A polynomial realization of the Hopf algebra of uniform block permutations.
Rémi Maurice

To cite this version:
Rémi Maurice. A polynomial realization of the Hopf algebra of uniform block permutations.. Advances in Applied Mathematics, 2013, 51 (2), pp.285-308. 10.1016/j.aam.2013.04.003. hal-00823472

HAL Id: hal-00823472
https://hal.science/hal-00823472
Submitted on 17 May 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
A POLYNOMIAL REALIZATION OF THE HOPF ALGEBRA OF
UNIFORM BLOCK PERMUTATIONS.

RÉMI MAURICE

Abstract. We provide a polynomial realization of the Hopf algebra $\text{UBP}$ of uniform block permutations defined by Aguiar and Orellana [J. Alg. Combin. 28 (2008), 115-138]. We describe an embedding of the dual of the Hopf algebra $\text{WQSym}$ into $\text{UBP}$, and as a consequence, obtain a polynomial realization of it.

Contents

1. Introduction 1
2. Notations and background 2
  2.1. Words 2
  2.2. Permutations 3
  2.3. The Hopf algebra of set partitions 3
  2.4. The Hopf algebras $\text{FQSym}$ and $\text{FQSym}^*$ 4
3. Uniform block permutations 5
  3.1. Definitions 5
  3.2. Decompositions 6
  3.3. Algebraic structures 6
  4. Polynomial realization 9
  5. Packed words 12
    5.1. The combinatorial Hopf algebra $\text{WQSym}^*$ 12
    5.2. $\text{WQSym}$ 18
References 20

1. Introduction

For several years, Hopf algebras based on combinatorial objects have been thoroughly investigated. Examples include the Malvenuto-Reutenauer Hopf algebra $\text{FQSym}$ whose bases are indexed by permutations [1,3,8], the Loday-Ronco Hopf algebra $\text{PBT}$ whose bases are indexed by planar binary trees, [6], the Hopf algebra of free symmetric functions $\text{FSym}$ whose bases are indexed by standard Young tableaux [3,12].

The product and the coproduct, which define the structure of a combinatorial Hopf algebra, are, in general, rules of composition and decomposition given by combinatorial algorithms such as the shuffle and the deconcatenation of permutations, as in $\text{FQSym}$ [3], of packed words, as in $\text{WQSym}$ [10] or of parking functions, as in $\text{PQSym}$ [11]. These can also be given by the disjoint union and the admissible cuts of rooted trees, as in the Connes-Kreimer Hopf algebra, [2], or by concatenation and subgraphs/contractions of graphs [9].

Computing with these structures can be difficult, and polynomial realizations can bring up important simplifications.

2000 Mathematics Subject Classification. Primary 05C05, Secondary 16W30.
Key words and phrases. Combinatorial Hopf algebras, Uniform block permutations.
The idea is to encode the combinatorial objects by polynomials, in such a way that the product of the combinatorial Hopf algebra become the ordinary product of polynomials, and that the coproduct be given by the disjoint union of alphabets, endowed with some extra structure as an order relation. This can be done, e.g., for $\mathbb{FQSym}$ [3], $\mathbb{WQSym}$ [10], $\mathbb{PQSym}$ [11], Connes-Kreimer [5] and the one of diagrams [4].

In this paper, we realize the Hopf algebra $\mathbb{UBP}$ of uniform block permutations introduced by Aguiar and Orellana [7]. We begin by recalling the preliminary notions on uniform block permutations in Section 2. We recall, in Section 3, the Hopf algebra structure of uniform block permutations, we translate this structure via some decomposition of uniform block permutations into pairs consisting of a permutation and a set partition, and we describe the dual structure. Using polynomial realizations of Hopf algebras based on permutations and set partitions, we realize this algebra in terms of noncommutative polynomials in infinitely many bi-letters in Section 4. Finally, we obtain the Hopf algebra $\mathbb{WQSym}$ as a quotient of $\mathbb{UBP}$ and its dual as a subalgebra of $\mathbb{UBP}^\ast$.

2. Notations and background

We denote by $[n]$ the set $\{1, 2, \cdots, n\}$. All algebras will be on some field $\mathbb{k}$ of characteristic zero.

2.1. Words. Let $A := \{a_1 < a_2 < \cdots \}$ be a totally ordered infinite alphabet and $A^*$ the free monoid generated by $A$. For a word $w \in A^*$, we denote by $|w|$ the length of $w$. The empty word (of length 0) is denoted by $\varepsilon$. The alphabet $\text{Alph}(w)$ of $w$ is the set of letters occurring in $w$.

We say that $(i, j)$ is an inversion of $w$ if $i < j$ and $w_i > w_j$.

For $u, v \in A^*$, the shuffle product $\shuffle$ is recursively defined by

\begin{equation}
\begin{aligned}
\shuffle uv := \begin{cases}
    u & \text{if } v = \varepsilon \\
    v & \text{if } u = \varepsilon \\
    u_1(u' \shuffle v) + v_1(u \shuffle v') & \text{otherwise}
\end{cases}
\end{aligned}
\end{equation}

where $u = u_1u'$ and $v = v_1v'$ with $u_1, v_1 \in A$ and $u', v' \in A^*$.

We shall need the following expression of the shuffle:

\textbf{Proposition 2.1.} For all $u, v \in A^*$ and for all $0 \leq k \leq |u| + |v|:

\begin{equation}
\begin{aligned}
\shuffle uv = \sum_{\substack{i+j=k \\ 0 \leq i \leq |u| \\ 0 \leq j \leq |v|}} (u_1 \cdots u_i \shuffle v_1 \cdots v_j) \cdot (u_{i+1} \cdots u_{|u|} \shuffle v_{j+1} \cdots v_{|v|})
\end{aligned}
\end{equation}

\textbf{Proof.} We may assume that $\text{Alph}(u)$ and $\text{Alph}(v)$ are disjoint. The proof proceeds by induction on the length of the prefix in $u \shuffle v$.

Let $k$ be an integer such that $0 \leq k \leq |u| + |v|$. Then the set of all prefixes of length $k$ of $u \shuffle v$ is equal to the set of words of the product $u' \shuffle v'$ with $u'$ (resp. $v'$) is a prefix of $u$ (resp. $v$) and $|u'| + |v'| = k$. \hfill $\Box$

More conceptually, if $\delta(w) = \sum_{uv=w} u \otimes v$ is the deconcatenation coproduct, and $\mu : u \otimes v \mapsto uv$ the concatenation product, then $\delta$ is a morphism for $\shuffle$. Define a projection by $\pi_{k,l}(u \otimes v) = u \otimes v$ if $|u| = k$ and $|v| = l$, and $\pi_{k,l}(u \otimes v) = 0$ otherwise. Obviously, $\pi_{k,l} \circ \delta(w) = w$ if $k + l = |w|$. The proposition is equivalent to $\mu \circ \pi_{k,n-k} \circ \delta(u \shuffle v) = \mu \circ \pi_{k,n-k}(\delta(u) \shuffle \delta(v)) = u \shuffle v$. 


Example 2.2.

\[ a_{12}a_{3}w_{a_{1}a_{3}} = a_{1}a_{2}a_{3}a_{1}a_{3} + 2a_{1}a_{2}a_{1}a_{3}a_{3} + 4a_{1}a_{1}a_{2}a_{3}a_{3} + 2a_{1}a_{1}a_{3}a_{2}a_{3} + a_{1}a_{3}a_{1}a_{2}a_{3} \]
\[ = a_{1}(a_{2}a_{3}w_{a_{1}a_{3}}) + a_{1}(a_{1}a_{3}a_{2}w_{a_{3}}) \]
\[ = a_{1}a_{2}(a_{3}w_{a_{1}a_{3}}) + 2a_{1}a_{1}(a_{2}a_{3}w_{a_{3}}) + a_{1}a_{3}a_{1}a_{2}a_{3} \]
\[ = a_{1}a_{2}a_{3}a_{3}a_{3} + 2(a_{1}a_{2}w_{a_{1}})a_{3}a_{3} + (a_{1}w_{a_{1}a_{3}})a_{2}a_{3} \]
\[ = (a_{1}a_{2}a_{3}w_{a_{1}a_{3}})a_{3} + (a_{1}a_{2}w_{a_{1}a_{3}})a_{3}. \]

2.2. Permutations. We denote by \( \mathcal{S}_n \) the set of permutations of size \( n \) and we set \( \mathcal{S} = \cup_{n \geq 0} \mathcal{S}_n \). The shifted shuffle product \( \overline{\sigma} \) is defined by

\[ \sigma_1 \overline{\sigma}_2 = \sigma_1 \omega \sigma_2[|\sigma_1|] \]

where \( \sigma[k] \) is the word obtained by adding \( k \) to each letter of \( \sigma \).

