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Local nonsmooth Lyapunov pairs for first-order evolution differential inclusions

Samir Adly · Abderrahim Hantoute · Michel Théra

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Abstract The general theory of Lyapunov's stability of first-order differential inclusions in Hilbert spaces has been studied by the authors in a previous work [2]. This new contribution focuses on the natural case when the maximally monotone operator governing the given inclusion has a domain with nonempty interior. This setting permits to have nonincreasing Lyapunov functions on the whole trajectory of the solution to the given differential inclusion. It also allows some more explicit criteria for Lyapunov's pairs. Some consequences to the viability of closed sets are given, as well as some useful cases relying on the continuity or/and convexity of the involved functions. Our analysis makes use of standard tools from convex and variational analysis.

Keywords Evolution differential inclusions, lower semi-continuous functions, invariant sets, proximal subdifferential, Clarke subdifferential, Fréchet subdifferential, limiting proximal subdifferential, maximally monotone operator, strong solution, weak solution, Lyapunov pair.

Mathematics Subject Classification (2000) 37B25, 47J35, 93B05

1 Introduction and notations

In various applications modeled by ODE's, one may be forced to work with systems that have non-differentiable solutions. Also, Lyapunov's functions, that is positive definite functions whose decay along the trajectories of the system, which are used to establish a stability property of the system, may be nondifferentiable. The need to extend the classical differentiable Lyapunov's stability to the nonsmooth case is unavoidable when studying stability properties of discontinuous systems. In practice, many systems in physics, engineering, biology etc exhibit generally nonsmooth energy functions, which are usually a typical candidates for Lyapunov functions; thus elements of nonsmooth analysis become essential [3, 16, 18, 25]. A typical example is given by the case of piecewise linear dynamical systems called *Linear Complementarity Systems* (LCS) for which the analysis of asymptotic and exponential stability uses a piecewise quadratic Lyapunov function [18]. Let us remind that LCS are defined as follows:

$$LCS(A, B, C, D) \quad \begin{cases} \dot{x}(t; x_0) = Ax(t) + Bu(t), & x(t_0) = x_0, \\ 0 \leq u(t) \perp Cx + Du \geq 0, \end{cases}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$ and $D \in \mathbb{R}^{m \times m}$ are real matrices, x_0 is the initial condition, \dot{x} is the time derivative of the trajectory $x(t)$ and $a \perp b$ means that the two vectors a and b are orthogonal. Linear and nonlinear complementarity problems belong to the more general mathematical formalism of *Differential Variational Inequalities* (DVI), introduced by J.S. Pang and D. Stewart [21]. It is a combination of an ordinary differential equation (ODE) with

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a variational inequality or a complementarity constraint. A DVI consists to find trajectories $t \mapsto x(t)$ and $t \mapsto u(t)$ such that

$$DVI(f, F, K) \quad \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & x(t_0) = x_0, \\ \langle F(t, x(t), u(t)), v - u(t) \rangle \geq 0, & \forall v \in K, u(t) \in K \text{ for a.e. } t \geq t_0, \end{cases}$$

where K is a closed convex subset of a Hilbert space H , f and F are given mappings. When K is a closed convex cone, then problem $DVI(f, F, K)$ is equivalent to a Differential Complementarity Problem (DCP):

$$DCP(f, F, K) \quad \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & x(t_0) = x_0, \\ K \ni u(t) \perp F(t, x(t), u(t)) \in K^*, & \text{a.e. } t \geq t_0. \end{cases}$$

Since DVI and DCP formalisms unify several known mathematical problems such that ordinary differential equations with discontinuous right-hand term, differential algebraic equations, dynamic complementarity problems etc .. (see [9, 10] for more details), it was proved to be powerful for the treatment of many problems in science and engineering such that: unilateral contact problems in mechanics, finance, traffic networks, electrical circuits etc According also to the fact that LCS formalism has many of applications in various areas including for instance robotics, economics, finance, non smooth mechanics, etc (see Camlibel, Pang and Shen, [18] and the monograph by Facchinei and Pang, [17]), it has received recently a great interest from the mathematical programming and control communities from the theoretical and numerical point of view.

Instead of considering LCSs or DVIs, throughout this contribution we are interested in the general framework of infinite-dimensional dynamical systems, that is systems of the form:

$$\dot{x}(t; x_0) \in f(x(\cdot; x_0)) - Ax(\cdot; x_0), \quad x_0 \in \text{cl}(\text{Dom } A) \text{ a.e. } t \geq 0. \quad (1.1)$$

Here, and thereafter, $\text{cl}(\text{Dom } A)$ is the closure of the domain of a maximally monotone operator $A : H \rightrightarrows H$ defined on a real Hilbert space H , possibly nonlinear and multivalued with domain $\text{Dom } A$ and f is a Lipschitz continuous mapping defined on $\text{cl}(\text{Dom } A)$.

A pair of proper lower semicontinuous (lsc for short) functions $V, W : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to form a Lyapunov pair for (1.1) if for all $x_0 \in \text{cl}(\text{Dom } A)$ the solution of (1.1), in a sense that will be precised in Section 3, denoted by $x(\cdot; \cdot, x_0)$ satisfies

$$V(x(t; x_0)) - V(x(s; x_0)) + \int_s^t W(x(\tau; x_0)) d\tau \leq 0 \text{ for all } t \geq s \geq 0. \quad (1.2)$$

Observe that when $W \equiv 0$ one recovers the classical notion of Lyapunov functions; e.g., [27]. More generally, instead of (1.2), we are going to consider functions V, W satisfying for some $a \geq 0$

$$e^{at}V(x(t; x_0)) - e^{as}V(x(s; x_0)) + \int_s^t W(x(\tau; x_0)) d\tau \text{ for all } t \geq s \geq 0.$$

In this case, the (weighted) pair (V, W) will be referred to as a a -Lyapunov pair. The main motivation in using a -Lyapunov pairs instead of simply functions is that many stability concepts for the equilibrium sets of (1.1), namely stability, asymptotic or finite-time stability, can be obtained just by choosing appropriate functions W in (1.2). The weight e^{at} is useful for instance when exponential stability is concerned. So, even in autonomous systems like those of (1.1), the function W or the weight e^{at} may be of a certain utility since, in some sense, it emphasizes the decreasing of the Lyapunov function V .

The method of Lyapunov functions is a corner stone of the study of the controllability and stabilizability of control systems. Its history is rich and has been described in several places and various seminal contributions has been made to the subject. We refer to Clarke [14, 15] for an overview of the recent developments of the theory where he pointed out that for nonlinear systems, Lyapunov's method turns out to be essential to consider nonsmooth Lyapunov functions, even if the underlying control dynamics are themselves smooth.

Over the years, among the various contributions, Kocan & Soravia [19], characterized Lyapunov's pairs in terms of viscosity solutions of a related partial differential inequality.

Another well-established approach consists of characterizing Lyapunov's pairs by means of the contingent derivative of the maximally monotone operator A , see for instance Cârjă & Motreanu [11], for the case of a maximally linear monotone operator and also when A is a multivalued m -accretive operator on an arbitrary Banach space [12]. In these approaches the authors used tangency and flow-invariance arguments combined with a priori estimates and approximation.

The starting point of this contribution is the paper by Adly & Goeleven [1] in which smooth Lyapunov functions were used in the framework of the second order differential equations, and non-linear mechanical systems with frictional unilateral constraints.

In this article we provide a different approach that don't make use of viscosity solutions or contingent derivatives associated to the operator A . Our objective is to emphasize our previous contribution [2] to the setting where the involved

maximally monotone operator has a domain with nonempty interior. This case includes the finite dimensional framework since in this case the relative interior of the domain of the operator is always nonempty. Moreover, the criteria for Lyapunov's pairs are checked only in the interior of the domain (or the relative interior) instead of the closure of the whole domain as in [1]. In contrary to [1], this setting also ensures obtaining global Lyapunov's pairs and permits in this way to control the whole trajectory of the solution to the given differential inclusion.

The summary of the paper is as follows. In Section 2 we introduce the main tools and basic results used in the paper. In Section 3 we give a new primal and dual criteria for lower semicontinuous Lyapunov pairs. This is achieved in Proposition 2 and Theorem 3.1. In Section 4, we make a review of some old and recent criteria for Lyapunov pairs. Section 5 is dedicated to complete the proofs of the main results given in Section 3.

2 Notation and main tools

Throughout the paper, H is a (real) Hilbert space endowed with the inner (or scalar) product $\langle \cdot, \cdot \rangle$ and the associated norm is denoted by $\|\cdot\|$. We identify H^* (the space of continuous linear functionals defined on H) to H , and we denote the weak limits (w -lim, for short) by the symbol \rightharpoonup to distinguish it from the usual symbol \rightarrow used for strong limits. The zero vector in H is denoted by θ .

We start this section by reviewing some notations used throughout the paper. Given a nonempty set $S \subset H$ (or $S \subset H \times \mathbb{R}$), by $\text{co}S$, $\text{cone}S$, and $\text{aff}S$, we denote the *convex hull*, the *conic hull*, and the *affine hull* of the set S , respectively. Moreover, $\text{Int}S$ is the topological *interior* of S , and $\text{cl}S$ and \overline{S} are indistinctly used for the *closure* of S (with respect to the norm topology on H). We also use $\text{cl}^w S$ or \overline{S}^w when we deal with the closure of S with respect to the weak topology. We note $\text{ri}S$ the (topological) *relative interior* of S , i.e., the interior of S in the topology relative to $\text{aff}S$ whatever this set is nonempty (see [23, Chapter 6] for more on this fundamental notion). For $x \in H$ (or $x \in H \times \mathbb{R}$), $\rho \geq 0$, $B_\rho(x)$ is the open ball with center x and radius ρ , and $\overline{B}_\rho(x)$ is the closure of $B_\rho(x)$, while $B := B_1(\theta)$ stands for the unit open ball. For $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ we denote $[a, b)$ the interval closed at a and open at b ($(a, b]$, (a, b) , ... are defined similarly); hence $\mathbb{R}_+ := [0, \infty)$. Finally, for $\alpha \in \mathbb{R}$, we note α^+ for $\max\{0, \alpha\}$.

Our notation is the standard one used in convex and variational analysis and in monotone operator theory; see, e.g., [8, 24]. The *indicator function* of S is the function defined as

$$I_S(x) := \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{otherwise.} \end{cases}$$

The *distance function* to S is denoted by

$$d(x, S) := \inf\{\|x - y\| \mid y \in S\},$$

and the *orthogonal projection* on S , π_S , is defined as

$$\pi_S(x) := \{y \in S \mid \|x - y\| = d(x, S)\}.$$

If S is closed and convex, $S_\infty \subset H$ (or $H \times \mathbb{R}$) denotes its *recession cone*:

$$S_\infty := \{y \mid x + \lambda y \in S \text{ for some } x \text{ and all } \lambda \geq 0\},$$

while, $S^\circ \subset H$ (or $H \times \mathbb{R}$) denotes the *polar* of S given by

$$S^\circ := \{y \mid \langle y, v \rangle \leq 1 \text{ for all } v \in S\}.$$

Given a function $\varphi : H \rightarrow \overline{\mathbb{R}}$, its (*effective*) *domain* and *epigraph* are defined by

$$\text{Dom } \varphi := \{x \in H \mid \varphi(x) < +\infty\},$$

$$\text{epi } \varphi := \{(x, \alpha) \in H \times \mathbb{R} \mid \varphi(x) \leq \alpha\}.$$

For $\lambda \in \mathbb{R}$, the *open sublevel set* of φ at λ is

$$[\varphi > \lambda] := \{x \in H \mid \varphi(x) > \lambda\};$$

$[\varphi \geq \lambda]$, $[\varphi \leq \lambda]$, and $[\varphi < \lambda]$ are defined similarly. We say that φ is *proper* if $\text{Dom } \varphi \neq \emptyset$ and $\varphi(x) > -\infty$ for all $x \in H$. We say that φ is *convex* if $\text{epi } \varphi$ is convex, and (weakly) *lower semicontinuous* (*lsc*, for short) if $\text{epi } \varphi$ is closed with respect to the (weak topology) norm-topology on H . We denote

$$\mathcal{F}(H) := \{\varphi : H \rightarrow \overline{\mathbb{R}} \mid \varphi \text{ is proper and lsc}\},$$

$$\mathcal{F}_w(H) := \{\varphi : H \rightarrow \overline{\mathbb{R}} \mid \varphi \text{ is proper and weakly lsc}\};$$

$\mathcal{F}(H; \mathbb{R}_+)$ and $\mathcal{F}_w(H; \mathbb{R}_+)$ stand for the subsets of nonnegative functions of $\mathcal{F}(H)$ and $\mathcal{F}_w(H)$, respectively.

