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Accelerated Life Testing Model for a Generalized Birnbaum-Saunders Distribution

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Abstract—Fatigue failures caused by cyclic stresses are commonly modeled by Birnbaum-Saunders (B-S) and Weibull distributions. Sometimes, materials with high cycle fatigue exhibit bimodal failure rates which are difficult to model with Weibull distribution. The objective of this paper is to investigate a general Birnbaum-Saunders (GB-S) distribution which covers diverse hazard rates such as increasing, upside down, multimodal and others. This paper also utilizes GB-S distribution to model fatigue failure data and validate its performance under different conditions. GB-S-based-ALT model is also developed to provide accurate reliability and lifetime prediction at operating conditions using data from accelerated life testing experiments.

***Index Terms*- Birnbaum-Saunders Distribution, SB-S, GB-S, ALT, inverse power-law, general log-linear model, MLE**

I. Introduction

Fatigue, recognized as one of the main causes of failures of mechanical and electrical components, is a class of structural damage that occurs when material is exposed to cyclic application of stress with varying or constant amplitudes. Failure caused by fatigue in metallic structures is a pervasive phenomenon and failure of structural materials under cyclic application of stress or strain is now a problem of increasing interests of industry because most of the mechanical components work under cyclic stresses with varying or constant amplitudes during their lifetime of operation. A single static stress or strain, which is far below the threshold of the structure and causes no damage to the structure if applied once, could induce fatigue failure if applied repeatedly. Thus, failure data is important, however, obtaining failure data under design stress level to predict components' reliability and lifetime directly is not always feasible. This paper models fatigue failure observed in materials which are subject to single-type-multiple-level mechanical stress with a general Birnbaum-Saunders (GB-S) distribution which is more flexible in modeling diverse failure rates. Additionally, this paper investigates the application of accelerated life testing models to the GB-S distribution in order to estimate reliability at normal operating conditions.

A. Birnbaum-Saunders Life Model

Birnbaum and Saunders [1] propose the standard Birnbaum-Saunders (SB-S) distribution to model fatigue failure time of units when a dominant crack, which is caused by cyclic or other types of stresses, surpasses or reaches a predetermined crack length threshold. If T denotes a specimen's lifetime, the cumulative density function (*cdf*) of T is approximately given by:

$$\Pr\left(T \le t\right) = 1 - \Phi\left(\frac{\omega}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma}\right) = \Phi\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\omega}{\sigma\sqrt{t}}\right) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right] \quad (1)$$

where

$$\alpha = \frac{\sigma}{\sqrt{\omega\mu}},\ \beta = \frac{\omega}{\mu},\quad \alpha>0,\ \beta>0$$

α is the shape parameter and β is the scale parameter of the distribution, ω is the failure threshold level, σ is the standard deviation of the failure time data and Φ is the cumulative normal distribution function. The expected time to failure, variance, skewness and kurtosis of this distribution are investigated by Birnbaum and Saunders [1]. Desmond [2] notes that the SB-S distribution applies even when the assumption of SB-S is relaxed, that is, crack increment in a certain cycle not only depends on the current loading but also is affected by the total crack size caused by previous cycles. The *k*th moment of SB-S distribution can be obtained by the moment generating function (MGF) as stated by Rieck [3]. It is shown that the SB-S hazard function is unimodally upside-down and the functional approximation of the changing point is given by Kundu [4].

Statistical analysis for the SB-S distribution is also developed. Since exact distributions of the MLEs are not available, Engelhardt and Wright [5] present asymptotic

distributions to construct confidence intervals of the parameters. Two modified moment estimators that both improve moment estimation (MME) and maximum likelihood estimation (MMLE) are proposed. MME and MMLE exhibit their own pros and cons under different conditions as studied by Kundu and Balakrishnan [6]. Dupuis and Mills [7] provide robust estimation of the parameters and quantiles of SB-S distribution since in practice the collected data do not always follow the SB-S model. The inference procedure for the SB-S distribution with symmetrically incomplete data is derived by Desmond [2] since in practice it is common to end a life testing before all units under test fail. Desmond [8] develops a log-linear model based on the SB-S distribution which considers random effects and studies the performance of various estimation and prediction methods.

SB-S distribution is considered as one of the normal distribution family and its relationship with similar distributions is discussed and investigated by Desmond [8], Kundu [9], etc. Desmond [2] states that SB-S distribution is more flexible than Inverse Gaussian (IG) distribution, whereas the IG distribution seems to have applications for incomplete data but SB-S distribution has difficulty in incorporating such data.

