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Light tails: Gibbs conditional principle under extreme deviation

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Abstract

Let X_1, \dots, X_n denote an i.i.d. sample with light tail distribution and S_1^n denote the sum of its terms; let a_n be a real sequence going to infinity with n . In a previous paper ([4]) it is proved that as $n \rightarrow \infty$, given $(S_1^n/n > a_n)$ all terms X_i concentrate around a_n with probability going to 1. This paper explores the asymptotic distribution of X_1 under the conditioning events $(S_1^n/n = a_n)$ and $(S_1^n/n \geq a_n)$. It is proved that under some regularity property, the asymptotic conditional distribution of X_1 given $(S_1^n/n = a_n)$ can be approximated in variation norm by the tilted distribution at point a_n , extending therefore the classical LDP case developed in ([9]). Also under $(S_1^n/n \geq a_n)$ the dominating point property holds.

It also considers the case when the X_i 's are \mathbb{R}^d -valued, f is a real valued function defined on \mathbb{R}^d and the conditioning event writes $(U_1^n/n = a_n)$ or $(U_1^n/n \geq a_n)$ with $U_1^n := (f(X_1) + \dots + f(X_n))/n$ and $f(X_1)$ has a light tail distribution. As a by-product some attention is paid to the estimation of high level sets of functions.

1 Introduction

Let X_1, \dots, X_n denote n independent unbounded real valued random variables and $S_1^n := X_1 + \dots + X_n$ be their sum. The purpose of this paper is to explore the limit distribution of the generic variable X_1 conditioned on extreme deviations (ED) pertaining to S_1^n . By extreme deviation we mean that S_1^n/n is supposed to take values which are going to infinity as n increases. Obviously such events are of infinitesimal probability. Our interest in this question stems from a first result which assesses that under appropriate conditions, when the sequence a_n is such that

$$\lim_{n \rightarrow \infty} a_n = \infty$$

then there exists a sequence ε_n which satisfies $\varepsilon_n/a_n \rightarrow 0$ as n tends to infinity such that

$$\lim_{n \rightarrow \infty} P(\cap_{i=1}^n (X_i \in (a_n - \varepsilon_n, a_n + \varepsilon_n)) | S_1^n/n \geq a_n) = 1 \quad (1.1)$$

which is to say that when the empirical mean takes exceedingly large values, then all the summands share the same behaviour; this result is useful when considering aggregate forming in large random media, or in the context of robust estimators in statistics. It requires a number of hypotheses, which we simply quote as of “light tail” type. We refer to [4] for this result and the connection with earlier related works, and name it ”democratic localization principle” (DLP), as referred to in [11].

The above result is clearly to be put in relation with the so-called Gibbs conditional Principle which we recall briefly in its simplest form.

Let a_n satisfy $a_n = a$, a constant with value larger than the expectation of X_1 and consider the behaviour of the summands when $(S_1^n/n \geq a)$, under a large deviation (LD) condition about the empirical mean. The asymptotic conditional distribution of X_1 given $(S_1^n/n \geq a)$ is the well known tilted distribution of P_X with parameter t associated to a . Let us introduce some notation. The hypotheses to be stated now together with notation are kept throughout the entire paper.

It will be assumed that P_X , which is the distribution of X_1 , has a density p with respect to the Lebesgue measure on \mathbb{R} . The fact that X_1 has a light tail is captured in the hypothesis that X_1 has a moment generating function

$$\phi(t) := E \exp tX_1$$

which is finite in a non void neighborhood \mathcal{N} of 0. This fact is usually referred to as a Cramer type condition.

Defined on \mathcal{N} are the following functions. The functions

$$t \rightarrow m(t) := \frac{d}{dt} \log \phi(t)$$

$$t \rightarrow s^2(t) := \frac{d^2}{dt^2} \log \phi(t)$$

and

$$t \rightarrow \mu_3(t) := \frac{d^3}{dt^3} \log \phi(t)$$

are the expectation, the variance and kurtosis of the r.v. \mathcal{X}_t with density

$$\pi^a(x) := \frac{\exp tx}{\phi(t)} p(x)$$

which is defined on \mathbb{R} and which is the tilted density with parameter t in \mathcal{N} defined through

$$m(t) = a. \quad (1.2)$$

. When ϕ is steep, meaning that

$$\lim_{t \rightarrow \partial \mathcal{N}} m(t) = \infty$$

then m parametrizes the convex hull $cvhull(P_X)$ of the support of P_X and (1.2) is defined in a unique way for all a in $cvhull(P_X)$. We refer to [1] for those properties.

We now come to some remark on the Gibbs conditional principle in the standard above setting. A phrasing of this principle is:

As n tends to infinity the conditional distribution of X_1 given $(S_1^n/n \geq a)$ approaches Π^a , the distribution with density π^a .

We state the Gibbs principle in a form where the conditioning event is a point condition $(S_1^n/n = a)$. The conditional distribution of X_1 given $(S_1^n/n = a)$ is a well defined distribution and Gibbs conditional principle states that it converges to Π^a as n tends to infinity. In both settings, this convergence holds in total variation norm. We refer to [9] for the local form of the conditioning event; we will mostly be interested in the extension of this form.

The present paper is also a continuation of [6] which contains a conditional limit theorem for the approximation of the conditional distribution of X_1, \dots, X_{k_n} given $S_1^n/n = a_n$ with $\limsup_{n \rightarrow \infty} k_n/n \leq 1$ and $\lim_{n \rightarrow \infty} n - k_n = \infty$. There, the sequence a_n is bounded, hence covering all cases from the LLN up to the LDP, and the approximation holds in the total variation distance. The resulting approximation, when restricted to the case $k_n = 1$, writes

$$\lim_{n \rightarrow \infty} \int |p_{a_n}(x) - g_{a_n}(x)| dx = 0 \tag{1.3}$$

where

$$g_{a_n}(x) := Cp(x) \mathbf{n}(a_n, s_n^2, x). \tag{1.4}$$

Hereabove $\mathbf{n}(a, s_n, x)$ denotes the normal density function at point x with expectation a_n , with variance s_n^2 , and $s_n^2 := s^2(t_n)(n - 1)$ and t_n such that $m(t_n) = a_n$; C is a normalizing constant. Obviously developing in display (1.4) yields

$$g_{a_n}(x) = \pi^{a_n}(x) (1 + o(1))$$

which proves that (1.3) is a form of Gibbs principle, with some improvement due to the second order term. In the present context the extension from $k = 1$ (or from fixed k) to the case when k_n approaches n requires a large burden of technicalities, mainly Edgeworth expansions of high order in the extreme value range; lacking a motivation for this task, we did not engage on this path.

The paper is organized as follows. Notation and hypotheses are stated in Section 2; a sharp Abelian result pertaining to the moment generating function and a refinement of a

local central limit theorem in the context of triangular arrays with non standard moments are presented in Section 3. Section 4 provides a local Gibbs conditional principle under EDP, namely producing the approximation of the conditional density of X_1 conditionally on $(S_1^n/n = a_n)$ for sequences a_n which tend to infinity. The first approximation is local. This result is extended to typical paths under the conditional sampling scheme, which in turn provides the approximation in variation norm for the conditional distribution. The method used here follows closely the approach developed in [6]. Extensions to other conditioning events are discussed. The differences between the Gibbs principles in LDP and EDP are also mentioned. Similar results in the case when the conditioning event is $(S_1^n/n \geq a_n)$ are stated in Section 5, where both EDP and DLP are simultaneously considered. This section also introduces some proposal for a stochastic approximation of high level sets of real valued functions defined on \mathbb{R}^d .