As maximally monotone set-valued operators play an important role in this work, it is useful to recall some of basic definitions and some of their properties. More generally, they have frequently shown themselves to be a key class of objects in both modern Optimization and Analysis; see, e.g., [4–6, 8, 24, 26].

For an operator $A : H \rightrightarrows H$, the *domain* and the *graph* of A are given respectively by

$$\text{Dom } A := \{z \in H \mid Az \neq \emptyset\} \text{ and } \text{gph } A := \{(x, y) \in H \times H \mid y \in Ax\};$$

for notational simplicity we identify the operator A to its graph. The *inverse operator* of A , denoted by A^{-1} , is defined as

$$(y, x) \in A^{-1} \iff (x, y) \in A.$$

We say that an operator A is *monotone* if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \text{ for all } (x_1, y_1), (x_2, y_2) \in A,$$

and *maximally monotone* if A is monotone and has no proper monotone extension (in the sense of graph inclusion). If A is maximally monotone, it is well known (e.g., [26]) that $\overline{\text{Dom } A}$ is convex, and Ax is convex and closed for every $x \in \text{Dom } A$. Moreover, if $\text{Int}(\text{Dom } A) \neq \emptyset$, then $\text{Int}(\text{Dom } A)$ is convex, $\text{Int}(\text{Dom } A) = \text{Int}(\overline{\text{Dom } A})$, and A is bounded locally on $\text{Int}(\text{Dom } A)$. Note that the domain or the range of a maximally monotone operator may fail to be convex, see, e.g., [24, page 555]. In particular, if A is the subdifferential $\partial\varphi$ of some lower semicontinuous (lsc for short) convex and proper function $\varphi : H \rightarrow \overline{\mathbb{R}}$, then A is a classical example of a maximally monotone operator, as is a linear operator with a positive symmetric part. We know that

$$\text{Dom } A \subset \text{Dom } \varphi \subset \overline{\text{Dom } \varphi} = \overline{\text{Dom } A}.$$

For $x \in \text{Dom } A$, we shall use the notation $(Ax)^\circ$ to denote the *principal section* of A , i.e., the set of points of minimal norm in Ax .

Nonsmooth and variational analysis play a central role in this study. Hence, we need to recall briefly some concepts used through the paper. More details can be found for instance in [7, 13, 16, 20, 24]. We assume that $\varphi \in \mathcal{F}(H)$, and take $x \in \text{Dom } \varphi$.

A vector $\xi \in H$ is called a *proximal subgradient* of φ at x , written $\xi \in \partial_P\varphi(x)$, if there are $\rho > 0$ and $\sigma \geq 0$ such that

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle - \sigma \|y - x\|^2 \text{ for all } y \in B_\rho(x);$$

the *domain* of $\partial_P\varphi$ is then given by

$$\text{Dom } \partial_P\varphi := \{x \in H \mid \partial_P\varphi(x) \neq \emptyset\}.$$

The set $\partial_P\varphi(x)$ is convex, possibly empty and not necessarily closed.

A vector $\xi \in H$ is called a *Fréchet subgradient* of φ at x , written $\xi \in \partial_F\varphi(x)$, if

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle + o(\|y - x\|).$$

Associated to proximal and Fréchet subdifferentials, limiting objects have been introduced. A vector $\xi \in H$ belongs to the *limiting proximal subdifferential* of φ at x , written $\partial_L\varphi(x)$, if there exist sequences $(x_k)_{k \in \mathbb{N}}$ and $(\xi_k)_{k \in \mathbb{N}}$ such that $x_k \xrightarrow[\varphi]{\varphi} x$ (that is, $x_k \rightarrow x$ and $\varphi(x_k) \rightarrow \varphi(x)$), $\xi_k \in \partial_P\varphi(x_k)$ and $\xi_k \rightarrow \xi$.

A vector $\xi \in H$ is called a *horisontal subgradient* of φ at x , written $\xi \in \partial_\infty\varphi(x)$, if there exist sequences $(\alpha_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$, $(x_k)_{k \in \mathbb{N}}$ and $(\xi_k)_{k \in \mathbb{N}}$ such that $\alpha_k \rightarrow 0^+$, $x_k \xrightarrow[\varphi]{\varphi} x$, $\xi_k \in \partial_P\varphi(x_k)$ and $\alpha_k \xi_k \rightarrow \xi$.

The *Clarke subdifferential* of φ at x is defined by the following so-called *representation formula*; see, e.g., Morukhovich [20] and Rockafellar [24],

$$\partial_C\varphi(x) = \overline{\text{co}}^w \{\partial_L\varphi(x) + \partial_\infty\varphi(x)\}.$$

From a geometrical point of view, if $S \subset H$ is closed and $x \in S$, the *proximal normal cone* to S at x is

$$N_S^P(x) := \partial_P I_S(x).$$

We also denote by $\tilde{N}_S^P(x)$ the subset of $N_S^P(x)$ given by

$$\tilde{N}_S^P(x) := \{\xi \in H \mid \langle \xi, y - x \rangle \leq \|y - x\|^2 \text{ for all } y \in S \text{ closed to } x\}.$$

It can be proved; e.g., [13], that

$$N_S^P(x) = \begin{cases} \text{cone}(\pi_S^{-1}(x) - x), & \text{if } \pi_S^{-1}(x) \neq \emptyset, \\ \{\theta\} & \text{if } \pi_S^{-1}(x) = \emptyset, \end{cases}$$

where $\pi_S^{-1}(x) := \{y \in H \setminus S \mid x \in \pi_S(y)\}$.

Similarly, $N_S^L(x) := \partial_L I_S(x)$ ($= \partial_\infty I_S(x)$) is the *limiting normal cone to S at x*, and $N_S^C(x) := \overline{\text{co}}^w \{N_S^L(x)\}$ is the *Clarke normal cone to S at x*.

In that way, the above subdifferentials of $\varphi \in \mathcal{F}(H)$ can be geometrically described as

$$\begin{aligned}\partial_P \varphi(x) &= \{\xi \in H \mid (\xi, -1) \in N_{\text{epi } \varphi}^P(x, \varphi(x))\}, \\ \partial_\infty \varphi(x) &= \{\xi \in H \mid (\xi, 0) \in N_{\text{epi } \varphi}^P(x, \varphi(x))\}.\end{aligned}$$

We call *contingent cone to S at x* (or the *Bouligand tangent cone*), written $T_S(x)$, the cone given by

$$T_S(x) := \{\xi \in H \mid x + \tau_k \xi_k \in S \text{ for some } \xi_k \rightarrow \xi \text{ and } \tau_k \rightarrow 0^+\}.$$

The Dini directional derivative of the function $\varphi \in \mathcal{F}(H)$ at $x \in \text{Dom } \varphi$ in the direction $v \in H$ is given by

$$\varphi'(x, v) = \liminf_{t \rightarrow 0^+, w \rightarrow v} \frac{\varphi(x + tw) - \varphi(x)}{t}.$$

Hence, $\text{epi } \varphi'(x, \cdot) = T_{\text{epi } \varphi}(x, \varphi(x))$. The *Gâteaux derivative* of φ at x is a linear continuous form on H , written $\varphi'_G(x)$, satisfying

$$\lim_{t \rightarrow 0^+} \frac{\varphi(x + tv) - \varphi(x)}{t} = \langle \varphi'_G(x), v \rangle \quad \text{for all } v \in H.$$

We close this section by giving some properties of the subdifferential sets defined above that will be used later on. First, it follows easily from the definitions that

$$\partial_P \varphi(x) \subset \partial_F \varphi(x) \subset \partial_L \varphi(x) \subset \partial_C \varphi(x).$$

If φ is convex, then

$$\partial_P \varphi(x) = \partial_C \varphi(x) = \partial \varphi(x),$$

where $\partial \varphi(x)$ is the usual *Moreau-Rockafellar* subdifferential of φ at x :

$$\partial \varphi(x) := \{\xi \in H \mid \varphi(y) - \varphi(x) \geq \langle \xi, y - x \rangle \text{ for all } y \in H\}.$$

If $\varphi \in \mathcal{F}(H)$ is Gâteaux-differentiable at $x \in \text{Dom } \varphi$, we have

$$\partial_P \varphi(x) \subset \{\varphi'_G(x)\} \subset \partial_C \varphi(x).$$

If φ is C^1 then

$$\partial_P \varphi(x) \subset \{\varphi'(x)\} = \partial_C \varphi(x) \text{ and } \partial_\infty \varphi(x) = \{\theta\}.$$

If φ is C^2 then

$$\partial_P \varphi(x) = \partial_C \varphi(x) = \{\varphi'(x)\}.$$

In particular, if $\varphi := d(\cdot, S)$ with $S \subset H$ closed, for $x \in S$ we have that

$$\partial_C \varphi(x) = N_S^C(x) \cap B,$$

while, for $x \notin S$ such that $\partial_P \varphi(x) \neq \emptyset$, $\pi_S(x)$ is a singleton and (e.g., [16])

$$\partial_P \varphi(x) = \frac{x - \pi_S(x)}{\varphi(x)};$$

hence

$$\partial_L \varphi(x) = \left\{ w - \lim_k \frac{x_k - \pi_S(x_k)}{\varphi(x)}; x_k \rightarrow x \right\}.$$

More generally, we have that

$$N_S^P(x) = \mathbb{R}_+ \partial_P d_S(x) \text{ and } N_S^C(x) = \overline{\mathbb{R}_+ \partial_C d_S(x)}^w$$

(with the convention that $0 \cdot \emptyset = \{\theta\}$).

Finally, we recall that $\varphi \in \mathcal{F}(\mathbb{R})$ is nonincreasing if and only if $\xi \leq 0$ for every $\xi \in \partial_P \varphi(x)$ and $x \in \mathbb{R}$, (e.g., [16]). We shall use the following version of the Gronwall Lemma (e.g., [1, Lemma 1]).

Lemma 1 *Given $t_2 > t_1 \geq 0$, $a \neq 0$, and $b \geq 0$, we assume that an absolutely continuous function $\psi : [t_1, t_2] \rightarrow \mathbb{R}_+$ satisfies*

$$\psi'(t) \leq a\psi(t) + b \quad \text{a.e. } t \in [t_1, t_2].$$

Then, for all $t \in [t_1, t_2]$,

$$\psi(t) \leq \left(\psi(t_1) + \frac{b}{a}\right)e^{a(t-t_1)} - \frac{b}{a}.$$

3 Local characterization of Lyapunov pairs on the interior of the domain of A

In this section we provide the desired explicit criterion for lower semicontinuous (weighted-) Lyapunov pairs associated to the differential inclusion (1.1):

$$\dot{x}(t; x_0) \in f(x(\cdot; x_0)) - Ax(\cdot; x_0), \quad x_0 \in \text{cl}(\text{Dom } A),$$

where $A : H \rightrightarrows H$ is a maximally monotone operator and $f : \text{cl}(\text{Dom } A) \subset H \rightarrow H$ is a Lipschitz continuous mapping. Recall that for fixed $T > 0$ and $x_0 \in \text{cl}(\text{Dom } A)$, a *strong solution* of (1.1), $x(\cdot; x_0) : [0, T] \rightarrow H$, is a uniquely defined absolute continuous function which satisfies $x(0; x_0) = x_0$ together with (see, e.g., [8])

$$\dot{x}(t; x_0) \in L_{\text{loc}}^\infty((0, T], H), \quad (3.3)$$

$$x(t; x_0) \in \text{Dom } A, \quad \text{for all } t > 0, \quad (3.4)$$

$$\dot{x}(t; x_0) \in f(x(t; x_0)) - Ax(t; x_0), \quad \text{a.e. } t \geq 0. \quad (3.5)$$

Existence of strong solutions is known to occur if for instance:

- $x_0 \in \text{Dom } A$, $\text{Int}(\text{co}(\text{Dom } A)) \neq \emptyset$;
- $\dim H < \infty$;
- or if $A \equiv \partial\varphi$ where $\varphi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lsc extended-real-valued convex proper function.