Example 2.3.

\[ 12 \overline{2} 1 = 2 \omega 43 \]
\[ = 1243 + 1423 + 4123 + 1432 + 4132 + 4312. \]

The standardization is a process that associates a permutation with a word. The standardized of \( w \) is defined as the permutation having the same inversions as \( w \).

Example 2.4.

\[ \text{std}(a_{2}a_{3}a_{2}a_{1}a_{3}a_{4}) = 3541726. \]

The convolution product * is defined by

\[ \sigma_1 * \sigma_2 = \sum_{w = u \in \mathcal{S} \mid \text{std}(u) = \sigma_1 \text{ and } \text{std}(v) = \sigma_2} w. \]

Example 2.5.

\[ 12 + 21 = 1243 + 1342 + 1432 + 2341 + 2431 + 3421. \]

2.3. The Hopf algebra of set partitions. We denote by \( \mathcal{P}_n \) the set of set partitions of \( [n] \) and we set \( \mathcal{P} = \cup_{n \geq 0} \mathcal{P}_n \). Given a collection of disjoint sets of integers, we denote by \( \text{std}(\mathcal{E}) \) the standardized of \( \mathcal{E} \) obtained by numbering each integer \( a \) in \( \mathcal{E} \) by one plus the number of integers in \( \mathcal{E} \) smaller than \( a \).

Example 2.6.

\[ \text{std} (\{\{1, 6\}, \{9, 13\}, \{3, 5, 12\}\}) = \{\{1, 4\}, \{5, 7\}, \{2, 3, 6\}\}. \]

The \( \mathbb{K} \)-vector space spanned by the set partitions \( \mathcal{P} \) can be endowed with a Hopf algebra structure. The product, denoted by \( \times \), is obtained by the shifted union of two set partitions and the coproduct is defined for a set partition \( \mathcal{A} = \{A_1, \ldots, A_k\} \) by

\[ \Delta(\mathcal{A}) = \sum_{H \subset \{1, 2, \ldots, k\}} \text{std} \left( \bigcup_{i \in H} \{A_i\} \right) \otimes \text{std} \left( \bigcup_{i \notin H} \{A_i\} \right). \]

Example 2.7.

\[ \{\{1, 3\}, \{2\}, \{4\}\} \times \{\{1\}, \{2, 3\}\} = \{\{1, 3\}, \{2\}, \{4\}, \{5\}, \{6, 7\}\}. \]

and, by setting \( \mathcal{A} = \{\{1, 3\}, \{2, 5\}, \{4\}\} \),

\[ \Delta(\mathcal{A}) = \{\{\} \otimes \mathcal{A} \} + \{\{1, 2\} \} \otimes \{\{1, 3\}, \{2\}\} + \{\{1, 2\} \} \otimes \{\{1, 3\}, \{2\}\} + \{\{1\} \} \otimes \{\{1, 3\}, \{2, 4\}\} + \{\{1, 3\}, \{2, 4\}\} \otimes \{\{1\}\} \]
\[ + \{\{1, 2\}, \{3\}\} \otimes \{\{1, 2\}\} + \{\{1, 3\}, \{2\}\} \otimes \{\{1, 2\}\} + \mathcal{A} \otimes \{\{\}\} \].
This algebra admits a polynomial realization [7]. Let $B$ be a noncommutative alphabet and let $A$ be a set partition. We say that a word $w \in B^*$ is $A$-compatible if

$$\forall A \in \mathcal{A}(\forall i,j \in A) \quad w_i = w_j.$$  

In other words, if $i$ and $j$ are in the same block in $A$, then $w_i = w_j$. We define

$$P_A(B) = \sum_{w \in B^*} w,$$  

where $w$ is $A$-compatible.

**Example 2.8.**

$$P_{\{1\}} = \sum b_i$$

$$P_{\{1,2\}} = \sum b_i b_j,$$

$$P_{\{1,2,3\}} = \sum b_i b_j b_k.$$  

**Proposition 2.9.** The polynomials $P_A(B)$ provide a realization of the Hopf algebra based on set partitions $\mathcal{P}$. That is to say, given two set partitions $A$ and $B$

$$P_A \cdot P_B = P_{A \times B}.$$  

Let $B'$ be an alphabet isomorphic to the alphabet $B$. If we allow $B$ and $B'$ to commute and identify $T(B)U(B')$ with $T \otimes U$,

$$\Delta(P_A) = P_A(B + B'),$$  

where $B + B'$ denotes the disjoint union of $B$ and $B'$.

**Example 2.10.**

$$P_{\{1,2,3\}}(B) \cdot P_{\{1,2,3\}}(B) = \sum_{i,j,k} b_i b_j b_k \sum_{i,j} b_i b_j b_j$$

$$= \sum_{i,j,k,l,m} b_i b_j b_k b_l b_m$$

$$= P_{\{1,2,3\}}(B).$$

We set $A_1 = \{1,3\}$, $A_2 = \{2,5\}$, $A_3 = \{4\}$, and $A = \{A_1, A_2, A_3\}$:

$$P_A(B + B') = \sum_{i,j,k} b_i b_j b_k b_j + \sum_{i,j,k} b_i b_j b_i b_j + \sum_{i,j,k} b_i b_j b_i b_j + \sum_{i,j,k} b_i b_j b_i b_j$$

$$+ \sum_{i,j,k} b_i b_j b_i b_j + \sum_{i,j,k} b_i b_j b_i b_j + \sum_{i,j,k} b_i b_j b_i b_j + \sum_{i,j,k} b_i b_j b_i b_j$$

$$= P_A(B) + P_{\{A_2,A_3\}}(B)P_{\{A_1\}}(B) + P_{\{A_1,A_3\}}(B)P_{\{A_2\}}(B)$$

$$+ P_{\{A_1,A_2\}}(B)P_{\{A_3\}}(B)P_{\{A_2\}}(B) + P_{\{A_2\}}(B)P_{\{A_1,A_3\}}(B) + P_{\{A_1\}}(B)P_{\{A_2,A_3\}}(B) + P_A(B').$$

2.4. The Hopf algebras $\text{FQSym}$ and $\text{FQSym}^*$. The Hopf algebra of free quasi-symmetric functions [3] is the $\mathbb{K}$-vector space generated by the family $\{F_\sigma\}_{\sigma \in \Theta}$

$$F_\sigma := \sum_{w, \text{std}(w) = \sigma^{-1}} w,$$

The product rule is

$$F_{\sigma_1} \cdot F_{\sigma_2} := \sum_{\tau \in \sigma_1 \boxplus \sigma_2} F_\tau,$$
and the coproduct is defined by

\[ \Delta(F_\sigma) := F_\sigma(A \oplus B) = \sum_{u \circ v = \sigma} F_{\text{std}(u)} \otimes F_{\text{std}(v)}, \]

where \( A \oplus B \) is the ordinal sum of two mutually commuting totally ordered alphabets.

This is a polynomial realization of the Malvenuto-Reutenauer Hopf algebra \( K[\mathcal{S}] \).

The Hopf algebra \( \mathbf{FQSym} \) is self-dual and the dual basis of \( F_\sigma \) is \( G_\sigma \) where \( G_\sigma = F_{\sigma^{-1}} \).

The convolution product of two permutations is obtained by the product of two polynomials

\[ G_{\sigma_1} \cdot G_{\sigma_2} = \sum_{\tau \in \sigma_1 \ast \sigma_2} G_\tau. \]

3. Uniform block permutations

3.1. Definitions.

Definition 3.1. A uniform block permutation of size \( n \) is a bijection between two set partitions of \([n]\) that maps each block to a block of the same cardinality.

Let us denote by \( \mathcal{UBP}_n \) the set of uniform block permutations of size \( n \) and by \( \mathcal{UBP} \) the set \( \bigcup_{n \geq 0} \mathcal{UBP}_n \). We will represent a uniform block permutation in the form of an array with two rows. Let \( A = \{ A_1, A_2, \ldots, A_k \} \) and \( A' = \{ A'_1, A'_2, \ldots, A'_k \} \) be two set partitions of the same size. We denote the uniform block permutation \( f : A \rightarrow A' \) for which the image of \( A_i \) is \( A'_i \) by

\[ \begin{array}{cccc}
A_1 & A_2 & \cdots & A_k \\
A'_1 & A'_2 & \cdots & A'_k 
\end{array} \]

Example 3.2. Here are all uniform block permutations of size 1 and 2:

\[ n = 1: \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad n = 2: \begin{pmatrix} 12 \\ 12 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}. \]

In the case where the set partitions consist of singletons (i.e., \( A = \{ \{1\}, \cdots, \{n\} \} \)), we find the permutations of size \( n \).