2 Notation and hypotheses

The density p of X_1 is uniformly bounded and writes

$$p(x) = c \exp \left(- (g(x) - q(x)) \right) \quad x \in \mathbb{R}_+, \quad (2.1)$$

where c is some positive normalizing constant. Define

$$h(x) := g'(x).$$

We assume that for some positive constant ϑ , for large x , it holds

$$\sup_{|v-x| < \vartheta x} |q(v)| \leq \frac{1}{\sqrt{xh(x)}}. \quad (2.2)$$

The function g is positive and satisfies

$$\lim_{n \rightarrow \infty} \frac{g(x)}{x} = \infty. \quad (2.3)$$

Not all positive g 's satisfying (2.3) are adapted to our purpose. Regular functions g are defined through the function h as follows. We define firstly a subclass R_0 of the family of *slowly varying* function. A function l belongs to R_0 if it can be represented as

$$l(x) = \exp \left(\int_1^x \frac{\epsilon(u)}{u} du \right), \quad x \geq 1, \quad (2.4)$$

where $\epsilon(x)$ is twice differentiable and $\epsilon(x) \rightarrow 0$ as $x \rightarrow \infty$.

We follow the line developed in [13] to describe the assumed regularity conditions of h .

The Class R_β : $x \rightarrow h(x)$ belongs to R_β , if, with $\beta > 0$ and x large enough, $h(x)$ can be represented as

$$h(x) = x^\beta l(x),$$

where $l(x) \in R_0$ and in (2.4) $\epsilon(x)$ satisfies

$$\limsup_{x \rightarrow \infty} x |\epsilon'(x)| < \infty, \quad \limsup_{x \rightarrow \infty} x^2 |\epsilon''(x)| < \infty. \quad (2.5)$$

The Class R_∞ : $x \rightarrow l(x)$ belongs to \widetilde{R}_0 , if, in (2.4), $l(x) \rightarrow \infty$ as $x \rightarrow \infty$ and

$$\lim_{x \rightarrow \infty} \frac{x\epsilon'(x)}{\epsilon(x)} = 0, \quad \lim_{x \rightarrow \infty} \frac{x^2\epsilon''(x)}{\epsilon(x)} = 0, \quad (2.6)$$

and, for some $\eta \in (0, 1/4)$

$$\liminf_{x \rightarrow \infty} x^\eta \epsilon(x) > 0. \quad (2.7)$$

We say that h belongs to R_∞ if h is increasing and strictly monotone and its inverse function ψ defined through

$$\psi(u) := h^\leftarrow(u) := \inf \{x : h(x) \geq u\} \quad (2.8)$$

belongs to \widetilde{R}_0 .

Denote $\mathfrak{R} := R_\beta \cup R_\infty$. The class \mathfrak{R} covers a large collection of functions, although, R_β and R_∞ are only subsets of the classes of *Regularly varying* and *Rapidly varying* functions, respectively.

Example 2.1. Weibull Density. Let p be a Weibull density with shape parameter $k > 1$ and scale parameter 1, namely

$$\begin{aligned} p(x) &= kx^{k-1} \exp(-x^k), \quad x \geq 0 \\ &= k \exp\left(-\left(x^k - (k-1) \log x\right)\right). \end{aligned}$$

Take $g(x) = x^k - (k-1) \log x$ and $q(x) = 0$. Then it holds

$$h(x) = kx^{k-1} - \frac{k-1}{x} = x^{k-1} \left(k - \frac{k-1}{x^k}\right).$$

Set $l(x) = k - (k-1)/x^k$, $x \geq 1$, then (2.4) holds, namely,

$$l(x) = \exp\left(\int_1^x \frac{\epsilon(u)}{u} du\right), \quad x \geq 1,$$

with

$$\epsilon(x) = \frac{k(k-1)}{kx^k - (k-1)}.$$

The function ϵ is twice differentiable and goes to 0 as $x \rightarrow \infty$. Additionally, ϵ satisfies condition (2.5). Hence we have shown that $h \in R_{k-1}$.

Example 2.2. A rapidly varying density. Define p through

$$p(x) = c \exp(-e^{x-1}), \quad x \geq 0.$$

Then $g(x) = h(x) = e^x$ and $q(x) = 0$ for all non negative x . We show that $h \in R_\infty$. It holds $\psi(x) = \log x + 1$. Since $h(x)$ is increasing and monotone, it remains to show that $\psi(x) \in R_0$. When $x \geq 1$, $\psi(x)$ admits the representation of (2.4) with $\epsilon(x) = \log x + 1$. Also conditions (2.6) and (2.7) are satisfied. Thus $h \in R_\infty$.

Throughout the paper we use the following notation. When a r.v. X has density p we write $p(X = x)$ instead of $p(x)$. For example $\pi^a(X = x)$ is the density at point x for the variable X generated under π^a , while $p(X = x)$ states for X generated under p .

For all α (depending on n or not) P_α designates the conditional distribution of the vector X_1, \dots, X_n given $(S_1^n = n\alpha)$. This distribution is degenerate on \mathbb{R}^n ; however its margins are a.c. w.r.t. the Lebesgue measure on \mathbb{R} . The function p_α denotes the density of a margin.

3 An Abelian Theorem and an Edgeworth expansion

In this short section we mention two Theorems to be used in the derivation of our main result. They deserve interest by themselves; see [5] for their proof.

3.1 An Abelian type result

We inherit of the definition of the tilted density π^a defined in Section 1, and of the corresponding definitions of the functions m , s^2 and μ_3 . Because of (2.1) and on the various conditions on g those functions are defined as $t \rightarrow \infty$. The proof of Corollary 3.1 is postponed to the Appendix.

Theorem 3.1. Let $p(x)$ be defined as in (2.1) and $h(x) \in \mathfrak{R}$. Denote by

$$m(t) = \frac{d}{dt} \log \phi(t), \quad s^2(t) = \frac{d}{dt} m(t), \quad \mu_3(t) = \frac{d^3}{dt^3} \log \Phi(t),$$

then with ψ defined as in (2.8) it holds as $t \rightarrow \infty$

$$m(t) \sim \psi(t), \quad s^2(t) \sim \psi'(t), \quad \mu_3(t) \sim \frac{M_6 - 3}{2} \psi''(t),$$

where M_6 is the sixth order moment of standard normal distribution.

Corollary 3.1. Let $p(x)$ be defined as in (2.1) and $h(x) \in \mathfrak{R}$. Then it holds as $t \rightarrow \infty$

$$\frac{\mu_3(t)}{s^3(t)} \longrightarrow 0. \quad (3.1)$$

For clearness we write m and s^2 for $m(t)$ and $s^2(t)$.

Example 3.1. *The Weibull case:* When $g(x) = x^k$ and $k > 1$ then $m(t) \sim Ct^{1/(k-1)}$ and $s^2(t) \sim C't^{(2-k)/(k-1)}$, which tends to 0 for $k > 2$.

