



HAL
open science

Homotopy BV-algebra structure on the double cobar construction

Alexandre Quesney

► **To cite this version:**

Alexandre Quesney. Homotopy BV-algebra structure on the double cobar construction. 2013. hal-00822303v2

HAL Id: hal-00822303

<https://hal.science/hal-00822303v2>

Preprint submitted on 12 Jun 2013 (v2), last revised 19 Jun 2014 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Homotopy BV-algebra structure on the double cobar construction

Alexandre Quesney

Abstract

We show that the double cobar construction, $\Omega^2 C_*(X)$, of a simplicial set X is a homotopy BV-algebra if X is a double suspension, or if X is 2-reduced and the coefficient ring contains the ring of rational numbers \mathbb{Q} . Indeed, Connes-Moscovici's operator defines the desired homotopy BV-algebra structure on $\Omega^2 C_*(X)$ when the antipode $S : \Omega C_*(X) \rightarrow \Omega C_*(X)$ is involutive. We proceed by defining a family of obstructions $O_n : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes n}$, $n \geq 2$ measuring the difference $S^2 - Id$. When X is a suspension, the only obstruction remaining is $O_2 := E^{1,1} - \tau E^{1,1}$ where $E^{1,1}$ is the dual of the \smile_1 -product. When X is a double suspension the obstructions vanish. We make explicit a homotopy BV-algebra structure on $\Omega^2 C_*(X)$.

Contents

Introduction	4
1 Notations and preliminaries	4
1.1 Conventions and notations	4
1.2 The bar and cobar constructions	5
1.3 Hopf algebras	5
2 Homotopy structures on the cobar construction	6
2.1 Homotopy G-algebra on the cobar construction	6
2.2 Homotopy BV-algebra on the cobar construction	7
3 Involutivity of the antipode of ΩC in terms of homotopy G-coalgebra C	9
4 Correspondance between Baues' coproduct and the McClure and Smith action on $C_*(X)$	12
4.1 Baues' coproduct	13
4.2 Action of the operad $F_2\chi$ on $C_*(X)$	13
4.3 Correspondance of two homotopy G-coalgebra structures	14
5 Obstruction to the involutivity of the antipode of $\Omega C_*(\Sigma X)$	15
6 Homotopy BV-algebra structure on $\Omega^2 C_*(\Sigma^2 X)$	17
7 Homotopy BV-algebra structure on $\Omega^2 C_*(X)$ over $R \supset \mathbb{Q}$	18

Introduction

Adams' cobar construction is a model of the loop space of a 1-connected topological space [Ada56]. The cobar construction is a functor from differential graded coalgebras to differential graded algebras; for iteration, a coproduct is thus needed. For a 1-reduced simplicial set X , Baues defined [Bau98] a DG-bialgebra structure on its first cobar construction $\Omega C_*(X)$. The resulting double cobar construction is an algebraic model for the double loop space. We show that the double cobar construction, $\Omega^2 C_*(X)$, of a simplicial set X is a homotopy BV-algebra if X is a double suspension, or if X is 2-reduced and the coefficient ring contains the ring of rational numbers \mathbb{Q} .

Baues' coproduct on $\Omega C_*(X)$ tantamounts to a homotopy G-coalgebra structure $\{E^{k,1}\}_{k \geq 1}$ on the DG-coalgebra $C_*(X)$, that is a family of operations

$$E^{k,1} : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes k} \otimes \tilde{C}_*(X),$$

for $k \geq 1$, satisfying some relations. This corresponds also to the coalgebra structure of $C_*(X)$ over the second stage filtration operad $F_2\chi$ of the surjection operad χ given in [MS01, BF04], see Section 4. Since a bialgebra structure determines the antipode (whenever it exists), the antipode

$$S : \Omega C_*(X) \rightarrow \Omega C_*(X)$$

on the cobar construction is then determined by the homotopy G-coalgebra structure on $C_*(X)$.

As a model of the chain complex of the double loop spaces, the double cobar construction is expected to be a Batalin-Vilkovisky algebra up to homotopy. Indeed, it is well-known that the circle action on the double loop space $\Omega^2 X$ by rotating the equator defines a BV operator; thereby the homology $H_*(\Omega^2 X)$ is a Batalin-Vilkovisky algebra [Get94].

The cobar construction $\Omega\mathcal{H}$ of an involutive Hopf algebra turns out to be the underlying complex in the Connes-Moscovici cyclic coHochschild homology of \mathcal{H} , [CM00]. The cyclic operator requires an involutive antipode on the underlying bialgebra \mathcal{H} . Under this hypothesis of an involutive antipode, Menichi proved [Men04], that for a unital (ungraded) Hopf algebra \mathcal{H} , Connes-Moscovici's operator induces a Batalin-Vilkovisky algebra on the homology of the cobar construction $H_*(\Omega\mathcal{H})$.

By defining a family of operations

$$O_n : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes n},$$

for $n \geq 2$, which measures the difference $S^2 - Id$, we provide a criterion for the involutivity of S in terms of the operations

$$E^{k,1} : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes k} \otimes \tilde{C}_*(X).$$

By Kadeishvili's work [Kad05] the double cobar construction is a homotopy G-algebra, that is endowed with a family of operations

$$E_{1,k} : \Omega^2 C_*(X) \otimes (\Omega^2 C_*(X))^{\otimes k} \rightarrow \Omega^2 C_*(X),$$

$k \geq 1$ satisfying some relations (dual from those of a homotopy G-coalgebra). This kind of structure induces a Gerstenhaber algebra on the homology $H_*(\Omega^2 C_*(X))$. The vanishing of operations

$$O_n : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes n},$$

gives a sufficient condition to the extension of the homotopy G-algebra $\Omega^2 C_*(X)$ given by Kadeishvili into the homotopy BV-algebra whose BV-operator is Connes-Moscovici's operator. We establish the following for a general homotopy G-coalgebra,

Proposition 3.3. *Let $(C, d, \nabla_C, E^{k,1})$ be a homotopy G-coalgebra.*

1. *The cobar construction ΩC is an involutive DG-Hopf algebra if and only if all the obstructions $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$ defined in (3.5) for $n \geq 2$ are zero.*
2. *Let C be 2-reduced i.e $C_0 = R$ and $C_1 = C_2 = 0$. If all the obstructions $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$ are zero, then the double cobar construction $\Omega^2 C$ is a homotopy BV-algebra given by Connes-Moscovici's operator.*

We apply this criterion to the homotopy G-coalgebra $C_*(X)$ of a 1-reduced simplicial set X . We show that when ΣX is a simplicial suspension, the family of operations

$$O_n : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes n}$$

reduces to

$$O_2 : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X) \otimes \tilde{C}_*(\Sigma X).$$

This operation is the deviation from the cocommutativity of the dual of the \smile_1 -product,

$$E^{1,1} : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X) \otimes \tilde{C}_*(\Sigma X),$$

the only non-trivial operation of the family $\{E^{k,1}\}$ defining the homotopy G-algebra structure on $C_*(\Sigma X)$, see Propositions 5.3 and 5.4. However in the case of a double simplicial suspension $\Sigma^2 X$, the homotopy G-coalgebra $C_*(\Sigma^2 X)$ becomes trivial and all the obstructions

$$O_n : \tilde{C}_*(\Sigma^2 X) \rightarrow \tilde{C}_*(\Sigma^2 X)^{\otimes n},$$

for $n \geq 2$ are zero. As a consequence, the cobar construction $\Omega C_*(\Sigma^2 X)$ is the free tensor DG-algebra with the shuffle coproduct. Thus we have,

Theorem 6.1. *Let $\Sigma^2 X$ be a double suspension. Then :*

- *the homotopy G-coalgebra structure on $C_*(\Sigma^2 X)$ corresponding to Baues' coproduct*

$$\nabla_0 : \Omega C_*(\Sigma^2 X) \rightarrow \Omega C_*(\Sigma^2 X),$$

is trivial ;

- *the double cobar construction $\Omega^2 C_*(\Sigma^2 X)$ is a homotopy BV-algebra with Connes-Moscovici's operator as BV-operator.*

On the other hand, if the ground ring contains the ring \mathbb{Q} , we deform (see Section 7 for the precise statement) $(\Omega C_*(X), \nabla_0, S)$ into a cocommutative DG-Hopf algebra $(\Omega C_*(X), \nabla'_0, S')$. Therefore, $(\Omega C_*(X), \nabla'_0, S')$ has an involutive antipode and the homotopy BV-algebra structure we consider on the deformed double cobar construction $\Omega(\Omega C_*(X), \nabla'_0, S')$ follows. Thus, we have

Theorem 7.1. *Let X be a 2-reduced simplicial set. Then the double cobar construction $\Omega(\Omega C_*(X), \nabla'_0, S')$ over $R \supset \mathbb{Q}$ coefficients is a homotopy BV-algebra with BV-operator Connes-Moscovici's operator.*

The paper is organized as follows.

In the first section we review background material on the bar/cobar constructions and Hopf algebras.

The second section is devoted to the structures of homotopy G-algebras [Kad05] and homotopy BV-algebras [Men04] on the cobar construction. These first two sections fix notations and sign conventions.

In the third section we define the family of operations $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$, $n \geq 2$, on a homotopy G-coalgebra $(C, E^{k,1})$.

In the fourth section we show that the homotopy G-coalgebra structure given by Baues' coproduct coincide with the one given in [BF04, MS01].