To generalize the SB-S model, two GB-S distributions are proposed by introducing a second shape parameter. One of the GB-S distributions proposed by Owen [10] considers the effect of sequence of loading and the crack extension. Owen [11] also proposes another GB-S distribution which is discussed in details in this paper. This GB-S distribution builds relationship between the lifetime *T* and standard normal variable *Z* as:

$$Z = \frac{1}{\alpha}\left[\left(\frac{T}{\beta}\right)^{\lambda} - \left(\frac{\beta}{T}\right)^{\lambda}\right] \tag{2}$$

where α and λ are the shape parameters and β is the scale parameter of this GB-S distribution.

Univariate and multivariate extensions of the SB-S distribution are given in Díaz-García and Dominguez-Molina [12]. Also, Díaz-García and Leiva-Sánchez [13] use a biological model to provide a more general derivation of SB-S distribution which yields a B-S distribution family incorporating lognormal and other distributions.

B. Acceleration Model

Statistics-based models are usually used to the failure times of fatigue failure data due to its ability in capturing the random variation of failure times due to different patterns or levels of cyclic forces. To model fatigue failure data, researchers usually utilize the normal distribution family to provide an accurate description of the failure time distribution. The SB-S distribution is derived from the normal distribution family. Compared with Weibull, lognormal and other distributions which fit failure data well especially within the central region of the distribution, SB-S distribution has been shown to provide an accurate description of failure data especially under low stress levels. Nevertheless, predicting reliability using fatigue data at normal operating conditions might not be feasible due to the extensive time and resource needed. Therefore, testing units at accelerated or other conditions and utilizing the failure times observed at different levels of stresses to predict the lifetime at operating levels of stress is an appropriate alternative approach. This type of test is termed an accelerated life test (ALT). A model which relates reliability and lifetime under severe conditions to normal environments is called an ALT model. The stress is not only referred to the mechanical force but also includes other types of “stresses”, such as humidity, voltage, temperature, etc.

Elsayed [14] classifies ALT models mainly into several groups. The most widely used are the parametric models (statistics-based) and the physics-statistics-based models. The parametric models assume that failure time at different stress levels are related to each other by a common failure time distribution with different parameters (usually a mean or scale parameters). The life-stress function, which is the function of applied stress, substitutes the scale parameters for different levels of stresses while the shape parameter remains the same. Accelerated failure time (AFT) model is one of typical parametric models.

The physics-statistics-based models explain the relationship between applied stress and failure rate by utilizing the parameters of the physics of the device in conjunction with the statistical parameters to obtain realistic models. The general log-linear relationship is a general life-stress relationship which incorporates other models, for example, the Arrhenius model, the inverse power law model and the Eyring model. Such life-stress relationships can be applied in a specified underlying distribution and has the effect of changing the mean or scale of the failure distribution, but the shape parameters remain the same. Often, applying life-stress relationships to a distribution increases the number of the unknown parameters.

The inverse power law model is applied to the SB-S distribution and the corresponding inference procedures are investigated by Owen [15].

II. Generalized Birnbaum-Saunders model

The hazard function of SB-S distribution is restricted to be unimodal which fails to cover a wide range of failure rate types as mentioned above. This paper investigates a generalized B-S model that overcomes the limitation of SB-S distribution in modeling diverse failure rates. As stated in Eq. 2, with the second shape parameter λ introduced, the *cdf* of the GB-S model is:

$$F(t;\alpha,\beta,\lambda) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\lambda} - \left(\frac{\beta}{t}\right)^{\lambda}\right]\right\} \tag{3}$$

Let $f(t;\alpha,\beta,\lambda)$ and $R(t;\alpha,\beta,\lambda)$ represent the probability density function (*pdf*) and reliability function of GB-S

distribution, respectively, the hazard function can be obtained as:

$$h(t;\alpha,\beta,\lambda)=\frac{f(t;\alpha,\beta,\lambda)}{R(t;\alpha,\beta,\lambda)}$$
$$=\frac{\frac{\lambda}{\sqrt{2\pi}\alpha t}\left[\left(\frac{t}{\beta}\right)^{\lambda}+\left(\frac{\beta}{t}\right)^{\lambda}\right]e^{\left\{-\frac{1}{2a^2}\left[\left(\frac{t}{\beta}\right)^{2\lambda}+\left(\frac{\beta}{t}\right)^{2\lambda}-2\right]\right\}}}{1-\Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\lambda}-\left(\frac{\beta}{t}\right)^{\lambda}\right]\right\}} \quad (4)$$

Note that SB-S is a special case of the GB-S distribution with $\lambda=0.5$.

