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Homotopy BV-algebra structure on the double cobar construction

Alexandre Quesney

Abstract

We show that the double cobar construction, $\Omega^2 C_*(X)$, of a simplicial set X is a homotopy BV-algebra if X is a double suspension, or if X is 2-reduced and the coefficient ring contains the ring of rational numbers \mathbb{Q} . Indeed, Connes' operator defines the desired hBVa-structure on $\Omega^2 C_*(X)$ when the antipode $S : \Omega C_*(X) \rightarrow \Omega C_*(X)$ is involutive. We proceed by defining a family of obstructions $O_n : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes n}$, $n \geq 2$ measuring the difference $S^2 - Id$. When X is a suspension, the only obstruction remaining is $O_2 := E^{1,1} - \tau E^{1,1}$ where $E^{1,1}$ is the dual of the \smile_1 -product. When X is a double suspension the obstructions vanish. We make explicit an hBVa-structure on $\Omega^2 C_*(X)$.

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Introduction

Adams' cobar construction is a model of the loop space of a 1-connected topological space [Ada56]. The cobar construction is a functor from differential graded coalgebras to differential graded algebras; for iteration, a coproduct is thus needed. For a 1-reduced simplicial set

X , Baues defined [Bau98] a DG-bialgebra structure on its first cobar construction $\Omega C_*(X)$. The resulting double cobar construction is an algebraic model for the double loop space. Baues' coproduct on $\Omega C_*(X)$ tantamounts to a homotopy G-coalgebra structure $\{E^{k,1}\}_{k \geq 1}$ on the DG-coalgebra $C_*(X)$. Thus this homotopy G-coalgebra structure on $C_*(X)$ determines an antipode $S : \Omega C_*(X) \rightarrow \Omega C_*(X)$ on the cobar construction. Under the hypothesis of an involutive antipode, Menichi proved [Men04], that for a unital (ungraded) Hopf algebra A , Connes' operator induces a Batalin-Vilkovisky algebra on the homology of the cobar construction $H_*(\Omega A)$.

By defining a family of operations $O_n : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes n}$, $n \geq 2$ which measures the difference $S^2 - Id$, we provide a criterion for the involutivity of S in terms of the operations $E^{k,1}$. Also, the circle action on $\Omega^2 X$ by rotating the equator defines a BV operator, and turns the Gerstenhaber algebra $H_*(\Omega^2 X)$ into a Batalin-Vilkovisky algebra [Get94]. The vanishing of operations $O_n : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes n}$ gives a sufficient condition to the extension of the homotopy G-algebra $\Omega^2 C_*(X)$ into the homotopy BV-algebra whose BV-operator is Connes' operator.

Proposition 3.0.6. *Let $(C, d, \nabla_C, E^{k,1})$ be a homotopy G-coalgebra.*

1. *The cobar construction ΩC is an iDGH if and only if all the obstructions $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$ defined by*

$$O_n := \sum_{s=1}^n \sum_{n_1 + \dots + n_s = n} \mu_{\Omega(12)}^{(s)}(F^{n_1} \otimes \dots \otimes F^{n_s}) F^s$$

for $n \geq 2$ are zero.

2. *Let C be 2-reduced.*

- (a) *If there exists an integer n such that $O_k = 0$ for $k \leq n$, then $\Omega^2(C_{\leq n})$ is a homotopy BV-algebra.*
- (b) *If all the obstructions $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$ are zero, then the double cobar construction $\Omega^2 C$ is a homotopy BV-algebra given by the BV-operator defined in (2.12).*

We apply this criterion to the homotopy G-coalgebra $C_*(X)$ of a 1-reduced simplicial set X . We show that when ΣX is a simplicial suspension, the family $O_n : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes n}$ reduces to $O_2 : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes 2}$ which is the deviation from cocommutativity of the dual of the \smile_1 -product $E^{1,1} : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes 2}$.