Moreover, we have that $\dot{x}(\cdot; x_0) \in L^\infty([0, T], H)$ if and only if $x_0 \in \text{Dom } A$. In this later case, $x(\cdot; x_0)$ is derivable from right at each $s \in [0, T]$ and

$$\frac{d^+ x(\cdot; x_0)}{t}(s) = f(x(s; x_0)) - \pi_{Ax(s; x_0)}(f(x(s; x_0))).$$

The strong solution also satisfies the so-called semi-group property,

$$x(s; x(t; x_0)) = x(s + t; x_0) \quad \text{for all } s, t \geq 0, \quad (3.6)$$

together with the relationship

$$\|x(t; x_0) - x(t; y_0)\| \leq e^{L_f t} \|x_0 - y_0\|$$

whenever $t \geq 0$ and $x_0, y_0 \in \text{cl}(\text{Dom } A)$; hereafter, L_f denotes the Lipschitz constant of the mapping f on $\text{cl}(\text{Dom } A)$.

In the general case, it is well established that (1.1) admits a unique *weak solution* $x(\cdot; x_0) \in C(0, T; H)$ which satisfies $x(t; x_0) \in \text{cl}(\text{Dom } A)$ for all $t \geq 0$. More precisely, there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \text{Dom } A$ converging to x_0 such that the strong solution $x_k(\cdot; z_k)$ of the equation

$$\dot{x}_k(t; z_k) \in f(x(t; z_k)) - Ax_k(t; z_k), \quad x_k(0, z_k) = z_k, \quad (3.7)$$

converges uniformly to $x(\cdot; x_0)$ on $[0, T]$. Moreover, we have that

$$x(s; x(t; x_0)) = x(s + t; x_0) \quad \text{for all } s, t \geq 0 \quad (3.8)$$

(called the semigroup property). If L_f denotes the Lipschitz constant of f on $\text{cl}(\text{Dom } A)$, then for every $t \geq 0$ and $x_0, y_0 \in \text{cl}(\text{Dom } A)$ we have that

$$\|x(t; x_0) - x(t; y_0)\| \leq e^{L_f t} \|x_0 - y_0\|.$$

In the remaining part of the paper, $x(\cdot; x_0)$ denotes the weak solution of Equation (1.1) (which is also, a strong one whenever a strong solution exists.)

From now on, we suppose throughout this section that

$$\boxed{\text{Int}(\text{co}(\text{Dom } A)) \neq \emptyset.}$$

Hence, $\text{Int}(\text{Dom } A)$ is convex, $\text{Int}(\text{Dom } A) = \text{Int}(\text{co}(\text{Dom } A)) = \text{Int}(\text{cl}(\text{Dom } A))$, and A is locally bounded on $\text{Int}(\text{Dom } A)$. Therefore, a (unique) strong solution of (1.1) always exists [8]. We have the following technical lemma, adding more information about the qualitative behavior of this solution.

Lemma 2 *Let $\bar{y} \in \text{Dom } A$ and $\rho > 0$ be such that $B_\rho(\bar{y}) \subset \text{Int}(\text{co}(\text{Dom } A))$. Then, $M := \sup_{z \in B_\rho(\bar{y})} \|(f(z) - Az)^\circ\| < \infty$ and for all $y \in B_\rho(\bar{y})$ and $t \leq 1$ we have that*

$$\left\| \frac{d^+ x(\cdot; y)}{dt}(t) \right\| \leq e^{L_f t} M.$$

Proof By virtue of the semi-group property (3.6), the following inequality holds for all $y \in \text{cl}(\text{Dom } A)$ and $0 \leq t < s$ (e.g., [8, Lemma 1.1])

$$\|x(t+s; y) - x(t; y)\| = \|x(t; x(s; y)) - x(t; y)\| \leq e^{L_f t} \|x(s; y) - y\|. \quad (3.9)$$

In particular, for $y \in B_\rho(\bar{y})$ and $t \leq 1$ we get that

$$\begin{aligned} \left\| \frac{d^+ x(\cdot; y)}{dt}(t) \right\| &= \lim_{s \downarrow 0} s^{-1} \|x(t+s; y) - x(t; y)\| \leq e^{L_f t} \lim_{s \downarrow 0} s^{-1} \|x(s; y) - y\| \\ &= e^{L_f t} \left\| \frac{d^+ x(\cdot; y)}{dt}(0) \right\| \\ &= e^{L_f t} \|(f(y) - Ay)^\circ\| \leq e^{L_f t} M. \end{aligned}$$

The fact that M is finite follows from the maximal monotonicity of A together with the Lipschitz continuity of f . \triangle

Definition 1 Let be given functions $V \in \mathcal{F}(H)$, $W \in \mathcal{F}(H; \overline{\mathbb{R}}_+)$ and a number $a \in \mathbb{R}_+$. We say that (V, W) forms a a -Lyapunov pair for (1.1) with respect to a set $D \subset \text{cl}(\text{Dom } A)$ if for all $y \in D$ we have that

$$e^{at} V(x(t; y)) + \int_0^t W(x(\tau; y)) d\tau \leq V(y) \text{ for all } t \geq 0. \quad (3.10)$$

a -Lyapunov pairs with respect to $\text{cl}(\text{Dom } A)$ are simply called a -Lyapunov pairs (see [2]); in addition, if $a = 0$ and $W = 0$, we recover the classical concept of Lyapunov functions. The case $D = \text{Int}(\text{Dom } A)$ (when nonempty), or $D = \text{ri}(\text{Dom } A)$ in the finite-dimensional setting, is useful too since it allows recovering the behaviour of V on the whole set $\text{cl}(\text{Dom } A)$ when, as in Proposition 1 below, some continuity conditions on V are known. More precisely, our characterization theorem, Theorem 3.1 below, provides criteria for Lyapunov pairs with respect to small sets, for instance balls, rather than the whole set $\text{Int}(\text{Dom } A)$. The lack of regularity properties of a -Lyapunov pairs (V, W) in Definition 1 is mainly due to the non-smoothness of the function V . Let us remind that inequality (3.10) also holds if instead of W one considers its Moreau-Yosida regularization, which is Lipschitz continuous on every bounded subset of H . This follows from the next Lemma 3 (e.g [2]).

Lemma 3 *For every $W \in \mathcal{F}(H; \overline{\mathbb{R}}_+)$, there exists a sequence of functions $(W_k)_{k \in \mathbb{N}} \subset \mathcal{F}(H, \mathbb{R}_+)$ converging to W (for instance, $W_k \uparrow W$) such that each W_k is Lipschitz continuous on every bounded subset of H , and satisfies $V(y) > 0$ if and only if $V_k(y) > 0$.*

Consequently, if $V, D \subset \text{cl}(\text{Dom } A)$, and $a \in \mathbb{R}_+$ are as in Definition 1 then, with respect to D , (V, W) forms an a -Lyapunov pair for (1.1) if and only if each pair (V, W_k) forms an a -Lyapunov pair for (1.1).

Proposition 1 *Let be given functions $V \in \mathcal{F}(H)$, $W \in \mathcal{F}(H; \overline{\mathbb{R}}_+)$ and a number $a \in \mathbb{R}_+$. If V verifies*

$$\liminf_{\text{Dom } A \ni z \rightarrow y} V(z) = V(y) \text{ for all } y \in \text{cl}(\text{Dom } A) \cap \text{Dom } V, \quad (3.11)$$

then it is equivalent to saying that (V, W) forms an a -Lyapunov pair with respect to either $\text{Dom } A$ or $\text{cl}(\text{Dom } A)$.

Property (3.11) has been already used in [19], and implicitly in [22], among other works. It holds, if for instance, $V \in F(H)$ is convex and its effective domain has a nonempty interior such that $\text{Int}(\text{Dom } V) \subset \text{Dom } A$.

Our starting point is the next result which characterizes a -Lyapunov pairs locally in $\text{Int}(\text{Dom } A)$. The general form corresponding to a -Lyapunov pairs in $\text{cl}(\text{Dom } A)$ was recently established in [2]. For the reader convenience we include here a sketch of the proof.

Proposition 2 *Assume that $\text{Int}(\text{co}\{\text{Dom } A\}) \neq \emptyset$. Let $V \in F_w(H)$ satisfy $\text{Dom } V \subset \text{cl}(\text{Dom } A)$, $W \in F(H; \overline{\mathbb{R}}_+)$, and $a \in \mathbb{R}_+$. Let $\bar{y} \in H$, $\bar{\lambda} \in [-\infty, V(\bar{y})]$, and $\bar{\rho} \in (0, +\infty]$ be such that*

$$\text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \subset \text{Int}(\text{Dom } A).$$

Then, the following statements are equivalent:

$$(i) \forall y \in \text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$$

$$\sup_{\xi \in \partial_P V(y)} \min_{v \in A y} \langle \xi, f(y) - v \rangle + aV(y) + W(y) \leq 0;$$

$$(ii) \forall y \in \text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$$

$$\sup_{\xi \in \partial_P V(y)} \langle \xi, f(y) - \pi_{A y}(f(y)) \rangle + aV(y) + W(y) \leq 0;$$

(iii) $\forall y \in B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ we have that

$$e^{at}V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) \quad \forall t \in [0, \rho(y)],$$

where

$$\rho(y) := \sup \left\{ \nu > 0 \left| \begin{array}{l} \exists \rho > 0 \text{ s.t. } B_\rho(y) \subset B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}], \text{ and for all } t \in [0, \nu] \\ 2 \|x(t; y) - y\| < \frac{\rho}{2} \text{ and} \\ |(e^{-at} - 1)V(y) - \int_0^t W(x(\tau; y))d\tau| < \frac{\rho}{2} \end{array} \right. \right\}. \quad (3.12)$$

Remark 1 (Before the proof) the constant $\rho(y)$ defined in (3.12) is positive whenever $y \in \text{cl}(\text{Dom } A) \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$. Hence, when $\bar{\rho} = -\bar{\lambda} = \infty$ one can easily show that (iii) is equivalent to (see [2, Proposition 3.2])

$$e^{at}V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) \quad \text{for all } t \geq 0;$$

that is, (V, W) forms a Lyapunov pair with respect to $\text{Int}(\text{Dom } A)$.

Proof For simplicity, we suppose that $W \equiv 0$.

(iii) \implies (ii) Let us fix $y \in B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ and $\xi \in \partial_P V(y)$ so that $y \in B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \cap \text{Dom } V \subset \text{Dom } A$ and there exist $\alpha > 0$ and $T \in (0, \rho(y))$ such that

$$\langle \xi, x(t; y) - y \rangle \leq V(x(t; y)) - V(y) + \alpha \|x(t; y) - y\|^2 \leq \alpha \|x(t; y) - y\|^2 \text{ for all } t \in [0, T].$$

But $y \in \text{Dom } A$ and so there exists a constant $l \geq 0$ such that

$$\langle \xi, t^{-1}(x(t; y) - y) \rangle \leq l \|x(t; y) - y\| \text{ for all } t \in [0, T];$$

hence, taking the limit as $t \rightarrow 0^+$ we obtain that

$$\langle \xi, f(y) - \pi_{A y}(f(y)) \rangle \leq 0;$$

that is, (ii) follows.

(i) \implies (iii) To simplify the proof of this part, we assume that $f \equiv 0$, $W \equiv 0$ and $a = 0$. For this aim we fix $y \in \text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ and let $\rho > 0$ and $\nu > 0$ be such that

$$B_\rho(y) \subset B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \text{ and} \quad (3.13)$$

$$\sup_{t \in [0, \nu]} 2 \|x(t; y) - y\| < \rho; \quad (3.14)$$

the existence of such scalars ρ and ν is a consequence of the lower semicontinuity of V and the Lipschitz continuity of $x(\cdot; \cdot)$ (see Lemma 2). Let $T < \nu$ be fixed and define the functions $z(\cdot) : [0, T] \subset \mathbb{R}_+ \rightarrow H \times \mathbb{R}$ and $\eta(\cdot) : [0, T] \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as

$$z(t) := (x(t; y), V(y)), \quad \eta(t) := \frac{1}{2} d^2(z(t), \text{epi } V); \quad (3.15)$$

observe that $z(\cdot)$ and $\eta(\cdot)$ are Lipschitz continuous on $[0, T]$. Now, using a standard chain rule (e.g. [13]), for fixed $t \in (0, T)$ it holds that

$$\partial_C \eta(t) = d(z(t), \text{epi } V) \partial_C d(z(\cdot), \text{epi } V)(t).$$

So, from one hand we get $\partial_C \eta(t) = \{\theta\}$ whenever $z(t) \in \text{epi } V$. On the other hand, when $z(t) \notin \text{epi } V$ we obtain that

$$\partial_C \eta(t) \subset \overline{\text{co}} \left[\bigcup_{(u, \mu) \in \Pi_{\text{epi } V}(z(t)), u \in B_\rho(y)} \langle x(t; y) - u, -Ax(t; y) \rangle \right]; \quad (3.16)$$

the fact that $u \in B_\rho(y)$ is a consequence of the following inequalities:

$$\begin{aligned} \|u - y\| &\leq \|x(t; y) - u\| + \|x(t; y) - y\| \\ &\leq \|(x(t; y), V(y)) - (u, \mu)\| + \|x(t; y) - y\| \\ &\leq \|(x(t; y), V(y)) - (y, V(y))\| + \|x(t; y) - y\| \\ &\leq 2 \|x(t; y) - y\| < \rho. \end{aligned}$$

Take now $\xi \in Ax(t; y)$ and $(u, \mu) \in \Pi_{\text{epi } V}(z(t))$ with $u \in B_\rho(y)$ so that $V(y) - \mu \leq 0$ and $u \in \text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$ (recall (3.13)).