A. Characteristics and Properties

With Eq. 2 and the general binomial theorem, the transformation between *T* and *Z* is achieved as:

For $Z=-\frac{2}{\alpha}$ or $\frac{2}{\alpha}$

$$\left(\frac{T}{\beta}\right)^{\gamma}=\sum_{k=0}^{\infty}\binom{\frac{r}{\lambda}}{k}2^{\frac{r}{2\lambda}-\frac{k}{2}}\left(\frac{\alpha Z}{2}\right)^{k}$$

For $-\frac{2}{\alpha}<Z<\frac{2}{\alpha}$

$$\left(\frac{T}{\beta}\right)^{\gamma}=\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\binom{\frac{r}{\lambda}}{k}\binom{\frac{r}{2\lambda}-\frac{k}{2}}{s}\left(\frac{\alpha}{2}\right)^{2s+k}Z^{2s+k}$$

For $Z<-\frac{2}{\alpha}$ or $Z>\frac{2}{\alpha}$

$$\left(\frac{T}{\beta}\right)^{\gamma}=\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\binom{\frac{r}{\lambda}}{k}\binom{\frac{r}{2\lambda}-\frac{k}{2}}{s}\left(\frac{\alpha}{2}\right)^{\frac{r}{\lambda}-2s}Z^{\frac{r}{\lambda}-2s}$$

Thus, the *r*th moment of the GB-S model is:

$$E\left(T^{r}\right)=\beta^{r}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\binom{\frac{r}{\lambda}}{k}\binom{\frac{r}{2\lambda}-\frac{k}{2}}{s}\cdot\left(I_1+I_2\right) \quad (5)$$

where

$$I_1=2^{\frac{-2s-k-1}{2}}\left[\Gamma\left(\frac{2s+k+1}{2}\right)-\Gamma\left(\frac{2s+k+1}{2},\frac{2}{a^2}\right)\right]\cdot\left[(-1)^{2s+k}+1\right]$$

$$I_2=\frac{\alpha^{\frac{r}{\lambda}-2s}}{2^{\frac{\frac{r}{\lambda}-2s+1}{2}}}\left[(-1)^{\frac{r}{\lambda}-2s}+1\right]\Gamma\left(\frac{\frac{r}{\lambda}-2s+1}{2},\frac{2}{\alpha^2}\right)$$

It is worth noting that when $Z<-\frac{2}{\alpha}$, the moments of *T* do not exist. The expectation, variance, skewness and kurtosis are obtained as the special case of moments. The skewness and kurtosis are not affected by the scale parameter β. Figure 1 reveals that when λ becomes smaller, the kurtosis increases sharply.

The *pdf* of GB-S distributions remains unimodal for different values of λ. Another important observation about the inverse of *T* is that: if *T* belongs to GB-S distribution with parameters α,β and λ, then T^{-1} also belongs to GB-S distribution with the corresponding parameters α,β^{-1} and λ, respectively. The skewness and kurtosis of T^{-1} are the same as the skewness and kurtosis of T.

B. GB-S Hazard Rate

Figure 2 shows GB-S hazard rates with constant α and varying λ since β, the scale parameter of the distribution has no effect on the shape so it is fixed at unity for all cases. Also for all value of α, the hazard rate function is always upside-down when $\lambda=0.5$. In general, the hazard function of GB-S distribution covers three types of failure conditions: the hazard function can be either increasing or multimodal when $\lambda>0.5$; the hazard function can be either upside-down or multimodal when $\lambda<0.5$ and the hazard function is always an upside down function of *t* when $\lambda=0.5$.

III. ACCELERATED MODELS

Although the GB-S distribution is more flexible in covering different types of failure rates, its performance in modeling failure data, especially accelerated failure data, needs to be investigated. This section begins with the development of GB-S-based accelerated models, as well as Weibull and SB-S accelerated models, either in specific or general forms. In section IV, these models are applied to the experimental data and their performances are compared.