Definition 3.3. Let \( f \) and \( g \) be two uniform block permutations. The concatenation of \( f \) and \( g \), denoted by \( f \times g \), is the uniform block permutation obtained by adding the size of \( f \) to all entries of \( g \) and concatenating the result to \( f \).

Example 3.4.

\[ \begin{pmatrix} 13 & 2 \\ 23 & 1 \end{pmatrix} \times \begin{pmatrix} 13 & 2 & 46 & 5 \\ 46 & 2 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 2 & 46 & 5 & 79 & 8 \\ 23 & 1 & 15 & 3 \end{pmatrix}. \]

Definition 3.5. Let \( f \) and \( g \) be two uniform block permutations of the same size \( n \), \( f : A \rightarrow A' \) and \( g : B \rightarrow B' \). The composition \( g \circ f : C \rightarrow C' \) of \( f \) and \( g \) is defined by the following process. The blocks \( C \) of the set partition \( C \) are the subsets of \([n]\) which are minimal for the two properties

1. \( C \) is a union of blocks of \( A \)
2. \( f(C) \) is a union of blocks \( B_i \) of \( B \).

The image of \( C \) is the union of images \( g(B_i) \).

Note that if \( f \) is a permutation, then the set partition \( C' \) is \( B' \), and if \( g \) is a permutation, then the set partition \( C \) is \( A \).
3.2. Decompositions. The data of a permutation and a set partition is enough to describe a uniform block permutation. Given a uniform block permutation $f : A \rightarrow A'$ of size $n$, there is a permutation $\sigma$ in $S_n$ such that

$$A' = \bigcup_{H \in A} \{ \bigcup_{x \in H} \{ \sigma(x) \} \}.$$  

Conversely, a set partition of the set $[n]$ and a permutation of $S_n$ determine a uniform block permutation of size $n$. Any uniform block permutation can thus be decomposed (non-uniquely) as

$$f = \sigma \circ \text{Id}_A \circ \text{Id}_{A'} \circ \sigma.$$  

The first equality expresses the fact that one can group the images $\sigma_i$ whose indices are in the same block in $A$, the second expresses the fact that one can group the images $\sigma_i$ whose values are in the same block in $A'$.  

Example 3.7.

$$\left( \begin{array}{cccc} 13 & 2 & 458 & 67 \\ 27 & 3 & 146 & 58 \end{array} \right) \circ \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 6 \\ 7 & 8 & 5 & 6 \end{array} \right) = \left( \begin{array}{cccc} 13 & 2 & 458 & 67 \\ 2 & 3 & 4 & 5 \end{array} \right).$$

3.3. Algebraic structures. In [7], Aguiar and Orellana introduced a Hopf algebra whose basis is the set of uniform block permutations. It will be denoted by $\text{UBP}$ in the sequel.

3.3.1. $\text{UBP}$. The product is defined as follows.

**Definition 3.8.** Let $f$ and $g$ be two uniform block permutations of respective sizes $m$ and $n$. The convolution, denoted by $f \ast g$, is the sum of all uniform block permutations of the form

$$\left( \begin{array}{cccc} 1 & \sigma_1 & \cdots & \sigma_m \\ \sigma_1 & \cdots & \cdots & \sigma_m + n \end{array} \right) \circ (f \times g)$$

where

\[\text{std}(\sigma_1 \cdots \sigma_m) = 12 \cdots m \text{ and } \text{std}(\sigma_m + 1 \cdots \sigma_m + n) = 12 \cdots n\]

i.e., $\sigma$ appears in $12 \cdots m * 12 \cdots n$.

**Example 3.9.**

$$\left( \begin{array}{c} 1 \\ 1 \end{array} \right) \ast \left( \begin{array}{c} 12 \\ 12 \end{array} \right) = \left[ \left( \begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \end{array} \right) + \left( \begin{array}{c} 1 \\ 2 \\ 2 \\ 3 \end{array} \right) + \left( \begin{array}{c} 1 \\ 3 \\ 1 \\ 2 \end{array} \right) \right] \circ \left( \begin{array}{c} 1 \\ 1 \\ 23 \end{array} \right) = \left( \begin{array}{c} 1 \\ 23 \end{array} \right) + \left( \begin{array}{c} 1 \\ 23 \end{array} \right) + \left( \begin{array}{c} 1 \\ 23 \end{array} \right).$$

This product can be rewritten in terms of the decomposition of uniform block permutations. Given $f : A \rightarrow A'$ and $g : B \rightarrow B'$ two uniform block permutations, $f = \sigma \circ \text{Id}_A$ and $g = \tau \circ \text{Id}_B$, we have

$$f \ast g = (\sigma \ast \tau) \circ \text{Id}_{A \times B}.$$
Example 3.10.
\[
\begin{pmatrix} 1 & 12 \\ 1 & 12 \end{pmatrix} = 1 \circ \text{Id}_{\{1\}} \ast 12 \circ \text{Id}_{\{1,2\}} \\
= (1 \ast 12) \circ \text{Id}_{\{1\},\{2,3\}} \\
= (123 + 213 + 312) \circ \text{Id}_{\{1\},\{2,3\}} \\
= \begin{pmatrix} 1 & 23 \\ 1 & 23 \end{pmatrix} + \begin{pmatrix} 2 & 23 \\ 2 & 13 \end{pmatrix} + \begin{pmatrix} 3 & 23 \\ 3 & 12 \end{pmatrix}
\]

Let $\mathbb{K}UBP_n$ be the $\mathbb{K}$-vector space spanned by $UBP_n$ and
\[
(3.5) \quad UBP = \bigoplus_{n \geq 0} \mathbb{K}UBP_n.
\]

We shall denote by $\{G_f; f \in UBP\}$ the basis of $UBP$. We endow $UBP$ with the product $\cdot$, defined, for uniform block permutations $f$ and $g$, by
\[
(3.6) \quad G_f \cdot G_g = \sum_{h \in f \ast g} G_h.
\]

Given a set partition $C$, we set
\[
(3.7) \quad C_{\{1,\cdots,i\}} = \left\{ H \in C; H \subset [i] \right\} \quad C_{\{i+1,\cdots,n\}} = \text{std} \left( C - C_{\{1,\cdots,i\}} \right).
\]

Let $B(C)$ be the set of integers $i$ such that
\[
(3.8) \quad \{1,2,\cdots,n\} = C_{\{1,\cdots,i\}} \times C_{\{i+1,\cdots,n\}}.
\]

Example 3.11. Let $C = \{\{1,3\}, \{2\}, \{4,6,7\}, \{5\}\}$. Then,
\[
\begin{align*}
C_{\{1,\cdots,2\}} &= \{\{2\}\} \\
C_{\{3,\cdots,7\}} &= \{\{1,3,4\}, \{2\}\} \\
C_{\{1,\cdots,3\}} &= \{\{1,3\}, \{2\}\} \\
C_{\{4,\cdots,7\}} &= \{\{1,3,4\}, \{2\}\}
\end{align*}
\]

$B(C) = \{0, 3, 7\}$

Definition 3.12. Let $f$ be a uniform block permutation of size $n$, $f : \Lambda \rightarrow \Lambda'$. By Lemma (2.2) in [7], $i$ is in $B(\Lambda')$ if and only if there exists a unique permutation $\sigma$ in $S_n$ appearing in the convolution product $12 \cdots i \ast 12 \cdots n - i$ and unique uniform block permutations $f(i)$ of size $i$ and $f'_{(n-i)}$ of size $n - i$ such that
\[
(3.9) \quad f = (f(i) \times f'_{(n-i)}) \circ \begin{pmatrix} 1 & 2 & \cdots & n \\ (\sigma^{-1})_1 & (\sigma^{-1})_2 & \cdots & (\sigma^{-1})_n \end{pmatrix}
\]

The coproduct is then
\[
(3.10) \quad \Delta(G_f) := \sum_{i \in B(\Lambda')} G_{f(i)} \otimes G_{f'_{(n-i)}}.
\]

Example 3.13.
\[
\Delta G \begin{pmatrix} 1 & 24 \\ 1 & 23 \end{pmatrix} = G \begin{pmatrix} 1 & 24 \\ 1 & 23 \end{pmatrix} \otimes G \begin{pmatrix} 1 & 24 \\ 1 & 23 \end{pmatrix} + G \begin{pmatrix} 1 & 13 \\ 1 & 12 \end{pmatrix} \otimes G \begin{pmatrix} 1 & 13 \\ 1 & 12 \end{pmatrix} + G \begin{pmatrix} 1 & 24 \\ 1 & 23 \end{pmatrix} \otimes 1
\]
Example 3.14. \( \Delta \) is a self-dual Hopf algebra. We can describe the dual structure. Let \( \text{UBP}^* \) be the basis of \( \text{UBP}^* \) dual to \( G_f \).