If $V(y) - \mu < 0$, we write $(\mu - V(y))^{-1}(x(t; y) - u) \in \partial V_P(u)$. Then, by the current assumption (i), select $v \in Au$ such that

$$\langle (\mu - V(y))^{-1}(x(t; y) - u), -v \rangle \leq 0.$$

Therefore, invoking the monotonicity of A we get

$$\langle x(t; y) - u, -\vartheta \rangle = \langle x(t; y) - u, -v \rangle + \langle x(t; y) - u, v - \vartheta \rangle \leq \langle x(t; y) - u, -v \rangle \leq 0.$$

Since $\xi \in Ax(t; y)$ is arbitrary and according to (3.16), we deduce that $\partial_C \eta(t) \subset \mathbb{R}_-$.

If $V(y) - \mu = 0$ so that $(x(t; y) - u, 0) \in N_{\text{epi } V}(u, V(u))$ and $x(t; y) - u \neq \theta$. Then, for a fixed $\varepsilon > 0$ such that $B_\varepsilon(u) \subset B_\rho(y) \cap \text{Int}(\text{Dom}(A))$ (recall that $u \in B_\rho(y) \cap \text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \subset \text{Int}(\text{Dom}(A))$), take $u_\varepsilon \in B_\varepsilon(u) \cap \text{Dom } V$ with $|V(u) - V(u_\varepsilon)| \leq \varepsilon$, $\alpha \in (0, \varepsilon)$ and $\xi \in B_\varepsilon(x(t; y) - u)$ such that $\alpha^{-1}\xi \in \partial V_P(u_\varepsilon)$ (see, e.g., [13, Theorem 2.4]). Hence, using the current assumption, select $\xi_\varepsilon \in Au_\varepsilon$ such that $\langle \xi, -\xi_\varepsilon \rangle \leq \alpha\varepsilon$. Hence,

$$\langle x(t; y) - u, -\xi_\varepsilon \rangle \leq \varepsilon \|\xi_\varepsilon\| + \langle \xi, -u_\varepsilon^* \rangle \leq \varepsilon \|\xi_\varepsilon\| + \alpha\varepsilon \leq \varepsilon \|\xi_\varepsilon\| + \varepsilon^2.$$

By the monotonicity of A this yields

$$\begin{aligned} \langle x(t; y) - u, -\vartheta \rangle &\leq \langle x(t; y) - u_\varepsilon, -\vartheta \rangle + \varepsilon \|\vartheta\| \\ &\leq \langle x(t; y) - u_\varepsilon, -\xi_\varepsilon \rangle + \varepsilon \|\vartheta\| \\ &\leq \langle x(t; y) - u, -\xi_\varepsilon \rangle + \|u_\varepsilon - u\| \|\xi_\varepsilon\| + \varepsilon \|\vartheta\| \\ &\leq 2\varepsilon \|\xi_\varepsilon\| + \varepsilon \|\vartheta\| + \varepsilon^2. \end{aligned}$$

Moreover, as $(u_\varepsilon)_{\varepsilon \leq 1}$ is bounded in $\text{Int}(\text{Dom}(A))$, the net $(\xi_\varepsilon)_\varepsilon$ is also bounded and passing to the limit as ε goes to 0 we get

$$\langle x(t; y) - u, -\vartheta \rangle \leq 0.$$

This gives the desired inclusion $\partial_C \eta(t) \subset \mathbb{R}_-$ (recall (3.16)) and so establishes the proof of (iii). \triangle

We are now ready to give the main result of this section, which provides a precise improvement of Proposition 2.

Theorem 3.1 *Assume that $\text{Int}(\text{co}\{\text{Dom } A\}) \neq \emptyset$. Let $V \in F_w(H)$ with $\inf V > -\infty$, $W \in F(H; \overline{\mathbb{R}}_+)$, and $a \in \mathbb{R}_+$ be given. Fix $\bar{y} \in \text{Dom } V$, $\bar{\lambda} \in (-\infty, V(\bar{y}))$ and let $\bar{\rho} > 0$ be such that*

$$\text{Dom } V \cap [V > \bar{\lambda}] \cap B_{\bar{\rho}}(\bar{y}) \subset \text{Int}(\text{Dom } A).$$

Then, the following statements are equivalent:

(i) $\forall y \in \text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$

$$\sup_{\xi \in \partial_P V(y)} \min_{v \in Ay} \langle \xi, f(y) - v \rangle + aV(y) + W(y) \leq 0;$$

(ii) *(If V is weakly continuous when restricted to $B_{\bar{\rho}}(\bar{y})$) $\forall y \in \text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}]$*

$$e^{at}V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) \text{ for all } t \geq 0.$$

Consequently, if (i)-(ii) holds on $\text{Int}(\text{Dom } A)$, the pair (V, W) is an a -Lyapunov pair for (1.1) with respect to $\text{cl}(\text{Dom } A)$.

Proof The consequence is immediate once we prove the main conclusion.

First, invoking Lemma 3 we may assume w.l.o.g. that W is Lipschitz continuous on every bounded subset of H . In the rest of the proof, we take \hat{y} in $\text{Dom } V \cap B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \subset \text{Int}(\text{Dom } A)$ and, taking into account the lsc of V , choose $\rho > 0$ such that $B_{2\rho}(\hat{y}) \subset B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \cap \text{Int}(\text{Dom } A)$ and

$$V(z) \geq V(\hat{y}) - 1 \quad \forall z \in B_{2\rho}(\hat{y}). \quad (3.17)$$

Also, by virtue of Lemma 2, we consider a positive constant M such that, for all $0 \leq t \leq 1$ and all $z \in B_{2\rho}(\hat{y})$,

$$\left\| \frac{d^+ x(\cdot; z)}{dt}(t) \right\| \leq M; \quad (3.18)$$

hence, $\|x(t; z) - z\| \leq Mt$ and so, by (3.17),

$$V(x(t; z)) \geq V(\hat{y}) - 1 \geq \bar{\lambda} - 1 \quad \forall z \in B_{\rho}(\hat{y}) \text{ and } \forall t \in \left[0, \frac{\rho}{M}\right]. \quad (3.19)$$

Let us fix $\gamma \geq 1$ and define the set

$$G(\hat{y}) := [|V| \leq |V(\hat{y})| + \gamma]. \quad (3.20)$$

Claim: there exists $T > 0$ such that

$$e^{at}V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) \quad \forall y \in B_{\rho}(\hat{y}) \cap G(\hat{y}), \forall t \in [0, T]. \quad (3.21)$$

Using the (L_W -)Lipschitz continuity of W on the (bounded) set $\{x(t; y) \mid 0 \leq t \leq 1, y \in B_{2\rho}(\hat{y})\}$, we write, for all $y \in B_{2\rho}(\hat{y}) \cap G(\hat{y}) \cap \text{Dom } V$ and $0 \leq t \leq 1$,

$$\begin{aligned} 2\|x(t; y) - y\| + \left| \left(e^{-at} - 1 \right) V(y) - \int_0^t W(x(\tau; y))d\tau \right| \\ \leq 2Mt + \left(1 - e^{-at} \right) (|V(\hat{y})| + \gamma) + (W(\hat{y}) + L_W(M + 2\rho))t. \end{aligned}$$

Therefore, we can choose $T > 0$ so that for all $y \in B_{2\rho}(\hat{y}) \cap G(\hat{y})$ we have that

$$\sup_{t \in [0, T]} 2\|x(t; y) - y\| + \left| \left(e^{-at} - 1 \right) V(y) - \int_0^t W(x(\tau; y))d\tau \right| < \frac{\rho}{2}.$$

We also observe that for any given $y \in B_{\rho}(\hat{y}) \cap G(\hat{y})$ we have that $B_{\rho}(y) \subset B_{2\rho}(\hat{y}) \cap [V > \bar{\lambda}]$. Therefore, since

$$B_{\rho}(\hat{y}) \cap G(\hat{y}) \subset B_{2\rho}(\hat{y}) \cap [V > \bar{\lambda}] \subset B_{\bar{\rho}}(\bar{y}) \cap [V > \bar{\lambda}] \cap \text{Dom } V,$$

the claim follows from Theorem 2.

To go further in the proof, we fix two parameters $\varepsilon, \delta > 0$ and we introduce the set $E_{\varepsilon, \delta} \subset \mathbb{R}_+$ defined as

$$E_{\varepsilon, \delta} := \left\{ \lambda \in \mathbb{R}_+ \left| \begin{array}{l} \exists \rho_1, \rho_2 \in (\frac{\rho}{2}, \rho), \rho_1 < \rho_2, \exists \rho_\lambda \in (\frac{\rho}{2}, \rho_2), \forall y \in B_{\rho_\lambda}(\hat{y}) \cap G(\hat{y}), \forall t \leq \lambda : \\ e^{at}V_\delta(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) \end{array} \right. \right\},$$

where $V_\delta : H \rightarrow \mathbb{R}$ is the function given by

$$V_\delta(y) := \inf_{z \in H} \left\{ V(z) + \frac{1}{\delta} \|y - z\|^2 \right\}.$$

V_δ is dominated by V and is Lipschitz continuous on the bounded sets of H . Then, we have that $[0, T] \subset E_{\varepsilon, \delta}$, that is, $E_{\varepsilon, \delta} \neq \emptyset$. Next, we shall show that $E_{\varepsilon, \delta} = \mathbb{R}_+$ or, equivalently, that $E_{\varepsilon, \delta}$ is closed and open with respect to the usual topology on \mathbb{R}_+ .

Claim: $E_{\varepsilon, \delta}$ is closed.

Let a sequence $(\lambda_n)_{n \in \mathbb{N}} \subset E_{\varepsilon, \delta}$ be such that $\lambda_n \rightarrow \tilde{\lambda}$ and, by the definition of $E_{\varepsilon, \delta}$, take $(\rho_{1,n})_{n \in \mathbb{N}}, (\rho_{2,n})_{n \in \mathbb{N}}, (\rho_{3,n})_{n \in \mathbb{N}} \subset (\frac{\rho}{2}, \rho)$ be such that

$$\rho_{1,n} < \rho_{2,n}, \rho_{3,n} \in \left(\frac{\rho}{2}, \rho_{2,n}\right),$$

together with the relation

$$e^{at}V_\delta(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) + \varepsilon(\rho_{1,n} - \frac{\rho}{2})(\rho - \rho_{2,n}) \quad \forall y \in B_{\rho_{3,n}}(\hat{y}) \cap G(\hat{y}),$$

valid for all $t \leq \lambda_n$. Because all the sequences $(\rho_{1,n})_{n \in \mathbb{N}}$, $(\rho_{2,n})_{n \in \mathbb{N}}$, and $(\rho_{3,n})_{n \in \mathbb{N}}$ are bounded, on relabeling if necessary, we may suppose that $\rho_{1,n} \rightarrow \rho_1 \in [\frac{\rho}{2}, \rho]$, $\rho_{2,n} \rightarrow \rho_2 \in [\frac{\rho}{2}, \rho_2]$, and $\rho_{3,n} \rightarrow \hat{\rho} \in [\frac{\rho}{2}, \rho_2]$. As well, it is enough to suppose that $\tilde{\lambda} > T$ and $\tilde{\lambda} > \lambda_n$ for all n because, otherwise, either $\tilde{\lambda} \leq \lambda_n$ for some n or $\tilde{\lambda} \leq T$; hence in both cases we have $\tilde{\lambda} \in E_{\varepsilon, \delta}$.