A. Inverse Power Law-Based-Accelerated Model

Once a baseline lifetime distribution with scale parameter or mean is adopted and an approporiate acceleration form is selected according to the applied stress type, the unknown parameters can be estimated by observing failure times at

elevated stress levels which are then used to predict reliability at normal operating conditions.

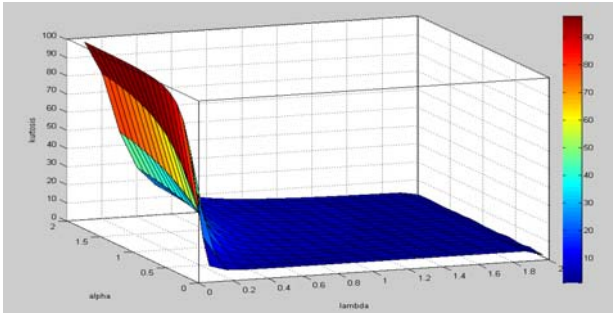

Fig.1. GB-S kurtosis for different α and λ

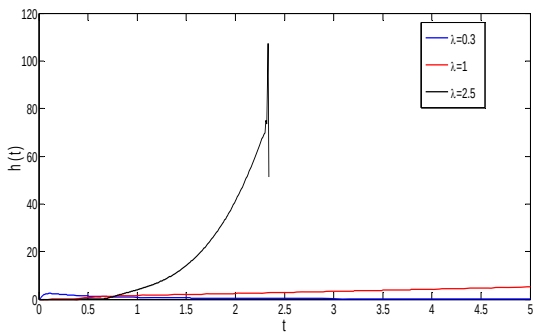

Fig.2. GB-S hazard rates with constant α and varied λ

The inverse power law model is commonly used for non-thermal accelerated stresses and is given as:

$$h(V) = \gamma \cdot V^{-\eta}, \quad \gamma > 0,\ \eta > 0 \tag{6}$$

where

h(V) is a quantifiable life measure, such as mean life or characteristic life

V represents the stress level

γ, η are model parameters.

By substituting the scale parameter β with the accelerated life model $h(V)$, the accelerated inverse power law GB-S model can then be written as:

$$F(t;V) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\gamma V^{-\eta}}\right)^{\lambda} - \left(\frac{t}{\gamma V^{-\eta}}\right)^{-\lambda}\right]\right\} \tag{7}$$

$$f(t;V) = \frac{\lambda}{\sqrt{2\pi}at}\left[\left(\frac{t}{\gamma V^{-\eta}}\right)^{\lambda} + \left(\frac{\gamma V^{-\eta}}{t}\right)^{\lambda}\right] \exp\left\{-\frac{1}{2a^2}\left[\left(\frac{t}{\gamma V^{-\eta}}\right)^{2\lambda} + \left(\frac{t}{\gamma V^{-\eta}}\right)^{-2\lambda} - 2\right]\right\} \tag{8}$$

The estimation of the unknown model parameters in Eq. 7 can be obtained by maximizing the likelihood function for the observed accelerated failure data. Assuming two stress levels V_1 and V_2 are applied and the corresponding two failure time data sets are obtained The likelihood function is obtained as:

$$L(\gamma,\eta,\alpha,\lambda;t_{ij},V_i) = \prod_{i=1}^{2}\prod_{j=1}^{n_i} f\left(\gamma,\eta,\alpha,\lambda;t_{ij},V_i\right) \tag{9}$$

where

j is the *j*th failure data in the corresponding data set

n_i is the number of observations in *i*th data set

t_{ij} represents the *j*th failure observation in the data set obtained under *i*th stress level

V_i is the *i*th stress level

In this paper, data are obtained from the Instrument Development Unit of the Physical Research Staff, Boeing Aircraft Company, by subjecting metal-coupons to repeated alternating stresses and strains. The three data sets obtained under three stress levels are listed below,

Sample 1 (Stress/cycle: 2.1×10^4 psi):