Sections 5 and 6 give applications to single and double simplicial suspensions. We reduce the obstructions $O_n : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes n}$, $n \geq 2$ to O_2 for a single suspension. We show that for a double suspension this last obstruction O_2 vanishes. We write explicitly the resulting homotopy BV-algebra structure on $\Omega^2 C_*(\Sigma^2 X)$.

The last section is devoted to the rational case. We prove that the double cobar construction of a 2-reduced simplicial set is a homotopy BV-algebra.

Acknowledgment I would like to thank Jean-Claude Thomas for his very helpful remarks, in particular for suggesting the study of suspensions. I am grateful to Muriel Livernet for the useful conversation at IHP and her many comments about the content of this paper.

1 Notations and preliminaries

1.1 Conventions and notations

Let R be a commutative ring. A graded R -module M is a family of R -modules $\{M_n\}$ where indices n run through the integers. The degree of $a \in M_n$ is denoted by $|a|$, so here $|a| = n$. The r -suspension s^r is defined by $(s^r M)_n = M_{n-r}$.

Algebras (respectively coalgebras) are understood as associative algebras (respectively coassociative coalgebras).

A unital R -algebra (A, μ, η) is called augmented if there is an algebra morphism $\epsilon : A \rightarrow R$. We denote by \overline{A} the augmentation ideal $\text{Ker}(\epsilon)$.

For a coalgebra (C, ∇) the n -iterated coproduct is denoted by

$$\nabla^{(n)} : C \rightarrow C^{\otimes n+1}$$

for $n \geq 1$. We use the Sweedler notation,

$$\nabla(c) = c^1 \otimes c^2 \quad \text{and} \quad \nabla^{(n)}(c) = c^1 \otimes c^2 \otimes \dots \otimes c^{n+1},$$

where we have omitted the sum. A counital R -coalgebra (C, ϵ) is called coaugmented if there is a coalgebra morphism $\eta : R \rightarrow C$. We denote by $\tilde{C} = \text{Ker}(\epsilon)$ the reduced coalgebra with the reduced coproduct

$$\bar{\nabla}(c) = \nabla(c) - c \otimes 1 - 1 \otimes c.$$

A (co)algebra A is called connected if it is both (co)augmented ($A_0 \cong R$) and $A_n = 0$ for $n \leq -1$. A (co)algebra, A is called n -reduced if it is both connected and $A_k = 0$ for $1 \leq k \leq n$.

1.2 The bar and cobar constructions

We refer to [LV12, Chapter 2] for the background material related to the bar and cobar constructions. We recall the bar-cobar adjunction,

Theorem 1.1. [LV12, Theorem 2.2.9] *For every augmented DG-algebra Λ and every conilpotent DG-coalgebra C there exist natural bijections*

$$\text{Hom}_{\text{DG-Alg}}(\Omega C; \Lambda) \cong \text{Tw}(C; \Lambda) \cong \text{Hom}_{\text{DG-Coalg}}(C; \mathcal{B}\Lambda).$$

Here :

- ΩC denotes the cobar construction of C ;
- $\mathcal{B}\Lambda$ denotes the bar construction of Λ ;
- $\text{Tw}(C; \Lambda)$ is the set of twisting cochains from C to Λ .

1.3 Hopf algebras

Definition 1.2. For a DG-bialgebra $(\mathcal{H}, d, \mu, \eta, \nabla, \epsilon)$ an **antipode** $S : \mathcal{H} \rightarrow \mathcal{H}$ is a chain map which is the inverse of the identity in the convolution algebra $\text{Hom}(\mathcal{H}, \mathcal{H})$; the convolution product being $f \smile g = \mu(f \otimes g) \nabla$. Explicitly, S satisfies

$$S(a^1)a^2 = \eta\epsilon(a) = a^1S(a^2)$$

for all $a \in \mathcal{H}$.

Definition 1.3. A **DG-Hopf algebra** $(\mathcal{H}, d, \mu, \eta, \nabla, \epsilon, S)$ is a DG-bialgebra \mathcal{H} endowed with an antipode S . If moreover, the antipode is involutive (i.e $S^2 = id$) we call \mathcal{H} an **involutive DG-Hopf algebra**.

An antipode satisfies the following properties.

Proposition 1.4. [Swe69, Proposition 4.0.1]

- i. $S(a^2) \otimes S(a^1) = (-1)^{|a^1||a^2|} S(a)^1 \otimes S(a)^2$ (coalgebra antimorphism);
- ii. $S(\eta(1)) = \eta(1)$ (unital morphism);

2. Homotopy structures on the cobar construction

iii. $S(ab) = (-1)^{|a||b|}S(b)S(a)$ (algebra antimorphism);

iv. $\epsilon(S(a)) = \epsilon(a)$ (counital morphism).

v. The following equations are equivalent:

(a) $S^2 = S \circ S = id$;

(b) $S(a^2)a^1 = \eta\epsilon(a)$;

(c) $a^2S(a^1) = \eta\epsilon(a)$.

vi. If \mathcal{H} is commutative or cocommutative, hence $S^2 = id$.

2 Homotopy structures on the cobar construction

2.1 Homotopy G-algebra on the cobar construction

We present the homotopy G-algebra structure on the cobar construction of a 1-reduced DG-bialgebra given in [Kad05]. We define the Hirsch algebras and the homotopy G-algebras via the bar construction.

Definition 2.1. A **Hirsch algebra** Λ is the data of a connected DG-algebra $(\Lambda, d, \mu_\Lambda)$ together with a map $\mu : \mathcal{B}\Lambda \otimes \mathcal{B}\Lambda \rightarrow \mathcal{B}\Lambda$ making $\mathcal{B}\Lambda$ into an associative unital DG-bialgebra.

For a connected DG-algebra Λ a DG-product $\mu : \mathcal{B}\Lambda \otimes \mathcal{B}\Lambda \rightarrow \mathcal{B}\Lambda$ corresponds to a twisting cochain $\tilde{E} \in \text{Tw}(\mathcal{B}\Lambda \otimes \mathcal{B}\Lambda, \Lambda)$, c.f Theorem 1.1. After a correct use of desuspensions (see sign convention below), the latter is a family of operations $\{E_{i,j}\}_{i,j \geq 1}$, $E_{i,j} : \overline{\Lambda}^{\otimes i} \otimes \overline{\Lambda}^{\otimes j} \rightarrow \overline{\Lambda}$ satisfying some relations, see [Kad05]. We denote by $\mu_E : \mathcal{B}\Lambda \otimes \mathcal{B}\Lambda \rightarrow \mathcal{B}\Lambda$ such a DG-product.

A Hirsch algebra is a particular B_∞ -algebra whose the underlying A_∞ -algebra structure is a DGA structure. These structures correspond to algebraic structures on the cofree tensor coalgebra $T^c(s\overline{\Lambda})$:

Λ	$T^c(s\overline{\Lambda})$
DGA (d_Λ, μ_Λ)	DGC d induced by d_Λ and μ_Λ
A_∞ $\{\mu_k\}_{k \geq 0}$	DGC d induced by μ_k for $k \geq 0$
Hirsch $(d_\Lambda, \mu, \{E_{i,j}\}_{i,j \geq 1})$	DG-bialgebra d induced by d_Λ and μ ; DG-product μ_E
B_∞ $(\{\mu_k\}_{k \geq 0}, \{E_{i,j}\}_{i,j \geq 1})$	DG-bialgebra d induced by μ_k for $k \geq 0$; DG-product μ_E

Definition 2.2. A **homotopy G-algebra** $(\Lambda, d_\Lambda, \mu_\Lambda, \{E_{i,j}\}_{j \geq 1})$ is a Hirsch algebra $(\Lambda, d_\Lambda, \mu_\Lambda, \{E_{i,j}\}_{i,j \geq 1})$ whose operations $E_{i,j} = 0$ for $i \geq 2$.

Sign convention

Let $\tilde{E} \in \text{Tw}(\mathcal{B}\Lambda \otimes \mathcal{B}\Lambda, \Lambda)$ be a twisting cochain. Its (i, j) -th component is $\tilde{E}_{i,j} : (\mathfrak{s}\overline{\Lambda})^{\otimes i} \otimes (\mathfrak{s}\overline{\Lambda})^{\otimes j} \rightarrow \overline{\Lambda}$. We denote by $E_{i,j} : \overline{\Lambda}^{\otimes i} \otimes \overline{\Lambda}^{\otimes j} \rightarrow \overline{\Lambda}$ the component

$$(2.1) \quad \begin{aligned} E_{i,j}(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_{i+j}) \\ := (-1)^{\sum_{s=1}^{i+j-1} |a_s|(i+j-s)} \tilde{E}_{i,j}(\mathfrak{s}^{\otimes i} \otimes \mathfrak{s}^{\otimes j})(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_{i+j}) \\ = \tilde{E}_{i,j}(\mathfrak{s} a_1 \otimes \dots \otimes \mathfrak{s} a_i \otimes \mathfrak{s} a_{i+1} \otimes \dots \otimes \mathfrak{s} a_{i+j}). \end{aligned}$$

The degree of $E_{i,j}$ is $i + j - 1$. We denote $E_{i,j}(a_1, \dots, a_i; b_1, \dots, b_j)$ for $E_{i,j}((a_1 \otimes \dots \otimes a_i) \otimes (b_1 \otimes \dots \otimes b_j))$.