Proposition 5.0.12. *The hGc-structure $E^{k,1}$ on $C_*(\Sigma X)$ defined in Proposition 4.0.9 is*

$$E^{1,1}(\sigma) = \sum_{1 < l < n} \pm \sigma(0, \dots, l) \otimes \sigma(0, l, l+1, \dots, n);$$

$$E^{k,1} = 0 \text{ for } k \geq 2.$$

Proposition 5.0.13. *Let $(C, d, \nabla_C, E^{k,1})$ be an hGc with $E^{k,1} = 0$ for $k \geq 2$. If $O_2 = E^{1,1} - \tau E^{1,1}$ is zero, then so is O_n for $n \geq 2$.*

In the case of a double simplicial suspension $\Sigma^2 X$, the homotopy G-coalgebra $C_*(\Sigma^2 X)$ becomes trivial and all obstructions $O_n : \tilde{C}_*(\Sigma^2 X) \rightarrow \tilde{C}_*(\Sigma^2 X)^{\otimes n}$, $n \geq 2$ are zero. As a consequence, the cobar construction $\Omega C_*(\Sigma^2 X)$ is the free tensor DG-algebra with the shuffle coproduct. We make explicit a homotopy BV-algebra structure on $\Omega^2 C_*(\Sigma^2 X)$.

Proposition 6.0.14. *Let $\Sigma^2 X$ be a double suspension. Then :*

- the hGc-structure on $C_*(\Sigma^2 X)$, defined in Proposition 4.0.9, is trivial;
- the double cobar construction $\Omega^2 C_*(\Sigma^2 X)$ is an hBVa with the BV-operator from (2.12).

On the other hand, if the ground ring contains the ring \mathbb{Q} , we deform $\Omega C_*(X)$ into a cocommutative DG-Hopf algebra. Consequently, $\Omega C_*(X)$ has an involutive antipode and the homotopy BV-algebra structure we consider on $\Omega^2 C_*(X)$ follows.

Proposition 7.0.16. *Let X be a 2-reduced simplicial set. Then the double cobar construction $(\Omega^2 C_*(X), d_{\Omega^2})$ over $R \supset \mathbb{Q}$ coefficients is an hBV-algebra with BV-operator defined in (2.12).*

The paper is organized as follows.

In the first section we define and recall some basics facts about the bar and cobar constructions, and antipode for Hopf algebras.

In the second section two homotopy structures on the cobar construction ΩA are introduced : a homotopy G-algebra structure when A is a DG-bialgebra and a homotopy BV-algebra structure when A is a DG-Hopf algebra with an involutive antipode.

In the third section we consider a homotopy G-coalgebra $(C, E^{k,1})$. We describe the antipode on the cobar construction ΩC by the operations $E^{k,1}$. We define a family of operations $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$, $n \geq 2$ which measures the difference $S^2 - Id$.

In the fourth section we recall the coproduct defined by Baues in [Bau98]. We show that the obtained homotopy G-coalgebra structure corresponds to the one given in [BF04, Kad03].

In the sections 5 and 6, we consider simple and double simplicial suspensions $\Sigma X, \Sigma^2 X$ of a simplicial set X . We reduce the obstructions $O_n : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes n}$, $n \geq 2$ to O_2 for a simple suspension; we show that all $O_n : \tilde{C}_*(\Sigma^2 X) \rightarrow \tilde{C}_*(\Sigma^2 X)^{\otimes n}$ are zero for a double suspension. As a consequence we obtain an hBVa-structure on $\Omega^2 C_*(\Sigma^2 X)$ we write explicitly.

The last section is devoted to the rational case. We deform step by step a DG-Hopf algebra with some good properties into a cocommutative DG-Hopf algebra. We prove that the double cobar construction of a 2-reduced simplicial set is a homotopy BV-algebra.

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1 Notations and preliminaries

Conventions and notations An element $a \in M_n$ where M is a graded object is said of degree n ; we denote $|a|$ its degree. By the r -suspension s^r we mean the r -shifting degree

1. Notations and preliminaries

$(s^r M)_n = M_{n-r}$. For graded modules M, M', N and N' and two linear maps

$$f : M \rightarrow M', \quad g : N \rightarrow N'$$

of degree $|f|$ and $|g|$ respectively, the tensor map

$$f \otimes g : M \otimes N \rightarrow M' \otimes N'$$

follows the Koszul sign rule

$$(f \otimes g)(m \otimes n) = (-1)^{|g||m|} f(m) \otimes g(n).$$

For $(A, \mu), (A', \mu')$ two graded algebras, we use the convention that the tensor product $A \otimes A'$ is an algebra with

$$(\mu \otimes \mu')(1 \otimes \tau \otimes 1)$$

as product where

$$\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a.$$

A unital algebra (A, η) is called augmented if there is an algebra morphism $\epsilon : A \rightarrow R$. We denote by \bar{A} the augmentation ideal $\text{Ker}(\epsilon)$. For a coalgebra (A, ∇) we use the Sweedler notation for the coproduct,