If $y \in B_{\hat{\rho}}(\hat{y}) \cap G(\hat{y})$ and $t < \tilde{\lambda}$, for all n large enough we get that $y \in B_{\rho_n}(\hat{y}) \cap G(\hat{y})$ and $t < \lambda_n$ and so,

$$e^{at}V_\delta(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) + \varepsilon(\rho_{1,n} - \frac{\rho}{2})(\rho - \rho_{2,n}).$$

As n goes to ∞ we obtain that

$$e^{at}V_\delta(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2);$$

this inequality also holds for $t = \tilde{\lambda}$ in view of the continuity of V_δ . It is also useful to notice here that for all $y \in \overline{B}_{\frac{\rho}{2}}(\hat{y}) \cap G(\hat{y})$ and $t \leq \tilde{\lambda}$

$$e^{at}V_\delta(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y). \quad (3.22)$$

Now, checking the possible values of ρ_1 , ρ_2 , and $\hat{\rho}$ we observe that only two cases may occur: the first corresponds to $(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) = 0$ and happens when $\rho_1 = \frac{\rho}{2}$, $\rho_2 = \rho$, or $\rho_2 = \rho$; this last equality implies that $\frac{\rho}{2} \leq \rho_1 \leq \rho_2 \leq \frac{\rho}{2}$ and, so, $(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) = 0$. While the second case corresponds to $(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) > 0$ and happens when $\rho_1, \rho_2 \in (\frac{\rho}{2}, \rho)$.

To begin with, we analyze the case $(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) > 0$. This necessarily implies that $\rho_2 > \frac{\rho}{2}$ in view of the inequality $\frac{\rho}{2} \leq \rho_1 \leq \rho_2$. We may suppose that $\hat{\rho} = \frac{\rho}{2}$ because otherwise $\hat{\rho} \in (\frac{\rho}{2}, \rho_2)$ trivially yields $\tilde{\lambda} \in E_{\varepsilon, \delta}$. So, in order to prove that $\tilde{\lambda} \in E_{\varepsilon, \delta}$, we only need to find some $\beta > 0$ such that $\frac{\rho}{2} + \beta \in (\frac{\rho}{2}, \rho_2)$ and for all $y \in B_{\frac{\rho}{2} + \beta}(\hat{y}) \cap G(\hat{y})$ and $t \leq \tilde{\lambda}$,

$$e^{at}V_\delta(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2). \quad (3.23)$$

Proceeding by contradiction, we assume that for each $k \geq 1$ verifying $\frac{\rho}{2} + \frac{1}{k} \in (\frac{\rho}{2}, \rho_2)$, there exist $y_k \in B_{\frac{\rho}{2} + \frac{1}{k}}(\hat{y}) \cap G(\hat{y})$ and $0 < t_k \leq \tilde{\lambda}$ such that

$$e^{at_k}V_\delta(x(t_k; y_k)) + \int_0^{t_k} W(x(\tau; y_k))d\tau > V(y_k) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2). \quad (3.24)$$

Because of (3.22) we must have $(y_k)_k \subset B_{\frac{\rho}{2} + \frac{1}{k}}(\hat{y}) \setminus \overline{B}_{\frac{\rho}{2}}(\hat{y})$. W.l.o.g. we may suppose that $t_k \rightarrow \tilde{t} \leq \tilde{\lambda}$. For each k , we denote by $\tilde{y}_k \in \overline{B}_{\frac{\rho}{2}}(\hat{y})$ the orthogonal projection of y_k onto $\overline{B}_{\frac{\rho}{2}}(\hat{y})$. Thus, from one hand, we may also suppose that $(\tilde{y}_k)_k$ weakly converges to some $\tilde{y} \in \overline{B}_{\frac{\rho}{2}}(\hat{y})$. Furthermore, from the inequality $\|y_k - \tilde{y}_k\| \leq \frac{1}{k}$ we infer that y_k also weakly converges to \tilde{y} and, so, by the weak continuity of V on $B_\rho(\hat{y})$,

$$V(\tilde{y}) = \lim_k V(\tilde{y}_k) = \lim_k V(y_k). \quad (3.25)$$

Hence,

$$|V(\tilde{y})| = \lim_k |V(\tilde{y}_k)| = \lim_k |V(y_k)| \leq |V(\hat{y})| + 1.$$

In particular, (w.l.o.g.) this implies that

$$(\tilde{y}_k)_k \cup \{\tilde{y}\} \subset \overline{B}_{\frac{\rho}{2}}(\hat{y}) \cap \{|V| \leq |V(\hat{y})| + 1\} = \overline{B}_{\frac{\rho}{2}}(\hat{y}) \cap G(\hat{y}).$$

On the other hand, the absolute continuity of $x(\cdot; \tilde{y}_k)$ yields

$$x(t_k; \tilde{y}_k) - \tilde{y}_k = \int_0^{t_k} \dot{x}(\tau; \tilde{y}_k)d\tau$$

and, since that $\dot{x}(\cdot; \tilde{y}_k) \in L^\infty([0, \tilde{\lambda}]; H)$, the following holds:

$$\begin{aligned} \|x(t_k; \tilde{y}_k) - \tilde{y}_k\| &\leq t_k \sup_{\tau \in [0, t_k]} \|\dot{x}(\tau; \tilde{y}_k)\| \leq t_k \sup_{\tau \in [0, t_k]} e^{L_f \tau} \|(f(\tilde{y}_k) - A\tilde{y}_k)^\circ\| \\ &\leq \tilde{\lambda} e^{L_f \tilde{\lambda}} \sup_{z \in \overline{B}_{\frac{\rho}{2}}(\hat{y})} e^{L_f \tau} \|(f(z) - Az)^\circ\| \leq M \tilde{\lambda} e^{L_f \tilde{\lambda}}. \end{aligned}$$

Hence, w.l.o.g. we may suppose that the bounded sequence $(x(t_k; \tilde{y}_k))_{k \in \mathbb{N}}$ weakly converges in H . Furthermore, the inequality

$$\|x(t_k; y_k) - x(t_k; \tilde{y}_k)\| \leq e^{L_f t_k} \|y_k - \tilde{y}_k\| \leq \frac{e^{L_f \tilde{t}}}{k},$$

infers that the both sequences $(x(t_k; y_k))_{k \in \mathbb{N}}$ and $(x(t_k; \tilde{y}_k))_{k \in \mathbb{N}}$ weakly converge to the same point in H .

On another hand, since the sequences $(x(t_k; y_k))_{k \in \mathbb{N}}$ and $(x(t_k; \tilde{y}_k))_{k \in \mathbb{N}}$ are bounded, there exists some $l \geq 0$ such that for all $t \leq \tilde{t}$

$$\begin{aligned} |W(x(t; y_k)) - W(x(t; \tilde{y}_k))| + |V_\delta(x(t; y_k)) - V_\delta(x(t; \tilde{y}_k))| &\leq l \|x(t; y_k) - x(t; \tilde{y}_k)\| \\ &\leq \frac{le^{L_f \tilde{t}}}{k} \end{aligned}$$

and, so, we deduce that (w.l.o.g.)

$$\lim_k V_\delta(x(t_k; y_k)) = \lim_k V_\delta(x(t_k; \tilde{y}_k)) \text{ and } \lim_k W(x(t_k; y_k)) = \lim_k W(x(t_k; \tilde{y}_k)). \quad (3.26)$$

Using Lebesgue's Theorem, this infers

$$\begin{aligned} \lim_k \left[e^{a\tilde{t}} V_\delta(x(t_k; \tilde{y}_k)) + \int_0^{\tilde{t}} W(x(\tau; \tilde{y}_k)) d\tau \right] \\ = e^{a\tilde{t}} \lim_k V_\delta(x(t_k; \tilde{y}_k)) + \int_0^{\tilde{t}} \lim_k W(x(\tau; \tilde{y}_k)) d\tau \\ = e^{a\tilde{t}} \lim_k V_\delta(x(t_k; y_k)) + \int_0^{\tilde{t}} \lim_k W(x(\tau; y_k)) d\tau \\ = \lim_k \left[e^{a\tilde{t}} V_\delta(x(t_k; y_k)) + \int_0^{\tilde{t}} W(x(\tau; y_k)) d\tau \right]. \end{aligned}$$

Consequently, taking limits in (3.24), and using (3.25) we obtain

$$\begin{aligned} \lim_k \left[e^{a\tilde{t}} V_\delta(x(t_k; \tilde{y}_k)) + \int_0^{\tilde{t}} W(x(\tau; \tilde{y}_k)) d\tau \right] \\ \geq \lim_k V(y_k) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) \\ = V(\tilde{y}) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) \\ = V(\lim_k \tilde{y}_k) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) \\ = \lim_k V(\tilde{y}_k) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2). \end{aligned} \quad (3.27)$$

In other words, for k large enough we have

$$e^{a\tilde{t}} V_\delta(x(t_k; \tilde{y}_k)) + \int_0^{\tilde{t}} W(x(\tau; \tilde{y}_k)) d\tau \geq V(\tilde{y}_k) + \frac{\varepsilon}{2}(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) > V(\tilde{y}_k),$$

and a contradiction to (3.22) as $\tilde{y}_k \in \overline{B_{\frac{\rho}{2}}(\hat{y})} \cap G(\hat{y})$, and $\tilde{t} \leq \tilde{\lambda}$. Hence, we conclude that some $\rho_{\tilde{\lambda}} \in (\frac{\rho}{2}, \rho_2)$ exists so that (3.23) holds for all $y \in B_{\frac{\rho}{2} + \beta}(\hat{y}) \cap G(\hat{y})$ and $t \leq \tilde{\lambda}$. This fact shows that $\tilde{\lambda} \in E_{\varepsilon, \delta}$.

It remains to analyse the other case corresponding to $(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) = 0$. If this happens, we choose $\tilde{\rho}_1, \tilde{\rho}_2 \in (\frac{\rho}{2}, \rho)$ such that $\tilde{\rho}_1 < \tilde{\rho}_2$ and $(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) > 0$. Thus, following the same argument as in the first case, taking into account (3.22) we can find some $\beta > 0$, with $\hat{\rho} + \beta \in (\frac{\rho}{2}, \rho_2)$, so that (3.23) holds for all $y \in B_{\frac{\rho}{2} + \beta}(\hat{y}) \cap G(\hat{y})$. This shows that $\tilde{\lambda} \in E_{\varepsilon, \delta}$ and, hence, establishes the proof of the closedness of $E_{\varepsilon, \delta}$.

Claim: $E_{\varepsilon, \delta}$ is open.

Fix $\lambda \in E_{\varepsilon, \delta}$ (it is sufficient to take $\lambda \geq \nu > 0$), and let $\rho_1, \rho_2 \in (\frac{\rho}{2}, \rho)$ and $\hat{\rho} \in (\frac{\rho}{2}, \rho_2)$ be such that $\rho_1 < \rho_2$ and, for all $y \in B_{\hat{\rho}}(\hat{y}) \cap G(\hat{y})$ and $t \leq \lambda$,

$$e^{at} V_\delta(x(t; y)) + \int_0^t W(x(\tau; y)) d\tau \leq V(y) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2). \quad (3.28)$$

We let $\hat{\nu} > 0$ verify

$$\hat{\nu} \leq \min\{\nu, \lambda\}, \quad \frac{\rho}{2} < \hat{\rho} - M\hat{\nu} < \rho_2, \text{ and } \frac{\rho}{2} < e^{a\hat{\nu}} \rho_1 < \rho_2.$$

So, from one hand, for all $0 \leq \alpha \leq \hat{\nu}$ and $y \in B_{\hat{\rho}-M\hat{\nu}}(\hat{y}) \cap G(\hat{y})$ it holds, by Lemma 2,

$$\|x(\alpha; y) - \hat{y}\| \leq M\alpha + \|y - \hat{y}\| < M\alpha + \hat{\rho} - M\hat{\nu} \leq \hat{\rho}, \quad (3.29)$$

where $M \geq 0$ is defined in (3.18). Hence, by the choice of ν ($\nu \leq \lambda$), from (3.28) we infer that

$$V(x(\alpha; y)) \leq V(y) \leq |V(\hat{y})| + 1.$$

Thus, taking into account (3.19) we obtain that

$$x(\alpha; y) \in B_{\hat{\rho}}(\hat{y}) \cap G(\hat{y}).$$

Now fix $y \in B_{\hat{\rho}-M\hat{\nu}}(\hat{y}) \cap G(\hat{y})$ and $t \in [0, \lambda]$. From above we have that

$$x(\hat{\nu}, y) \in B_{\hat{\rho}}(\hat{y}) \cap G(\hat{y}) \quad (3.30)$$

and, so, applying (3.28) we get that

$$e^{at}V_{\delta}(x(t; x(\hat{\nu}, y))) + \int_0^t W(x(\tau; x(\hat{\nu}, y)))d\tau \leq V(x(\hat{\nu}, y)) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2).$$