Let B be a 1-reduced DG-bialgebra. For $a \in B$ and $\bar{b} := \mathfrak{s}^{-1} b_1 \otimes \dots \otimes \mathfrak{s}^{-1} b_s \in \Omega B$, we set

$$a \diamond \bar{b} = \mathfrak{s}^{-1}(a^1 b_1) \otimes \dots \otimes \mathfrak{s}^{-1}(a^s b_s).$$

Proposition 2.3. [*Kad05*] *Let B be a 1-reduced DG-bialgebra. Then the cobar construction $(\Omega B, d_\Omega)$ together with the operations $E_{1,k}$, $k \geq 1$ defined by*

$$(2.2) \quad \begin{aligned} E_{1,k}(\mathfrak{s}^{-1} a_1 \otimes \dots \otimes \mathfrak{s}^{-1} a_n; \bar{b}_1, \dots, \bar{b}_k) = \\ \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} \pm \mathfrak{s}^{-1} a_1 \otimes \dots \otimes \mathfrak{s}^{-1} a_{i_1-1} \otimes a_{i_1} \diamond \bar{b}_1 \otimes \dots \otimes \mathfrak{s}^{-1} a_{i_k-1} \otimes a_{i_k} \diamond \bar{b}_k \otimes \mathfrak{s}^{-1} a_{i_k+1} \otimes \dots \otimes \mathfrak{s}^{-1} a_n \end{aligned}$$

when $n \geq k$ and zero when $n < k$ is a homotopy G -algebra.

In particular the operation $E_{1,1} : \overline{\Omega B} \otimes \overline{\Omega B} \rightarrow \Omega B$ is given by

$$(2.3) \quad \begin{aligned} E_{1,1}(\mathfrak{s}^{-1} a_1 \otimes \dots \otimes \mathfrak{s}^{-1} a_m; \mathfrak{s}^{-1} b_1 \otimes \dots \otimes \mathfrak{s}^{-1} b_n) \\ := \sum_{l=1}^m (-1)^\gamma \mathfrak{s}^{-1} a_1 \otimes \dots \otimes (a_l^1 \cdot b_1) \otimes \dots \otimes \mathfrak{s}^{-1}(a_l^n \cdot b_n) \otimes \dots \otimes \mathfrak{s}^{-1} a_m \end{aligned}$$

with

$$\begin{aligned} \gamma = \sum_{u=1}^n |b_u| \left(\sum_{s=l+u}^m |a_s| + m - l - u + 1 \right) + \kappa_n(a_l) + \sum_{u=2}^n (|a_l^u| - 1) \left(\sum_{s=1}^{u-1} |b_s| - u + 1 \right) \\ + \sum_{s=1}^n (|a_l^s| - 1)(2n - 2s + 1) + \sum_{s=1}^{n-1} (|a_l^s| + |b_s|)(n - s), \end{aligned}$$

where

$$\kappa_n(a) := \begin{cases} \sum_{1 \leq 2s+1 \leq n} |a^{2s+1}| & \text{if } n \text{ is even;} \\ \sum_{1 \leq 2s \leq n} |a^{2s}| & \text{if } n \text{ is odd.} \end{cases}$$

2.2 Homotopy BV-algebra on the cobar construction

We define a homotopy BV-algebra as a homotopy G -algebra together with a BV-operator. For a 1-reduced involutive DG-Hopf algebra \mathcal{H} the cobar construction $\Omega \mathcal{H}$ is a homotopy BV-algebra [*Men04*]; the BV-operator is Connes-Moscovici's operator [*CM00*].

2. Homotopy structures on the cobar construction

Definition 2.4. A **homotopy BV-algebra** Λ is a homotopy G-algebra together with a DG-operator $B : \Lambda \rightarrow \Lambda$ subject to the relations:

1. $B^2 = 0$;
2. $\{a; b\} = (-1)^{|a|}(B(a \cdot b) - B(a) \cdot b - (-1)^{|a|}a \cdot B(b)) + d_\Lambda H(a; b) + H(d_\Lambda a; b) + (-1)^{|a|}H(a; d_\Lambda b)$
for all $a, b \in \Lambda$,

where $\{-, -\}$ is the Gerstenhaber bracket $\{a, b\} = E_{1,1}(a; b) - (-1)^{(|a|-1)(|b|-1)}E_{1,1}(b; a)$ and $H : \Lambda \otimes \Lambda \rightarrow \Lambda$ is a degree 2 linear map.

The BV-operator we describe below is Connes-Moscovici's operator [CM00] ; it is a square zero chain map defined on the cobar construction of an involutive Hopf algebra. Together with Connes-Moscovici's operator B the cobar construction $\Omega\mathcal{H}$ of an involutive Hopf algebra \mathcal{H} form a bicomplex. It is the underlying bicomplex of the cyclic coHochschild homology of \mathcal{H} .

Let $(\mathcal{H}, d, \mu, \eta, \nabla, \epsilon, S)$ be an involutive DG-Hopf algebra, that is with an involutive antipode S . Connes-Moscovici's operator for Hopf algebras $(\mathcal{H}, d, \mu, \eta, \nabla, \epsilon, S)$ defined in [CM00] induces a BV-operator on $H_*(\Omega\mathcal{H})$ provided that the antipode S is involutive. With our sign convention Connes-Moscovici's operator

$$B : \Omega\mathcal{H} \rightarrow \Omega\mathcal{H}$$

is zero on $R = (s^{-1}\overline{\mathcal{H}})^{\otimes 0}$ and $(s^{-1}\overline{\mathcal{H}})^{\otimes 1}$ and is given on the n -th component $(s^{-1}\overline{\mathcal{H}})^{\otimes n}$, $n \geq 2$ by :

$$(2.4) \quad B(s^{-1}a_1 \otimes s^{-1}a_2 \otimes \dots \otimes s^{-1}a_n) = \sum_{i=0}^{n-1} (-1)^i \pi_n \tau_n^i (s^{-1}a_1 \otimes s^{-1}a_2 \otimes \dots \otimes s^{-1}a_n),$$

where τ_n is the cyclic permutation

$$\tau_n(s^{-1}a_1 \otimes s^{-1}a_2 \otimes \dots \otimes s^{-1}a_n) := (-1)^{(|a_1|-1)(\sum_{i=2}^n |a_i|-1)} s^{-1}a_2 \otimes s^{-1}a_3 \otimes \dots \otimes s^{-1}a_n \otimes s^{-1}a_1,$$

and

$$\begin{aligned} \pi_n &:= (s^{-1}\mu)^{\otimes n-1} \tau_{n-1, n-1} (S^{\otimes n-1} \otimes 1^{\otimes n-1}) ((\tau\nabla)^{(n-2)} \otimes 1^{n-1}) s^{\otimes n}, \\ \tau_{n,n}(a_1 \otimes a_2 \otimes \dots \otimes a_n \otimes b_1 \otimes \dots \otimes b_n) &:= (-1)^{\sum_{i=1}^{n-1} |b_i| (\sum_{j=i+1}^n |a_j|)} a_1 \otimes b_1 \otimes a_2 \otimes b_2 \otimes \dots \otimes a_n \otimes b_n. \end{aligned}$$

More picturally, this gives

$$(2.5) \quad B(s^{-1}a_1 \otimes s^{-1}a_2 \otimes \dots \otimes s^{-1}a_n) = \sum_{k=1}^n \pm s^{-1} S(a_k^{n-1}) a_{k+1} \otimes s^{-1} S(a_k^{n-2}) a_{k+2} \otimes \dots \otimes s^{-1} S(a_k^1) a_{k-1}.$$

The involutivity of the antipode of \mathcal{H} makes B into a square zero chain map. Menichi proved [Men04, Proposition 1.9] that for a unital (ungraded) Hopf algebra \mathcal{H} , Connes-Moscovici's operator induces a Batalin-Vilkovisky algebra on the homology of the cobar construction $H_*(\Omega\mathcal{H})$. In our context,

Proposition 2.5. [Men04] Let \mathcal{H} be a 1-reduced involutive DG-Hopf algebra. Then the cobar construction $(\Omega\mathcal{H}, d_\Omega)$ is a homotopy BV-algebra with BV-operator defined in (2.4).

Proof. Let us define Menichi's homotopy $H(\bar{a}; \bar{b}) := H_1(\bar{a}; \bar{b}) - (-1)^{(|\bar{a}|-1)(|\bar{b}|-1)} H_1(\bar{b}; \bar{a})$.

$$(2.6) \quad H_1(s^{-1} a_1 \otimes \dots \otimes s^{-1} a_m; s^{-1} b_1 \otimes \dots \otimes s^{-1} b_n) \\ := \sum_{1 \leq j \leq p \leq m-1} (-1)^{\xi_j + n + 1} \pi_{m+n-1}^s \tau_{m+n-1}^{s, n+m-1-j} \rho_{n+m}^{(p-j+1)} (s^{-1} a_1 \otimes \dots \otimes s^{-1} a_m \otimes s^{-1} b_1 \otimes \dots \otimes s^{-1} b_n)$$

with

$$\xi_j = \begin{cases} \sum_{s=1}^m |a_s| & \text{for } j = 1; \\ \sum_{s=m-j+1}^m |a_s| & \text{for } j > 1. \end{cases}$$

Where,

$$\begin{aligned} \pi_m^s &= \pi_m(s^{-1})^{\otimes m} \\ \tau_m^{s,i} &:= (\tau_m^s)^i \\ \tau_m(a_1 \otimes a_2 \otimes \dots \otimes a_m) &:= (-1)^{|a_1|(\sum_{i=2}^m |a_i|)} a_2 \otimes a_3 \otimes \dots \otimes a_m \otimes a_1 \\ \rho_{m+1}^{(i)} &:= (1^{\otimes i-1} \otimes \mu \otimes 1^{\otimes m-i})(1^{\otimes i-1} \otimes \tau_{m-i+1,1}) s^{\otimes m+1} \\ \tau_{k,1}(a_1 \otimes \dots \otimes a_k \otimes b) &:= (-1)^{|b|(\sum_{i=2}^k |a_i|)} a_1 \otimes b \otimes a_2 \otimes \dots \otimes a_k. \end{aligned}$$