$$\nabla(a) = a^1 \otimes a^2$$

where we have omitted the sum. A counital coalgebra (A, ϵ) is called coaugmented if there is a coalgebra morphism $\eta : R \rightarrow A$. We denote by $\tilde{A} = \text{Ker}(\epsilon)$ the reduced coalgebra. The reduced coalgebra is endowed with the reduced coproduct

$$\bar{\nabla}(a) = \nabla(a) - a \otimes 1 - 1 \otimes a.$$

Let (A, ∇_A) and (B, ∇_B) be two coalgebras. Their tensor product $A \otimes B$ is still a coalgebra with the coproduct

$$\nabla_{A \otimes B} = (1 \otimes \tau \otimes 1)(\nabla_A \otimes \nabla_B).$$

An algebra or a coalgebra A is called n -reduced if it is both connected and $A_k = 0$ for $1 \leq k \leq n$.

The bar and cobar constructions Let (C, d, ∇_C) be a connected DG-coalgebra. Adam's cobar construction [Ada56] is the free DG-algebra

$$\Omega C := (T(s^{-1} \tilde{C}), d_\Omega)$$

whose differential is induced by

$$d_\Omega = -s^{-1} ds + (s^{-1} \otimes s^{-1}) \bar{\nabla}_C s.$$

We will denote $d_\Omega = d_0 + d_1$ where d_0 is the linear differential induced by d_C as a derivation and d_1 is the quadratic differential induced by the coproduct.

Let $(\Lambda, d_\Lambda, \mu_\Lambda)$ to be a DG-algebra. The bar construction is the cofree coalgebra

$$B\Lambda = (T^c(s\bar{\Lambda}), d_{B\Lambda}, \nabla_{B\Lambda})$$

with the differential whose component on $s\bar{\Lambda}$ is

$$d_{B\Lambda} = -s d s^{-1} + s \mu_{\bar{\Lambda}}(s^{-1} \otimes s^{-1}).$$

We denote $d_{B\Lambda} = d_0 + d_1$ where d_0 is the linear differential induced by d_Λ and d_1 is the quadratic differential induced by the product.

Let (C, d_C, ∇_C) and $(\Lambda, d_\Lambda, \mu_\Lambda)$ to be respectively a conilpotent¹ DG-coalgebra and an augmented DG-algebra. We denote by $\text{Hom}(C, \Lambda)$ the set of linear morphisms from C to Λ . We have the commutative diagram

$$\begin{array}{ccccc} \text{Hom}_{\text{Alg}}(\Omega C; \Lambda) & \longleftrightarrow & \text{Hom}(\tilde{C}; \bar{\Lambda}) & \longleftrightarrow & \text{Hom}_{\text{Coalg}}(C; B\Lambda) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Hom}_{\text{DG-Alg}}(\Omega C; \Lambda) & \longleftrightarrow & \text{Tw}(C; \Lambda) & \longleftrightarrow & \text{Hom}_{\text{DG-Coalg}}(C; B\Lambda) \end{array}$$

where horizontal maps are natural bijections. We denote by $\text{Tw}(C; \Lambda)$ the set of twisting cochains : a map $f : C \rightarrow \Lambda$ is a twisting cochain if it satisfies the Brown's condition $\partial f = -f \smile f$ in the convolution DG-algebra $(\text{Hom}(C; \Lambda), \smile, \partial)$. The differential ∂ is given by $\partial f = d_\Lambda f - (-1)^{|f|} f d_C$ and the \smile -product by $f \smile g := \mu_\Lambda(f \otimes g) \nabla_C$.

Hopf algebras

Definition 1.0.1. For a DG-bialgebra $(\mathcal{H}, d, \mu, \eta, \Delta, \epsilon)$ an **antipode** $S : \mathcal{H} \rightarrow \mathcal{H}$ is a linear chain map which is the inverse of the identity in the convolution algebra $\text{Hom}(\mathcal{H}, \mathcal{H})$; the convolution product being $f \smile g = \mu(f \otimes g) \Delta$. Explicitly, S satisfies

$$S(a^1) a^2 = \eta \epsilon(a) = a^1 S(a^2)$$

for all $a \in \mathcal{H}$.