The product and coproduct are, for \( f : \mathcal{A} \rightarrow \mathcal{A}' \) and \( g : \mathcal{B} \rightarrow \mathcal{B}' \) two uniform block permutations with decompositions \( f = Id_{\mathcal{A}} \circ \sigma \) and \( g = Id_{\mathcal{B}} \circ \tau \)
\[
\Delta(G_f) = \sum_{i \in \text{Bi}(\mathcal{A}')} G_{f_1(\ldots,i)} \otimes G_{f_i(1,i+\ldots,n)}
\]
where
\[
f_1(\ldots,i) = Id_{\mathcal{A}_1(\ldots,i)} \circ \sigma_1(\ldots,i)
\]
\[
f_i(1,i+\ldots,n) = Id_{\mathcal{A}_i(1,i+\ldots,n)} \circ \sigma_i(1,i+\ldots,n)
\]

Example 3.15.
We set \( A = \{ \{1\}, \{2, 4\}, \{3\} \} \). Then, we have
\[
\Delta F \left( \begin{array}{c|c} 1 & 24 \\ \hline 3 & 4 \end{array} \right) = \Delta F_{1243 \otimes Id_A} \\
= 1 \otimes F_{1243 \otimes Id_A} + F_{1243 \otimes Id_A} \otimes F_{132 \otimes Id_{\{1,3\}}} + F_{1243 \otimes Id_A} \otimes 1 \\
= 1 \otimes F \left( \begin{array}{c|c} 1 & 24 \\ \hline 3 & 4 \end{array} \right) + F \left( \begin{array}{c|c} 1 & 24 \\ \hline 3 & 4 \end{array} \right) \otimes F \left( \begin{array}{c|c} 13 & 2 \\ \hline 3 & 4 \end{array} \right) + F \left( \begin{array}{c|c} 1 & 24 \\ \hline 3 & 4 \end{array} \right) \otimes 1 \\
\]
Setting \( G_f = F_{f^{-1}} \), we can identify \( \text{UBP} \) and \( \text{UBP}^\ast \). If \( \langle , \rangle \) denotes the scalar product such that \( \langle G_f, F_g \rangle = \delta_{fg} \), we have
\[
\begin{align*}
\langle \Delta G_f, F_g \otimes F_h \rangle &= \langle G_f, F_g \cdot F_h \rangle \\
\langle G_f \cdot G_g, F_h \rangle &= \langle G_f \otimes G_g, \Delta F_h \rangle,
\end{align*}
\]
which shows that \( \text{UBP} \) is indeed self-dual.

4. POLYNOMIAL REALIZATION

Given two alphabets \( B = \{ b_i; 0 \leq i \} \) and \( C = \{ c_i; 0 \leq i \} \), with \( C \) totally ordered, we consider an alphabet \( A = \{ \begin{pmatrix} b_i \\ c_j \end{pmatrix}; 0 \leq i, j \} \) of bi-letters endowed with two relations
\- for all \( i \) and \( k \), \( \begin{pmatrix} b_i \\ c_j \end{pmatrix} \leq \begin{pmatrix} b_k \\ c_l \end{pmatrix} \) if and only if \( c_j \leq c_l \),
\- for all \( j \) and \( l \), \( \begin{pmatrix} b_i \\ c_j \end{pmatrix} \equiv \begin{pmatrix} b_k \\ c_l \end{pmatrix} \) if and only if \( b_i = b_k \).

The product of two biwords is the concatenation
\[
\left( \begin{array}{c} u_1 \\ v_1 \end{array} \right) \cdot \left( \begin{array}{c} u_2 \\ v_2 \end{array} \right) = \left( \begin{array}{c} u_1 \cdot u_2 \\ v_1 \cdot v_2 \end{array} \right).
\]

We extend the definition of the standardization to biwords. Let \( w = \begin{pmatrix} u \\ v \end{pmatrix} \) be a biword. The \textit{standardized} of \( w \) is the permutation \( \text{std}(v) \).

\textbf{Example 4.1.}

\[
\text{std} \left( \begin{pmatrix} b_1 b_1 b_3 b_2 \\ c_2 c_3 c_2 c_1 c_6 \end{pmatrix} \right) = 24315.
\]

For any uniform block permutation \( f : A \rightarrow A' \), we denote by \( \xi_f \) the minimum permutation (for the lexicographic order) such that
\[
f = \xi_f \circ \text{Id}_A.
\]
We say that a biword \( w \) is \( f \)-compatible, and we write \( w \vdash f \), if
- \( \text{std}(w) = \xi_f \)
and
- if \( i \) and \( j \) are in the same block in \( A \), then the bi-letters \( w_i \) and \( w_j \) satisfy \( w_i \equiv w_j \).

We define
\[
G_f(A) = \sum_{w \vdash f} w.
\]

\textbf{Example 4.2.} Let \( f = \left( \begin{array}{c|c|c|c} 13 & 256 & 4 \\ \hline 15 & 234 & 6 \end{array} \right) \). The permutation \( \sigma = 125634 \) is the minimum permutation associated with \( f \) and we have
\[
G_f(A) = \sum_{\text{std}(c_1 c_2 \cdots c_6) = \sigma} \left( \begin{array}{c|c|c} b_1 b_1 b_3 b_2 \\ c_2 c_3 c_4 c_1 c_6 \end{array} \right).
\]
Theorem 4.3. The polynomials $G_f(A)$ provide a realization of the Hopf algebra $UBP$. That is,
\begin{equation}
G_f(A) \cdot G_g(A) = \sum_{h \in f \ast g} G_h(A)
\end{equation}
\begin{equation}
G_f(A + A') = \Delta(G_f),
\end{equation}
where $A + A'$ is defined as follows. $A'$ is an alphabet isomorphic to $A$ such that, for any bi-letter $w'$ of $A'$ and for any bi-letter $w$ of $A$, we have $w \preceq w'$.

Then, $A + A'$ denotes the disjoint union of $A$ and $A'$ endowed with the relations $\preceq$ and $\equiv$. As usual, we allow the bi-letters of $A$ and $A'$ to commute and identify $P(A)Q(A')$ with $P \otimes Q$.

Proof. Let us first prove that
\begin{equation*}
G_f(A) \cdot G_g(A) = \sum_{h \in f \ast g} G_h(A).
\end{equation*}
Writing $f = \xi_f \circ \text{Id}_A$ and $g = \xi_g \circ \text{Id}_B$, we have
\begin{align*}
G_f(A) \cdot G_g(A) &= \sum_{w : w \preceq f} w \sum_{w : w \preceq g} w \\
&= \sum_{u_1 \in B', v_1 \in C^*} \left( \begin{array}{c} u_1 \\ v_1 \end{array} \right) \sum_{u_2 \in B', v_2 \in C^*} \left( \begin{array}{c} u_2 \\ v_2 \end{array} \right) \\
&= \sum_{\sigma \in \xi_f \ast \xi_g} \sum_{u \in \mathcal{A} \times \mathcal{B} \text{-compatible}} \left( \begin{array}{c} u \\ v \end{array} \right).
\end{align*}

But, given two decompositions $f = \sigma \circ \text{Id}_A$ and $g = \tau \circ \text{Id}_B$, if $\sigma$ (resp. $\tau$) is the minimum permutation associated with $f$ (resp. $g$), then each permutation $\nu$ occurring in the product $\sigma \ast \tau$ is the minimum permutation associated with $\nu \circ \text{Id}_A \times \text{Id}_B$.