By the Proposition [Men04, Proposition 1.9] we have

$$\{a; b\} = (-1)^{|a|} (B(a \cdot b) - B(a) \cdot b - (-1)^{|a|} a \cdot B(b)) + d_1 H(a; b) + H(d_1 a; b) + (-1)^{|a|} H(a; d_1 b)$$

on the cobar construction $(\Omega\mathcal{H}, d_0 + d_1)$, where d_1 is the quadratic part of the differential d_Ω . The operators involved in the above equation commute with d_0 . Therefore we can replace d_1 by d_Ω in the previous equation. \blacksquare

3 Involutivity of the antipode of ΩC in terms of the homotopy G-coalgebra C

A Hirsch coalgebra C is the formal dual of a Hirsch algebra, that is it corresponds to a DG-coproduct $\nabla : \Omega C \rightarrow \Omega C \otimes \Omega C$ making ΩC into a DG-bialgebra. Baues' coproduct [Bau98] $\nabla_0 : \Omega C_*(X) \rightarrow \Omega C_*(X)$ defined on the cobar construction of a simplicial set X corresponds to a homotopy G-coalgebra structure $E^{k,1}$ on $C_*(X)$, special kind of Hirsch coalgebra structure. With this example in mind we consider a homotopy G-coalgebra $(C, E^{k,1})$. Then there exists a unique antipode $S : \Omega C \rightarrow \Omega C$ on its cobar construction. The purpose of this section is to give a criterion for the involutivity of the antipode S in terms of the operations $E^{k,1}$. This takes the form of a family of operations $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$, $n \geq 2$.

Definition 3.1. A **Hirsch coalgebra** C is the data of a 1-reduced DG-coalgebra (C, d, ∇_C) together with a map $\nabla : \Omega C \rightarrow \Omega C \otimes \Omega C$ making ΩC into a coassociative¹ counital DG-bialgebra. The corresponding operations on C are denoted by $E^{i,j} : \tilde{C} \rightarrow \tilde{C}^{\otimes i} \otimes \tilde{C}^{\otimes j}$. A **homotopy G-coalgebra** is a Hirsch coalgebra whose operations $E^{i,j} = 0$ for $j \geq 2$.

Let the cobar construction $(\Omega C, d, \mu_\Omega, \nabla)$ be a DG-bialgebra. We can define the antipode

$$S : \Omega C \rightarrow \Omega C$$

by $S(1) = 1$ and for $\sigma \in \tilde{C}_n$ with $n \geq 2$ by

$$(3.1) \quad S(\sigma) := -\sigma - \mu_\Omega(S \otimes 1)\bar{\nabla}(\sigma)$$

which makes sense since

$$\bar{\nabla}(\sigma) \subset \bigoplus_{\substack{i+j=n \\ 0 < i, j < n}} (\Omega C)_i \otimes (\Omega C)_j.$$

Remark 3.2. Equivalently, we can define S as the geometric series $(Id) \smile^{-1} = \sum_{n \geq 0} (\eta\epsilon - Id) \smile^{-n}$ with $(\eta\epsilon - Id) \smile^0 = \eta\epsilon$ and where \smile denote the convolution product from Definition 1.2. Note that this sum is finite when it is evaluated on an element. This presentation is combinatorial and gives for $[\sigma] \in (s^{-1}C)_n$,

$$(3.2) \quad S([\sigma]) = \eta\epsilon([\sigma]) + (\eta\epsilon - Id)([\sigma]) + \mu_\Omega \bar{\nabla}([\sigma]) - \mu_\Omega^{(2)} \bar{\nabla}^{(2)}([\sigma]) + \dots + (-1)^{n-1} \mu_\Omega^{(n)} \bar{\nabla}^{(n)}([\sigma]).$$

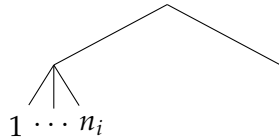
By the **iii** of Proposition 1.4 the antipode is an algebra antimorphism that is an algebra morphism from $(\Omega C, \mu_\Omega)$ to $\Omega C_{(12)} := (\Omega C, \mu_{\Omega_{(12)}} := \tau\mu_\Omega)$. Moreover, it is also a DG-map, therefore it corresponds to a twisting cochain $F \in \text{Tw}(C, \Omega C_{(12)})$. An antipode is determined by the underlying bialgebra structure. Here the latter is equivalent to a homotopy G-algebra structure $E^{k,1}$ on C . We make explicit the twisting cochain F in terms of $E^{k,1}$.

We write $F^i : \tilde{C} \rightarrow \tilde{C}^{\otimes i}$ the i -th component of F . The relation² $\mu_\Omega(1 \otimes S_F)\nabla = \eta\epsilon$ gives

$$(3.3) \quad F^1 = -Id_{\tilde{C}};$$

$$(3.4) \quad F^n = (-1)^n \sum_{\substack{1 \leq s \leq n-1 \\ n_1 + \dots + n_s = n-1 \\ n_i \geq 1}} (1^{\otimes n_1 + \dots + n_{s-1}} \otimes E^{n_s, 1}) \dots (1^{\otimes n_1} \otimes E^{n_2, 1}) E^{n_1, 1}, \quad n \geq 2.$$

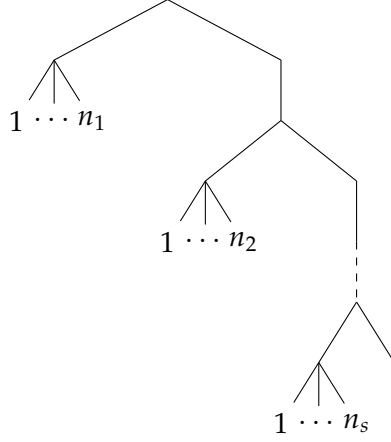
By representing the operation $E^{n_i, 1} : \tilde{C} \rightarrow \tilde{C}^{\otimes n_i} \otimes \tilde{C}$ as the tree



¹We precise the coassociativity condition to avoid confusions with the definition of Hirsch coalgebra presented in [KS02]. Their main Hirsch coalgebra structure appears as a coproduct on a double cobar construction seen as a chain complex of a permutahedral set. This coproduct is induced by a diagonal on the permutahedra and is not coassociative.

²We can also consider $\mu_\Omega(S_F \otimes 1)\nabla = \eta\epsilon$. It gives equivalent F^n but with a more complicated description because of the apparition of permutations. For example $F^3 = -(E^{1,1} \otimes 1)E^{1,1} - (\tau \otimes 1)E^{2,1}$.

a summand of the equation (3.4) is represented as



The three first terms are $F^1 = -Id_{\tilde{C}}$, $F^2 = E^{1,1}$ and $F^3 = -(1 \otimes E^{1,1})E^{1,1} - E^{2,1}$.

Formulated in this terms, the n -th component of the difference $S^2 - Id$ is zero for $n = 1$ and

$$(3.5) \quad O_n := \sum_{s=1}^n \sum_{n_1+\dots+n_s=n} \mu_{\Omega(12)}^{(s)} (F^{n_1} \otimes \dots \otimes F^{n_s}) F^s$$

for $n \geq 2$. The terms $(F^{n_1} \otimes \dots \otimes F^{n_s}) F^s$ arrive into $\tilde{C}^{\otimes n_1} \otimes \tilde{C}^{\otimes n_2} \otimes \dots \otimes \tilde{C}^{\otimes n_s}$. The s -iterated product $\mu_{\Omega(12)}^{(s)}$ permutes these s blocs as the permutation $(s, s-1, \dots, 2, 1) \in S_s$, where S_s denotes the symmetric group on s objects. The two first terms we obtain are

$$O_2 = F^2 - \tau F^2 = E^{1,1} - \tau E^{1,1}$$

and

$$\begin{aligned} O_3 &= (1 + \tau_3)F^3 + \tau^{2,1}(F^2 \otimes 1)F^2 + \tau^{1,2}(1 \otimes F^2)F^2 \\ &= (\tau^{1,2} - 1 - \tau_3)(1 \otimes E^{1,1})E^{1,1} + \tau^{2,1}(E^{1,1} \otimes 1)E^{1,1} - (1 + \tau_3)E^{2,1}, \end{aligned}$$

where the permutations are $\tau^{1,2}(a \otimes b \otimes c) = \pm b \otimes c \otimes a$, $\tau^{2,1}(a \otimes b \otimes c) = \pm c \otimes a \otimes b$ and $\tau_3(a \otimes b \otimes c) = \pm c \otimes b \otimes a$; the signs being given by the Koszul sign rule.

