\mathcal{X}$. The ℓ_2 extension of $\tilde{d}$ to the product $\mathcal{X}^n$ is*

$$\tilde{d}_2(x,y) = \left[\sum_{i=1}^n \theta(ad(x_i,y_i))\right]^{1/2}, \qquad x,y\in\mathcal{X}^n,$$

and it holds

$$\widetilde{A}_{a,r} = \{x\in\mathcal{X}^n; \tilde{d}_2(x,A)\le\sqrt{r}\}, \qquad \forall A\subset\mathcal{X}^n.$$

Therefore, statement (2) can be restated by saying that μ satisfies $\mathbf{CI}_2^\infty(\alpha)$ (with respect to the distance $\tilde{d}$) with the Gaussian concentration profile $\alpha(r) = be^{-r^2}$. Applying Theorem 1.8, we conclude that μ satisfies the 2-Talagrand transport entropy inequality with the constant 1 with respect to the distance $\tilde{d}$, which is exactly (3) with $C=a$.

An immediate consequence of Theorem 1.3 and of Bobkov-Ledoux Theorem (1) $\Rightarrow$ (2) above is the following result showing the equivalence between the two forms of dimension free exponential concentration.

Theorem 2.10. *Let μ be a probability measure on $\mathcal{X}$. The following are equivalent.*

(1) *The probability measure μ satisfies the Talagrand exponential type dimension free concentration inequality (2.7) with constants a and b.*
(2) *The probability measure μ satisfies $\mathbf{CI}_2^\infty(\alpha)$ with a profile $\alpha(u) = b'e^{-a'u}$, $u\ge 0$.*

In these conditions, the constants are related as follows: (1) $\Rightarrow$ (2) *with $a' = 2a$ and $b' = eb$, and* (2) $\Rightarrow$ (1) *with $a = a'/\sqrt{\log(2b')}$ and $b=1$.*

We do not know if there is a direct proof of the implication (2) $\Rightarrow$ (1).

Proof. We have already proved that (1) implies (2). Let us prove the converse. According to Theorem 1.3 we conclude from (2) that μ satisfies Poincaré inequality with a constant $C \le \left(\frac{u}{\overline{\Phi}^{-1}(\alpha(u))}\right)^2$, for all u such that $\alpha(u) < 1/2$. A classical inequality gives $\overline{\Phi}(t) \le \frac{1}{2}e^{-t^2/2}$, $t\ge 0$. Therefore, $\overline{\Phi}^{-1}(t) \ge 2\sqrt{-\log(2t)}$, for all $t\in(0,1/2)$ and so taking $u = 2\log(2b')/a'$ yields to $C \le \frac{\log(2b')}{a'^2}$. According to the implication (1) $\Rightarrow$ (2) in Theorem 2.9 we conclude that μ satisfies Talagrand concentration inequality (2.7) with $a = \kappa a'/\sqrt{\log(2b')}$. □

3. Some properties of inf-convolution operators

In this section, we recall properties of inf-convolution operators related to Hamilton-Jacobi equations and to concentration of measure.

3.1. Link with Hamilton-Jacobi equations.

We recall that $(\mathcal{X}, d)$ is a metric space. The following proposition collects some basic observations about the operators Q_t, $t > 0$.

Proposition 3.1. *Let $f : \mathcal{X} \to \mathbb{R}$ be a bounded Lipschitz function. For all $x \in \mathcal{X}$, $Q_t f(x) \to f(x)$, when $t \to 0^+$ and for all $\nu \in \mathcal{P}(\mathcal{X})$,*

$$\limsup_{t\to 0^+} \frac{1}{t} \int f(x) - Q_t f(x)\, \nu(dx) \leq \frac{1}{4} \int |\nabla^- f(x)|^2\, \nu(dx). \tag{3.2}$$

Before giving the proof of Proposition 3.1, let us complete the picture by recalling the following theorem of [2] and [14] (improving preceding results of [19] and [3]). This result will not be used in the sequel.