Let us now show the second point. Let $w$ be a biword occurring in $G_f(A + A')$. Let $i$ be the number of bi-letters of $w$ from the alphabet $A$ and let $\left( \begin{array}{c} u_1 u_2 \cdots u_i \\ v_1 v_2 \cdots v_i \end{array} \right) \in A^*$ be the sub-biword of $w$ on these letters. Then
\begin{align*}
\text{std}(v_1 v_2 \cdots v_i) &= (\xi_f)_{\{1,2,\ldots,i\}} \\
\text{and} \quad u_1 u_2 \cdots u_i &= \text{std}(A'_{\{\xi_f^{-1}(1),\xi_f^{-1}(2),\ldots,\xi_f^{-1}(i)\}})\text{-compatible}.
\end{align*}
This partition and the permutation $(\xi_f)_{\{1,2,\ldots,i\}}$ define a uniform block permutation $g$ whose minimum permutation is $(\xi_f)_{\{1,2,\ldots,i\}}$. Hence, the biword $\left( \begin{array}{c} u_1 u_2 \cdots u_i \\ v_1 v_2 \cdots v_i \end{array} \right)$ is a term appearing in $G_g(A)$. Similarly, the biword $w$ defined on the alphabet $A'$ appears in $G_h(A')$ for some uniform block permutation $h$. By construction, $G_g \otimes G_h$ is a term appearing in $\Delta(G_f)$.

Conversely, let $P \otimes Q$ be a term occurring in $\Delta(G_f)$. Let $w = \left( \begin{array}{c} u \\ v \end{array} \right)$ be the biword in $P(A) \omega Q(A')$ such that $\text{std}(w) = \xi_f$. The biword $u$ is $A'_{\{\xi_f^{-1}(1),\xi_f^{-1}(2),\ldots,\xi_f^{-1}(i)\}}\text{-compatible}$. Thus $u$ is $A$-compatible and $w$ appears in $G_f(A + A')$. 

A POLYNOMIAL REALIZATION OF THE HOPF ALGEBRA OF UNIFORM BLOCK PERMU TATIONS. 11

It remains to show that the family $G_f(A)$ is independent. To this aim, we need an order relation on the set partitions of $[n]$. Let $A = \{A_1, \cdots, A_k\}$ and $B = \{B_1, \cdots, B_l\}$ be two set partitions. We say that $B$ is finer than $A$, and we write $A \preceq B$, if and only if

$$\forall 1 \leq i \leq k \ (\exists H \subset \{1, \cdots, l\}, A_i = \bigcup_{j \in H} B_j).$$

Let $u \in B^*$ be a word. We set

$$P(u) = \bigcup_{a \in \text{Alph}(u)} \{\{i; u_i = a\}\}.$$  \ (4.5)

For example,

$$P(121212) = \{\{1, 3, 5\}, \{2, 4, 6\}\} \quad P(12\cdots n) = \{\{1\}, \{2\}, \cdots, \{n\}\}.$$

Let $f$ be a uniform block permutation. We denote by $w_f$ a maximal $f$-compatible biword for the relation $\preceq$. We define

$$\tilde{G}_f(A) = \sum_{w \in A^*: \text{std}(w) = \xi_f, \ P(w) = P(w)} w.$$  \ (4.6)

The family $\{\tilde{G}_f(A)\}_f$ is independent since any biword $w = \begin{pmatrix} u \\ v \end{pmatrix}$ appearing in $\tilde{G}_f(A)$ allows to reconstruct $f$:

$$f = \text{std}(v) \circ \text{Id}_{P(u)}.$$  \ (4.7)

But

$$G_f(A) = \sum_{w \in A^*: \text{std}(w) = \xi_f} w = \sum_{B \in \mathcal{P}: B \preceq P(w_f)} \sum_{w \in A^*: \ P(w) = B \ \text{and} \ \text{std}(w) = \xi_f} w.$$  

The partition $B \in \mathcal{P}$ and the permutation $\xi_f \in \mathcal{S}$ define a uniform block permutation $g$. Thus, we have

$$\tilde{G}_g(A) = \sum_{w \in A^*: \ P(w) = B \ \text{and} \ \text{std}(w) = \xi_f} w.$$  

Hence

$$G_f(A) = \sum_{B \preceq P(w_f)} \tilde{G}_g(A).$$  

Since the family $\{\tilde{G}_g(A)\}$ is independent, so is the family $\{G_f(A)\}$. \qed
Example 4.4.

\[
G\left(\begin{array}{c}1 \\ 1\end{array}\right)(A) \cdot G\left(\begin{array}{c}1 \\ 1\end{array}\right)(A) = \sum_{b \in B, c \in C} \left(\begin{array}{c}b \\ c\end{array}\right)^2 = \sum_{b_1, b_2 \in B, c_1, c_2 \in C} \left(\begin{array}{c}b_1 b_2 \\ c_1 c_2\end{array}\right)
\]
\[
= \sum_{b_1, b_2 \in B, c_1, c_2 \in C, c_1 \leq c_2} \left(\begin{array}{c}b_1 b_2 \\ c_1 c_2\end{array}\right) + \sum_{b_1, b_2 \in B, c_1, c_2 \in C, c_1 > c_2} \left(\begin{array}{c}b_1 b_2 \\ c_1 c_2\end{array}\right)
\]
\[
= G\left(\begin{array}{c}1 \\ 2\end{array}\right)(A) + G\left(\begin{array}{c}2 \\ 1\end{array}\right)(A).
\]

\[
G\left(\begin{array}{c}12 \\ 12\end{array}\right)(A) \cdot G\left(\begin{array}{c}1 \\ 1\end{array}\right)(A) = \sum_{b \in B, \text{std}(v) = 12} \left(\begin{array}{c}b \\ v\end{array}\right) \sum_{c \in C} \left(\begin{array}{c}b \\ c\end{array}\right) = \sum_{b_1, b_2, v_1, v_2} \left(\begin{array}{c}b_1 b_2 \\ v_1 v_2\end{array}\right)
\]
\[
= \sum_{c_1 \leq c_2 \leq c_3} \left(\begin{array}{c}b_1 b_1 b_2 \\ c_1 c_2 c_3\end{array}\right) + \sum_{c_1 \leq c_2 \leq c_3} \left(\begin{array}{c}b_1 b_1 b_2 \\ c_1 c_2 c_3\end{array}\right) + \sum_{c_1 \leq c_2 \leq c_3} \left(\begin{array}{c}b_1 b_1 b_2 \\ c_1 c_2 c_3\end{array}\right)
\]
\[
= G\left(\begin{array}{c}12 \\ 3\end{array}\right) + G\left(\begin{array}{c}12 \\ 3\end{array}\right) + G\left(\begin{array}{c}12 \\ 3\end{array}\right).
\]

\[
G\left(\begin{array}{c}13 \\ 13\end{array}\right)\left(\begin{array}{c}24 \\ 24\end{array}\right)(A + A') = \sum_{i, j, k} \left(\begin{array}{c}b_i b_j b_k \\ c_1 c_2 c_3 c_4\end{array}\right) + \sum_{i, j, k} \left(\begin{array}{c}b_i' b_j' b_k' \\ c_i' c_2' c_3' c_4'\end{array}\right)
\]
\[
= G\left(\begin{array}{c}13 \\ 24\end{array}\right)(A) + G\left(\begin{array}{c}13 \\ 24\end{array}\right)(A').
\]

5. Packed words

We recall the definitions of packed words and of the Hopf algebra \textbf{WQSym} (see, e.g., [10]). Let \( A := \{a_1 < a_2 < \cdots \} \) be a totally ordered infinite alphabet. The packed word \( \text{pack}(w) \) associated with a word \( w \in A^* \) is the word \( u \) obtained by the following process. If \( b_1 < b_2 < \cdots < b_r \) are the letters occurring in \( w \), \( u \) is the image of \( w \) by the homomorphism \( b_i \rightarrow a_i \). A word \( w \) is said to be packed if \( \text{pack}(w) = w \).

Example 5.1.

\( \text{pack}(51735514) = 41524413 \quad \text{pack}(772335) = 441223. \)

5.1. The combinatorial Hopf algebra \textbf{WQSym}. 
5.1.1. WQSym*. The Hopf algebra WQSym* is the $\mathbb{K}$-vector space generated by packed words endowed with the shifted shuffle product and the coproduct given by, using notations in [5]

\begin{align}
M_u \cdot M_v &:= \sum_{w \in u \cup v[\max(u)]} M_w, \\
\Delta(M_w) &:= \sum_{u \cdot v = w} M_{\text{pack}(u)} \otimes M_{\text{pack}(v)}.
\end{align}

Let $\psi$ be the map that associates with a uniform block permutation $f = \text{Id}_{A'} \circ \sigma$ of size $n$ a word $w$ of length $n$ constructed by packing the word $u$ obtained by the following process. The $i$th letter $u_i$ of $u$ is the minimum of the letters $\sigma_j$ which are in the same block of $A$ as $\sigma_i$.