3.70, 7.06, 7.06, 7.46, 7.85, 7.97, 8.44, 8.55, 8.58, 8.86, 8.86, 9.30, 9.60, 9.88, 9.90, 10.00, 10.10, 10.16, 10.18, 10.20, 10.55, 10.85, 11.02, 11.02, 11.08, 11.15, 11.20, 11.34, 11.40, 11.99, 12.00, 12.00, 12.03, 12.22, 12.35, 12.38, 12.52, 12.58, 12.62, 12.69, 12.70, 12.90, 12.93, 13.00, 13.10,13.13, 13.15, 13.30, 13.55, 13.90, 14.16, 14.19, 14.20, 14.20, 14.50, 14.52, 14.75, 14.78, 14.81, 14.85, 15.02, 15.05, 15.13, 15.22, 15.22, 15.30, 15.40, 15.60, 15.67, 15.78, 15.94, 16.02, 16.04, 16.08, 16.30, 16.42, 16.74, 17.30, 17.50, 17.50, 17.63, 17.68, 17.81, 17.82, 17.92, 18.20, 18.68, 18.81, 18.90, 18.93, 18.95, 19.10, 19.23, 19.40, 19.45, 20.23, 21.00, 21.30, 22.15, 22.68, 24.40

Sample 2 (Stress/cycle: 2.6×10^4 psi):

2.33,2.58,2.68,2.76,2.90,3.10,3.12,3.15,3.18,3.21,3.21,3.29,
3.35,3.36,3.38,3.38,3.42,3.42,3.42,3.44,3.49,3.50,3.50,3.51,
3.51,3.52,3.52,3.56,3.58,3.58,3.60,3.62,3.63,3.66,3.67,3.70,
3.70,3.72,3.72,3.74,3.75,3.76,3.79,3.79,3.80,3.82,3.89,3.89,
3.95,3.96,4.00,4.00,4.00,4.03,4.04,4.06,4.08,4.08,4.10,4.12,
4.14,4.16,4.16,4.16,4.20,4.22,4.23,4.26,4.28,4.32,4.32,4.33,
4.33,4.37,4.38,4.39,4.39,4.43,4.45,4.45,4.52,4.56,4.56,4.60,
4.64,4.66,4.68,4.70,4.70,4.73,4.74,4.76,4.76,4.86,4.88,4.89,
4.90,4.91,5.03,5.17,5.40,5.60

Sample 3 (Stress/cycle: 3.1×10^4 psi):

0.7,0.9,0.96,0.97,0.99,1.00,1.03,1.04,1.04,1.05,1.07,1.08,
1.08,1.08,1.09,1.09,1.12,1.121.13,1.14,1.14,1.14,1.16,1.19,
1.20,1.20,1.20,1.21,1.21,1.23,1.24,1.24,1.24,1.24,1.24,1.28,
1.28,1.29,1.29,1.30,1.30,1.30,1.31,1.31,1.31,1.31,1.31,1.32,
1.32,1.32,1.33,1.34,1.34,1.34,1.34,1.34,1.36,1.36,1.37,1.38,
1.38,1.38,1.39,1.39,1.41,1.41,1.42,1.42,1.42,1.42,1.42,1.42,
1.44,1.44,1.45,1.46,1.48,1.48,1.49,1.51,1.51,1.52,1.55,1.56,
1.57,1.57,1.57,1.57,1.58,1.59,1.62,1.63,1.63,1.64,1.66,1.66,
1.68, 1.70,1.74,1.96,2.12

To examine the performance of inverse power law GB-S model, failure data from any two of the three samples can be utilized to estimate the unknown parameters of the model and these estimated parameters can be used to predict the reliability under design stress. The estimated reliability is then compared with the theoretical reliability (observed data set).

We use data sets 1 and 2 to estimate the unknown parameters. The log-likelihood function of the inverse power law GB-S model can be written as:

$$l = 203\log\lambda - 203\log\alpha + \sum_{j=1}^{101}\log\left[\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{\lambda} + \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-\lambda}\right] + \sum_{j=1}^{102}\log\left[\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{\lambda} + \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-\lambda}\right] - \sum_{j=1}^{101}\frac{\left[\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{\lambda} - \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-\lambda}\right]^2}{2\alpha^2} - \sum_{j=1}^{102}\frac{\left[\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{\lambda} - \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-\lambda}\right]^2}{2\alpha^2}$$