Definition 1.0.2. A **DG-Hopf algebra** $(\mathcal{H}, d, \mu, \eta, \Delta, \epsilon, S)$ is a DG-bialgebra \mathcal{H} endowed with an antipode S . If moreover, the antipode is involutive (i.e $S^2 = id$) we call \mathcal{H} an **involutive DG-Hopf algebra** (or iDGH for short).

An antipode satisfies the following properties.

Proposition 1.0.3. [Swe69, Proposition 4.0.1]

$$i. S(a^2) \otimes S(a^1) = (-1)^{|a^1||a^2|} S(a)^1 \otimes S(a)^2 \quad (\text{coalgebra antimorphism});$$

¹The conilpotency hypothesis of C is only necessary for the right hand part of the diagram, that is to extend a map from $\text{Hom}(\tilde{C}; \bar{\Lambda})$ (resp. $\text{Tw}(C; \Lambda)$) to a map in $\text{Hom}_{\text{Coalg}}(C; B\Lambda)$ (resp. $\text{Hom}_{\text{DG-Coalg}}(C; B\Lambda)$). We will only use this to the coalgebra $C = B\Lambda \otimes B\Lambda$ which is conilpotent.

2. Homotopy structures on the cobar construction

- ii. $S(\eta(1)) = \eta(1)$ (unital morphism);
- iii. $S(ab) = (-1)^{|a||b|}S(b)S(a)$ (algebra antimorphism);
- iv. $\epsilon(S(a)) = \epsilon(a)$ (counital morphism).
- v. The following equations are equivalent:
 - (a) $S^2 = S \circ S = id$;
 - (b) $S(a^2)a^1 = \eta\epsilon(a)$;
 - (c) $a^2S(a^1) = \eta\epsilon(a)$.
- vi. If \mathcal{H} is commutative or cocommutative, hence $S^2 = id$.

2 Homotopy structures on the cobar construction

2.1 Homotopy G-algebra on the cobar construction

In this section we present the homotopy G-algebra structure on the cobar construction of a 1-reduced DG-bialgebra given in [Kad05]. A homotopy G-algebra is a particular Hirsch algebra and has the property to induce in homology a Gerstenhaber algebra structure. We define the Hirsch algebras and homotopy G-algebras via the bar construction. Also we recall,

Definition 2.1.1. A **Gerstenhaber algebra** A is a graded commutative algebra (A, \cdot) together with a degree one bracket $\{ ; \} : A \otimes A \rightarrow A$ which satisfies the three following properties

$$(2.1) \quad \{a, b\} = -(-1)^{(|a|-1)(|b|-1)}\{b, a\} \quad (\text{The degree 1 antisymmetry});$$

$$(2.2) \quad \{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{(|a|-1)|b|}b \cdot \{a, c\} \quad (\text{The degree 1 Poisson identity});$$

$$(2.3) \quad \{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|-1)(|b|-1)}\{b, \{a, c\}\} \quad (\text{The degree 1 Jacobi identity}).$$

To an associative DG-product (of degree 0) $\mu_E : B\Lambda \otimes B\Lambda \rightarrow B\Lambda$ which is also a coalgebra map corresponds a twisting cochain $\tilde{E} \in \text{Tw}(B\Lambda, \Lambda)$. Its (i, j) -th component is $\tilde{E}_{i,j} : (s\bar{\Lambda})^{\otimes i} \otimes (s\bar{\Lambda})^{\otimes j} \rightarrow \bar{\Lambda}$. We denote by $E_{i,j} : \bar{\Lambda}^{\otimes i} \otimes \bar{\Lambda}^{\otimes j} \rightarrow \bar{\Lambda}$ the component

$$(2.4) \quad \begin{aligned} E_{i,j}(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_{i+j}) \\ := (-1)^{\sum_{s=1}^{i+j-1} |a_s|(i+j-s)} \tilde{E}_{i,j}(s^{\otimes i} \otimes s^{\otimes j})(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_{i+j}) \\ = \tilde{E}_{i,j}(s a_1 \otimes \dots \otimes s a_i \otimes s a_{i+1} \otimes \dots \otimes s a_{i+j}). \end{aligned}$$

The degree of $E_{i,j}$ is $i + j - 1$.