Theorem 3.3. *If f is a bounded function on a polish metric space $\mathcal{X}$, then $(t, x) \mapsto Q_t f(x)$ satisfies the following Hamilton-Jacobi (in)equation*

$$\frac{d}{dt_+} Q_t f(x) \leq -\frac{1}{4} |\nabla Q_t f|^2(x), \qquad \forall t > 0,\ \forall x \in \mathcal{X} \tag{3.4}$$

where d/dt_+ stands for the right derivative, and $|\nabla g|(x) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{d(y,x)}$. Moreover, if the space $\mathcal{X}$ is geodesic (i.e for all $x, y \in \mathcal{X}$ there exists at least one curve $(z_t)_{t\in[0,1]}$ such that $z_0 = x$, $z_1 = y$ and $d(z_s, z_t) = |t - s| d(x, y)$) then (3.4) holds with equality.

Proof of Proposition 3.1. Let $M = \sup |f|$; since $Q_t f \leq f$ one has in particular, $Q_t f \leq M$. Therefore

$$Q_t f(x) = \inf_{y\in B(x,\sqrt{2Mt})} \left\{ f(y) + \frac{1}{t} d^2(x, y) \right\}.$$

So, it holds

$$\begin{aligned} 0 \leq \frac{f(x) - Q_t f(x)}{t} &= \sup_{y\in B(x,\sqrt{2Mt})} \left\{ \frac{f(x) - f(y)}{t} - \frac{d^2(x, y)}{t^2} \right\} \\ &\leq \sup_{y\in B(x,\sqrt{2Mt})} \left\{ \frac{[f(x) - f(y)]_+}{d(x, y)} \frac{d(x, y)}{t} - \frac{d^2(x, y)}{t^2} \right\} \\ &\leq \frac{1}{4} \sup_{y\in B(x,\sqrt{2Mt})} \frac{[f(x) - f(y)]_+^2}{d^2(x, y)}. \end{aligned}$$

We conclude from this that $0 \leq (f - Q_t f)/t \leq L^2/4$, where L is the Lipschitz constant of f. This implies in particular that $Q_t f \to f$ when $t \to 0$. Taking the lim sup when $t \to 0^+$ gives

$$\limsup_{t\to 0^+} \frac{f(x) - Q_t f(x)}{t} \leq \frac{1}{4} |\nabla^- f(x)|^2. \tag{3.5}$$

Inequality (3.2) follows from (3.5) using Fatou's Lemma in its lim sup version. The application of Fatou's Lemma is justified by the fact that the family of functions $\{(f - Q_t f)/t\}_{t>0}$ is uniformly bounded. □

Remark 3.6. *The proof of (3.5) can also be found in* [31, Theorem 22.46] *(see also* [14, Proposition A.3], [19], [3], [2]*).*

3.2. **Link with concentration of measure.** For the reader's convenience, let us restate Proposition 1.13.

Proposition 3.7. *Let $\mu \in \mathcal{P}(\mathcal{X})$; μ satisfies the dimension free concentration property with the concentration profile α if and only if for all $n \in \mathbb{N}^*$ and for all measurable function $f : \mathcal{X}^n \to \mathbb{R} \cup \{+\infty\}$ bounded from below and such that $\mu^n(f = +\infty) < 1/2$, it holds*

$$\mu^n(Q_t f > m(f) + r) \leq \alpha(\sqrt{tr}), \quad \forall r, t > 0, \tag{3.8}$$

where $m(f)$ is any number such that $\mu^n(f \leq m(f)) \geq 1/2$.

Proof. We recall the short proof of Proposition 1.13 for the sake of completeness. Suppose that μ satisfies the dimension free concentration property with the profile α, and define $A = \{f \leq m(f)\}$. By definition of $m(f)$, $\mu^n(A) \geq 1/2$ and so $\mu^n(\mathcal{X}^n \setminus A_u) \leq \alpha(u)$, for all $u \geq 0$. Observe that

$$Q_t f(x) \leq m(f) + \frac{1}{t} d_2^2(x, A), \qquad \forall x \in \mathcal{X}^n.$$

So $\{Q_t f > m(f) + r\} \subset \{d_2(\cdot, A) > \sqrt{tr}\} = A^c_{\sqrt{tr},2}$, which proves (3.8).