Example 3.2. *A rapidly varying density.* Define p through

$$p(x) = c \exp(-e^{x-1}), \quad x \geq 0.$$

Since $\psi(x) = \log x + 1$ it follows that $m(t) \sim \log t$ and $s^2(t) \sim 1/t \rightarrow 0$.

3.2 Edgeworth expansion under extreme normalizing factors

With π^{a_n} defined through

$$\pi^{a_n}(x) = \frac{e^{tx}p(x)}{\phi(t)},$$

and t determined by $a_n = m(t)$, define the normalized density of π^{a_n} by

$$\bar{\pi}^{a_n}(x) = s\pi^{a_n}(sx + a_n),$$

and denote the n -convolution of $\bar{\pi}^{a_n}(x)$ by $\bar{\pi}_n^{a_n}(x)$. Denote by ρ_n the normalized density of n -convolution $\bar{\pi}_n^{a_n}(x)$,

$$\rho_n(x) := \sqrt{n}\bar{\pi}_n^{a_n}(\sqrt{n}x).$$

The following result extends the local Edgeworth expansion of the distribution of normalized sums of i.i.d. r.v's to the present context, where the summands are generated under the density $\bar{\pi}^{a_n}$. Therefore the setting is that of a triangular array of row wise independent summands; the fact that $a_n \rightarrow \infty$ makes the situation unusual. The proof of this result follows Feller's one (Chapter 16, Theorem 2 [10]). With $m(t) = a_n$ and $s^2 := s^2(t)$, the following Edgeworth expansion holds.

Theorem 3.2. *With the above notation, uniformly upon x it holds*

$$\rho_n(x) = \phi(x) \left(1 + \frac{\mu_3}{6\sqrt{n}s^3}(x^3 - 3x) \right) + o\left(\frac{1}{\sqrt{n}}\right). \quad (3.2)$$

where $\phi(x)$ is standard normal density.

4 Gibbs' conditional principles under extreme events

We now explore Gibbs conditional principles under extreme events. For Y_1, \dots, Y_n a random vector generated according to the conditional distribution of the X_i 's given $(S_1^n = na_n)$

and with the density p_{a_n} defined as its marginal density we first provide a density g_{a_n} on \mathbb{R} such that

$$p_{a_n}(Y_1) = g_{a_n}(Y_1)(1 + R_n)$$

where R_n is a function of the vector (Y_1, \dots, Y_n) which goes to 0 as n tends to infinity. The above statement may also be written as

$$p_{a_n}(y_1) = g_{a_n}(y_1)(1 + o_{P_{a_n}}(1)) \quad (4.1)$$

where P_{a_n} is the joint probability measure of the vector (Y_1, \dots, Y_n) under the condition $(S_1^n = na_n)$; note that we designate p_{a_n} the marginal density of P_{a_n} , which is well defined although P_{a_n} is restricted on the plane $y_1 + \dots + y_n = a_n$. This statement amounts to provide the approximation on typical realizations under the conditional sampling scheme. We will deduce from (4.1) that the L^1 distance between p_{a_n} and g_{a_n} goes to 0 as n tends to infinity. We first derive a local marginal Gibbs result, from which (4.1) is easily obtained.

4.1 A local result in \mathbb{R}

Fix y_1 in \mathbb{R} and define t through

$$m(t) := a_n. \quad (4.2)$$

Define $s^2 := s^2(t)$.

Consider the following condition

$$\lim_{t \rightarrow \infty} \frac{\psi(t)^2}{\sqrt{n\psi'(t)}} = 0, \quad (4.3)$$

which amounts to state a growth condition on the sequence a_n .

Define z_1 through

$$z_1 = \frac{na_n - y_1}{s\sqrt{n-1}}. \quad (4.4)$$

Lemma 4.1. *Assume that $p(x)$ satisfies (2.1) and $h(x) \in \mathfrak{R}$. Let t be defined in (4.2). Assume that $a_n \rightarrow \infty$ as $n \rightarrow \infty$ and that (4.3) holds. Then*

$$\lim_{n \rightarrow \infty} z_1 = 0.$$

Proof: When $n \rightarrow \infty$, it holds

$$z_1 \sim m(t)/(s(t)\sqrt{n}).$$

From Theorem 3.1, it holds $m(t) \sim \psi(t)$ and $s(t) \sim \sqrt{\psi'(t)}$. Hence we have

$$z_1 \sim \frac{\psi(t)}{\sqrt{n\psi'(t)}}.$$

Hence

$$z_1^2 \sim \frac{\psi(t)^2}{n\psi'(t)} = \frac{\psi(t)^2}{\sqrt{n}\psi'(t)} \frac{1}{\sqrt{n}} = o\left(\frac{1}{\sqrt{n}}\right).$$

Theorem 4.1. *With the above notation and hypotheses denoting $m := m(t)$, assuming (4.3), it holds*

$$p_{a_n}(y_1) = p(X_1 = y_1 | S_1^n = na_n) = g_m(y_1) \left(1 + o(1)\right).$$

with

$$g_m(y_1) = \pi^m(X_1 = y_1).$$

Proof:

But for limit arguments which are specific to the extreme deviation context, the proof of this local result is classical; see the LDP case in [9].

We make use of the following invariance property:

For all y_1 and all α in the range of X_1

$$p(X_1 = y_1 | S_1^n = na_n) = \pi^\alpha(X_1 = y_1 | S_1^n = na_n)$$

where on the LHS, the r.v.'s X_i 's are sampled i.i.d. under p and on the RHS, sampled i.i.d. under π^α .

It thus holds

$$\begin{aligned} p(X_1 = y_1 | S_1^n = na_n - y_1) &= \pi^m(X_1 = y_1 | S_1^n = na_n) \\ &= \pi^{m_1}(X_1 = y_1) \frac{\pi^{m_1}(S_2^n = na_n - y_1)}{\pi^m(S_1^n = na_n)} \\ &= \frac{\sqrt{n}}{\sqrt{n-1}} \pi^{m_1}(X_1 = y_1) \frac{\widetilde{\pi}_{n-1}\left(\frac{m-y_1}{s\sqrt{n-1}}\right)}{\widetilde{\pi}_n(0)} \\ &= \frac{\sqrt{n}}{\sqrt{n-1}} \pi^{m_1}(X_1 = y_1) \frac{\widetilde{\pi}_{n-1}(z_1)}{\widetilde{\pi}_n(0)}, \end{aligned}$$

where $\widetilde{\pi}_{n-1}$ is the normalized density of S_2^n under i.i.d. sampling under π^m ; correspondingly, $\widetilde{\pi}_n$ is the normalized density of S_1^n under the same sampling. Note that a r.v. with density π^m has expectation $m = a_n$ and variance s^2 .