Example 5.2.

\begin{align}
\psi(\text{Id}_{\{1,2\}} \circ 12) &= 11 \\
\psi(\text{Id}_{\{1,2\}} \circ 1324) &= 1212 \\
\psi(\text{Id}_{\{1,5\}} \circ 231) &= 221 \\
\psi(\text{Id}_{\{1,5\}} \circ 23415) &= 23411
\end{align}

The map $\psi$ induces a linear map from the vector space $\text{UBP}^*$ spanned by uniform block permutations to the one spanned by packed words WQSym* defined by:

\begin{equation}
\Psi: \text{UBP}^* \rightarrow \text{WQSym}^* 
F_f \mapsto M_{\psi(f)}
\end{equation}

Example 5.3.

\begin{align}
\psi(F_{12}) &= M_{11}^* \\
\psi(F_{123}) &= M_{221}^* \\
\psi(F_{1324}) &= M_{1212}^* \\
\psi(F_{12345}) &= M_{23411}^*
\end{align}

Proposition 5.4. Let $f$ and $g$ be two uniform block permutations. Then

\begin{equation}
\Psi(F_f \cdot F_g) = \Psi(F_f) \cdot \Psi(F_g).
\end{equation}

Proof. Let $M_w$ be a term in the left-hand side of (5.4) and $M_w = \Psi(F_h)$ for some uniform block permutation $h$ in the product $f \circ g$. The $i$th letter of $w$ is obtained from a permutation $\sigma$ and comes from a subset $E$ of $A \times B$ containing $\sigma_i$. But, if $w_i$ is less than the number of blocks in $A$, then $E$ is contained in $A$. So the word extracted from $w$ whose values are less than the number of blocks in $A$ is $\Psi(F_f)$. Similarly, the packed word extracted from $w$ whose values are strictly greater than the number of blocks in $A$ is $\Psi(F_g)$. Hence, each term in the left-hand side of (5.4) is in the right-hand side. Since the number of terms and the multiplicities are the same, we have equality. \qed

Example 5.5.

\begin{align}
\psi(F_{123}) \cdot F_{123} &= M_{11234} + M_{11324} + M_{13124} + M_{13142} + M_{3124} + M_{3142} + M_{31412} + M_{34112} \\
&= \Psi(F_{123}) \cdot \Psi(F_{123})
\end{align}
The map $\Psi$ cannot be a morphism of coalgebras as shown in the example
\[
\Psi \otimes \Psi \circ \Delta \left( F \left( \begin{array}{cc} 12 \\ 23 \\ 3 \\ 1 \end{array} \right) \right) = \Psi \otimes \Psi \left( 1 \otimes F \left( \begin{array}{cc} 12 \\ 23 \end{array} \right) + F \left( \begin{array}{c} 1 \\ 3 \end{array} \right) \right) + F \left( \begin{array}{cc} 12 \\ 23 \end{array} \right) \otimes 1 \\
= 1 \otimes M'_{M_{221}} + M'_{M_1} \otimes M'_{M_1} + M'_{M_{221}} \otimes 1 \\
\neq 1 \otimes M_{M_{221}} + M_1 \otimes M_{M_1} + M_1 \otimes M_{M_{221}} + M'_{M_{221}} \otimes 1 \\
\neq \Delta (M_{M_{221}}).
\]

The idea is that the coproduct on packed words deconcatenates the word in all suffixes and prefixes, whereas the coproduct on uniform block permutations preserves the blocks. This suggests to define another coproduct
\[
\Delta'(M'_u) := \sum_{u \in \text{Alph}(u) \cap \text{Alph}(v) = \emptyset} M'_v \otimes M'_{\text{pack}(v)}.
\]

Example 5.6.

\[
\Delta'(M'_{M_{221}}) = 1 \otimes M_{M_{221}} + M_1 \otimes M_{M_1} + M_{M_{221}} \otimes 1 \\
\Delta'(M'_{M_{331412}}) = 1 \otimes M_{M_{331412}} + M_1 \otimes M_{M_{14312}} + M_{M_{22131}} \otimes M_{M_1} + M_{M_331412} \otimes 1
\]

5.1.2. $\text{WQSym}^{++}$. We introduce then a combinatorial Hopf algebra based on packed words, endowed with the product and with the coproduct $\Delta'$, which will be denoted by $\text{WQSym}^{++}$.

Indeed, the coproduct $\Delta'$ is coassociative and, by (2.2), is compatible with the shifted shuffle. Indeed, let $u$ and $v$ be two packed words of respective length $m$ and $n$. Let $w$ be a packed word in the product $u \overline{\otimes} v$. We shall denote by $I(w)$ the set
\[
\{0 \leq i \leq m + n; \text{Alph}(w_1 \cdots w_i) \cap \text{Alph}(w_{i+1} \cdots w_{m+n}) = \emptyset\}.
\]

Since the letters appearing in $u$ and $v$ are distinct, if $k$ is in $I(w)$, then there exists $i$ in $I(u)$ and $j$ in $I(v)$ whose sum is $k$. Hence
\[
\Delta'(M_u \cdot M_v) = \sum_{w \in \text{Alph}(u) \otimes \text{Alph}(v) \cap I(w)} M_{\text{pack}(w_1 \cdots w_k)} \otimes M_{\text{pack}(w_{k+1} \cdots w_{m+n})}
\]
\[
= \sum_{0 \leq k \leq m+n} M_{\text{pack}(w_1 \cdots w_k)} \otimes M_{\text{pack}(w_{k+1} \cdots w_{m+n})}
\]
\[
= \sum_{0 \leq k \leq m+n} M_{\text{pack}(w)} \otimes M_{\text{pack}(w')}
\]
\[
= \Delta'(M_u) \cdot \Delta'(M_v).
\]

This Hopf algebra $\text{WQSym}^{++}$ is isomorphic to $\text{WQSym}^*$. This will be shown later. To this end, we introduce the following relation on packed words which will allow us to construct an isomorphism.

We denote, given a packed word $u$ and an integer $1 \leq i < \max(u)$, by $f(u, i)$ the packed word $v$ defined by
\[
v_j := \begin{cases} u_j - 1 & \text{if } u_j > i \\ u_j & \text{otherwise} \end{cases}
\]

Example 5.7.

\[
f(121, 1) = 111 \quad f(3122, 2) = 2111 \\
f(11122344, 2) = 1122233 \quad f(42315, 1) = 31214
\]
We extend the definition of iterated images by \( f \) by setting, given a packed word \( u \) and a subset \( I = \{ i_1 > i_2 > \cdots > i_k \} \) of \( \max (u) \):

\[
f(u, I) := \begin{cases} u & \text{if } I = \emptyset \\ f(\cdots f(u, i_1), \cdots, i_k) & \text{otherwise.}
\end{cases}
\]

We consider the relation, denoted by \( \preceq \), on packed words of the same size with the cover relation, given two packed words \( u \) and \( v \):

\[
(5.5) \quad u \preceq v \iff \begin{cases} \std(u) = \std(v) \\ \text{there exists } i \text{ such that } u = f(v, i) \end{cases}
\]

**Example 5.8.** Here is the Hasse diagram of packed words of size 3:

111

121 122 123 211 212 213 312 321

112 211 212 213 312 321

Let \( \Gamma \) defined by

\[
\Gamma : WQSym^* \longrightarrow WQSym^*
\]

\[
M_w^* \longmapsto \sum_{u \succeq w} M_u^*.
\]

**Example 5.9.** The image of a permutation is a permutation, the image of a word of length \( n \) consisting only of 1s is the sum of all non-decreasing packed words of length \( n \) and

\[
\Gamma(M_{2441235}^*) = M_{2441235}^* + M_{2551346}^* + M_{2451236}^* + M_{2561347}^*.
\]

**Proposition 5.10.** The linear map \( \Gamma \) is an isomorphism of Hopf algebras from \( WQSym^* \) to \( WQSym^* \).

**Proof.** To show that \( \Gamma \) is a morphism of algebras, that is

\[
(5.6) \quad \Gamma(M_w^* \cdot M_{w'}^*) = \Gamma(M_w^*) \cdot \Gamma(M_{w'}^*),
\]

we shall show that every packed word in the left-hand side of \( (5.6) \) is in the right-hand side. Since these sums are multiplicity free, it suffices to show the equality of the underlying sets. Consider two packed words \( w \) and \( w' \). Let \( u \succeq w \) and \( u' \succeq w' \). Then the set \( u \succeq w \) is equal to the set of the packed words \( v \succeq w \cdot w' \) such that

\[
(5.7) \quad v|\{1, \ldots, \max(u)\} = u \quad \text{pack}(v|\{\max(u) + 1, \ldots, \max(v)\}) = u'.
\]

Thus, the set \( \Gamma(M_w^*) \cdot \Gamma(M_{w'}^*) \) is equal to \( \Gamma(M_u^* \cdot M_{u'}^*) \).