We conclude,

Proposition 3.3. *Let $(C, d, \nabla_C, E^{k,1})$ be a homotopy G-coalgebra.*

1. *The cobar construction ΩC is an involutive DG-Hopf algebra if and only if all the obstructions $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$ defined in (3.5) for $n \geq 2$ are zero.*
2. *Let C be 2-reduced. If all the obstructions $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$ are zero, then the double cobar construction $\Omega^2 C$ is a homotopy BV-algebra given by the BV-operator defined in (2.4).*

In fact we can make more precise the second point of the previous proposition. Let M be a DG-module and let $M_{\leq n}$ be the submodule of elements $m \in M$ of degree $|m| \leq n$. The homotopy G-coalgebra structure on a connected coalgebra C preserves the filtration $C_{\leq n}$. Indeed the degree of the operations $E^{k,1} : \tilde{C} \rightarrow \tilde{C}^{\otimes k} \otimes \tilde{C}$ is k . We have,

4. Correspondance between Baues' coproduct and the McClure and Smith action on $C_*(X)$

Proposition 3.4. *Let $(C, d, \nabla_C, E^{k,1})$ be an i -reduced homotopy G -coalgebra, $i \geq 2$. If there exists an integer n such that $O_k = 0$ for $ki \leq n - 1$, then $\Omega^2(C_{\leq n})$ is a homotopy BV-algebra.*

Proof. The degree of $O_k : \tilde{C} \rightarrow \tilde{C}^{\otimes k}$ is $k - 1$. Then on $\tilde{C}_{\leq n}$ the only eventually non-zero operations $O_k : \tilde{C}_{\leq n} \rightarrow (\tilde{C}_{\leq n})^{\otimes k}$ are those with $ki \leq n - 1$. ■

Remark 3.5. The two previous propositions work for a Hirsch coalgebra C instead of a homotopy G -coalgebra : the operations from (3.5) have the same definition in terms of F_n ; only the F_n 's defining the antipode differ. However, the involved technics above are quite similar for Hirsch coalgebras.

Remark 3.6. For a general Hirsch coalgebra $(C, E^{i,j})$ the condition of cocommutativity of the coproduct on ΩC is $E^{i,j} = \tau E^{j,i}$ for all (i, j) . Accordingly, with Proposition 1.4 we see (already in component three) that condition for the antipode to be involutive is weaker than condition for coproduct to be cocommutative. When the Hirsch coalgebra is a homotopy G -coalgebra, the cocommutativity of the coproduct means that the homotopy G -coalgebra is quasi trivial : $E^{1,1} = \tau E^{1,1}$ and $E^{k,1} = 0$ for $k \geq 2$.

4 Correspondance between Baues' coproduct and the McClure and Smith action on $C_*(X)$

In this section we recall the explicit formula of Baues' coproduct [Bau98] in subsection 4.1 and we show in the Proposition 4.1 that the corresponding homotopy G -coalgebra structure on the chain complex of a simplicial set coincides with the one given in [MS01, BF04], see subsection 4.2. The latter action will be use in the two next sections to study the obstruction to the involutivity of the antipode on the cobar construction $\Omega C_*(X)$.

The chain complex $C_*(X)$ of a topological space or a simplicial set X has a rich algebraic structure, it is an E_∞ -coalgebra. This was treated in many papers including V. Smirnov [Smi02], J. McClure and J. Smith [MS01], C. Berger and B. Fresse [BF04]. The E_∞ operad given by V. Smirnov is universal in the sense that if \mathcal{O} is an operad acting on $C_*(X)$ then this action factorizes through this universal operad. However there exist some other "smaller" operads giving an E_∞ -structure on $C_*(X)$.

For example, the surjection operad χ introduced in [MS01] acts on the normalized chain complex $C_*(X)$ of a simplicial set X making it into a coalgebra over χ , see [MS01, BF04]. It has a filtration of suboperads

$$F_1\chi \subset F_2\chi \subset \cdots \subset F_{n-1}\chi \subset F_n\chi \subset \cdots \subset \chi.$$

T. Kadeishvili showed that an action of the suboperad $F_2\chi$ on $C_*(X)$ corresponds to a homotopy G -coalgebra structure on $C_*(X)$, [Kad05, Kad03]. This structure leads to a coproduct $\nabla_0 : \Omega C_*(X) \rightarrow \Omega C_*(X) \otimes \Omega C_*(X)$ first defined by Baues in [Bau98]. The third stage filtration $F_3\chi$ gives a homotopy cocommutativity $\nabla_1 : \Omega C_*(X) \rightarrow \Omega C_*(X) \otimes \Omega C_*(X)$ to Baues' coproduct, see [Bau98]. In turn the operation $\nabla_1 : \Omega C_*(X) \rightarrow \Omega C_*(X) \otimes \Omega C_*(X)$ is cocommutative up to a homotopy $\nabla_2 : \Omega C_*(X) \rightarrow \Omega C_*(X) \otimes \Omega C_*(X)$ and so on. The resulting structure is known as a structure of DG-bialgebra with Steenrod coproduct ∇_i . This was

4. Correspondance between Baues' coproduct and the McClure and Smith action on $C_*(X)$

achieved by Kadeishvili in [Kad03] where the corresponding operations in χ are given.

We fix the base ring k to be any commutative ring. For a simplicial set X we denote by DX the set of degenerate elements and we set $C_*(X) := \text{span}_k(X)/\text{span}_k(DX)$ to be the normalized chain complex.

4.1 Baues' coproduct

Let us fix X to be a 1-reduced simplicial set, that is $X_0 = X_1 = *$. We recall the augmentation

$$(4.1) \quad \begin{aligned} \epsilon : \Omega C_*(X) &\rightarrow k \\ s^{-1} a_1 \otimes \dots \otimes s^{-1} a_n &\mapsto \begin{cases} 0 & \text{for } n \geq 1 \\ id_k & \text{for } n = 0. \end{cases} \end{aligned}$$

To give an explicit formulation of the coproduct we need to introduce some notations. For $\sigma_i \in X$, $[\sigma_1|\sigma_2|\dots|\sigma_n]$ denote the tensor $s^{-1}\sigma_{i_1} \otimes s^{-1}\sigma_{i_2} \otimes \dots \otimes s^{-1}\sigma_{i_r}$ where indices i_j are such that $\sigma_{i_j} \in X_{n_{i_j}}$ with $n_{i_j} \geq 2$. For a subset $b = \{b_0 < b_1 < \dots < b_r\} \subset \{0, 1, \dots, n\}$ we denote by i_b the unique order preserving injective function $i_b : \{0, 1, \dots, r\} \rightarrow \{0, 1, \dots, n\}$ such that $\text{Im}(i_b) = b$. Let $\sigma \in X_n$ and $0 \leq b_0 < b_1 \leq n$. By $\sigma(b_0, \dots, b_1)$ we mean $i_b^* \sigma \in X_{b_1 - b_0}$ where $b = \{b_0, b_0 + 1, \dots, b_1 - 1, b_1\}$. Let $b \subset \{1, \dots, n - 1\}$. By $\sigma(0, b, n)$ we mean $i_{b'}^* \sigma$ where $b' = \{0\} \cup b \cup \{n\}$.

We define the coproduct ∇_0 on elements $\sigma \in X_n$, $n \geq 2$ by :

$$(4.2) \quad \begin{aligned} \nabla_0 : \Omega C_*(X) &\rightarrow \Omega C_*(X) \otimes \Omega C_*(X) \\ s^{-1} \sigma = [\sigma] &\mapsto \sum_{\substack{b = \{b_1 < b_2 < \dots < b_r\} \\ b \subset \{1, \dots, n-1\}}} (-1)^\zeta [\sigma(0, \dots, b_1) | \sigma(b_1, \dots, b_2) | \dots | \sigma(b_r, \dots, n)] \otimes [\sigma(0, b, n)] \end{aligned}$$

where

$$\zeta = r |\sigma(0, \dots, b_1)| + \sum_{i=2}^r (r+1-i) |\sigma(b_{i-1}, \dots, b_i)| - r(r+1)/2,$$

and extend it as an algebra morphism.

4.2 Action of the operad $F_2\chi$ on $C_*(X)$

Let us first define succinctly the surjection operad χ and its coaction on $C_*(X)$. We refer to [BF04, Section 1.2] (see also [MS01]) for the complete definition.

Let $r \geq 1$, $d \geq 0$ be two integers. Let $u : \{1, \dots, r+d\} \rightarrow \{1, \dots, r\}$ be a surjective map. We represent it by a sequence $(u(1), u(2), \dots, u(r+d))$. Such a map is degenerate if there exists an $1 \leq i < r+d$ such that $u(i) = u(i+1)$. We define $\chi(r)_d$ to be the module freely generated by the non-degenerate surjective maps $u : \{1, \dots, r+d\} \rightarrow \{1, \dots, r\}$. It is an S_r -module in the following sense : to a permutation $\sigma \in S_r$ and an element $u \in \chi(r)_d$ the surjective map $\sigma \cdot u$ is given by the sequence $(\sigma(u(1)), \dots, \sigma(u(r+d)))$.

The suboperad $F_2\chi \subset \chi$ is generated by $(1, 2)$ and $(1, 2, 1, 3, 1, \dots, 1, k, 1, k+1, 1)$ for $k \geq 1$.