Definition 2.1.2. A **Hirsch algebra** Λ is the data of a connected DG-algebra $(\Lambda, d, \mu_\Lambda)$ together with a map $\mu_E : B\Lambda \otimes B\Lambda \rightarrow B\Lambda$ making $B\Lambda$ into an associative unital DG-bialgebra. We denote it by $(\Lambda, E_{i,j})$ where the operations $E_{i,j}$ are the ones corresponding to the product μ_E by using the sign convention (2.4). A **homotopy G-algebra** (hGa for short) is a Hirsch algebra whose operations $E_{i,j} = 0$ for $i \geq 2$.

The unitary condition gives

$$(2.5) \quad E_{0,i} = E_{i,0} = 0 \text{ for every } i \neq 1 \quad \text{and } E_{0,1} = E_{1,0} = Id_{\Lambda}.$$

We denote $E_{i,j}(a_1, \dots, a_i; b_1, \dots, b_j)$ for $E_{i,j}((a_1 \otimes \dots \otimes a_i) \otimes (b_1 \otimes \dots \otimes b_j))$. Over $\overline{\Lambda}^{\otimes i} \otimes \overline{\Lambda}^{\otimes j} \otimes \overline{\Lambda}^{\otimes k}$ the associativity condition reads

$$(2.6) \quad E_{1,k}(E_{i,j}(a_1, \dots, a_i; b_1, \dots, b_j); c_1, \dots, c_k) + \\ \sum_{n=1}^{i+j} \sum_{\substack{0 \leq i_1 \leq \dots \leq i_n \leq i \\ 0 \leq j_1 \leq \dots \leq j_n \leq j}} (-1)^{\alpha_1} E_{n+1,k}(E_{i_1, j_1}(a_1, \dots, a_{i_1}; b_1, \dots, b_{j_1}), \dots \\ \dots, E_{i-i_n, j-j_n}(a_{i_n+1}, \dots, a_i; b_{j_n+1}, \dots, b_j); c_1, \dots, c_k) \\ = \sum_{m=1}^{j+k} \sum_{\substack{0 \leq j_1 \leq \dots \leq j_m \leq j \\ 0 \leq k_1 \leq \dots \leq k_m \leq k}} (-1)^{\alpha_2} E_{i, m+1}(a_1, \dots, a_i; E_{j_1, k_1}(b_1, \dots, b_{j_1}; c_1, \dots, c_{k_1}), \dots \\ \dots, E_{j-j_m, k-k_m}(b_{j_m+1}, \dots, b_j; c_{k_m+1}, \dots, c_k)) \\ + E_{i,1}(a_1, \dots, a_i; E_{j,k}(b_1, \dots, b_j; c_1, \dots, c_k))$$

where

$$\alpha_1 = \sum_{u=1}^n \left(\sum_{s=i_u+1}^{i_{u+1}} |a_s| + i_{u+1} - i_u \right) \left(\sum_{s=1}^{j_u} |b_s| + j_u \right) \\ \alpha_2 = \sum_{u=1}^m \left(\sum_{s=j_u+1}^{j_{u+1}} |b_s| + j_{u+1} - j_u \right) \left(\sum_{s=1}^{k_u} |c_s| + k_u \right)$$

with

$$i_0 = 0; \quad j_0 = 0; \quad k_0 = 0; \\ i_{n+1} = i; \quad j_{n+1} = j_{m+1} = j; \quad k_{m+1} = k.$$

For example when $i = j = k = 1$ the relation (2.6) gives :

$$(2.7) \quad E_{1,1}(E_{1,1}(a; b); c) = E_{1,1}(a; E_{1,1}(b; c)) + E_{1,2}(a; b, c) + (-1)^{(|b|-1)(|c|-1)} E_{1,2}(a; c, b) \\ - E_{2,1}(a, b; c) - (-1)^{(|a|-1)(|b|-1)} E_{2,1}(b, a; c).$$