Let us now prove that \( \Gamma \) is a morphism of coalgebras, that is to say

\[
(5.8) \quad \Gamma \odot \Delta(M_w^*) = \Delta' \odot \Gamma(M_w^*).
\]

To do this, we shall show that every packed word in the left-hand side of \( (5.8) \) is in the right-hand side and conversely. We shall conclude with an argument of multiplicity.

We consider a packed word \( w \). We set

\[
E_1 := \bigcup_{u \cdot v' = w} \{ u' \otimes v'; u' \geq \text{pack}(u), v' \geq \text{pack}(v) \},
\]

\[
E_2 := \bigcup_{u' \succeq w, v \cdot u' = w' \atop \text{Alph}(u') \cap \text{Alph}(v) = \emptyset} \{ u' \otimes v'; \text{pack}(u) = u', \text{pack}(v) = v' \}.
\]
so that
\[ \Gamma \otimes \Gamma \circ \Delta(M_u^*) = \sum_{u' \otimes v' \in E_1} M_{u'}^* \otimes M^*_v, \]
\[ \Delta' \circ \Gamma(M_u^*) = \sum_{u' \otimes v' \in E_2} M_{u'}^* \otimes M^*_v. \]

Show \( E_2 \subset E_1 \). Define \( F(w,u) \) as the word \( u \) given by \( u_i = w_j \) if \( w_j \leq n \) and \( u_i = w_j - 1 \) otherwise. Let \( u' \otimes v' \) in \( E_2 \). There exists \( w \preceq u' \) and \( w' = u \cdot v \) with \( \text{pack}(u) = u' \) and \( \text{pack}(v) = v' \). Then there exists \( I \) such that
\[ w = f(w', I) = F(w', I) = F(u, I) \cdot F(v, I). \]

But, there exists \( I_1 \) and \( I_2 \) (roughly, \( I_1 \) and \( I_2 \) are a relabelling of \( I \) caused by the packing) such that
\[ \text{pack}(F(u, I)) = f(\text{pack}(u), I_1) \quad \text{pack}(F(v, I)) = f(\text{pack}(v), I_2). \]

In particular, \( \text{pack}(F(u, I)) \preceq u' \) and \( \text{pack}(F(v, I)) \preceq v' \). Hence, \( u' \otimes v' \) is in \( E_1 \).

Show \( E_1 \subset E_2 \). Let \( u' \otimes v' \) in \( E_2 \). There exists two words \( u \) and \( v \) such that \( w = u \cdot v \) and \( u' \preceq \text{pack}(u), \, v' \preceq \text{pack}(v) \). It remains to be seen that there exists two words \( U \) and \( V \) such that
\[ \text{pack}(U) = u', \quad \text{pack}(V) = v', \]
\[ \text{Alph}(U) \cap \text{Alph}(V) = \emptyset, \]
\[ U \cdot V \succeq w. \]

We illustrate the algorithm step by step to find \( U \) and \( V \) on the following example. Let \( u = 311434141, \, v = 22441144 \) and \( w = u \cdot v \). Let \( u' = 411546263 \) and \( v' = 23451156 \) be two packed words such that \( \text{pack}(u) \preceq u' \) and \( \text{pack}(v) \preceq v' \).

The first step is to have two words \( U' \) and \( V' \) such that
\[ \text{pack}(u) = \text{pack}(U') \preceq u', \quad \text{pack}(v) = \text{pack}(V') \preceq v', \]
\[ \text{Alph}(U') \cap \text{Alph}(V') = \emptyset, \]
\[ U' \cdot V' \succeq w. \]

This is iteratively obtained by the following process. Given a word \( u \) and an integer \( i \), let \( g(u, i) \) be the word \( v \) defined by
\[ v_j = \begin{cases} u_j + 1 & \text{if } u_j \geq i \\ u_j & \text{otherwise} \end{cases}. \]

If \( i \) is a letter occurring in \( \text{Alph}(u) \cap \text{Alph}(v) \), then we consider the words \( \tilde{u} = g(u, i + 1) \) and \( \tilde{v} = g(v, i) \) and repeat this process with these words \( \tilde{u} \) and \( \tilde{v} \). At the end of the process, we have two words \( U' \) and \( V' \) satisfying the desired properties since \( g \) does not modify the packing. For example:
\[ u \cdot v = 311434141 \cdot 22441144 \]
\[ i = 1: \quad \tilde{u} \cdot \tilde{v} = 411545151 \cdot 33552255 \]
\[ i = 5: \quad U' \cdot V' = 411545151 \cdot 33662266 \]

The second step, from the words \( U' \) and \( V' \), is to have the words \( U'' \) and \( V'' \) such that
\[ \text{pack}(U'') = u', \quad \text{pack}(v) = \text{pack}(V'') \preceq v', \]
\[ \text{Alph}(U'') \cap \text{Alph}(V'') = \emptyset, \]
\[ U'' \cdot V'' \succeq w. \]

This is iteratively obtained by the following process. Since \( \text{pack}(U') \preceq u' \), there exists \( I \) such that \( f(\text{pack}(U'), I) = u' \). If \( I \) is empty, then \( \text{pack}(U') = u' \). Otherwise, let \( i \in I \). By incrementing the letters \( i \) occurring in \( g(U', i + 1) \) at the position \( j \) such that \( u'_j \neq i \), we obtain a word \( \tilde{u} \) satisfying \( \text{pack}(U') \preceq \text{pack}(\tilde{u}) \preceq u' \). We set \( \tilde{v} = g(V', i) \). We repeat this process with
these words $\tilde{u}$ and $\tilde{v}$. At the end of the process, we have two words $U''$ and $V''$ satisfying the desired properties since $g$ does not modify the packing. For example:

\[
\begin{array}{ccc}
U' \cdot V' & = & 411545151 \cdot 33662266 \\
i = 1 & : & \tilde{u} \cdot \tilde{v} = 511656262 \cdot 44733777 \\
i = 2 & : & \tilde{u} \cdot \tilde{v} = 611767273 \cdot 55884488 \\
i = 7 & : & U'' \cdot V'' = 611768283 \cdot 55994499
\end{array}
\]

The third step is similar to the second step. From the words $U''$ and $V''$, we obtain the words $U$ and $V$ satisfying the desired properties by referring the act on the word $U'$ (resp. $V'$) in the second step to $U''$ (resp. $U''$). For example:

\[
\begin{array}{ccc}
U'' \cdot V'' & = & 611768283 \cdot 55994499 \\
i = 5 & : & \tilde{u} \cdot \tilde{v} = 711879293 \cdot 56AA44AA \\
i = A & : & \tilde{u} \cdot \tilde{v} = 711879293 \cdot 56AB44BC \\
i = B & : & U \cdot V = 711879293 \cdot 56AB44BC
\end{array}
\]

where $A$, $B$ and $C$ respectively means the integer 10, 11 and 12. This proves the equality $E_1 = E_2$.

Since both members of the equality (5.8) are multiplicity free, we have equality.

Finally, by the M"obius inversion, $\Gamma$ is an isomorphism. The inverse, denoting by $\mu$ the M"obius function, is given by:

\[
\Gamma^{-1} : M_w^* \longrightarrow \sum_{u \geq w} \mu(\text{std}(u), u) M_u^* .
\]

Example 5.11.

\[
\Gamma(M_{i1} \cdot M_{i2}) = \Gamma(M_{i12} + M_{i21} + M_{211}) \\
= M_{i12} + M_{i21} + M_{i132} + M_{311} + M_{312} \\
= M_{i1} \cdot M_{i2} + M_{i2} \cdot M_{i1} \\
= \Gamma(M_{i1}) \cdot \Gamma(M_{i2})
\]

\[
\Gamma \circ \Gamma(\Delta(M_{i21})) = M_{i12} \otimes 1 + M_{i32} \otimes 1 + M_{i12} \otimes M_{i1} + M_{i2} \otimes M_{i21} + 1 \otimes M_{i21} + 1 \otimes M_{i32} \\
= \Delta'(M_{i32}) \\
= \Delta'(\Gamma(M_{i21})).
\]

\[
\Gamma^{-1}(M_{i11}) = M_{i11}^* - M_{i12}^* - M_{i22}^* + M_{i23}^* \\
\Gamma^{-1}(M_{i2134}) = M_{i2134}^* - M_{i21345}^* - M_{i2135}^* + M_{i2136}^*
\]

We can now return to the main result of this section, the linear map $\Psi$, defined in (5.3), is a surjective Hopf algebra morphism from $\text{UBP}^*$ to $\text{WQSym}^*$.