4. Correspondance between Baues' coproduct and the McClure and Smith action on $C_*(X)$

Let us describe shortly how the surjection operad χ acts on $C_*(X)$. Let us consider an element $u \in \chi(r)_d$ and a n -simplex σ . Integers r, d and n being fixed, we cut the interval $[0, n] = \{0, 1, \dots, n\} \subset \mathbb{N}$ into $r + d$ subintervals, $0 = n_0 \leq n_1 \leq \dots \leq n_{r+d} = n$. For $u = (u(1), \dots, u(r+d))$ we attribute to the subinterval $[n_{i-1}, n_i]$ the label $u(i)$.

$$\begin{array}{ccccccccccc} 0 = n_0 & & n_1 & & n_2 & & n_{i-1} & & n_i & & n_{r+d-1} & & n_{r+d} = n \\ \times & & \times & & \times & \cdots & \times & & \times & \cdots & \times & & \times \\ & & u(1) & & u(2) & & & & u(i) & & & & u(r+d) \end{array}$$

We denote by $I(j)$ the concatenation of subintervals with value j that is

$$I(j) := [n_{j_1-1}, n_{j_1}], [n_{j_2-1}, n_{j_2}], \dots, [n_{j_l-1}, n_{j_l}]$$

where $j_1 < j_2 < \dots < j_l$ are the indices such that $u(j_1) = u(j_2) = \dots = u(j_l) = j$. Finally, for such a $u = (u(1), \dots, u(r+d)) \in \chi(r)_d$, we define the operation

$$E(u) : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes r}$$

to be

$$E(u)(\sigma) := \sum \pm \sigma(I(1)) \otimes \dots \otimes \sigma(I(r)),$$

the sum being over all cuttings.

For example, let $(1, 2) \in \chi(2)_0$ then we obtain

$$E((1, 2))(\sigma) = \sum_{0 \leq i \leq n} \pm \sigma(0, \dots, i) \otimes \sigma(i, \dots, n),$$

that is the Alexander-Withney coproduct.

To the elements $u = (1, 2, 1, 3, 1, \dots, 1, k, 1, k+1, 1)$ with $k \geq 1$ corresponds the operations $E^{1,k}$ defining the homotopy G-coalgebra structure on $C_*(X)$. For an n -simplex σ we have

$$\begin{aligned} E^{1,k}(\sigma) = & \sum_{0 \leq j_1 < j_2 < \dots < j_{2k} \leq n} \\ & \pm \sigma(0, \dots, j_1, j_2, \dots, j_3, j_4, \dots, j_{2k-1}, j_{2k}, \dots, n) \otimes \sigma(j_1, \dots, j_2) \otimes \sigma(j_3, \dots, j_4) \otimes \dots \otimes \sigma(j_{2k-1}, \dots, j_{2k}). \end{aligned}$$

4.3 Correspondance of two homotopy G-coalgebra structures

Proposition 4.1. *Let X be a 1-reduced simplicial set. Then the homotopy G-coalgebra structure on $C_*(X)$ given by Baues' coproduct (4.2) and the homotopy G-coalgebra structure on $C_*(X)$ given in the previous paragraph are the same.*

Proof. We use the same notations as above. We denote by $E_0^{k,1}$ the operations defined by Baues' coproduct ∇_0 . To show the correspondance between these two homotopy G-coalgebra structures it is sufficient to show that each $(k, 1)$ -component are equal i.e $E_0^{k,1} = \tau^{1,k} E^{1,k}$. Let σ be an n -simplex, the Baues' coproduct is a sum over all subsets $b = \{b_1 < b_2 < \dots < b_r\} \subset \{1, \dots, n-1\}$. For such a subset b , let $b_0 := 0$ and $\beta_l \in b \cup \emptyset$ defined as follow. For $l = 1$ we set

$$\beta_1 := \min_{1 \leq i \leq r} \{b_i \mid b_i - b_{i-1} \geq 2\},$$

and η_1 the index such that $b_{\eta_1} = \beta_1$; for $l \geq 2$, we set

$$\beta_l := \min_{\eta_{l-1}+1 \leq i \leq r} \{b_i \mid b_i - b_{i-1} \geq 2\}$$

where η_l is the index such that $b_{\eta_l} = \beta_l$. Additionally, we set $\alpha_1 := b_{\eta_1-1}$. We obtain that

$$E_0^{k,1}(\sigma) = \sum \pm \sigma(\alpha_1, \dots, \beta_1) \otimes \sigma(\alpha_2, \dots, \beta_2) \otimes \dots \otimes \sigma(\alpha_k, \dots, \beta_k) \otimes \sigma(0, b, n)$$

Then the k left terms correspond to those of $\tau^{1,k} E^{1,k}(\sigma)$. Indeed since X is 1-reduced, the elements of $E^{1,k}(\sigma)$, $\sigma(j_l, \dots, j_{l+1})$ where $j_{l+1} - j_l = 1$ belong to $X_1 = * = s_0(*)$ and are degenerate. To establish the complete correspondance it remains to compare $\sigma(0, b, n) = \sigma(0, b_1, b_2, \dots, b_r, n)$ with $\sigma(0, \dots, \alpha_1, \beta_1, \dots, \alpha_2, \dots, \alpha_k, \beta_k, \dots, n)$. But they are equal by minimality of β_l . ■

5 Obstruction to the involutivity of the antipode of $\Omega C_*(\Sigma X)$

Let ΣX be a simplicial suspension of a simplicial set X . We show that the family of obstructions $O_n : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes n}$ defined in (3.5) can be reduced to $O_2 : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes 2}$; the latter being governed by the (lack of) cocommutativity of the operation $E^{1,1}$.

Definition 5.1. [May92, Definition 27.6 p.124] A **simplicial suspension** ΣX of a simplicial set X with $X_0 = *$ is the simplicial set given by $(\Sigma X)_0 = a_0$ and $(\Sigma X)_n = \{(i, x) \in \mathbb{N}^{\geq 1} \times X_{n-i}\} / \left((i, s_0^n(*)) = s_{n+i}(a_0) = a_{n+i} \right)$. The face and degeneracy operators are generated by :

- $d_0(1, x) = a_n$ for all $x \in X_n$;
- $d_1(1, x) = a_0$ for all $x \in X_0$;
- $d_{i+1}(1, x) = (1, d_i(x))$ for all $x \in X_n, n > 0$;
- $s_0(i, x) = (i+1, x)$;
- $s_{i+1}(1, x) = (1, s_i(x))$.

Proposition 5.2. [HPS07] *The differential of $(\Omega C_*(\Sigma X), d_0 + d_1)$ is reduced to its linear part d_0 .*

Proof. The only non degenerate elements in ΣX are $(1, x)$ and we have that $d_0(1, x) = a_n$ is degenerate. Therefore the Alexander-Whitney coproduct on $C_*(\Sigma X)$ is primitive (we recall that C_* are the normalized chains). Then, the reduced coalgebra $\tilde{C}_*(\Sigma X)$ is a trivial coalgebra; so the quadratic part d_1 of the differential $d_\Omega = d_0 + d_1$ on the cobar construction is trivial. ■

A natural coproduct to define is the shuffle coproduct (primitive on cogenerators) which gives a cocommutative DG-Hopf structure to $\Omega C_*(\Sigma X)$; applying Proposition 2.5 to X 1-reduced, we obtain on $\Omega^2 C_*(\Sigma X)$ a homotopy BV-algebra structure. However, this coproduct does not correspond to the one defined in (4.2). Indeed, the hGa-structure on $C_*(\Sigma X)$ is not completely trivial. Because of the cocommutativity of the Alexander-Whitney coproduct, the operation $E^{1,1}$ must be a chain map but not necessary the zero map. We obtain,

Proposition 5.3. *The homotopy G -coalgebra structure $E^{k,1}$ on $C_*(\Sigma X)$ defined in Proposition 4.1 is*

$$E^{1,1}(\sigma) = \sum_{1 < l < n} \pm \sigma(0, \dots, l) \otimes \sigma(0, l, l+1, \dots, n);$$

$$E^{k,1} = 0 \text{ for } k \geq 2.$$

Proof. For a non degenerate $\sigma \in (\Sigma X)_n$, we have $d_0\sigma = \sigma(1, \dots, n)$ is degenerate. Consequently, the operation defined in Proposition 4.1

$$E^{1,1}(\sigma) = \sum_{k < l} \pm \sigma(k, \dots, l) \otimes \sigma(0, \dots, k, l, l+1, \dots, n)$$

is reduced to

$$E^{1,1}(\sigma) = \sum_{1 < l < n} \pm \sigma(0, \dots, l) \otimes \sigma(0, l, l+1, \dots, n).$$

The higher operations $E^{k,1}$ for $k \geq 2$ given by

$$E^{1,k}(\sigma) = \sum_{0 \leq j_1 < j_2 < \dots < j_{2k} \leq n} \pm \sigma(0, \dots, j_1, j_2, \dots, j_3, j_4, \dots, j_{2k-1}, j_{2k}, \dots, n) \otimes \sigma(j_1, \dots, j_2) \otimes \sigma(j_3, \dots, j_4) \otimes \dots \otimes \sigma(j_{2k-1}, \dots, j_{2k})$$

are zero since the terms $\sigma(j_l, \dots, j_{l+1})$ for $l \leq 3$ are degenerate. \blacksquare

The triviality of higher operations $E^{k,1}$ for $k \geq 2$ does not imply the vanishing of higher obstructions O_n since the $E^{1,1}$ operation appears in all the O_n 's. However, we can reduce the family of obstructions O_n to only O_2 in this case.