Thus, using the semi-group property together with (3.28) and (3.30), we infer that

$$\begin{aligned} e^{a(\hat{\nu}+t)}V_{\delta}(x(\hat{\nu}+t, y)) &+ \int_0^{\hat{\nu}+t} W(x(\tau; y))d\tau \\ &= e^{a\hat{\nu}}e^{at}V_{\delta}(x(t; x(\hat{\nu}, y))) + \int_0^t W(x(\tau; x(\hat{\nu}, y)))d\tau + \int_0^{\hat{\nu}} W(x(\tau; y))d\tau \\ &\leq e^{a\hat{\nu}} \left[e^{at}V_{\delta}(x(t; x(\hat{\nu}, y))) + \int_0^t W(x(\tau; x(\hat{\nu}, y)))d\tau \right] + \int_0^{\hat{\nu}} W(x(\tau; y))d\tau \\ &\leq e^{a\hat{\nu}}V(x(\hat{\nu}, y)) + \int_0^{\hat{\nu}} W(x(\tau, y))d\tau + \varepsilon e^{a\hat{\nu}}(\rho_1 - \frac{\rho}{2})(\rho - \rho_2). \end{aligned}$$

At this step, for the choice that we made on $\hat{\nu}$ ($\hat{\nu} \leq \nu$), the last inequality above reads, for all $y \in B_{\hat{\rho}-M\hat{\nu}}(\hat{y})$ and $t \in [0, \hat{\nu} + \lambda]$,

$$\begin{aligned} e^{a(\hat{\nu}+t)}V_{\delta}(x(\hat{\nu}+t, y)) + \int_0^{\hat{\nu}+t} W(x(\tau; y))d\tau &\leq V(y) + \varepsilon e^{a\hat{\nu}}(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) \\ &\leq V(y) + \varepsilon(e^{a\hat{\nu}}\rho_1 - \frac{\rho}{2})(\rho - \rho_2). \end{aligned}$$

Consequently, since that $\hat{\rho} - M\hat{\nu} \in (\frac{\rho}{2}, \rho_2)$ and $e^{a\hat{\nu}}\rho_1 \in (\frac{\rho}{2}, \rho_2)$ it follows that $[0, \lambda + \hat{\nu}] \subset E_{\varepsilon, \delta}$ and, so, the openness of $E_{\varepsilon, \delta}$ follows.

In order to conclude the proof, let $y \in \overline{B_{\frac{\rho}{2}}}(\hat{y}) \cap G(\hat{y})$ be given. Then, for every $t \geq 0$ we have that $t \in \cap_{\varepsilon > 0} E_{\varepsilon, \delta}$; that is for all $\varepsilon > 0$ it holds

$$e^{at}V_{\delta}(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) + \varepsilon(\rho - \frac{\rho}{2})(\rho - \frac{\rho}{2}) = V(y) + \varepsilon\frac{\rho^2}{4}.$$

Hence, letting $\varepsilon \rightarrow 0$ it follows that

$$e^{at}V_{\delta}(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y),$$

which as $\delta \rightarrow 0$ yields (using the fact that $\lim_{\delta \rightarrow 0} V_{\delta}(x(t; y)) = V(x(t; y))$)

$$e^{at}V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y).$$

Now, if $\bar{z} \in B_{\rho}(\hat{y}) \cap \text{Dom } V$, then similarly as above, we can find $\rho_{\bar{z}} > 0$ such that for every $z \in \overline{B_{\frac{\rho_{\bar{z}}}{2}}}(\bar{z}) \cap G(\bar{z})$ (where $G(\bar{z})$ is defined as in (3.20)) we have that

$$e^{at}V(x(t; z)) + \int_0^t W(x(\tau; z))d\tau \leq V(z) \text{ for all } t \geq 0.$$

Thus, the main conclusion of the current theorem follows since that the last inequality obviously holds when $\bar{z} \notin \text{Dom } V$. \triangle

Remark 2 The conclusion of Theorem 3.1 also holds if, instead of V being weak continuous on $B_\rho(\bar{y})$, we assume that either H is finite-dimensional or V is convex.

Proof The only difference with the proof of Theorem 3.1 arises in showing (3.23).

(a) Assume that H is finite-dimensional. Let us show that (3.23) holds. Assuming the contrary, we find bounded sequence $y_k \in B_{\frac{\rho}{2} + \frac{1}{k}}(\bar{y}) \cap G(\bar{y})$ and $0 < t_k \leq \tilde{\lambda}$ such that (3.24) holds. W.l.o.g. we may suppose that $t_k \rightarrow \tilde{t} \leq \tilde{\lambda}$ and $y_k \rightarrow \tilde{y} \in \overline{B_{\frac{\rho}{2}}(\bar{y})}$. Furthermore, we have that

$$V(\tilde{y}) \leq \liminf_k V(y_k) \leq |V(\bar{y})| + 1,$$

while (3.19) guarantees that $V(\tilde{y}) \geq V(\bar{y}) - 1$. Hence, we also have that $\tilde{y} \in [|V| \leq |V(\bar{y})| + 1]$. Now, recalling that $x(t_k; y_k)$ converges to $x(\tilde{t}, \tilde{y})$ in this case, it follows that

$$\begin{aligned} e^{a\tilde{t}}V_\delta(x(\tilde{t}, \tilde{y})) + \int_0^{\tilde{t}} W(x(\tau; \tilde{y}))d\tau &= e^{a\tilde{t}}V_\delta(\lim_k x(t_k, y_k)) + \int_0^{\tilde{t}} W(\lim_k x(\tau; y_k))d\tau \\ &= \lim_k \left[e^{at_k}V_\delta(x(t_k; y_k)) + \int_0^{\tilde{t}} W(x(\tau; y_k))d\tau \right] \\ &\geq \liminf_k V(y_k) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) \\ &\geq V(\tilde{y}) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) > V(\tilde{y}), \end{aligned}$$

which contradicts (3.22).

(b) Assume that V is convex. We consider again the sequences of the proof of Theorem 3.1, $(y_k)_{k \in \mathbb{N}} \subset B_{\frac{\rho}{2} + \frac{1}{k}}(\bar{y}) \cap G(\bar{y}) \setminus \overline{B_{\frac{\rho}{2}}(\bar{y})}$ and $(\tilde{y}_k)_{k \in \mathbb{N}} \subset \overline{B_{\frac{\rho}{2}}(\bar{y})}$, which both converge to $\tilde{y} \in \overline{B_{\frac{\rho}{2}}(\bar{y})} \cap G(\bar{y})$. Since that each $\tilde{y}_k \in [y_k, \bar{y}]$, with $k \geq 1$, we find $\beta_k \in [0, 1]$ such that $\tilde{y}_k := \beta_k y_k + (1 - \beta_k)\bar{y}$, this yields

$$V(\tilde{y}_k) \leq \beta_k V(y_k) + (1 - \beta_k)V(\bar{y}).$$

We notice that $1 \geq \beta_k \geq \frac{k\rho}{k\rho + 2}$ since by construction, \tilde{y}_k is on the boundary of $\overline{B_{\frac{\rho}{2}}(\bar{y})}$ and $y_k \in B_{\frac{\rho}{2} + \frac{1}{k}}(\bar{y})$. Thus, we may suppose that $\beta_k \rightarrow 1$. Consequently, taking limits in the inequality above,

$$\liminf_k V(\tilde{y}_k) \leq \lim_k \beta_k \liminf_k V(y_k) = \liminf_k V(y_k).$$

Hence, as in (3.27), using (3.26) we obtain that

$$\begin{aligned} \lim_k \left[e^{a\tilde{t}}V_\delta(x(t_k; \tilde{y}_k)) + \int_0^{\tilde{t}} W(x(\tau; \tilde{y}_k))d\tau \right] &= e^{a\tilde{t}} \lim_k V_\delta(x(t_k; \tilde{y}_k)) + \int_0^{\tilde{t}} \lim_k W(x(\tau; \tilde{y}_k))d\tau \\ &= e^{a\tilde{t}} \lim_k V_\delta(x(t_k; y_k)) + \int_0^{\tilde{t}} \lim_k W(x(\tau; y_k))d\tau \\ &= \lim_k \left[e^{at_k}V_\delta(x(t_k; y_k)) + \int_0^{t_k} W(x(\tau; y_k))d\tau \right] \\ &\geq \liminf_k V(y_k) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2) \\ &\geq \liminf_k V(\tilde{y}_k) + \varepsilon(\rho_1 - \frac{\rho}{2})(\rho - \rho_2), \end{aligned}$$

which contradicts (3.22). △

Corollary 1 Assume that $\text{Int}(\text{co}\{\text{Dom } A\}) \neq \emptyset$. Let $V \in \mathcal{F}(H)$ be convex, and let $W \in \mathcal{F}(H; \overline{\mathbb{R}}_+)$ and $a \in \mathbb{R}_+$ be given. Fix $\bar{y} \in \text{Int}(\text{Dom } A) \cap \text{Dom } V$, and let $\rho > 0$ be such that $B_{2\rho}(\bar{y}) \subset \text{Int}(\text{Dom } A)$. For all $y \in B_{2\rho}(\bar{y}) \cap \text{Dom } V$ we assume that

$$\sup_{\xi \in \partial_P V(y)} \inf_{v \in Ay} \langle \xi, f(y) - v \rangle + aV(y) + W(y) \leq 0.$$

Then, for all $y \in B_\rho(\bar{y})$ we have that

$$e^{at}V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) \text{ for all } t \geq 0.$$

Proof According to Theorem 3.1 and Remark 2, it suffices to show that the current assumption implies that, for every given $y \in B_{2\rho}(\bar{y}) \cap \text{Dom } V$ and $\xi \in \partial_\infty V(y) = N_{\text{Dom } V}(y)$ (if any), there exists $v \in Ay$ such that

$$\langle \xi, f(y) - v \rangle \leq 0. \quad (3.31)$$

To prove this fact, by the lsc of V we let $\varepsilon > 0$ be such that

$$B_{\sqrt{\varepsilon}}(y) \subset \text{Int}(\text{cl}(\text{Dom } A)), \quad V(B_{\sqrt{\varepsilon}}(y)) \geq V(y) - 1.$$

Pick $y_\varepsilon \in \partial_\varepsilon V(y)$; this last set is not empty since that $V \in \mathcal{F}(H)$ is a convex function. Then, from the relationship $N_{\text{Dom } V}(y) = (\partial_\varepsilon V(y))_\infty$ (e.g.), for every $k \in \mathbb{N}$ we have that

$$y_\varepsilon + k\xi \in \partial_\varepsilon V(y).$$

According to the Brøndsted-Rockafellar Theorem, there are $y_k \in B_{\sqrt{\varepsilon}}(y)$ and $u_k \in B_{\sqrt{\varepsilon}}(\theta)$ such that

$$y_\varepsilon + k\xi \in \partial_\varepsilon V(y_k) + u_k;$$

that is, in particular, $y_k \in \text{Dom } V$. Consequently, by the current assumption we get that

$$\begin{aligned} k\langle \xi, f(y) - \pi_{Ay_k}(f(y_k)) \rangle &\leq \langle u_k - y_\varepsilon, f(y_k) - \pi_{Ay_k}(f(y_k)) \rangle - aV(y_k) - W(y_k) \\ &\leq \langle u_k - y_\varepsilon, f(y_k) - \pi_{Ay_k}(f(y_k)) \rangle - aV(y) + a \\ &\quad + k\langle \xi, f(y) - f(y_k) \rangle \\ &\leq \langle u_k - y_\varepsilon, f(y_k) - \pi_{Ay_k}(f(y_k)) \rangle - aV(y) + a + kL_f\sqrt{\varepsilon} \|\xi\|. \end{aligned}$$

Since that $\langle u_k - y_\varepsilon, f(y_k) - \pi_{Ay_k}(f(y_k)) \rangle$ is bounded independently of k , for $k_\varepsilon \geq 1$ big enough we get that

$$\langle \xi, f(y) - \pi_{Ay_{k_\varepsilon}}(f(y_{k_\varepsilon})) \rangle \leq \sqrt{\varepsilon} + L_f\sqrt{\varepsilon} \|\xi\|.$$

Moreover, as $y_{k_\varepsilon} \in B_{\sqrt{\varepsilon}}(y)$ and $\zeta_\varepsilon := \pi_{Ay_{k_\varepsilon}}(f(y_{k_\varepsilon})) \in Ay_{k_\varepsilon}$ is bounded independently of k_ε , we may suppose as $\varepsilon \rightarrow 0$ that (ζ_ε) weakly converges to some $v \in Ay$. Thus, taking limits in the last inequality above we get that $\langle \xi, f(y) - v \rangle \leq 0$; that is (3.31) follows. \triangle