Let us prove the converse. Take a Borel set $A \subset \mathcal{X}^n$ such that $\mu^n(A) \geq 1/2$ and consider the function f_A equals to 0 on A and $+\infty$ on A^c. For this function, $Q_t f_A = (1/t) d_2^2(x, A)$ and one can choose $m(f) = 0$. Applying (3.8) gives the result. □

4. Poincaré inequality and concentration of measure

This section contains the proof of our main result Theorem 1.3.

4.1. **From dimension free concentration to Poincaré inequality.**

Proof of Theorem 1.3. Let $h : \mathcal{X} \to \mathbb{R}$ be a bounded Lipschitz function such that $\int h \, d\mu = 0$. For all $n \in \mathbb{N}^*$, define $f_n : \mathcal{X}^n \to \mathbb{R}^+$ by

$$f_n(x) = h(x_1) + \cdots + h(x_n), \qquad \forall x = (x_1, \ldots, x_n) \in \mathcal{X}^n.$$

Applying (3.8) to f_n with $t = 1/\sqrt{n}$ and $r = \sqrt{n}u$, for some $u > 0$, we easily arrive at

$$\mu^n\left(\frac{1}{\sqrt{n}\sigma_n}\sum_{i=1}^n \left[Q_{1/\sqrt{n}}h(x_i) - \mu(Q_{1/\sqrt{n}}h)\right] > \frac{1}{\sigma_n\sqrt{n}} m(f_n) + \frac{\sqrt{n}}{\sigma_n}\mu\left(h - Q_{1/\sqrt{n}}h\right) + \frac{u}{\sigma_n}\right) \leq \alpha(\sqrt{u}),$$

where $\sigma_n^2 = \mathrm{Var}_\mu(Q_{1/\sqrt{n}}h)$ and $m(f_n)$ is a median of f_n under μ^n, that is to say any number $m \in \mathbb{R}$ such that $\mu^n(f \geq m) \geq 1/2$ and $\mu^n(f_n \leq m) \geq 1/2$. According to the Berry-Esseen Theorem 1.15, we conclude that $\overline{\Phi}(x) = \frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-u^2/2}\, du$, $x \in \mathbb{R}$ satisfies the inequality

$$\overline{\Phi}\left(\frac{1}{\sigma_n\sqrt{n}} m\left(f_n\right) + \frac{\sqrt{n}}{\sigma_n}\mu\left(h - Q_{1/\sqrt{n}}h\right) + \frac{u}{\sigma_n}\right) \leq \alpha(\sqrt{u}) + \kappa\frac{\rho_n}{\sigma_n^3\sqrt{n}},$$

where κ is some universal constant and $\rho_n = \int (Q_{1/\sqrt{n}}h - \mu(Q_{1/\sqrt{n}}h))^3 \, d\mu$. According to Point (1) of Proposition 3.1, $\sigma_n \to \sqrt{\mathrm{Var}_\mu(h)}$, when n goes to ∞ and according to Point (2) of Proposition 3.1,

$$\limsup_{n\to+\infty} \sqrt{n}\mu\left(h - Q_{1/\sqrt{n}}h\right) \leq \frac{1}{4}\int |\nabla^- h|^2 \, d\mu.$$