2. Homotopy structures on the cobar construction

The condition $d\mu_E = \mu_E(d \otimes 1 + 1 \otimes d)$ reads over $\overline{\Lambda}^{\otimes i} \otimes \overline{\Lambda}^{\otimes j}$:

$$\begin{aligned}
(2.8) \quad d_\Lambda E_{i,j}(a_1, \dots, a_i; b_1, \dots, b_j) \\
+ \sum_{\substack{0 \leq i_1 \leq i \\ 0 \leq j_1 \leq j}} (-1)^{\beta_2} E_{i_1, j_1}(a_1, \dots, a_{i_1}; b_1, \dots, b_{j_1}) \cdot E_{i-i_1, j-j_1}(a_{i_1+1}, \dots, a_i; b_{j_1+1}, \dots, b_j) = \\
= \sum_{l=1}^i (-1)^{\beta_3} E_{i,j}(a_1, \dots, d_\Lambda(a_l), \dots, a_i); b_1, \dots, b_j) \\
+ \sum_{l=1}^{i-1} (-1)^{\beta_4} E_{i-1, j}(a_1, \dots, a_l \cdot a_{l+1}, \dots, a_i); b_1, \dots, b_j) \\
+ \sum_{l=1}^j (-1)^{\beta_5} E_{i,j}(a_1, \dots, a_i); b_1, \dots, d_\Lambda(b_l), \dots, b_j) \\
+ \sum_{l=1}^{j-1} (-1)^{\beta_6} E_{i, j-1}(a_1, \dots, a_i); b_1, \dots, b_l \cdot b_{l+1}, \dots, b_j)
\end{aligned}$$

where

$$\begin{aligned}
\beta_2 &= \sum_{s=1}^{i_1} |a_s| + \sum_{s=1}^{j_1} |b_s| + i_1 + j_1 + \left(\sum_{s=i_1+1}^i |a_s| + i - i_1 \right) \left(\sum_{s=1}^{j_1} |b_s| + j_1 \right) \\
\beta_3 &= \sum_{s=1}^{l-1} |a_s| + l \\
\beta_4 &= \sum_{s=1}^{l-1} |a_s| + l + |a_l| \\
\beta_5 &= \sum_{s=1}^i |a_s| + i + \sum_{s=1}^{l-1} |b_s| + l \\
\beta_6 &= \sum_{s=1}^i |a_s| + i + \sum_{s=1}^{l-1} |b_s| + l + |b_l|.
\end{aligned}$$

We consider now $\Lambda := \Omega A = T(s^{-1} \tilde{A})$ where A is a 1-reduced DG-bialgebra. We write $\bar{a} := s^{-1} a_1 \otimes \dots \otimes s^{-1} a_m$ a typical element in ΩA . The degree $|\bar{a}|$ of $\bar{a} = s^{-1} a_1 \otimes \dots \otimes s^{-1} a_m$ is $\sum_{s=1}^m |a_s| - m$. We construct a \smile_1 -product $E_{1,1} : \overline{\Lambda} \otimes \overline{\Lambda} \rightarrow \Lambda$ making Λ into a homotopy pre-Gerstenhaber algebra. That is, the bracket defined by

$$(2.9) \quad \{\bar{a}; \bar{b}\} = E_{1,1}(\bar{a}; \bar{b}) - (-1)^{(|\bar{a}|-1)(|\bar{b}|-1)} E_{1,1}(\bar{b}; \bar{a})$$

which induces a Gerstenhaber bracket on the homology. The \smile_1 -product is a homotopy for the product of Λ . We define

$$\begin{aligned}
(2.10) \quad E_{1,1}(s^{-1} a_1 \otimes \dots \otimes s^{-1} a_m; s^{-1} b_1 \otimes \dots \otimes s^{-1} b_n) \\
:= \sum_{l=1}^m (-1)^{\gamma_3} s^{-1} a_1 \otimes \dots \otimes (a_l^1 \cdot b_1) \otimes \dots \otimes s^{-1} (a_l^n \cdot b_n) \otimes \dots \otimes s^{-1} a_m
\end{aligned}$$

with

$$\begin{aligned} \gamma_3 = & \sum_{u=1}^n |b_u| \left(\sum_{s=l+u}^m |a_s| + m - l - u + 1 \right) + \kappa_n(a_l) + \sum_{u=2}^n (|a_l^u| - 1) \left(\sum_{s=1}^{u-1} |b_s| - u + 1 \right) \\ & + \sum_{s=1}^n (|a_l^s| - 1)(2n - 2s + 1) + \sum_{s=1}^{n-1} (|a_l^s| + |b_s|)(n - s), \end{aligned}$$

where

$$\kappa_n(a) := \begin{cases} \sum_{1 \leq 2s+1 \leq n} |a^{2s+1}| & \text{if } n \text{ is even;} \\ \sum_{1 \leq 2s \leq n} |a^{2s}| & \text{if } n \text{ is odd.} \end{cases}$$

Let us define the others operations $E_{1,k} : \overline{\Lambda} \otimes \overline{\Lambda}^{\otimes k} \rightarrow \Lambda$ giving the hGa-structure of $\Lambda = \Omega A$.