**Proposition 5.12.** The map $\Psi$ is a surjective Hopf algebra morphism from $\text{UBP}^*$ to $\text{WQSym}^*$.

**Proof.** That $\Psi$ is a morphism of algebras follows from Proposition 5.4.

Let us show that $\Psi$ is a morphism of coalgebras, that is

\[
\Psi \otimes \Psi \circ \Delta(F_f) = \Delta' \circ \Psi(F_f).
\]

Let $f$ be a uniform block permutation of size $n$, $f : A \longrightarrow A'$ and let $w$ be the word defined by $w_i = \min \{\{(\xi_f)_j) : \exists A \in A \text{ containing } i \text{ and } j\}$. 

Let us write \( w \) as the concatenation \( u \cdot v \) and \( \text{pack}(w) \) as the concatenation \( u' \cdot v' \) such that \( u \) and \( u' \) are of the same length. The sets of letters in \( u \) and \( v \) are disjoint if and only if \( |u| \) is in \( I(A) \). Moreover, by definition of \( w \), we have

\[
\Psi(F_{f_1 \cdots f_{|u|}}) = M_{\text{pack}(u)}^* = M_{\text{pack}(v')}^* \\
\Psi(F_{f_{|u|+1} \cdots f_n}) = M_{\text{pack}(v)}^* = M_{\text{pack}(v')}^*.
\]

Hence

\[
\Delta'(\Psi(F_f)) = \sum_{u' \cdot v' = \text{pack}(w)} M_{\text{pack}(u')}^* \otimes M_{\text{pack}(v')}^* \\
= \sum_{i \in I(A)} \Psi(F_{f_1 \cdots f_i}) \otimes \Psi(F_{f_{i+1} \cdots f_n}) \\
= \Psi \otimes \Psi \circ \Delta(F_f).
\]

Finally, let us prove that \( \Psi \) is surjective. Let \( w \) be a packed word. We denote by \( \mathcal{P}(w) \) the set partition of the set \( \{1, \ldots, n\} \)

\[
\mathcal{P}(w) = \bigcup_{0 \leq i \leq \max(w)} \{\{j; w_j = i\}\}.
\]

Then, the image, for the uniform block permutation \( f = \text{std}(w) \circ Id_{\mathcal{P}(w)} \), of \( F_f \) is \( M_{w}^* \).

**Example 5.13.**

\[
\Psi(F_{\left(\begin{array}{c} 12 \\ 12 \end{array}\right)} \cdot F_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)}) = \Psi(F_{\left(\begin{array}{c} 12 \\ 12 \end{array}\right)} + F_{\left(\begin{array}{c} 13 \\ 12 \end{array}\right)} + F_{\left(\begin{array}{c} 23 \\ 12 \end{array}\right)}) = M_{112} + M_{1}^{*121} + M_{211} \\
= M_{11}^* \cdot M_1^* \\
= \Psi(F_{\left(\begin{array}{c} 12 \\ 12 \end{array}\right)}) \cdot \Psi(F_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)}).
\]

We set \( f = \left(\begin{array}{c} 1 & 24 & 3 & 5 \\ 5 & 12 & 3 & 4 \end{array}\right)\).

\[
\Psi \otimes \Psi \circ \Delta(F_f) = 1 \otimes \Psi(F_f) + \Psi(F_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)}) \otimes \Psi(F_{\left(\begin{array}{c} 13 \\ 12 \end{array}\right)}) \\
+ \Psi(F_{\left(\begin{array}{c} 1 \\ 4 \\ 24 \\ 12 \end{array}\right)}) \otimes \Psi(F_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)}) + \Psi(F_f) \otimes 1 \\
= 1 \otimes M_{11213}^* + M_1^* \otimes M_{11213}^* + M_{3121}^* \otimes M_1^* + M_{41213}^* \otimes 1 \\
= \Delta'(M_{41213}^*) \\
= \Delta'(\Psi(F_f)).
\]

**5.2. WQSym.** The Hopf algebra \( \text{WQSym} \) is the \( K \)-vector space spanned by packed words endowed with the convolution product and the coproduct given by

\[
M_u \cdot M_v = \sum_{\text{pack}(u') = u \atop \text{pack}(v') = v} M_w,
\]

\[
\Delta(M_u) = \sum_{0 \leq k \leq \max(u)} M_{u_1(1, \ldots, k)} \otimes M_{\text{pack}(u_{(k+1, \ldots, \max(u))})}
\]
The dual $WQSym'$ of $WQSym^*$ is the Hopf algebra endowed with the product

\begin{equation}
M_u \cdot M_v = \sum_{\substack{w = u' \cdot v' \\
\text{pack}(u') = u \\
\text{pack}(v') = v \\
\text{Alph}(u') \cap \text{Alph}(v') = \emptyset}} M_w
\end{equation}

for any packed words $u$ and $v$, and with the coproduct $\Delta$. Indeed, if we denote by $w \in WQSym^*$ the dual basis to $w \in WQSym$, since the deconcatenation coproduct $\Delta'$ is multiplicity free, we have, by duality

\begin{equation}
M_u \cdot M_v = \sum_{u \otimes v \in \Delta'(w)} M_w.
\end{equation}

This is equivalent to Equation (5.12). The Hopf algebra $WQSym'^*$ and $WQSym^*$ are isomorphic, Proposition 5.10, so is for $WQSym'$ and $WQSym$. We can explicitly give this isomorphism. The map $\Gamma$ induces a dual map. This is

$\Gamma^*: M_w \mapsto -\sum_{u \preceq w} M_u$.

Indeed, if we denote by $M_\ast^\ast \in WQSym^*$ the dual basis to $M_\ast \in WQSym$, since $\Gamma(M_\ast^\ast)(u) = 1$ if $u \succeq w$, 0 otherwise, so is for $M_\ast \circ \Gamma^*$. For example,

$\Gamma^*(M_{1212334}) = M_{1212334} + M_{1212223} + M_{1212333} + M_{1212222}$.

**Proposition 5.14.** The map

$\Phi: WQSym' \rightarrow UBP$

$M_w \mapsto \sum_{f: \Psi(f) = w} G_f$

is an injective morphism of Hopf algebras.

**Proof.** The map $\Phi$ is the dual to $\Psi$. Since $\Psi(F_f)(M_w) = 1$ if $\Psi(f) = M_w$, $\Psi(F_f)(M_w) = 0$ otherwise. Hence, $F_f \circ \Phi(M_w) = 1$ if $\Psi(f) = M_w$ and $F_f \circ \Phi(M_w) = 0$ otherwise. $\square$

**Example 5.15.**

$\Phi(M_{121} \cdot M_1) = \Phi(M_{1213} + M_{1312} + M_{2321})$

$= (G_{(13 \ 12 \ 2 \ 4 \ 4)} + G_{(13 \ 12 \ 2 \ 4 \ 3)} + G_{(13 \ 12 \ 2 \ 4 \ 4)} + G_{(13 \ 12 \ 2 \ 4 \ 3)} + G_{(13 \ 12 \ 4 \ 4)})$

$+ (G_{(13 \ 13 \ 2 \ 4 \ 2)} + G_{(13 \ 14 \ 2 \ 3 \ 4)} + G_{(13 \ 23 \ 2 \ 4 \ 1)} + G_{(13 \ 24 \ 2 \ 3 \ 4)}),$

$= (G_{(13 \ 12 \ 2 \ 3)} + G_{(13 \ 13 \ 2 \ 2)} \cdot G_{(1)}$, $= \Phi(M_{121}) \cdot \Phi(M_1).$
\[ \Phi \otimes \Phi(\Delta(M_{2131})) = \Phi \otimes \Phi(1 \otimes M_{2131} + M_{11} \otimes M_{12} + M_{211} \otimes M_1 + M_{2131} \otimes 1) \\
= 1 \otimes \left( G_{124}^{234} + G_{124}^{124} \right) + \left( G_{124}^{234} + G_{124}^{124} \right) \otimes G_{124}^{124} \\
+ \left( G_{124}^{234} + G_{124}^{124} \right) \otimes 1 \\
= \Delta \left( G_{124}^{234} + G_{124}^{124} \right) \otimes 1 \\
= \Delta(\Phi(M_{2131})). \]

REFERENCES