Taking partial derivatives of the log-likelihood function with respect to α, λ, γ and η yields the following four equations:

$$\frac{\partial l}{\partial \alpha} = -\frac{203}{\alpha} + \frac{1}{\alpha^3}\sum_{j=1}^{101}\left[\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{\lambda} - \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-\lambda}\right]^2 + \frac{1}{\alpha^3}\sum_{j=1}^{102}\left[\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{\lambda} - \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-\lambda}\right]^2$$

$$\frac{\partial l}{\partial \lambda} = \frac{203}{\lambda} + \sum_{j=1}^{101}\frac{\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{\lambda} - \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-\lambda}}{\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{\lambda} + \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-\lambda}}\log\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right) + \sum_{j=1}^{102}\frac{\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{\lambda} - \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-\lambda}}{\left(\frac{t_{1j}}{\gamma V_2^{-\eta}}\right)^{\lambda} + \left(\frac{t_{1j}}{\gamma V_2^{-\eta}}\right)^{-\lambda}}\log\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right) - \sum_{j=1}^{101}\frac{\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{2\lambda} - \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-2\lambda}}{\alpha^2}\log\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right) - \sum_{j=1}^{102}\frac{\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{2\lambda} - \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-2\lambda}}{\alpha^2}\log\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)$$

$$\frac{\partial l}{\partial \gamma} = -\frac{\lambda}{\gamma}\sum_{j=1}^{101}\frac{\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{\lambda} - \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-\lambda}}{\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{\lambda} + \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-\lambda}} - \frac{\lambda}{\gamma}\sum_{j=1}^{102}\frac{\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{\lambda} - \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-\lambda}}{\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{\lambda} + \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-\lambda}} + \frac{\lambda}{\alpha^2\gamma}\sum_{j=1}^{101}\left[\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{2\lambda} - \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-2\lambda}\right] + \frac{\lambda}{\alpha^2\gamma}\sum_{j=1}^{102}\left[\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{2\lambda} - \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-2\lambda}\right] \quad (10)$$

$$\frac{\partial l}{\partial \eta} = \lambda \log V_1 \sum_{j=1}^{101} \frac{\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{\lambda} - \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-\lambda}}{\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{\lambda} + \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-\lambda}}$$
$$+ \lambda \log V_2 \sum_{j=1}^{102} \log\left[\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{\lambda} + \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-\lambda}\right]$$
$$- \frac{\lambda \log V_1}{\alpha^2} \sum_{j=1}^{101} \left[\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{2\lambda} - \left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^{-2\lambda}\right]$$
$$- \frac{\lambda \log V_2}{\alpha^2} \sum_{j=1}^{102} \left[\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{2\lambda} - \left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^{-2\lambda}\right]$$

Newton's iterative method is applied to solve the partial derivatives of the log likelihood functions. The iteration ends when the estimate converges. Usually for a nonlinear equation, there exists more than one local optimal solution. These solutions are returned to the likelihood function and the global optimal value is obtained accordingly.

To compare the performance of GB-S distribution with other models, the inverse power law Weibull accelerated model and the inverse power law SB-S accelerated model are developed as follow.

The Weibull accelerated model:

$$F(t;V) = 1 - e^{-\left(\frac{t}{\gamma V^{-\eta}}\right)^k} \tag{11}$$

$$f(t;V) = \left(\frac{t}{\gamma V^{-\eta}}\right)^{k-1} \frac{k}{\gamma V^{-\eta}} e^{-\left(\frac{t}{\gamma V^{-\eta}}\right)^k} \tag{12}$$

Taking the logarithm of the likelihood function and partial derivates with respect to the unknown parameters of Weibull accelerated model we obtain as:

$$\begin{aligned} l &= 203\log k - 203\log\gamma + 101\eta\log V_1 \\ &+ 102\log V_2 + (k-1)\sum_{i=1}^{2}\sum_{j=1}^{n_1} t_{ij} - 203(k-1)\log\gamma \\ &+ 101(k-1)\eta\log V_1 + 102(k-1)\eta\log V_2 \\ &- \sum_{j=1}^{101}\left(\frac{t_{1j}}{\gamma V_1^{-\eta}}\right)^k - \sum_{j=1}^{102}\left(\frac{t_{2j}}{\gamma V_2^{-\eta}}\right)^k \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{\partial l}{\partial k} &= \frac{203}{k} + \sum_{i=1}^{2}\sum_{j=1}^{n_1} t_{ij} - 203\log\gamma \\ &+ 101\eta\log V_1 + 102\log\eta\log V_2 \\ &- \sum_{j=1}^{101}\log\left(\frac{t_{1j}V_1^{\eta}}{\gamma}\right)\cdot\left(\frac{t_{1j}V_1^{\eta}}{\gamma}\right)^k - \sum_{j=1}^{102}\log\left(\frac{t_{2j}V_2^{\eta}}{\gamma}\right)\cdot\left(\frac{t_{2j}V_2^{\eta}}{\gamma}\right)^k \end{aligned}$$