Perform a third-order Edgeworth expansion of $\widetilde{\pi}_{n-1}(z_1)$, using Theorem 3.2. It follows

$$\widetilde{\pi}_{n-1}(z_1) = \phi(z_1) \left(1 + \frac{\mu_3}{6s^3\sqrt{n-1}}(z_1^3 - 3z_1)\right) + o\left(\frac{1}{\sqrt{n}}\right),$$

The approximation of $\widetilde{\pi}_n(0)$ is obtained from (3.2) through

$$\widetilde{\pi}_n(0) = \phi(0) \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right).$$

It follows that

$$\begin{aligned}
& p(X_1 = y_1 | S_1^n = na_n) \\
&= \frac{\sqrt{n}}{\sqrt{n-1}} \pi^m(X_1 = y_1) \frac{\phi(z_1)}{\phi(0)} \left[1 + \frac{\mu_3}{6s^3\sqrt{n-1}} (z_1^3 - 3z_1) + o\left(\frac{1}{\sqrt{n}}\right) \right] \\
&= \frac{\sqrt{2\pi n}}{\sqrt{n-1}} \pi^m(X_1 = y_1) \phi(z_1) (1 + R_n + o(1/\sqrt{n})),
\end{aligned}$$

where

$$R_n = \frac{\mu_3}{6s^3\sqrt{n-1}} (z_1^3 - 3z_1).$$

Under condition (4.3), using Lemma 4.1, it holds $z_1 \rightarrow 0$ as $a_n \rightarrow \infty$, and under Corollary (3.1), $\mu_3/s^3 \rightarrow 0$. This yields

$$R_n = o(1/\sqrt{n}),$$

which gives

$$\begin{aligned}
p(X_1 = y_1 | S_1^n = na_n) &= \frac{\sqrt{2\pi n}}{\sqrt{n-1}} \pi^m(X_1 = y_1) \phi(z_1) (1 + o(1/\sqrt{n})) \\
&= \frac{\sqrt{n}}{\sqrt{n-1}} \pi^m(X_1 = y_1) (1 - z_1^2/2 + o(z_1^2)) (1 + o(1/\sqrt{n})),
\end{aligned}$$

where we used a Taylor expansion in the second equality. Using once more Lemma 4.1, under conditions (4.3), we have as $a_n \rightarrow \infty$

$$z_1^2 = o(1/\sqrt{n}),$$

whence we get

$$\begin{aligned}
p(X_1 = y_1 | S_1^n = na_n) &= \left(\frac{\sqrt{n}}{\sqrt{n-1}} \pi^m(X_1 = y_1) (1 + o(1/\sqrt{n})) \right) \\
&= \left(1 + o\left(\frac{1}{\sqrt{n}}\right) \right) \pi^m(X_1 = y_1),
\end{aligned}$$

which completes the proof.

4.2 Gibbs conditional principle in variation norm

4.2.1 Strengthening the local approximation

We now turn to a stronger approximation of p_{a_n} . Consider Y_1, \dots, Y_n with distribution P_{a_n} and Y_1 with density p_{a_n} and the resulting random variable $p_{a_n}(Y_1)$. We prove the following result

Theorem 4.2. *With all the above notation and hypotheses it holds*

$$p_{a_n}(Y_1) = g_{a_n}(Y_1)(1 + R_n)$$

where

$$g_{a_n} = \pi^{a_n}$$

the tilted density at point a_n , and where R_n is a function of Y_1, \dots, Y_n such that $P_{a_n}(|R_n| > \delta\sqrt{n}) \rightarrow 0$ as $n \rightarrow \infty$ for any positive δ .

This result is of greater relevance than the previous one. Indeed under P_{a_n} the r.v. Y_1 may take large values as n tends to infinity. At the contrary the approximation of p_{a_n} by g_{a_n} on any y_1 in \mathbb{R}_+ only provides some knowledge on p_{a_n} on sets with smaller and smaller probability under p_{a_n} as n increases. Also it will be proved that as a consequence of the above result, the L^1 norm between p_{a_n} and g_{a_n} goes to 0 as $n \rightarrow \infty$, a result out of reach through the aforementioned result.

In order to adapt the proof of Theorem 4.1 to the present setting it is necessary to get some insight on the plausible values of Y_1 under P_{a_n} . It holds

Lemma 4.2. *Under P_{a_n} it holds*

$$Y_1 = O_{P_{a_n}}(a_n).$$

Proof: Without loss of generality we can assume $Y_1 > 0$. By Markov Inequality:

$$P(Y_1 > u | S_1^n = na_n) \leq \frac{E(Y_1 | S_1^n = na_n)}{u} = \frac{a_n}{u}$$

which goes to 0 for all $u = u_n$ such that $\lim_{n \rightarrow \infty} u_n/a_n = \infty$.

We now turn back to the proof of Theorem 4.2. Define

$$Z_1 := \frac{na_n - Y_1}{s\sqrt{n-1}} \tag{4.5}$$

the natural counterpart of z_1 as defined in (4.4).

It holds

$$Z_1 = O_{p_{a_n}}(1/\sqrt{n}).$$

$$P(X_1 = Y_1 | S_1^n = na_n) = P(X_1 = Y_1) \frac{P(S_2^n = na_n - Y_1)}{P(S_1^n = na_n)}$$

in which the tilting substitution of measures is performed, with tilting density π^{a_n} , followed by normalization. Now following verbatim the proof of Theorem 4.1 if the growth condition (4.3) holds, it follows that

$$P(X_1 = Y_1 | S_1^n = na_n) = \pi^{a_n}(Y_1)(1 + R_n)$$

as claimed where the order of magnitude of R_n is $o_{P_{a_n}}(1/\sqrt{n})$. We have proved Theorem 4.2.

Denote the conditional probabilities by P_{a_n} and G_{a_n} on \mathbb{R}^n , which correspond to the marginal density functions p_{a_n} and g_{a_n} on \mathbb{R} , respectively.

4.2.2 From approximation in probability to approximation in variation norm

We now consider the approximation of the margin of P_{a_n} by G_{a_n} in variation norm.

The main ingredient is the fact that in the present setting approximation of p_{a_n} by g_{a_n} in probability plus some rate implies approximation of the corresponding measures in variation norm. This approach has been developed in [6]; we state a first lemma which states that whether two densities are equivalent in probability with small relative error when measured according to the first one, then the same holds under the sampling of the second.

Let \mathfrak{R}_n and \mathfrak{S}_n denote two p.m.'s on \mathbb{R}^n with respective densities \mathfrak{r}_n and \mathfrak{s}_n .

Lemma 4.3. *Suppose that for some sequence ε_n which tends to 0 as n tends to infinity*

$$\mathfrak{r}_n(Y_1^n) = \mathfrak{s}_n(Y_1^n) (1 + o_{\mathfrak{R}_n}(\varepsilon_n)) \quad (4.6)$$

as n tends to ∞ . Then

$$\mathfrak{s}_n(Y_1^n) = \mathfrak{r}_n(Y_1^n) (1 + o_{\mathfrak{S}_n}(\varepsilon_n)). \quad (4.7)$$

Proof. Denote

$$A_{n,\varepsilon_n} := \{y_1^n : (1 - \varepsilon_n)\mathfrak{s}_n(y_1^n) \leq \mathfrak{r}_n(y_1^n) \leq \mathfrak{s}_n(y_1^n) (1 + \varepsilon_n)\}.$$

It holds for all positive δ

$$\lim_{n \rightarrow \infty} \mathfrak{R}_n(A_{n,\delta\varepsilon_n}) = 1.$$

Write

$$\mathfrak{R}_n(A_{n,\delta\varepsilon_n}) = \int \mathbf{1}_{A_{n,\delta\varepsilon_n}}(y_1^n) \frac{\mathfrak{r}_n(y_1^n)}{\mathfrak{s}_n(y_1^n)} \mathfrak{s}_n(y_1^n) dy_1^n.$$

Since

$$\mathfrak{R}_n(A_{n,\delta\varepsilon_n}) \leq (1 + \delta\varepsilon_n)\mathfrak{S}_n(A_{n,\delta\varepsilon_n})$$

it follows that

$$\lim_{n \rightarrow \infty} \mathfrak{S}_n(A_{n,\delta\varepsilon_n}) = 1,$$

which proves the claim. □

Applying this Lemma to the present setting yields

$$g_{a_n}(Y_1) = p_{a_n}(Y_1) (1 + o_{G_{a_n}}(1/\sqrt{n}))$$

as $n \rightarrow \infty$.