Proposition 5.4. *Let $(C, d, \nabla_C, E^{k,1})$ be a homotopy G -coalgebra with $E^{k,1} = 0$ for $k \geq 2$. If $O_2 = E^{1,1} - \tau E^{1,1}$ is zero, then so is O_n for $n \geq 2$.*

Proof. By making explicit the coassociativity of the coproduct $\nabla : \Omega C \rightarrow \Omega C$ we obtain in particular the equation

$$(1 \otimes E^{1,1})E^{1,1} - (E^{1,1} \otimes 1)E^{1,1} = (\tau \otimes 1)E^{2,1} - (1 \otimes 1)E^{2,1}.$$

Therefore the triviality of the higher operation $E^{2,1}$ implies that $E^{1,1}$ is coassociative. Together with the vanishing of O_2 we obtain a coassociative and cocommutative operation $E^{1,1}$. We use this fact to vanish the O_n 's : by (3.4), the terms F^i , $i \leq n$ involved in O_n are

$$F^n = (-1)^n (1^{\otimes n-2} \otimes E^{1,1}) \dots (1 \otimes E^{1,1}) E^{1,1}.$$

Now because of the coassociativity of $E^{1,1}$ we can write each term $\mu_{\Omega(12)}^{(s)} (F^{n_1} \otimes \dots \otimes F^{n_s}) F^s$ of the sum

$$O_n = \sum_{s=1}^n \sum_{n_1 + \dots + n_s = n} \mu_{\Omega(12)}^{(s)} (F^{n_1} \otimes \dots \otimes F^{n_s}) F^s$$

as $\pm \sigma \circ F^n$ where σ is a permutation (depending of the term we consider). Now using the cocommutativity and again the coassociativity of $E^{1,1}$ we can remove all transpositions to obtain $\pm F^n$. A direct counting shows that positive and negative terms are equal in number : the sign of $(F^{n_1} \otimes \dots \otimes F^{n_s}) F^s$ is $(-1)^{n_1 + \dots + n_s + s} = (-1)^{n+s}$ and the number of partitions of n into s integers ≥ 1 is $\binom{n-1}{s-1}$. \blacksquare

Therefore we can consider O_2 as the only obstruction to the involutivity of the antipode on the cobar construction.

6 Homotopy BV-algebra structure on $\Omega^2 C_*(\Sigma^2 X)$

We prove here the homotopy G-coalgebra structure on $C_*(\Sigma^2 X)$ is trivial and then that $\Omega^2 C_*(\Sigma^2 X)$ is an involutive DG-Hopf algebra with the shuffle coproduct.

Theorem 6.1. *Let $\Sigma^2 X$ be a double suspension. Then :*

- the homotopy G-coalgebra structure on $C_*(\Sigma^2 X)$, defined in Proposition 4.1, is trivial;
- the double cobar construction $\Omega^2 C_*(\Sigma^2 X)$ is a homotopy BV-algebra with the BV-operator from (2.4).

Proof. By the Proposition 5.3 the operations $E^{k,1}$ for $k \geq 2$ are trivial. Therefore to prove the first statement it remains to prove that $E^{1,1}$ is trivial. This follows from the fact that we have $d_1(1, (1, x)) = (1, d_0(1, x)) = (1, a_n) = s_{n+1}(1, a_0)$ is degenerate. The last operation

$$E^{1,1}(\sigma) = \sum_{1 < l < n} \pm \sigma(0, \dots, l) \otimes \sigma(0, l, l+1, \dots, n)$$

becomes trivial since $l > 1$.

For the second statement since the homotopy G-coalgebra structure is trivial, all the obstructions O_n are zero and so, by the Proposition 3.3 we obtain the announced homotopy BV-algebra structure on $\Omega^2 C_*(\Sigma^2 X)$. \blacksquare

We make explicit the homotopy BV-algebra structure on $\Omega^2 C_*(\Sigma^2 X)$ by first observing that the coproduct ∇_0 from (4.2) is a shuffle coproduct ; we write it as

$$\nabla_0([a_1 | \dots | a_n]) = \sum_{I_0 \sqcup I_1} \pm [a_{I_0}] \otimes [a_{I_1}]$$

where the sum is taken over all partitions $I_0 \sqcup I_1$ of $I = \{1, \dots, n\}$ with $I_0 = \{i_1 < i_2 < \dots < i_k\}$ and $I_1 = \{j_1 < j_2 < \dots < j_{n-k}\}$. We denote a_{I_0} to be $a_{i_1} \otimes a_{i_2} \otimes \dots \otimes a_{i_k}$. Also we denote $a_{I_0^{-1}}$ to be $a_{i_k} \otimes \dots \otimes a_{i_2} \otimes a_{i_1}$ and similarly for a_{I_1} .

We make explicit now the BV-operator on a 2 and 3-tensor. For the sake of simplicity we do not keep track of signs ; also to avoid confusion we write $\underline{a} = [a_1 | a_2 | \dots | a_k]$ an element in $\Omega C_*(\Sigma^2 X)$ and $[\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n]$ an element in $\Omega^2 C_*(\Sigma^2 X)$.

Thus $[\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n]$ is $[[a_{1,1} | a_{1,2} | \dots | a_{1,k_1}], [a_{2,1} | a_{2,2} | \dots | a_{2,k_2}], \dots, [a_{n,1} | a_{n,2} | \dots | a_{n,k_n}]]$.

With these conventions, the BV-operator B is given by

$$\begin{aligned} B([\underline{a}, \underline{b}]) &= B([a_1 | \dots | a_m], [b_1 | \dots | b_n]) = [[a_m | \dots | a_1 | b_1 | \dots | b_n]] + [[b_n | \dots | b_1 | a_1 | \dots | a_m]]. \\ B([\underline{a}_1 | \dots | \underline{a}_m], [\underline{b}_1 | \dots | \underline{b}_n], [\underline{c}_1 | \dots | \underline{c}_r]) &= [[a_{I_1^{-1}} | b_1 | \dots | b_n], [a_{I_0^{-1}} | c_1 | \dots | c_r]] + [[b_{I_1^{-1}} | c_1 | \dots | c_r], [b_{I_0^{-1}} | a_1 | \dots | a_m]] \\ &\quad + [[c_{I_1^{-1}} | a_1 | \dots | a_m], [c_{I_0^{-1}} | b_1 | \dots | b_n]]. \end{aligned}$$

Remark 6.2. In topological point of view, let us consider X to be a connected (countable) CW-complex with one vertex. The James/Milgram's models $J_i(X)$ are H-spaces homotopically equivalent to $\Omega^i \Sigma^i X$, see [Mil66, theorem 5.2]. Moreover, for $i \geq 1$ the cellular chain complex

$C_*(J_i(\Sigma X))$ is a cocommutative, primitively generated DG-Hopf algebra, and $C_*(J_{i+1}(X))$ is isomorphic to $\Omega C_*(J_i(\Sigma X))$, [Mil66, theorem 6.1 and 6.2]. Then using Proposition 2.5 we get a homotopy BV-algebra structure on it which is similar to the one obtained in the simplicial context. Also, we have a DGA-quasi-isomorphism

$$\Omega C_*(J_1(\Sigma X)) \longrightarrow C_*(J_2(X)) \xrightarrow{C_*(j_2)} C_*(\Omega^2 \Sigma^2 X).$$

7 Homotopy BV-algebra structure on $\Omega^2 C_*(X)$ over $R \supset \mathbb{Q}$

In [Bau98], Baues gives an explicit construction of the cobar construction of a 1-reduced simplicial set X over the integer coefficient ring as a cocommutative up to homotopy DG-bialgebra. By using methods of [Ani89], he shows that over a ring R containing \mathbb{Q} as a subring, this DG-bialgebra can be deformed into a strictly cocommutative DG-bialgebra [Bau98, Theorem 4.7]. The latter is isomorphic as DG-Hopf algebras to the universal enveloping algebra of the Lie algebra $L(s^{-1} \tilde{C}_* X)$ generated by the desuspension of the reduced coalgebra of normalized chain complex, combine [Bau98, Theorem 4.8] and [Kas95, Proposition V.2.4]. Hence, over such a ring R , the cobar construction $\Omega C_*(X)$ is (can be deformed into) an involutive DG-Hopf algebra. By applying the Proposition 2.5 to the resulting involutive cobar construction $(\Omega C_*(X), \nabla'_0, S')$, we obtain

Theorem 7.1. *Let X be a 2-reduced simplicial set. Then the double cobar construction $\Omega(\Omega C_*(X), \nabla'_0, S')$ over $R \supset \mathbb{Q}$ coefficients is a homotopy BV-algebra with BV-operator Connes-Moscovici's operator.*

In fact, we can obtain this theorem in a slightly different way. Over the integer ring, the cobar construction of a 1-reduced simplicial set, $(\Omega C_*(X), \nabla_0, S)$ is a DG-Hopf algebra (not involutive in general). Using the structure of the rational ring \mathbb{Q} , we deform step by step this DG-Hopf algebra into a cocommutative (hence involutive) DG-Hopf algebra over a ring $R \supset \mathbb{Q}$. We extend the deformation (given in [Bau98]) of the DG-bialgebra structure of the cobar construction to a deformation of the DG-Hopf structure. The resulting cobar construction $(\Omega C_*(X), \nabla'_0, S')$ comes with homotopy derivations (see below) connecting the obtained DG-Hopf structure (∇'_0, S') with the initial one (∇_0, S) .

We present first how to deform a DG-Hopf algebra with some good properties into a cocommutative one and we apply this deformation to the cobar construction.

Let \mathbf{DGA}_0 be the category of connected DG-algebras (associative). For a free DG-module V , we denote by $L(V)$ the DG-Lie algebra which is the free graded Lie algebra on the free module V . We set $V_{\leq n}$ (resp. $V_{< n}$) the sub DG-module of elements $v \in V$ such that $|v| \leq n$ (resp. $|v| < n$). Similarly, $V_{\geq n}$ (resp. $V_{> n}$) denotes the sub graded set of elements $v \in V$ such that $|v| \geq n$ (resp. $|v| > n$).