We use the notation $a \diamond \overline{b} = s^{-1}(a^1 b_1) \otimes \dots \otimes s^{-1}(a^s b_s)$ where $\overline{b} := s^{-1} b_1 \otimes \dots \otimes s^{-1} b_s$. We have

$$(2.11) \quad E_{1,k}(s^{-1} a_1 \otimes \dots \otimes s^{-1} a_n; \overline{b}_1, \dots, \overline{b}_k) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} \pm s^{-1} a_1 \otimes \dots \otimes s^{-1} a_{i_1-1} \otimes a_{i_1} \diamond \overline{b}_1 \otimes \dots \otimes s^{-1} a_{i_k-1} \otimes a_{i_k} \diamond \overline{b}_k \otimes s^{-1} a_{i_k+1} \otimes \dots \otimes s^{-1} a_n$$

when $n \geq k$ and zero when $n < k$. We have,

Proposition 2.1.3. [Kad05] *Let A be a 1-reduced DG-bialgebra. Then the cobar construction $(\Omega A, d_\Omega)$ together with $E_{1,k}$ defined in (2.10) and (2.11) is a homotopy G-algebra.*

2.2 Homotopy BV-algebra on the cobar construction

We describe here the homotopy BV-structure we obtain on $\Omega \mathcal{H}$ when \mathcal{H} is a 1-reduced involutive DG-Hopf algebra. This structure induces a structure of Batalin-Vilkovisky algebra in homology.

Definition 2.2.1. A **Batalin-Vilkovisky algebra** Λ is a Gerstenhaber algebra $(\Lambda, \{ , \})$ together with a degree +1 operator $B : \Lambda \rightarrow \Lambda$ satisfying

1. $B^2 = 0$;
2. $\{a; b\} = (-1)^{|a|}(B(a \cdot b) - B(a) \cdot b - (-1)^{|a|} a \cdot B(b))$ for all $a, b \in \Lambda$.

Definition 2.2.2. A **homotopy BV-algebra** Λ (hBVa for short) is a homotopy G-algebra together with a DG-operator $B : \Lambda \rightarrow \Lambda$ subject to relations:

1. $B^2 = 0$;
2. $\{a; b\} = (-1)^{|a|}(B(a \cdot b) - B(a) \cdot b - (-1)^{|a|} a \cdot B(b)) + d_\Lambda H(a; b) + H(d_\Lambda a; b) + (-1)^{|a|} H(a; d_\Lambda b)$ for all $a, b \in \Lambda$,

where $\{-, -\}$ is the Gerstenhaber bracket in the form (2.9) and $H : \Lambda \otimes \Lambda \rightarrow \Lambda$ is a degree 2 linear map.

2. Homotopy structures on the cobar construction

Connes' operator for Hopf algebras A defined in [CM00] induces a BV-operator on $H_*(\Omega A)$ provided that the antipode is involutive. With our sign convention it is given by :

$$(2.12) \quad B(s^{-1} a_1 \otimes s^{-1} a_2 \otimes \dots \otimes s^{-1} a_n) = \sum_{i=0}^{n-1} (-1)^i \pi_n \tau_n^i (s^{-1} a_1 \otimes s^{-1} a_2 \otimes \dots \otimes s^{-1} a_n).$$

where τ_n is the cyclic permutation

$$\tau_n(s^{-1} a_1 \otimes s^{-1} a_2 \otimes \dots \otimes s^{-1} a_n) := (-1)^{(|a_1|-1)(\sum_{i=2}^n |a_i|-1)} s^{-1} a_2 \otimes s^{-1} a_3 \otimes \dots \otimes s^{-1} a_n \otimes s^{-1} a_1$$

and

$$\begin{aligned} \pi_n &:= (s^{-1} \mu)^{\otimes n-1} \tau_{n-1, n-1} (S^{\otimes n-1} \otimes 1^{\otimes n-1}) ((\tau \Delta)^{(n-2)} \otimes 1^{n-1}) s^{\otimes n} \\ \tau_{n,n}(a_1 \otimes a_2 \otimes \dots \otimes a_n \otimes b_1 \otimes \dots \otimes b_n) &:= (-1)^{\sum_{i=1}^{n-1} |b_i| (\sum_{j=i+1}^n |a_j|)} a_1 \otimes b_1 \otimes a_2 \otimes b_2 \otimes \dots \otimes a_n \otimes b_n. \end{aligned}$$