Corollary 2 Assume that $\dim H < \infty$. Let $V \in \mathcal{F}(H)$, $W \in \mathcal{F}(H; \overline{\mathbb{R}}_+)$, and $a \in \mathbb{R}_+$ be given. Fix $\bar{y} \in \text{Int}(\text{Dom } A)$, and let $\rho > 0$ be such that $B_{2\rho}(\bar{y}) \subset \text{Int}(\text{Dom } A)$. For all $y \in B_{2\rho}(\bar{y}) \cap \text{Dom } V$ we assume that

$$\sup_{\xi \in \partial_P V(y)} \inf_{v \in Ay} \langle \xi, f(y) - v \rangle + aV(y) + W(y) \leq 0.$$

Then, for all $y \in B_\rho(\bar{y})$ we have that

$$e^{at}V(x(t; y)) + \int_0^t W(x(\tau; y))d\tau \leq V(y) \text{ for all } t \geq 0.$$

Proof As in the proof of Corollary 1, given $y \in B_{2\rho}(\bar{y}) \cap \text{Dom } V$ and $\xi \in \partial_\infty V(y)$ (if any), we only to find some $v \in Ay$ such that

$$\langle \xi, f(y) - v \rangle \leq 0.$$

Fix $\varepsilon > 0$. By definition, we let $\xi_k \in \partial_P V(y_k)$ and $\alpha_k \downarrow 0$ such that $y_k \rightarrow y$, $V(y_k) \rightarrow V(y)$, and $\alpha_k \xi_k \rightarrow \xi$. Then, by the current assumption, for each k there exists $y_k^* \in Ay_k$ such that

$$\langle \xi_k, f(y_k) - y_k^* \rangle + aV(y_k) + W(y_k) \leq \varepsilon.$$

Because $\dim H < \infty$ and y_k, y_k^* are bounded, we may suppose that y_k^* converges to some $v \in Ay$. Thus, multiplying the equation above by α_k and next passing to the limit as $\varepsilon \rightarrow 0$ and finally invoking the lsc of V and the Lipschitz continuity of f , we obtain that

$$\langle \xi, f(y) - v \rangle \leq \liminf_k \langle \alpha_k \xi_k, f(y_k) - y_k^* \rangle + a \liminf_k \alpha_k V(y_k) \leq \lim_k \alpha_k \varepsilon = 0.$$

The conclusion follows. \triangle

Remark 3 Let $V \in \mathcal{F}(H)$, and let $W \in \mathcal{F}(H; \overline{\mathbb{R}}_+)$ be Lipschitz continuous on $\text{cl}(\text{Dom } A)$. Define the mapping $\tilde{f} : H \times \mathbb{R}^2 \rightarrow H \times \mathbb{R}^2$ and the operator $\tilde{A} : H \times \mathbb{R}^2 \rightrightarrows H \times \mathbb{R}^2$ respectively as

$$\tilde{f}(y, \alpha, \gamma) := \begin{pmatrix} f(y) \\ W(y) \\ 0 \end{pmatrix} \text{ and } \tilde{A}(y, \alpha, \gamma) := \begin{pmatrix} Ay \\ 0 \\ 0 \end{pmatrix},$$

and denote $\tilde{V} : H \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ and $\tilde{V} : H \times \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ the functions given respectively as

$$\tilde{V}(y, \alpha) := V(y) + \alpha \text{ and } \tilde{V}(y, \alpha, \gamma) := \mathbf{I}_{\text{epi } \tilde{V} \cap \text{cl}(\text{Dom } \tilde{A})}(y, \alpha, \gamma).$$

Consider the differential inclusion

$$\dot{z}(t; (y, \alpha, \gamma)) \in \tilde{f}(z(t; (y, \alpha, \gamma))) - \tilde{A}z(t; (y, \alpha, \gamma)), \quad z(0, (y, \alpha, \gamma)) = (y, \alpha, \gamma), \quad (3.32)$$

the solution of which is the function $z(t; (y, \alpha, \gamma)) : [0, \infty) \rightarrow H \times \mathbb{R}^2$ given by

$$z(t; (y, \alpha, \gamma)) = \begin{pmatrix} x(t; y) \\ \int_0^t W(x(\tau; y)) d\tau + \alpha \\ \gamma \end{pmatrix}.$$

Then, (V, W) is a Lyapunov pair for (1.1) if and only if the function $\mathbf{I}_{\text{epi } \tilde{V} \cap \text{cl}(\text{Dom } \tilde{A})}$ is a Lyapunov function for this new differential inclusion (3.32).

4 Characterizations of finite-dimensional nonsmooth Lyapunov pairs

This section is devoted to the finite-dimensional setting. Assuming that $\dim H < \infty$, we give multiple primal and dual characterizations for nonsmooth a -Lyapunov pairs for the differential inclusion (1.1), with respect to the set $\text{rint}(\text{cl}(\text{Dom } A))$. Naturally, these conditions turn out to be sufficient for nonsmooth a -Lyapunov functions with respect to every given set $D \subset \text{cl}(\text{Dom } A)$ verifying condition (3.11).

Further, in this setting, the dual characterization does not depend on the choice of the subdifferential operator which can be either the proximal, the Fréchet, the Limiting (which coincides with the viscosity subdifferential (see Borwein [7]), or, more generally, every subdifferential operator $\partial V : H \rightrightarrows H$ satisfying

$$\partial_P V \subset \partial V \subset \partial_L V, \quad (4.33)$$

where $V \in \mathcal{F}(H)$ is the first part of Lyapunov's candidate pairs.

Proposition 3 Assume that $\dim H < \infty$. Let $V \in \mathcal{F}(H)$, $W \in \mathcal{F}(H; \overline{\mathbb{R}}_+)$, and $a \in \mathbb{R}_+$ be given, and let ∂ be as in (4.33). Fix $\bar{y} \in \text{rint}(\text{cl}(\text{Dom } A))$ and let $\rho > 0$ be such that $B_{2\rho}(\bar{y}) \cap \text{aff}(\text{cl}(\text{Dom } A)) \subset \text{Dom } A$. Then, the following assertions (i)–(v) are equivalent:

(i) for every $y \in \text{Dom } A \cap \text{Dom } V \cap B_\rho(\bar{y})$

$$e^{at}V(x(t; y)) + \int_0^t W(x(\tau; y)) d\tau \leq V(y) \text{ for all } t \geq 0;$$

(ii) for every $y \in \text{Dom } A \cap \text{Dom } V \cap B_\rho(\bar{y})$

$$\sup_{\xi \in \partial_P V(y)} \langle \xi, f(y) - \pi_{Ay}(f(y)) \rangle + aV(y) + W(y) \leq 0;$$

(iii) for every $y \in \text{Dom } A \cap \text{Dom } V \cap B_\rho(\bar{y})$

$$\sup_{\xi \in \partial V(y)} \inf_{v \in Ay} \langle \xi, f(y) - y^* \rangle + aV(y) + W(y) \leq 0;$$

(iv) for every $y \in \text{Dom } A \cap \text{Dom } V \cap B_\rho(\bar{y})$

$$V'(y; f(y) - \pi_{Ay}(f(y))) + aV(y) + W(y) \leq 0;$$

(v) for every $y \in \text{Dom } A \cap \text{Dom } V \cap B_\rho(\bar{y})$

$$\inf_{v \in Ay} V'(y; f(y) - v) + aV(y) + W(y) \leq 0.$$

If V is nonnegative, each one of the statements above is equivalent to

(vi) for every $y \in \text{Dom } A \cap \text{Dom } V \cap B_\rho(\bar{y})$

$$V(x(t; y)) + a \int_0^t V(x(\tau; y)) d\tau + \int_0^t W(x(\tau; y)) d\tau \leq V(y) \text{ for all } t \geq 0.$$

Proof (iii with $\partial \equiv \partial_P$) \implies (i): Let $H_0 := \text{lin}(\text{cl}(\text{Dom } A))$ denote the linear hull of $\text{Dom } A$; we may suppose that $\theta \in \text{Dom } A$. Let $A_0 : H_0 \rightrightarrows H_0$ be the operator given by

$$A_0 y = Ay \cap H_0, \quad (4.34)$$

and define the Lipschitz continuous mapping $f_0 : H_0 \rightarrow H_0$ as

$$f_0(y) = \pi_{H_0}(f(y)), \quad (4.35)$$

where π_{H_0} denotes the orthogonal projection onto H_0 . According to the Minty Theorem, it follows that A_0 is also a maximally monotone operator. Further, for every $y \in \text{Dom } A$ we have $Ay + N_{\text{cl}(\text{Dom } A)}(y) = Ay$, and therefore $Ay + H_0^\perp = Ay$. Hence,

$$Ay = (Ay \cap H_0) + H_0^\perp = A_0 y + H_0^\perp. \quad (4.36)$$

From this inequality we deduce that $\text{Dom } A_0 = \text{Dom } A$ and, so,

$$\text{rint}(\text{cl}(\text{Dom } A)) = \text{Int}(\text{cl}(\text{Dom } A_0)) = \text{Int}(\text{Dom } A_0);$$

for the last equality see, e.g., [8, Remark 2.1- Page 33]. Further, since for $y \in \text{cl}(\text{Dom } A)$ we have that

$$f_0(y) - A_0 y \subset f(y) - A_0 y + H_0^\perp = f(y) - Ay,$$

from which it follows that $x(\cdot; y)$ is the unique solution of the the differential inclusion

$$\dot{x}(t; y) \in f_0(x(t; y)) - A_0 x(t; y), \quad x(0, y) = y.$$

Next, we are going to show the assumption of Corollary 2 (which is the same as Conditions (i) of Theorem 3.1) holds with respect to the pair (A_0, f_0) . Fix $y \in \text{Dom } A \cap \text{Dom } V \cap B_\rho(\bar{y})$ and $\xi \in \partial V(y)$ (if any). For fixed $\varepsilon > 0$, by assumption take $v \in Ay$ in such a way that

$$\langle \xi, f(y) - v \rangle + aV(y) + W(y) \leq \varepsilon.$$

Since $f(y) \in f_0(y) + H_0^\perp$ and $v + H_0^\perp \in Ay + H_0^\perp = A_0 y$, we have

$$\inf_{v \in A_0 y} \langle \xi, f_0(y) - v \rangle \leq \inf_{v \in Ay} \langle \xi, f(y) - v \rangle \leq \varepsilon - aV(y) - W(y), \quad (4.37)$$

and the assumption of Corollary 2 follows as $\varepsilon \rightarrow 0$.

(i) \implies (iv): Fix $y \in \text{Dom } A \cap \text{Dom } V \cap B_\rho(\bar{y})$. Then, as shown in the paragraph above, the solution $x(t; y)$ of (1.1) is also the unique strong solution of the equation

$$\dot{x}(t; y) \in f_0(x(t; y)) - A_0 x(t; y), \quad x(0; y) = y \in \text{cl}(\text{Dom } A),$$

where A_0 and f_0 are defined in (4.34) and (4.35), respectively. Let $(t_n)_{n \in \mathbb{N}} \subset (0, T)$ be such that $t_n \rightarrow 0^+$ and set

$$w_n := \frac{x(t_n; y) - y}{t_n}.$$

Because $x(\cdot; y)$ is derivable from the right at 0 ($y \in \text{Dom } A$) and

$$\frac{d^+ x(\cdot; y)}{dt}(0) = (f(y) - Ay)^\circ = f(y) - \pi_{Ay}(f(y)),$$

we infer that

$$w_n \rightarrow f(y) - \pi_{Ay}(f(y)).$$

Therefore, using the current assumption (i),

$$\begin{aligned} & \frac{V(y + t_n w_n) - V(y)}{t_n} \\ &= \frac{V(x(t_n, y)) - V(y)}{t_n} \\ &\leq \frac{e^{-at_n}(1 - e^{at_n})}{t_n} V(y) - \frac{e^{-at_n}}{t_n} \int_0^{t_n} W(x(s; y)) ds, \end{aligned}$$

and taking limits yields

$$\begin{aligned} V'(y; f(y) - \pi_{Ay}(f(y))) &\leq \liminf_n \frac{e^{-at_n}(1 - e^{at_n})}{t_n} V(y) - \frac{e^{-at_n}}{t_n} \int_0^{t_n} W(x(s; y)) ds \\ &= -aV(y) - W(y); \end{aligned}$$

this proves (iv).

(iv) \implies (v) is trivial.