$$\begin{aligned} \frac{\partial l}{\partial \gamma} &= -\frac{203}{\gamma} - 203(k-1)\frac{1}{\gamma} \\ &+ \sum_{j=1}^{101}\frac{k}{\gamma}\cdot\left(\frac{t_{1j}V_1^{\eta}}{\gamma}\right)^k + \sum_{j=1}^{102}\frac{k}{\gamma}\cdot\left(\frac{t_{2j}V_2^{\eta}}{\gamma}\right)^k \end{aligned}$$

$$\begin{aligned} \frac{\partial l}{\partial \eta} &= 101k\log V_1 + 102k\log V \\ &- \sum_{j=1}^{101}\frac{k\eta}{V_1}\cdot\left(\frac{t_{1j}V_1^{\eta}}{\gamma}\right)^k - \sum_{j=1}^{102}\frac{k\eta}{V_2}\cdot\left(\frac{t_{2j}V_2^{\eta}}{\gamma}\right)^k \end{aligned}$$

The parameters estimation procedure is similar to that of GB-S. The details of the power law SB-S accelerated model are given in Owen [15].

B. A General Log-linear Acceleration Form

The inverse power law accelerated model is limited to modeling the relationship between lifetime and mechanical stress. To examine the GB-S accelerated model in a more general case, an exponential form of life-stress relationship, incorporating the inverse power law model, is considered:

$$h(\underset{\sim}{z}) = \exp(a_0 + \underset{\sim}{a}\underset{\sim}{z}) \tag{14}$$

where

$\underset{\sim}{z}$: the vector of stresses

a_0 : the model parameter

$\underset{\sim}{a}$: vector of regression coeffients

When single stress type is applied, above acceleration form can be written as:

$$h(z_1) = \exp(a_0 + a_1 z_1) \tag{15}$$

Furthermore, when $\exp(a_0) = \gamma$, $\exp(a_1 z) = V^{-\eta}$, the exponential model yields the inverse-power accelerated model. Substituting the scale parameter of each baseline life distribution, we obtain the general acceleration form based Weibull, SB-S and GB-S distributions respectively, as

$$F_{Weibull}(t;V) = 1 - e^{-\left(\frac{t}{\exp(a_0 + a_1 z_1)}\right)^k} \tag{16}$$

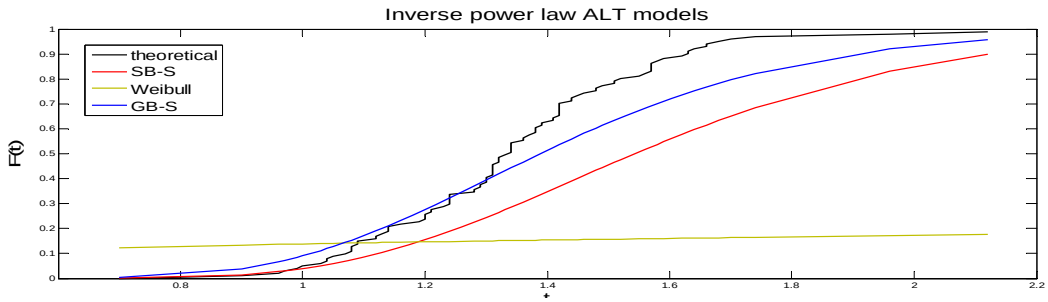

Fig.3.Predicted *cdf* of inverse power law based Weibull, SB-S, GB-S models and theoretical *cdf*

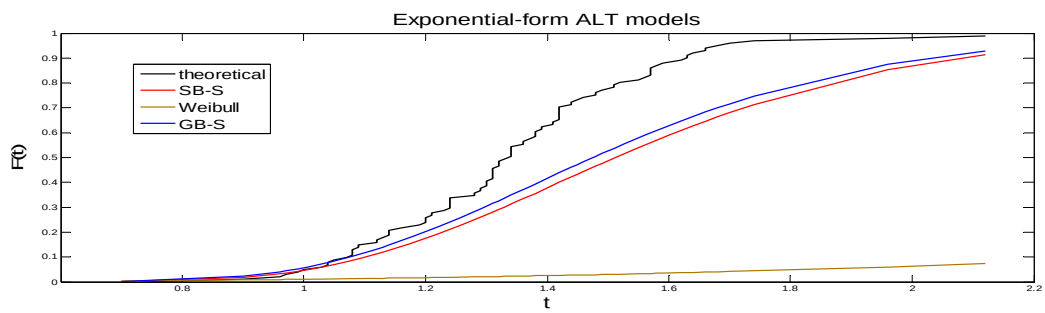

Fig.4. Predicted *cdf* of exponential form based Weibull, SB-S, GB-S models and theoretical *cdf*