This fact entails, as in [6]

Theorem 4.3. *Under all the notation and hypotheses above the total variation norm between the marginal distribution of P_{a_n} and G_{a_n} goes to 0 as $n \rightarrow \infty$.*

The proof goes as follows

For all $\delta > 0$, let

$$E_\delta := \left\{ y \in \mathbb{R} : \left| \frac{p_{a_n}(y) - g_{a_n}(y)}{g_{a_n}(y)} \right| < \delta \right\}$$

which, turning to the marginal distribution of P_{a_n} writes

$$\lim_{n \rightarrow \infty} P_{a_n}(E_\delta) = \lim_{n \rightarrow \infty} G_{a_n}(E_\delta) = 1. \quad (4.8)$$

It holds

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P_{a_n}(C \cap E_\delta) - G_{a_n}(C \cap E_\delta)| \leq \delta \sup_{C \in \mathcal{B}(\mathbb{R})} \int_{C \cap E_\delta} g_{a_n}(y) dy \leq \delta.$$

By the above result (4.8)

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P_{a_n}(C \cap E_\delta) - P_{a_n}(C)| < \eta_n$$

and

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |G_{a_n}(C \cap E_\delta) - G_{a_n}(C)| < \eta_n$$

for some sequence $\eta_n \rightarrow 0$; hence

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P_{a_n}(C) - G_{a_n}(C)| < \delta + 2\eta_n$$

for all positive δ , which proves the claim.

As a consequence, applying Scheffé's Lemma we have proved

$$\int |p_{a_n} - g_{a_n}| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

as sought.

Remark 4.1. *This result is to be paralleled with Theorem 1.6 in Diaconis and Freedman [9] and Theorem 2.15 in Dembo and Zeitouni [8] which provide a rate for this convergence in the LDP range.*

4.3 The asymptotic location of X under the conditioned distribution

This section intends to provide some insight on the behaviour of X_1 under the condition ($S_1^n = na_n$). It will be seen that conditionally on ($S_1^n = na_n$) the marginal distribution of the sample concentrates around a_n . Let \mathcal{X}_t be a r.v. with density π^{a_n} where $m(t) = a_n$ and a_n satisfies (4.3). Recall that $E\mathcal{X}_t = a_n$ and $Var\mathcal{X}_t = s^2$. We evaluate the moment generating function of the normalized variable $(\mathcal{X}_t - a_n)/s$. It holds

$$\log E \exp \lambda (\mathcal{X}_t - a_n) / s = -\lambda a_n / s + \log \phi \left(t + \frac{\lambda}{s} \right) - \log \phi (t) .$$

A second order Taylor expansion in the above display yields

$$\log E \exp \lambda (\mathcal{X}_t - a_n) / s = \frac{\lambda^2 s^2 \left(t + \frac{\theta \lambda}{s} \right)}{2 s^2}$$

where $\theta = \theta(t, \lambda) \in (0, 1)$. The following Lemma, which is proved in the Appendix, states that the function $t \rightarrow s(t)$ is self neglecting. Namely

Lemma 4.4. *Under the above hypotheses and notation, for any compact set K*

$$\lim_{n \rightarrow \infty} \sup_{u \in K} \frac{s^2 \left(t + \frac{u}{s} \right)}{s^2} = 1 .$$

Applying the above Lemma it follows that the normalized r.v.'s $(\mathcal{X}_t - a_n)/s$ converge to a standard normal variable $N(0, 1)$ in distribution, as $n \rightarrow \infty$. This amount to say that

$$\mathcal{X}_t = a_n + sN(0, 1) + o_{\Pi^{a_n}}(1) . \tag{4.9}$$

In the above formula (4.9) the remainder term $o_{\Pi^{a_n}}(1)$ is of smaller order than $sN(0, 1)$ since the variance of a r.v. with distribution Π^{a_n} is s^2 . When $\lim_{t \rightarrow \infty} s(t) = 0$ then \mathcal{X}_t concentrates around a_n with rate $s(t)$. Due to Theorem 4.3 the same holds for X_1 under ($S_1^n = na_n$). This is indeed the case when g is a regularly varying function with index γ larger than 2, plus some extra regularity conditions captured in the fact that h belongs to $R_{\gamma-1}$. The gaussian tail corresponds to $\gamma = 2$ and the variance of the tilted distribution Π^{a_n} has a non degenerate variance for all a_n (in the standard gaussian case, $\Pi^{a_n} = N(a_n, 1)$), as does the conditional distribution of X_1 given ($S_1^n = na_n$).

4.4 Extension to other conditioning events

Let X_1, \dots, X_n be n i.i.d. r.v.'s with common density p defined on \mathbb{R}^d and let f denote a measurable function from \mathbb{R}^d onto \mathbb{R} such that $f(X_1)$ has a density p_f . We assume that p_f enjoys all properties stated in Section 2, i.e. $p_f(x) = \exp - (g(x) - q(x))$ and we denote accordingly $\phi_f(t)$ its moment generating function, and $m_f(t)$ and $s_f^2(t)$ the corresponding first and second derivatives of $\log \phi_f(t)$. The following extension of Theorem 4.2 holds. Denote $\Sigma_1^n := f(X_1) + \dots + f(X_n)$.

Denote for all a in the range of f

$$\pi_f^a(x) := \frac{\exp tf(x)}{\phi_f(t)} p(x) \quad (4.10)$$

where t is the unique solution of $m(t) := (d/dt) \log \phi_f(t) = a$ and Π_f^a the corresponding probability measure. Denote P_{f,a_n} the conditional distribution of (X_1, \dots, X_n) given $(\Sigma_1^n = na_n)$. The sequence a_n is assumed to satisfy (4.3) where ψ is defined with respect to p_f .