Moreover letting $\sigma = \sqrt{\mathrm{Var}_\mu(h)}$ and $m_n = m(f_n)/(\sqrt{n}\sigma)$, it follows from the Berry-Esseen inequality that

$$\frac{1}{2} - \delta_n \leq \mu^n\left(\frac{f_n}{\sqrt{n}\sigma} \geq m_n\right) - \delta_n \leq \overline{\Phi}(m_n) \leq \mu^n\left(\frac{f_n}{\sqrt{n}\sigma} > m_n\right) + \delta_n \leq 1/2 + \delta_n,$$

where $\delta_n \to 0$ when $n \to \infty$. So $\overline{\Phi}(m_n) \to 1/2$ which implies that $m_n \to 0$, and also that $m(f_n)/(\sqrt{n}\sigma_n) \to 0$. Since $\overline{\Phi}$ is decreasing and continuous and ρ_n is bounded, we get

$$\overline{\Phi}\left(\frac{\int |\nabla^- h|^2 \, d\mu}{4\sqrt{\mathrm{Var}_\mu(h)}} + \frac{u}{\sqrt{\mathrm{Var}_\mu(h)}}\right) \leq \alpha(\sqrt{u}),$$

for all $u \geq 0$. Let $u \geq 0$ be such that $\alpha(\sqrt{u}) < 1/2$, then letting $k(u) = \overline{\Phi}^{-1}(\alpha(\sqrt{u})) > 0$, we easily get from the inequality above the following

$$k(u)\sqrt{\mathrm{Var}_\mu(h)} \leq u + \frac{1}{4}\int |\nabla^- h|^2 \, d\mu.$$

Replacing h by λh, $\lambda > 0$, we arrive at

$$k(u)\sqrt{\mathrm{Var}_\mu(h)} \leq \inf_{\lambda > 0}\left\{\frac{u}{\lambda} + \frac{\lambda}{4}\int |\nabla^- h|^2 \, d\mu\right\} = \sqrt{u\int |\nabla^- h|^2 \, d\mu},$$

which completes the proof. □

4.2. **Poincaré inequality and boundedness of observable diameters of product probability spaces.** In this section we prove Corollary 1.11.

Proof of Corollary 1.11. First assume that μ satisfies Poincaré inequality (1.2) with the optimal constant λ. Then according to Theorem 1.4, μ satisfies $\mathbf{CI}_2^\infty(\alpha)$ with the concentration profile $\alpha(r) = be^{-\sqrt{\lambda}r}$, where a, b are universal constants ($b \geq 1/2$). According to the first part of Lemma 1.10 (applied to the metric probability space $(\mathcal{X}^n, d_2, \mu^n)$), it follows that for all $n \in \mathbb{N}^*$, $\mathrm{Obs\,Diam}(\mathcal{X}^n, d_2, \mu^n, t) \leq 2\frac{\log(4b/t)}{a\sqrt{\lambda}}$, for all $t \leq 1$ and thus

$$r_\infty(t)\sqrt{\lambda} \leq a' \log(b'/t), \qquad \forall t \leq 1$$

for some universal constant a', b'.

Conversely, assume that $0 < r_\infty(t_o) < \infty$ for some $t_o \in (0, 1/2)$. According to the second part of Lemma 1.10, μ satisfies $\mathbf{CI}_2^\infty(\beta_{t_o, r_\infty(t_o)})$, where the minimal profiles β are defined in (1.6). According to Theorem 1.3, we conclude that μ satisfies Poincaré inequality with an optimal constant $\lambda > 0$ such that

$$\sqrt{\lambda} r_\infty(t_o) \geq \overline{\Phi}^{-1}(t_o).$$

According to the first step, we conclude that $r_\infty(t) < \infty$ for all $t \leq 1$, and so the inequality above is true for all $t \in (0, 1/2)$. □

4.3. **Self improvement of dimension free concentration inequalities.** The following result shows that a non-trivial dimension free concentration inequality can always be upgraded into an inequality with an exponential decay. This observation goes back to Talagrand [28, Proposition 5.1].

Corollary 4.1. *If μ satisfies $\mathbf{CI}_2^\infty(\alpha)$ with a profile α such that $\alpha(r_o) < 1/2$ for some r_o, then it satisfies $\mathbf{CI}_2^\infty$ with an exponential concentration. More explicitly, it satisfies the dimension free concentration property with the profile $\tilde{\alpha}(r) = be^{-a\sqrt{\lambda}r}$, where a, b are universal constants and*

$$\sqrt{\lambda} = \sup\left\{\frac{\overline{\Phi}^{-1}(\alpha(r))}{r}; r > 0 \ s.t\ \alpha(r) < 1/2\right\}.$$

This result is an immediate corollary of Theorem 1.3 and Theorem 1.4.