Definition 7.2. Let $f, g : A \rightarrow B$ be two maps between two DG-modules A, B . A **derivation homotopy** between f and g is a map $F : A \rightarrow B$ satisfying

$$(7.1) \quad dF + Fd = f - g$$

$$(7.2) \quad F(ab) = F(a)g(b) + (-1)^{|F||a|} f(a)F(b).$$

Definition 7.3. A homotopy DG-bialgebra (A, ∇, G_1, G_2) is an object A in \mathbf{DGA}_0 together with a coproduct ∇ in \mathbf{DGA}_0 making (A, ∇) into a coalgebra in \mathbf{DGA}_0 cocommutative up to G_1 , coassociative up to G_2 , both being derivation homotopies.

Definition 7.4. Let (A, ∇, G_1, G_2) be a homotopy DG-bialgebra such that $A = TV$, V being a DG-module with $V_0 = 0$. We call (A, ∇, G_1, G_2, V) n -good if the following conditions (7.3), (7.4), (7.5) hold.

$$(7.3) \quad \nabla = \tau \nabla \text{ and } (\nabla \otimes 1) \nabla = (1 \otimes \nabla) \nabla \text{ on } V_{\leq n}$$

$$(7.4) \quad d(V_{\leq n+1}) \subset L(V_{\leq n}) \subset T(V) = A$$

$$(7.5) \quad V_{\leq n} \subset \ker(\overline{\nabla}).$$

In [Ani89], Anick showed when $R \supset \mathbb{Q}$ that a homotopy DG-bialgebra satisfying the n -good condition is also $(n+1)$ -good. That is there exist an extension of this homotopy DG-bialgebra which is $(n+1)$ -good. We consider the case of an n -good coassociative homotopy DG-bialgebra with an antipode S and show we can deform it into a $(n+1)$ -good one with antipode S^{n+1} .

Lemma 7.5. Let $\mathcal{A} := (A, \nabla, G_1, 0, S, V)$ be a homotopy DG-bialgebra with antipode S . We suppose it is n -good. Then there exists a DG-Hopf algebra $(A, \nabla^{n+1}, G_1^{n+1}, 0, S^{n+1}, V^{n+1})$ which both extends \mathcal{A} and is $(n+1)$ -good. Moreover, there are derivation homotopies $F^{n+1} : \nabla \simeq \nabla^{n+1}$ and $\Gamma^{n+1} : S \simeq S^{n+1}$ with $F^{n+1}(a) = 0$ and $\Gamma^{n+1}(a) = 0$ for $|a| < n$, $a \in A$.

Proof. The part $(n+1)$ -good as bialgebra is already done in [Bau98, Theorem 4.5] where $\nabla^{n+1}, G_1^{n+1}, G_2^{n+1}, F^{n+1}$ and V^{n+1} are defined. Then we are just interested by the construction of S^{n+1} and Γ^{n+1} .

Let $S^{n+1} = S - R^{n+1}$ where $R^{n+1} = \mu(1 \otimes S) \nabla^{n+1} - \eta \epsilon$. Then $R^{n+1} = 0$ on $V_{\leq n}$. On V_{n+1} we have

$$\begin{aligned} \mu(1 \otimes S^{n+1}) \nabla^{n+1} &= \mu(1 \otimes S) \nabla^{n+1} - \mu(1 \otimes R^{n+1}) \nabla^{n+1} \\ &= \mu(1 \otimes S) \nabla^{n+1} - R^{n+1} \\ &= \mu(1 \otimes S) \nabla^{n+1} - \mu(1 \otimes S) \nabla^{n+1} + \eta \epsilon \\ &= \eta \epsilon. \end{aligned}$$

Let F^{n+1} be the homotopy between ∇ and ∇^{n+1} , we have $dF^{n+1} + F^{n+1}d = \nabla - \nabla^{n+1}$. Setting $\Gamma^{n+1} = -\mu(1 \otimes S)F^{n+1}$ we obtain the desired homotopy. Indeed,

$$d\Gamma^{n+1} + \Gamma^{n+1}d = -\mu(1 \otimes S)(dF^{n+1} + F^{n+1}d) = \mu(1 \otimes S)(\nabla^{n+1} - \nabla) = R^{n+1} = S - S^{n+1}.$$

By construction F^{n+1} is such that $F^{n+1} = 0$ on $V_{<n} \cup V_{>n+1}$ then Γ^{n+1} so.

The same argument is true for $\tilde{S}^{n+1} = S - \tilde{R}^{n+1}$ with $\tilde{R}^{n+1} = \mu(S \otimes 1) \nabla^{n+1} - \eta \epsilon$ and $\tilde{\Gamma}^{n+1} = -\mu(S \otimes 1)F^{n+1}$. Because of coassociativity, both S and \tilde{S} are equal. \blacksquare

Lemma 7.6. Let $(A = TV, \nabla, G_1, S)$ be a DG-Hopf algebra cocommutative up to the homotopy G_1 and with S as antipode. Suppose that (A, ∇, G_1, S, V) is 1-good. Then there are a coproduct ∇' and an antipode S' such that (A, ∇', S') is a cocommutative DG-Hopf algebra. Moreover there are derivation homotopies $F : \nabla \simeq \nabla'$ and $\Gamma : S \simeq S'$.

Proof. By iterating the previous Lemma 7.5 we obtain a n -good deformation as DG-Hopf algebra for each $n \geq 1$. Hence we define $S' = S^n$. We have $\mu(1 \otimes S')\nabla' = \eta\epsilon$ and $S' = \tilde{S}'$. The homotopy Γ is $\sum_{n \geq 1} \Gamma^n$. ■

Proposition 7.7. *Let $\mathbb{Q} \subset R$ and let X be a 1-reduced simplicial set. Then there is both a coproduct ∇' and an antipode S' on the cobar construction $\Omega C_* X$ such that $(\Omega C_*(X), \nabla', S')$ is a cocommutative DG-Hopf algebra. Moreover there are derivation homotopies $F : \nabla_0 \simeq \nabla'$ and $\Gamma : S_0 \simeq S'$, where ∇_0 and S_0 are respectively the coproduct from (4.2) and the resulting antipode.*

Proof. We denote by G_1 the homotopy to the cocommutativity of the coproduct ∇_0 . It is defined in [Bau98, (4)]. We apply the Lemma 7.6 to $(\Omega C_*(X), \nabla_0, G_1, 0, S, s^{-1} \tilde{C}_* X)$ which is 1-good. ■

Finally we obtain

Theorem 7.1. *Let X be a 2-reduced simplicial set. Then the double cobar construction $\Omega(\Omega C_*(X), \nabla'_0, S')$ over $R \supset \mathbb{Q}$ coefficients is a homotopy BV-algebra with BV-operator Connes-Moscovici's operator.*

References

- [Ada56] J. F. Adams. On the cobar construction. *Proc. Nat. Acad. Sci. U.S.A.*, 42:409–412, 1956.
- [Ani89] David J. Anick. Hopf algebras up to homotopy. *J. Amer. Math. Soc.*, 2(3):417–453, 1989.
- [Bau98] Hans-Joachim Baues. The cobar construction as a Hopf algebra. *Invent. Math.*, 132(3):467–489, 1998.
- [BF04] Clemens Berger and Benoit Fresse. Combinatorial operad actions on cochains. *Math. Proc. Cambridge Philos. Soc.*, 137(1):135–174, 2004.
- [CM00] Alain Connes and Henri Moscovici. Cyclic cohomology and Hopf algebra symmetry. *Lett. Math. Phys.*, 52(1):1–28, 2000. Conference Moshé Flato 1999 (Dijon).
- [Get94] E. Getzler. Batalin-Vilkovisky algebras and two-dimensional topological field theories. *Comm. Math. Phys.*, 159(2):265–285, 1994.
- [HPS07] Kathryn Hess, Paul-Eugène Parent, and Jonathan Scott. A chain coalgebra model for the James map. *Homology Homotopy Appl.*, 9(2):209–231, 2007.
- [Kad03] T. Kadeishvili. Cochain operations defining Steenrod \smile_i -products in the bar construction. *Georgian Math. J.*, 10(1):115–125, 2003.
- [Kad05] T. Kadeishvili. On the cobar construction of a bialgebra. *Homology Homotopy Appl.*, 7(2):109–122, 2005.
- [Kas95] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

-
- [KS02] T. Kadeishvili and S. Saneblidze. The twisted Cartesian model for the double path fibration. *ArXiv Mathematics e-prints*, October 2002.
- [LV12] J.L. Loday and B. Vallette. *Algebraic Operads*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Springer Berlin Heidelberg, 2012.
- [May92] J.P. May. *Simplicial Objects in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1992.
- [Men04] Luc Menichi. Batalin-Vilkovisky algebras and cyclic cohomology of Hopf algebras. *K-Theory*, 32(3):231–251, 2004.
- [Mil66] R. James Milgram. Iterated loop spaces. *Ann. of Math. (2)*, 84:386–403, 1966.
- [MS01] J. E. McClure and J. H. Smith. Multivariable cochain operations and little n-cubes. *ArXiv Mathematics e-prints*, June 2001.
- [Smi02] Vladimir A. Smirnov. The homology of iterated loop spaces (appendix by Francis Sergeraert). *Forum Math.*, 14(3):345–381, 2002.
- [Swe69] Moss E. Sweedler. *Hopf algebras*. W.A.Benjamin, Inc., 1969.

LABORATOIRE DE MATHÉMATIQUES JEAN LERAY
2 RUE DE LA HOUSSINIÈRE - BP 92208 F-44322 NANTES CEDEX 3, FRANCE
E-mail address : alexandre.quesney@univ-nantes.fr