Theorem 4.4. *Under the current hypotheses of this section the variation distance between the margin of P_{f,a_n} and $\Pi_f^{a_n}$ goes to 0 as n tends to infinity.*

Proof: It holds, for Y_1 a r.v. with density p ,

$$\begin{aligned} p(X_1 = Y_1 | \Sigma_1^n = na_n) &= p(X_1 = Y_1) \frac{p(\Sigma_2^n = na_n - f(Y_1))}{p(\Sigma_1^n = na_n)} \\ &= \frac{p(X_1 = Y_1)}{p(f(X_1) = f(Y_1))} p(f(X_1) = f(Y_1)) \frac{p(\Sigma_2^n = na_n - f(Y_1))}{p(\Sigma_1^n = na_n)} \\ &= \frac{p(X_1 = Y_1)}{p(f(X_1) = f(Y_1))} \pi_f^m(f(X_1) = f(Y_1)) \frac{\pi_f^m(\Sigma_2^n = na_n - f(Y_1))}{\pi_f^m(\Sigma_1^n = na_n)} \\ &= p(X_1 = Y_1) \frac{e^{tf(Y_1)} \pi_f^m(\Sigma_2^n = na_n - f(Y_1))}{\phi_f(t) \pi_f^m(\Sigma_1^n = na_n)} \end{aligned}$$

where $m := (d/dt) \log \phi_f(t) = a_n$. The proof then follows verbatim that of Theorem 4.2 with X_i substituted by $f(X_i)$.

4.5 Differences between Gibbs principle under LDP and under extreme deviation

It is of interest to confront the present results with the general form of the Gibbs principle under linear constraints in the LDP range.

Consider the application of the above result to r.v's Y_1, \dots, Y_n with $Y_i := (X_i)^2$ where the X_i 's are i.i.d. and are such that the density of the i.i.d. r.v's Y_i 's satisfy (2.1) with all the hypotheses stated in this paper, assuming that Y has a Weibull distribution with parameter larger than 2. By the Gibbs conditional principle under a point conditioning (see e.g. [9]), for *fixed* a , conditionally on $(\sum_{i=1}^n Y_i = na)$ the generic r.v. Y_1 has a non degenerate limit distribution

$$p_Y^*(y) := \frac{\exp ty}{E \exp tY_1} p_Y(y)$$

and the limit density of X_1 under $(\sum_{i=1}^n X_i^2 = na)$ is

$$p_X^*(x) := \frac{\exp tx^2}{E \exp tX_1^2} p_X(x)$$

a non degenerate distribution, with $m_Y(t) = a$. As a consequence of the above result, when $a_n \rightarrow \infty$ the distribution of X_1 under the condition $(\sum_{i=1}^n X_i^2 = na_n)$ concentrates sharply at $-\sqrt{a_n}$ and $+\sqrt{a_n}$.

5 EDP under exceedances

5.1 DLP and EDP

This section extends the previous Theorem 4.2 when the conditioning event has non null measure and writes $A_n = (S_1^n \geq na_n)$. It also provides a bridge between the "democratic localization principle (DLP)" (1.1) and the conditional Gibbs principle. We denote the density of X_1 given A_n by p_{A_n} to differentiate it from p_{a_n} .

In the LDP case, when $a_n = a > EX_1$ then the distribution of X_1 given $(S_1^n \geq na)$ or given $(S_1^n = na)$ coincide asymptotically, both converging to the tilted distribution at point a , the dominating point of $[a, \infty)$. This result follows from the local approach derived in [9], which uses limit results for sums of i.i.d. r.v's, and from Sanov Theorem which amounts to identify the Kullback-Leibler projection of the p.m. P on the set of all p.m.'s with expectation larger or equal a . When a is allowed to tend to infinity with n no Sanov-type result is available presently, and the first approach, although cumbersome, has to be used. The "dominating point property" still holds.

Most applications of extreme deviation principles deal with phenomenons driven by multiplicative cascade processes with power laws; see e.g. [11] for fragmentation models and turbulence. For sake of simplicity we restrict to this case, assuming therefore that h belongs to R_{k-1} for some $k > 1$. All conditions and notation of Section 2 are assumed to hold. Furthermore, since we consider the asymptotics of the marginal distribution of the sample under A_n , it is natural to assume that a_n is such that the democratic localization principle holds together with the local Gibbs conditional principle.

Specialized to the present setting it holds (see Theorem 6 in [4])

Theorem 5.1. Assume that for some $\delta > 0$

$$\liminf_{n \rightarrow \infty} \frac{\log g(a_n)}{\log n} > \delta \quad (5.1)$$

and define a sequence ϵ_n such that

$$\lim_{n \rightarrow \infty} \frac{n \log a_n}{a_n^{k-2} \epsilon_n^2} = 0. \quad (5.2)$$

Then

$$\lim_{n \rightarrow \infty} P \left(\bigcap_{i=1}^n (a_n - \epsilon_n < X_i < a_n + \epsilon_n) \middle| S_1^n/n \geq a_n \right) = 1. \quad (5.3)$$

Furthermore, when $1 < k \leq 2$ then ϵ_n can be chosen to satisfy $\epsilon_n/a_n \rightarrow 0$, and when $k > 2$, ϵ_n can be chosen to satisfy $\epsilon_n \rightarrow 0$.

We assume that both conditions (5.1) and (4.3) hold. Therefore we assume that the range of a_n makes both the DLP and EDP hold. This is the case for example when

$$a_n = n^\alpha$$

together with

$$\alpha < \frac{2}{2+k}.$$

We state the following extension of Theorems 5.1 and 4.3.

Theorem 5.2. Assume (5.1) and (5.2). Then for any family of Borel sets B_n such that

$$\liminf_{n \rightarrow \infty} P_{A_n}(B_n) > 0$$

it holds

$$P_{A_n}(B_n) = (1 + o(1))G_{a_n}(B_n)$$

as $n \rightarrow \infty$.

Remark 5.1. This Theorem is of the same kind as those related to conditioning on thin sets, in the range of the LDP; see [7].

Proof of Theorem 5.2:

For the purpose of the proof, we need the following lemma, based on Theorem 6.2.1 of Jensen [12], in order to provide the asymptotic estimation of the tail probability $P(S_1^n \geq na_n)$. Its proof is deferred to the Appendix.

Define

$$I(x) := xm^{-1}(x) - \log \Phi(m^{-1}(x)). \quad (5.4)$$

and let t_n be defined through

$$m(t_n) = a_n.$$

Lemma 5.1. *Let X_1, \dots, X_n be i.i.d. random variables with density p defined in (2.1) and $h \in \mathfrak{R}$. Suppose that when $n \rightarrow \infty$, it holds*

$$\frac{\psi(t_n)^2}{\sqrt{n}\psi'(t_n)} \rightarrow 0.$$

Then it holds

$$P(S_1^n \geq na_n) = \frac{\exp(-nI(a_n))}{\sqrt{2\pi}\sqrt{nt_n}s(t_n)} \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right). \quad (5.5)$$

Denote $\Delta_n := (a_n - \epsilon_n, a_n + \epsilon_n)$ and $\Delta_n^+ := (a_n, a_n + \epsilon_n)$. It holds

$$P(S_1^n/n \in \Delta_n | S_1^n/n \geq a_n) \geq P\left(\bigcap_{i=1}^n (X_i \in \Delta_n) \mid S_1^n/n \geq a_n\right)$$

hence by Theorem 5.1

$$P(S_1^n/n \in \Delta_n^+ | S_1^n/n \geq a_n) = P(S_1^n/n \in \Delta_n | S_1^n/n \geq a_n) \rightarrow 1$$

which yields

$$\frac{P(S_1^n/n \in \Delta_n^+)}{P(S_1^n/n > a_n)} \rightarrow 1.$$

Since $(S_1^n/n \in \Delta_n^+) \subset (S_1^n/n > a_n)$ it follows that for any Borel set B (depending on n or not)

$$P(X_1 \in B | S_1^n/n > a_n) = (1 + o(1))P(X_1 \in B | S_1^n/n \in \Delta_n^+) + C_n$$

where

$$0 < C_n < \frac{P(S_1^n/n \geq a_n + \epsilon_n)}{P(S_1^n/n \geq a_n)}.$$

Using Lemma 5.1 both in the numerator and the denominator, with some control using (5.1) and (5.2) proving that (5.5) holds with a_n substituted by $a_n + \epsilon_n$, we get $C_n \rightarrow 0$.