(v) \implies (iii). Use $\partial \equiv \partial_L$.: Take $y \in \text{Dom } A \cap \text{Dom } V \cap B_\rho(\bar{y})$. For fixed $\varepsilon > 0$, by (v) we let $v \in Ay$ be such that

$$V'(y; f(y) - v) \leq \varepsilon - aV(y) - W(y);$$

that is

$$(f(y) - v, \varepsilon - aV(y) - W(y)) \in \text{epi } V'(y, \cdot) = \text{T}_{\text{epi } V}(y, V(y)) \subset \left[\text{N}_{\text{epi } V}^p(y, V(y)) \right]^\circ.$$

If $\xi \in \partial_P V(y)$, since that $(\xi, -1) \in \text{N}_{\text{epi } V}^p(y, V(y))$ the last above inequality leads us to

$$\begin{aligned} \langle \xi, f(y) - v \rangle &\leq \langle (\xi, -1), (f(y) - v, \varepsilon - aV(y) - W(y)) \rangle + \varepsilon - aV(y) - W(y) \\ &\leq \varepsilon - aV(y) - W(y) \end{aligned}$$

so that (ii) follows when $\varepsilon \rightarrow 0$.

If $\xi \in \partial_L V(y)$, then there are sequences $y_n \rightarrow y, \xi_n \rightarrow \xi$ such that $V(\xi_n) \rightarrow V(\xi)$ and $\xi_n \in V(y_n)$ for every integer n sufficiently large. As just shown above, given an $\varepsilon > 0$, for each n there exists $y_n^* \in Ay_n$ such that

$$\langle \xi_n, f(y_n) - y_n^* \rangle \leq \varepsilon - aV(y_n) - W(y_n).$$

Because $(y_n)_n \subset B_\rho(\bar{y}) \subset \text{Int}(\text{Dom } A_0) \subset H_0$ (the ball $B_\rho(\bar{y})$ is with respect to H_0), then we may suppose that $y_n^* \rightarrow v \in Ay$. Thus, passing to the limit in the above inequality, and taking into account the lsc of V and the continuity of W ,

$$\langle \xi, f(y) - v \rangle \leq \varepsilon - aV(y) - W(y).$$

showing that (iii) holds with $\partial \equiv \partial_L$.

At this point we have proved that (i) \iff (iii with $\partial \equiv \partial_L$) \iff (iv) \iff (v). To see that (ii) is also equivalent to the other statements we observe that, from one hand, (ii) \implies (iii) holds obviously. On the other hand, the implication (iv) \implies (ii) follows in a similar way as in the proof of the statement (v) \implies (iii). This finishes the proof of the equivalences of (i) through (v).

Finally, if V is nonnegative, (vi) is nothing else but (i) with a and W replaced by θ and $aV + W$, respectively. Thus, (vi) is equivalent to (iii). \triangle

The following Theorem, which is an immediate consequence of Proposition 4.33, establishes primal and dual characterizations of Lyapunov pairs for (1.1) with respect to $\text{rint}(\text{cl}(\text{Dom } A))$. Sufficient conditions for Lyapunov pairs with respect to other sets are then deduced under (3.11).

Theorem 4.2 *Assume that $\dim H < \infty$. Let $V \in \mathcal{F}(H), W \in \mathcal{F}(H; \overline{\mathbb{R}}_+)$, and $a \in \mathbb{R}_+$ be given, and let ∂ be as in (4.33). Then, (V, W) forms an a -Lyapunov pair for (1.1), with respect to $\text{rint}(\text{cl}(\text{Dom } A))$, if and only if one of the following assertions holds:*

(i) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap \text{Dom } V$

$$\sup_{\xi \in \partial_P V(y)} \langle \xi, f(y) - \pi_{Ay}(f(y)) \rangle + aV(y) + W(y) \leq 0;$$

(ii) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap \text{Dom } V$

$$\sup_{\xi \in \partial V(y)} \inf_{v \in Ay} \langle \xi, f(y) - y^* \rangle + aV(y) + W(y) \leq 0;$$

(iii) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap \text{Dom } V$

$$V'(y; f(y) - \pi_{Ay}(f(y))) + aV(y) + W(y) \leq 0;$$

(iv) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap \text{Dom } V$

$$\inf_{v \in Ay} V'(y; f(y) - v) + aV(y) + W(y) \leq 0.$$

Consequently, if V satisfies (3.11) for a given set $D \subset \text{cl}(\text{Dom } A)$, then any of the conditions (i)-(iv) above implies that (V, W) is an a -Lyapunov pair for (1.1) with respect to D .

In contrast to the (analytic) Definition 1, Lyapunov stability can also be approached from a geometrical point of view using the concept of invariance:

Definition 2 Let be given a set $D \subset \text{cl}(\text{Dom } A)$. A non-empty closed set $S \subset H$ is said invariant for (1.1) with respect to D if for all $y \in S \cap D$ one has that

$$x(t; y) \in S \text{ for all } t \geq 0.$$

This fact, which was already mentioned in the **infinite-dimensional** setting in Corollary 1, is explicitly characterized here in the **finite-dimensional** setting. This characterization is also valid in the infinite-dimensional setting provided that $S \cap \text{cl}(\text{Dom } A)$ is a convex set, according to Remark 2 and Corollary 1.

Corollary 3 Assume that $\dim H < \infty$. A closed set $\emptyset \neq S \subset H$ is invariant for (1.1), with respect to $\text{rint}(\text{cl}(\text{Dom } A))$, if and only if one of the following assertions are satisfied:

(i) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap S$

$$\sup_{\xi \in N_{S \cap \text{cl}(\text{Dom } A)}^P(y)} \langle \xi, f(y) - \pi_{Ay}(f(y)) \rangle \leq 0;$$

(ii) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap S$

$$\sup_{\xi \in N_{S \cap \text{cl}(\text{Dom } A)}^P(y)} \inf_{v \in Ay} \langle \xi, f(y) - v \rangle \leq 0;$$

(iii) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap S$

$$f(y) - \pi_{Ay}(f(y)) \in T_{S \cap \text{cl}(\text{Dom } A)}(y);$$

(iv) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap S$

$$[f(y) - Ay] \cap T_{S \cap \text{cl}(\text{Dom } A)}(y) \neq \emptyset;$$

(v) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap S$

$$[f(y) - Ay] \cap \overline{\text{co}} [T_{S \cap \text{cl}(\text{Dom } A)}(y)] \neq \emptyset.$$

Consequently, S is invariant for (1.1) with respect to a given set $D \subset \text{cl}(\text{Dom } A)$ if

$$S \cap D \subset \text{cl}(S \cap \text{rint}(\text{cl}(\text{Dom } A))).$$

Proof It is an immediate fact that, with respect to $\text{rint}(\text{cl}(\text{Dom } A))$, S is invariant if and only if $I_{S \cap \text{cl}(\text{Dom } A)}$ is a Lyapunov function. Then, the current assertions (i) and (ii) come from statements (i) and (ii) of Proposition 3, respectively. Similarly, always with respect to $\text{rint}(\text{cl}(\text{Dom } A))$, S is invariant if and only if $d(\cdot, S \cap \text{cl}(\text{Dom } A))$ is a Lyapunov function. Thus, by virtue of the relationship

$$T_{S \cap \text{cl}(\text{Dom } A)}(y) = \{w \in H \mid d'(\cdot, S \cap \text{cl}(\text{Dom } A))(w) = 0\},$$

the current assertions (iii) and (iv) follow from statements (iii) and (iv) of Proposition 3, respectively. This shows that (i) \iff (ii) \iff (iii) \iff (iv).

It remains to show that (v) is equivalent to the other statements. We obviously have that (iv) \implies (v) and so (i) \implies (v). To prove the reverse implication it suffices to show that (v) \implies (ii). Indeed, fix $y \in S \cap \text{Dom } A$ and $\xi \in N_{S \cap \text{cl}(\text{Dom } A)}^P$. Then, by (v) there exists $v \in Ay$ such that

$$f(y) - v \in \overline{\text{co}} [T_{S \cap \text{cl}(\text{Dom } A)}(y)] \subset \left[N_{S \cap \text{cl}(\text{Dom } A)}^P \right]^\circ.$$

Therefore, $\langle \xi, f(y) - v \rangle \leq 0$; that is (ii) follows. \triangle

The following corollary follows from Theorem 4.2.

Corollary 4 Assume that $\dim H < \infty$. Let $V \in \mathcal{F}(H)$, $W \in \mathcal{F}(H, \mathbb{R}_+)$, and $a \in \mathbb{R}_+$ be given, and let ∂ be as in (4.33). Then, the following statements are equivalent provided that V is continuous relative to $\text{cl}(\text{Dom } A)$:

- (i) (V, W) is an a -Lyapunov pair for (1.1) with respect to $\text{cl}(\text{Dom } A)$;
- (ii) (V, W) is an a -Lyapunov pair for (1.1) with respect to $\text{rint}(\text{cl}(\text{Dom } A))$;
- (iii) for every $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap \text{Dom } V$

$$\sup_{\xi \in \partial V(y)} \langle \xi, f(y) - \pi_{Ay}(f(y)) \rangle + aV(y) + W(y) \leq 0;$$

- (iv) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap \text{Dom } V$

$$\inf_{v \in Ay} V'(y; f(y) - v) + aV(y) + W(y) \leq 0.$$

The characterization of Gâteaux differentiable Lyapunov functions is a special case of the following corollary.

Corollary 5 Assume that $\dim H < \infty$. Let $V \in \mathcal{F}(H)$, $W \in \mathcal{F}(H, \mathbb{R}_+)$, and $a \in \mathbb{R}_+$ be given. If V is Gâteaux differentiable, then the following statements are equivalent:

- (i) (V, W) is an a -Lyapunov pair for (1.1) with respect to $\text{cl}(\text{Dom } A)$;
- (ii) (V, W) is an a -Lyapunov pair for (1.1) with respect to $\text{rint}(\text{cl}(\text{Dom } A))$;
- (iii) for every $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap \text{Dom } V$

$$V'_G(y)(f(y) - \pi_{Ay}(f(y))) + aV(y) + W(y) \leq 0;$$

- (iv) for all $y \in \text{rint}(\text{cl}(\text{Dom } A)) \cap \text{Dom } V$

$$\inf_{v \in Ay} V'_G(y)(f(y) - v) + aV(y) + W(y) \leq 0.$$

In order to fix ideas, let us discuss the simple case when $A \equiv 0$ so that our inclusion (1.1) becomes an ordinary differential equation which reads: for every $y \in H$ there exists a unique $x(\cdot; y) \in C^1(0, \infty; H)$ such that $x(0, y) = y$ and

$$\dot{x}(t; y) = f(x(t; y)) \text{ for all } t \geq 0. \quad (4.38)$$

In this case, Theorem 2 gives in a simplified form the characterization of the associated a -Lyapunov pairs.

Corollary 6 Assume that $\dim H < \infty$. Let be given $V \in \mathcal{F}(H)$, $W \in \mathcal{F}(H; \overline{\mathbb{R}}_+)$, and $a \in \mathbb{R}_+$. The following statements are equivalent:

- (i) (V, W) is an a -Lyapunov pair for (4.38) (with respect to H);
- (ii) for every $y \in \text{Dom } V$

$$V'(y; f(y)) + aV(y) + W(y) \leq 0;$$

- (iii) for all $y \in \text{Dom } V$

$$\sup_{\xi \in \partial V(y)} \langle \xi, f(y) \rangle + aV(y) + W(y) \leq 0,$$

where ∂V stands for any subdifferential operator verifying $\partial_P V \subset \partial V \subset \partial_C V$.

Proof By Theorem 4.2 the conclusion holds for all the subdifferentials ∂V such that $\partial_P V \subset \partial V \subset \partial_L V$. To show that (iii) is also a characterization when $\partial \equiv \partial_C$ it suffices, in view of the relationship $\partial_L \subset \partial_C$, to show that (iii with $\partial \equiv \partial_L$) implies (iii with $\partial \equiv \partial_C$). Indeed, fix $y \in \text{Dom } V$ so that

$$\sup_{\xi \in \partial_\infty V(y)} \langle \xi, f(y) \rangle \leq 0.$$

So, according to [24], (iii with $\partial \equiv \partial_C$) follows since that

$$\sup_{\xi \in \partial_C V(y)} \langle \xi, f(y) \rangle + aV(y) + W(y) = \sup_{\xi \in \overline{\text{co}}\{\partial_L V(y) + \partial_\infty V(y)\}} \langle \xi, f(y) \rangle + aV(y) + W(y) \leq 0.$$

△

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