In [28] this result was stated and proved only for probability measures on $\mathbb{R}$. We thank E. Milman for mentioning to us that the argument was in fact more general. For the sake of completeness, we recall below the argument of Talagrand. It yields to the following extension of Corollary 4.1 (with slightly less accurate constants in the case $p = 2$).

Proposition 4.2. *Suppose that a probability μ on $\mathcal{X}$ satisfies $\mathbf{CI}_p^\infty(\beta_{a_o,r_o})$ for some $p \geq 1, r_o > 0, a_o \in [0,1/2)$. Then, for any $\gamma \in (-\log(1-a_o)/\log(2), 1)$, there exists $c \in [1/2,1)$ depending only on γ and a_o such that for all $n \in \mathbb{N}^*$,*

$$\mu^n(A_{r,p}) \geq 1 - \frac{1-c}{\gamma}\gamma^{r/r_o}, \qquad \forall r \geq 0, \qquad \forall A \subset \mathcal{X}^n \text{ s.t. } \mu^n(A) \geq c.$$

Proof. Take $A \subset \mathcal{X}$ then $(A^n)_{r_o,p} \subset (A_{r_o})^n$. Therefore, if $\mu(A) \geq (1/2)^{1/n}$, it holds $\mu(A_{r_o}) \geq (1-a_o)^{1/n}$. Let $A \subset \mathcal{X}$ be such that $\mu(A) \geq 1/2$ and let n_A be the greatest integer $n \in \mathbb{N}^*$ such that $\mu(A) \geq (1/2)^{1/n}$. By definition of n_A, $\frac{\log(2)}{\log(1/\mu(A))} - 1 < n_A \leq \frac{\log(2)}{\log(1/\mu(A))}$. According to what precedes,

$$\mu(A_{r_o}^c) \leq 1 - (1-a_o)^{1/n_A} \leq 1 - \exp\left(\frac{\log(1-a_o)\log(1/\mu(A))}{\log(2) - \log(1/\mu(A))}\right).$$

The function $\varphi(u) = \exp\left(\frac{\log(1-a_o)\log(1/u)}{\log(2)-\log(1/u)}\right)$ satisfies

$$\varphi(u) = 1 - \frac{\log(1-a_o)}{\log(2)}(u-1) + o(u-1),$$

when $u \to 1$. So $\frac{1-\varphi(\mu(A))}{1-\mu(A)} \to -\frac{\log(1-a_o)}{\log(2)} \in (0,1)$, when $\mu(A) \to 1$. Therefore, if γ is any number in the interval $(-\frac{\log(1-a_o)}{\log(2)}, 1)$, there exists $c > 1/2$ (depending only on γ) such that for all $A \subset \mathcal{X}$ with $\mu(A) \geq c$ it holds

$$\mu(A_{r_o}^c) \leq \gamma\mu(A^c).$$

Iterating yields

$$\mu(A_{kr_o}^c) \leq \gamma^k \mu(A^c), \qquad \forall k \in \mathbb{N}^*.$$

It follows easily that for all $u \geq 0$,

$$\mu(A_u^c) \leq ((1-c)/\gamma)\gamma^{u/r_o}.$$

Applying the argument above to the product measure μ^p, $p \in \mathbb{N}^*$, gives the conclusion. □

Appendix: From Poincaré inequality to exponential concentration

In this section, we give a proof of Theorem 1.4. The conclusion of Theorem 1.4 is very classical in say a Euclidean framework, but to deal with the general metric space framework requires some additional technical ingredients.