By Theorem 5.1, with $B = \Delta_n$

$$P(X_1 \in \Delta_n | S_1^n/n \geq a_n) \rightarrow 1$$

which in turn implies that

$$P(X_1 \in \Delta_n | S_1^n/n \in \Delta_n^+) \rightarrow 1$$

which proves that the conditional distribution of X_1 given $S_1^n/n \geq a_n$ concentrates on Δ_n , as does its distribution given $S_1^n/n \in \Delta_n^+$. Furthermore both distributions are asymptotically equivalent in the sense that

$$\frac{P(X_1 \in B_n | S_1^n/n \in \Delta_n^+)}{P(X_1 \in B_n | S_1^n/n \geq a_n)} \rightarrow 1$$

for all sequence of Borel sets B_n such that

$$\liminf_{n \rightarrow \infty} P(X_1 \in B_n | S_1^n/n \geq a_n) > 0.$$

We now prove Theorem 5.2.

For a Borel set $B = B_n$ it holds

$$\begin{aligned} P_{A_n}(B) &= \int_{a_n}^{\infty} P_v(B) p(S_1^n/n = v | S_1^n/n > a_n) dv \\ &= \frac{1}{P(S_1^n/n > a_n)} \int_{a_n}^{\infty} P_v(B) p(S_1^n/n = v) dv \\ &= (1 + o(1)) \frac{1}{P(S_1^n/n \in \Delta_n^+)} \int_{a_n}^{\infty} P_v(B) p(S_1^n/n = v) dv \\ &= (1 + o(1)) \frac{1}{P(S_1^n/n \in \Delta_n^+)} \int_{a_n}^{a_n + \epsilon_n} P_v(B) p(S_1^n/n = v) dv \\ &\quad + (1 + o(1)) \frac{1}{P(S_1^n/n \in \Delta_n^+)} \int_{a_n + \epsilon_n}^{\infty} P_v(B) p(S_1^n/n = v) dv \\ &= (1 + o(1)) \int_{a_n}^{a_n + \epsilon_n} P_v(B) p(S_1^n/n = v | S_1^n/n \in \Delta_n^+) dv \\ &\quad + (1 + o(1)) \frac{1}{P(S_1^n/n \in \Delta_n^+)} \int_{a_n + \epsilon_n}^{\infty} P_v(B) p(S_1^n/n = v) dv \\ &= (1 + o(1)) P_{a_n + \theta_n \epsilon_n}(B) \\ &\quad + (1 + o(1)) \frac{1}{P(S_1^n/n \in \Delta_n^+)} \int_{a_n + \epsilon_n}^{\infty} P_v(B) p(S_1^n/n = v) dv \\ &= (1 + o(1)) P_{a_n + \theta_n \epsilon_n}(B) + C_n \end{aligned}$$

for some θ_n in $(0, 1)$. The term C_n is less than $P(S_1^n/n > a_n + \epsilon_n)/P(S_1^n/n > a_n)$ which tends to 0 as n tends to infinity, under (5.1) and (5.2), using Lemma 5.1.

Hence when B_n is such that $\liminf_{n \rightarrow \infty} P_{a_n + \theta_n \epsilon_n}(B_n) > 0$ using Theorem 4.3 it holds, since $a'_n := a_n + \theta_n \epsilon_n$ satisfies (4.3)

$$P_{A_n}(B_n) = (1 + o(1)) G_{a_n + \theta_n \epsilon_n}(B_n). \quad (5.6)$$

It remains to prove that

$$G_{a_n + \theta_n \epsilon_n}(B_n) = (1 + o(1)) G_{a_n}(B_n). \quad (5.7)$$

Make use of (4.9) to prove the claim. Define t through $m(t) = a_n$ and t' through $m(t') = a'_n$. It is enough to obtain

$$s(t) = (1 + o(1)) s(t') \quad (5.8)$$

as $n \rightarrow \infty$. In the present case when h belongs to R_{k-1} , making use of Theorem 3.1 it holds

$$s^2(t) = C(1 + o(1))m(t)^{2-k}$$

for some constant C ; a similar formula holds for $s^2(t')$ which together yield (5.8), whatever $k > 1$.

This proves (5.7) and concludes the proof of Theorem 5.2.

Remark 5.2. *By (4.9) and Theorem 5.2 the r.v. X_1 conditioned upon $(S_1^n/n \geq a_n)$ has a standard deviation of order Ca_n^{2-k} , much smaller than ϵ_n which by (5.2) is larger than $\sqrt{n \log a_n/a_n^{(k-2)/2}}$; this proves that (5.3) might be improved. However the qualitative bound $k = 2$ appears both in the DLP and in the EDP.*

5.2 High level sets of functions

We explore the consequences of Theorem 5.2 in connection with (1.1) in the context when the conditioning event writes

$$A_n := \frac{1}{n} \sum_{i=1}^n f(X_i) \geq a_n$$

where the i.i.d. r.v's X_1, \dots, X_n belong to \mathbb{R}^d , and f is a real valued function such that the density p_f of $f(X_1)$ satisfies all the hypotheses of Section 2. We denote P_{A_n} the distribution of (X_1, \dots, X_n) given A_n . We assume that p^f satisfies (2.1) with h in R_{k-1} , $k > 1$.

Using the DLP stated in Section 5.1 it holds

$$P_{A_n} \left(\bigcap_{i=1}^n (f(X_i) \in (a_n - \epsilon_n, a_n + \epsilon_n)) \right) \rightarrow 1$$

which implies that

$$P_{A_n} (f(X_1) \in (a_n - \epsilon_n, a_n + \epsilon_n)) \rightarrow 1.$$

In turn, using Theorem 5.2

$$G_{a_n} (f(X_1) \in (a_n - \epsilon_n, a_n + \epsilon_n)) \rightarrow 1$$

where G_{a_n} has density g_{a_n} defined through

$$g_{a_n}(y) := \frac{e^{ty}}{\phi^f(t)} p^f(y)$$

with $\phi^f(t) := Ee^{tf(X)}$ and $m^f(t) = a_n$. Note that following (4.9), G_{a_n} is nearly gaussian with expectation a_n and standard deviation $s^f(t) := (d/dt)m^f(t)$.