$$F_{SB-S}(t;V) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\exp(a_0 + a_1 z_1)}\right)^{\frac{1}{2}} - \left(\frac{t}{\exp(a_0 + a_1 z_1)}\right)^{-\frac{1}{2}}\right]\right\} \quad (17)$$

$$F_{GB-S}(t;V) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\exp(a_0 + a_1 z_1)}\right)^{\lambda} - \left(\frac{t}{\exp(a_0 + a_1 z_1)}\right)^{-\lambda}\right]\right\} \quad (18)$$

The likelihood functions and partial derivates with respect to each model's unknown parameters can be found respectively.

IV. COMPARISON

As discussed earlier, an ALT model can be used to estimate reliability performance under the desired stress level by utilizing failure data obtained at different stress levels to obtain the parameters of the model. In this section, we consider all the scenarios where any two of the data sets are utilized to estimate the parameters of each proposed model and the predicted reliabilities of each model are compared with the third data set.

Applying the data sets 1 and 2 to each model and comparing with the third data set, we obtain the following estimated parameters and the sum of squared errors (SSE) between the observed and estimated reliabilities for each model are obtained as summarized in Table I:

Table I. Parameters and SSEs of each accelerated model

Inverse power law Weibull accelerated model	
$\hat{k} = 0.347,\ \hat{\gamma} = 1775.6,\ \hat{\eta}=1.756$	SSE=19.918
Inverse power law SB-S accelerated model	
$\hat{\alpha} = 0.249,\ \hat{\gamma} = 821.186,\ \hat{\eta}=5.548$	SSE=5.089
Inverse power law GB-S accelerated model	
$\hat{\alpha} = 0.164, \hat{\lambda}=0.333, \hat{\gamma} = 1045.834, \hat{\eta}=5.855$	SSE=1.003
General Weibull accelerated model	
$\hat{k} = 2.618,\ \hat{a}_0 = 3.852,\ \hat{a}_1 = -1.868$	SSE=30.328
General SB-S accelerated model	
$\hat{\alpha} = 0.248,\ \hat{a}_0 = 6.674,\ \hat{a}_1 = -5.533$	SSE=4.032
General GB-S accelerated model	
$\hat{\alpha} = 0.258, \hat{\lambda}=0.522, \hat{a}_0 = 6.758, \hat{a}_1 = -5.629$	SSE=2.896

The Weibull accelerated model results in the largest SSEs for all scenarios implying that its prediction as an Accelerated Failure Time (AFT) model for these fatigue data is inaccurate. The GB-S accelerated model has the smallest SSEs for both inverse power law and the general cases.

For the general accelerated models, the estimates of the SB-S and GB-S's shape parameters are almost identical (estimates of λ are close to 0.5). For the inverse power law accelerated models, there exist significant differences among the three models in terms of SSE and estimates of parameters. Clearly,

GB-S model provides the most accurate prediction among all models.

Similarly, data sets 1 and 3 are used to obtain the parameters of the models which are then used for reliability prediction at the stress level of data set 2. Likewise, data sets 2 and 3 are used to obtain the parameters of the models which are then used for reliability prediction at the stress level of data set 1. The GB-S shows slightly better performance than the SB-S in terms of SSE. The estimates of the parameters of the GB-S and SB-S accelerated models are almost identical for the general (exponential) case. The ratios of the estimated shape parameters (α / λ) of the SB-S and GB-S accelerated models are close to the inverse-power-law case.

V. Conclusions

SB-S models are widely accepted to model fatigue failure data. However, it is limited in modeling different failure types of failure rates. In this paper, we generalize BS distribution and develop two GB-S-based-ALT models. Their performances are compared with SB-S-based-ALT models and Weibull-based-ALT models using several multiple sets of experiment data. The results show that the GB-S-based-ALT model outperforms the three accelerated. Extensive analysis of different data sets show that developed GB-S ALT model can be used to provide accurate reliability prediction for fatigue and wear out data.

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