Proposition 4.3. *There are universal constants $a, b > 0$ such that if μ satisfies Poincaré inequality (1.2) with the constant $\lambda > 0$, then for all $n \in \mathbb{N}^*$, it holds*

$$\mu^n\left(f > \int f\, d\mu^n + r\right) \leq b\exp(-a\sqrt{\lambda}r), \qquad \forall r \geq 0$$

for all bounded function $f : \mathcal{X}^n \to \mathbb{R}$ such that

$$\sum_{i=1}^n |\nabla_i^- f|^2(x) \leq 1, \qquad \forall x \in \mathcal{X}^n, \tag{4.4}$$

where

$$|\nabla_i^- f|(x) = \limsup_{y \to x_i} \frac{[f(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n) - f(x)]_-}{d(y,x_i)}$$

Proof. It is well known that the Poincaré inequality tensorizes well (see [18]): if μ satisfies (1.2), then for all $n \in \mathbb{N}^*$, the product probability μ^n satisfies

$$\lambda \mathrm{Var}_\mu(g) \leq \int \sum_{i=1}^n |\nabla_i^- g|^2(x)\, \mu^n(dx), \tag{4.5}$$

for all function $g : \mathcal{X}^n \to \mathbb{R}$ that is locally Lispchitz in each coordinate.

Let $f : \mathcal{X}^n \to \mathbb{R}$ be bounded and such that (4.4) holds, and define $Z(s) = \log \int e^{sf}\, d\mu^n$, for all $s \geq 0$. Applying (4.5) to $g = e^{sf}$ and using (4.4) yields easily to

$$\lambda \left[\int e^{2sf} d\mu^n - \left(\int e^{sf}\, d\mu^n \right)^2 \right] \leq s^2 \int e^{2sf}\, d\mu^n,$$

and thus

$$\log(1 - s^2/\lambda) + Z(2s) \leq 2Z(s), \qquad \forall 0 \leq s \leq \sqrt{\lambda}.$$

According to Hölder inequality, the function Z is convex. Therefore,

$$Z(2s) \geq Z(s) + Z'(s)s.$$

As a result,

$$\log(1 - s^2/\lambda) + Z'(s)s \leq Z(s), \qquad \forall 0 \leq s \leq \sqrt{\lambda},$$

and so

$$\frac{d}{ds}\left(\frac{Z(s)}{s}\right) \leq -\frac{\log(1 - s^2/\lambda)}{s^2}, \qquad \forall 0 < s \leq \sqrt{\lambda}.$$

Since $Z(s)/s \to \int f\, d\mu^n$ when $s \to 0$, we conclude that

$$\int e^{s(f - \int f\, d\mu^n)} d\mu^n \leq \exp\left(\frac{s}{\sqrt{\lambda}} \int_0^{s/\sqrt{\lambda}} \frac{-\log(1 - v^2)}{v^2}\, dv \right), \qquad \forall s \leq \sqrt{\lambda}.$$

Taking $s = \sqrt{\lambda}/2$, we easily get

$$\mu^n \left(f - \int f\, d\mu^n > r \right) \leq b e^{-\frac{\sqrt{\lambda}}{2} r}, \qquad \forall r \geq 0,$$

with $b = \frac{1}{2} \int_0^{1/2} \frac{-\log(1-v^2)}{v^2}\, dv$.

□

Lemma 4.6. *Let $f : \mathcal{X}^n \to \mathbb{R}$ be a 1-Lipschitz function for the distance d_2. For all compact subset K of $\mathcal{X}^n$, and $\varepsilon > 0$, the function $f_{K,\varepsilon}$ defined by*

$$f_{K,\varepsilon}(x) = \sup_{z \in K} \left\{ f(z) - \sqrt{\varepsilon^2 + d_2^2(x, z)} \right\}, \qquad \forall x \in \mathcal{X}^n,$$

is bounded and satisfies the condition $\sum_{i=1}^n |\nabla_i^- f_{K,\varepsilon}|^2 \leq 1$ and the following inequality

$$f(x) - \varepsilon - 2d_2(x, K) \leq f_{K,\varepsilon}(x) \leq f(x), \qquad \forall x \in \mathcal{X}^n.$$