Turning back to Theorem 4.4 it then holds

Theorem 5.1. *When the above hypotheses hold let a_n satisfies*

$$\liminf_{n \rightarrow \infty} \frac{\log g(a_n)}{\log n} > \delta$$

with h in R_{k-1} , $k > 1$ and

$$g_{a_n}(x) := \frac{e^{tf(x)}}{\phi^f(t)} p(x)$$

where $m^f(t) = a_n$. Then there exists a sequence ϵ_n satisfying $\lim_{n \rightarrow \infty} \epsilon_n/n = 0$ such that a r.v. X in \mathbb{R}^d with density g_{a_n} is a solution of the inequation

$$a_n - \epsilon_n < f(x) < a_n + \epsilon_n$$

with probability 1 as $n \rightarrow \infty$. The sequence ϵ_n satisfies

$$\lim_{n \rightarrow \infty} \frac{n \log a_n}{a_n^{k-2} \epsilon_n^2} = 0.$$

6 Appendix

6.1 Proof of Lemma 4.4

Case 1: if $h \in R_\beta$. By Theorem 3.1, it holds $s^2 \sim \psi'(t)$ with $\psi(t) \sim t^{1/\beta} l_1(t)$, where l_1 is some slowly varying function. It also holds $\psi'(t) = 1/h'(\psi(t))$; therefore

$$\begin{aligned} \frac{1}{s^2} &\sim h'(\psi(t)) = \psi(t)^{\beta-1} l_0(\psi(t)) (\beta + \epsilon(\psi(t))) \\ &\sim \beta t^{1-1/\beta} l_1(t)^{\beta-1} l_0(\psi(t)) = o(t), \end{aligned}$$

which implies that for any $u \in K$

$$\frac{u}{s} = o(t).$$

It follows that

$$\begin{aligned} \frac{s^2(t+u/s)}{s^2} &\sim \frac{\psi'(t+u/s)}{\psi'(t)} = \frac{\psi(t)^{\beta-1} l_0(\psi(t)) (\beta + \epsilon(\psi(t)))}{(\psi(t+u/s))^{\beta-1} l_0(\psi(t+u/s)) (\beta + \epsilon(\psi(t+u/s)))} \\ &\sim \frac{\psi(t)^{\beta-1}}{\psi(t+u/s)^{\beta-1}} \sim \frac{t^{1-1/\beta} l_1(t)^{\beta-1}}{(t+u/s)^{1-1/\beta} l_1(t+u/s)^{\beta-1}} \rightarrow 1. \end{aligned}$$

Case 2: if $h \in R_\infty$. Then we have $\psi(t) \in \widetilde{R}_0$; hence it holds

$$\frac{1}{st} \sim \frac{1}{t\sqrt{\psi'(t)}} = \sqrt{\frac{1}{t\psi(t)\epsilon(t)}} \longrightarrow 0.$$

Hence for any $u \in K$, we get as $n \rightarrow \infty$

$$\frac{u}{s} = o(t),$$

thus using the slowly varying propriety of $\psi(t)$ we have

$$\begin{aligned} \frac{s^2(t+u/s)}{s^2} &\sim \frac{\psi'(t+u/s)}{\psi'(t)} = \frac{\psi(t+u/s)\epsilon(t+u/s)}{t+u/s} \frac{t}{\psi(t)\epsilon(t)} \\ &\sim \frac{\epsilon(t+u/s)}{\epsilon(t)} = \frac{\epsilon(t) + O(\epsilon'(t)u/s)}{\epsilon(t)} \longrightarrow 1, \end{aligned}$$

where we used a Taylor expansion in the second line. This completes the proof.

6.2 Proof of Lemma 5.1

For the density p defined in (2.1), we show that g is convex when x is large. If $h \in R_\beta$, it holds for x large

$$g''(x) = h'(x) = \frac{h(x)}{x}(\beta + \epsilon(x)) > 0.$$

If $h \in R_\infty$, its reciprocal function $\psi(x)$ belongs to \widetilde{R}_0 . Set $x := \psi(u)$; for x large

$$g''(x) = h'(x) = \frac{1}{\psi'(u)} = \frac{u}{\psi(u)\epsilon(u)} > 0,$$

where the inequality holds since $\epsilon(u) > 0$ under condition (2.7), for large u . Therefore g is convex for large x .

Therefore, the density p with $h \in \mathfrak{R}$ satisfies the conditions of Jensen's Theorem 6.2.1 ([12]). A third order Edgeworth expansion in formula (2.2.6) of ([12]) yields

$$P(S_1^n \geq na_n) = \frac{\Phi(t_n)^n \exp(-nt_n a_n)}{\sqrt{nt_n s(t_n)}} \left(B_0(\lambda_n) + O\left(\frac{\mu_3(t_n)}{6\sqrt{n}s^3(t_n)} B_3(\lambda_n)\right) \right), \quad (6.1)$$

where $\lambda_n = \sqrt{nt_n s(t_n)}$, and $B_0(\lambda_n)$ and $B_3(\lambda_n)$ are defined by

$$B_0(\lambda_n) = \frac{1}{\sqrt{2\pi}} \left(1 - \frac{1}{\lambda_n^2} + o\left(\frac{1}{\lambda_n^2}\right) \right), \quad B_3(\lambda_n) \sim -\frac{3}{\sqrt{2\pi}\lambda_n}.$$

We show that, as $a_n \rightarrow \infty$,

$$\frac{1}{\lambda_n^2} = o\left(\frac{1}{n}\right). \quad (6.2)$$

Since $n/\lambda_n^2 = 1/(t_n^2 s^2(t_n))$, (6.2) is equivalent to show that

$$t_n^2 s^2(t_n) \rightarrow \infty. \quad (6.3)$$

By Theorem 3.1, $m(t_n) \sim \psi(t_n)$ and $s^2(t_n) \sim \psi'(t_n)$; this entails that $t_n \sim h(a_n)$.

If $h \in R_\beta$, notice that

$$\psi'(t_n) = \frac{1}{h'(\psi(t_n))} = \frac{\psi(t_n)}{h(\psi(t_n))(\beta + \epsilon(\psi(t_n)))} \sim \frac{a_n}{h(a_n)(\beta + \epsilon(\psi(t_n)))}.$$

holds. Hence we have

$$t_n^2 s^2(t_n) \sim h(a_n)^2 \frac{a_n}{h(a_n)(\beta + \epsilon(\psi(t_n)))} = \frac{a_n h(a_n)}{\beta + \epsilon(\psi(t_n))} \rightarrow \infty.$$

If $h \in R_\infty$, then $\psi(t_n) \in \widetilde{R}_0$, it follows that

$$t_n^2 s^2(t_n) \sim t_n^2 \frac{\psi(t_n)\epsilon(t_n)}{t_n} = t_n \psi(t_n)\epsilon(t_n) \rightarrow \infty,$$

where the last step holds from condition (2.7). We have shown (6.2). Therefore

$$B_0(\lambda_n) = \frac{1}{\sqrt{2\pi}} \left(1 + o\left(\frac{1}{n}\right)\right).$$

By (6.3), λ_n goes to ∞ as $a_n \rightarrow \infty$, which implies further that $B_3(\lambda_n) \rightarrow 0$. On the other hand, by Theorem 3.1, $\mu_3/s^3 \rightarrow 0$. Hence we obtain from (6.1)

$$P(S_1^n \geq na_n) = \frac{\Phi(t_n)^n \exp(-nt_n a_n)}{\sqrt{2\pi n t_n s(t_n)}} \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right),$$

which together with (5.4) proves the claim.

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