Proof of Lemma 4.6. The inequality is left to the reader. Let us show that $f_{K,\varepsilon}$ satisfies the condition $\sum_{i=1}^n |\nabla_i^- f_{K,\varepsilon}|^2 \leq 1$. Fix $x \in \mathcal{X}^n$, and for all $y \in \mathcal{X}$ and $i \in \{1, 2, \ldots, n\}$, set $\bar{x}^i y = (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$. Since K is compact, there exists $a \in K$, such that $f_{K,\varepsilon}(x) = f(a) - \sqrt{\varepsilon^2 + d_2^2(x, a)}$. It also holds $f_{K,\varepsilon}(\bar{x}^i y) \geq f(a) - \sqrt{\varepsilon^2 + d_2^2(\bar{x}^i y, a)}$. Therefore,

$$\begin{aligned} f_{K,\varepsilon}(\bar{x}^i y) - f_{K,\varepsilon}(x) &\geq \sqrt{\varepsilon^2 + d_2^2(x, a)} - \sqrt{\varepsilon^2 + d_2^2(\bar{x}^i y, a)} \\ &\geq -d(y, x_i) \frac{d(y, a_i) + d(x_i, a_i)}{\sqrt{\varepsilon^2 + d_2^2(\bar{x}^i y, a)} + \sqrt{\varepsilon^2 + d_2^2(x, a)}} \end{aligned}$$

Since the function $u \mapsto [u]_-$ is non increasing, one concludes that $|\nabla_i^- f_{K,\varepsilon}|(x) \leq \frac{d(x_i, a_i)}{\sqrt{\varepsilon^2 + d_2^2(x,a)}}$ and so $\sum_{i=1}^n |\nabla_i^- f_{K,\varepsilon}|^2(x) \leq 1$. □

Corollary 4.7. *There are universal constants $a, b > 0$ such that if μ satisfies Poincaré inequality (1.2) with the constant $\lambda > 0$, then for all $n \in \mathbb{N}^*$, it holds*

$$\mu^n \left(f > \int f\, d\mu^n + r \right) \leq b \exp(-a\sqrt{\lambda} r), \qquad \forall r \geq 0$$

for all bounded function $f : \mathcal{X}^n \to \mathbb{R}$ which is 1-Lipschitz with respect to the distance d_2.

Proof. Take a 1-Lipschitz function f for the distance d_2 on $\mathcal{X}^n$, $\varepsilon > 0$ and K a compact of $\mathcal{X}^n$. Proposition 4.3 applied to the function $f_{K,\varepsilon}$ of Lemma 4.6 implies that $\mu^n(f_{K,\varepsilon} > \int f_{K,\varepsilon}\, d\mu^n + r) \leq be^{-a\sqrt{\lambda}r}$, for all $r \geq 0$. Using the inequality given by Lemma 4.6, one gets

$$\mu^n\left(f - \varepsilon - 2d_2(\,\cdot\,, K) > \int f\, d\mu^n + r\right) \leq be^{-a\sqrt{\lambda}r}, \qquad \forall r \geq 0,$$

and so

$$\mu^n\left(\{f > \int f\, d\mu^n + r + \varepsilon\} \cap K\right) \leq be^{-a\sqrt{\lambda}r}, \qquad \forall r \geq 0.$$

Since the space $\mathcal{X}^n$ is polish, the probability μ^n is tight. So there is a nondecreasing sequence of compact sets K_p such that $\mu^n(K_p^c) \to 0$, when $p \to \infty$. Applying the inequality above, with K_p and a sequence ε_p tending to 0, one gets using the monotone convergence theorem

$$\mu^n\left(f > \int f\, d\mu^n + r\right) \leq be^{-a\sqrt{\lambda}r}, \qquad \forall r \geq 0.$$

□

Proof of Theorem 1.4. Having established in Corollary 4.7 (dimension free) deviations inequality for bounded 1-Lipschitz functions with respect to their mean, Theorem 1.4 now follows at once from [18, Proposition 1.7]. □

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