For example,

$$B(s^{-1} a \otimes s^{-1} b) = (-1)^{|a|-1} s^{-1} (S(a)b) + (-1)^{|a|(|b|-1)+1} s^{-1} (S(b)a).$$

The involutivity of the antipode of A makes B into a square zero chain map. The following proposition is essentially an adaptation of [Men04, Proposition 1.9] to our context.

Proposition 2.2.3. *Let A be a 1-reduced iDGH. Then the cobar construction $(\Omega A, d_\Omega)$ is a homotopy BV-algebra with BV-operator defined in (2.12).*

Proof. Let us define the homotopy $H(\bar{a}; \bar{b}) := H_1(\bar{a}; \bar{b}) - (-1)^{(|\bar{a}|-1)(|\bar{b}|-1)} H_1(\bar{b}; \bar{a})$.

$$(2.13) \quad \begin{aligned} &H_1(s^{-1} a_1 \otimes \dots \otimes s^{-1} a_m; s^{-1} b_1 \otimes \dots \otimes s^{-1} b_n) \\ &:= \sum_{1 \leq j \leq p \leq m-1} (-1)^{\xi_j+n+1} \pi_{m+n-1}^s \tau_{m+n-1}^{s, n+m-1-j} \rho_{n+m}^{(p-j+1)} (s^{-1} a_1 \otimes \dots \otimes s^{-1} a_m \otimes s^{-1} b_1 \otimes \dots \otimes s^{-1} b_n) \end{aligned}$$

with

$$\xi_j = \begin{cases} \sum_{s=1}^m |a_s| & \text{for } j = 1; \\ \sum_{s=m-j+1}^m |a_s| & \text{for } j > 1. \end{cases}$$

Where,

$$\begin{aligned} \pi_m^s &= \pi_m (s^{-1})^{\otimes m} \\ \tau_m^{s,i} &:= (\tau_m^s)^i \\ \tau_m(a_1 \otimes a_2 \otimes \dots \otimes a_m) &:= (-1)^{|a_1|(\sum_{i=2}^m |a_i|)} a_2 \otimes a_3 \otimes \dots \otimes a_m \otimes a_1 \\ \rho_{m+1}^{(i)} &:= (1^{\otimes i-1} \otimes \mu \otimes 1^{\otimes m-i}) (1^{\otimes i-1} \otimes \tau_{m-i+1,1}) s^{\otimes m+1} \\ \tau_{k,1}(a_1 \otimes \dots \otimes a_k \otimes b) &:= (-1)^{|b|(\sum_{i=2}^k |a_i|)} a_1 \otimes b \otimes a_2 \otimes \dots \otimes a_k. \end{aligned}$$

By the Proposition [Men04, Proposition 1.9] we have

$$\{a; b\} = (-1)^{|a|} (B(a \cdot b) - B(a) \cdot b - (-1)^{|a|} a \cdot B(b)) + d_1 H(a; b) + H(d_1 a; b) + (-1)^{|a|} H(a; d_1 b).$$

on the cobar construction $(\Omega A, d_0 + d_1)$. The operators involved in the above equation commute with d_0 . Therefore we can replace d_1 by d_Ω in the previous equation. \blacksquare

3 Involutivity of the antipode of ΩC in terms of the homotopy G-coalgebra C

Let us consider a homotopy G-coalgebra $(C, E^{k,1})$. We describe the antipode on the cobar construction ΩC in terms of the operations $E^{k,1}$. We define a family of operations $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$, $n \geq 2$ which measures the difference $S^2 - Id$.

Definition 3.0.4. A **Hirsch coalgebra** C is the data of a 1-reduced DG-coalgebra (C, d, ∇_C) together with a map $\nabla : \Omega C \rightarrow \Omega C \otimes \Omega C$ making ΩC into a coassociative counital DG-bialgebra. The corresponding operations on C are denoted by $E^{i,j} : \tilde{C} \rightarrow \tilde{C}^{\otimes i} \otimes \tilde{C}^{\otimes j}$. A **homotopy G-coalgebra** (hGc for short) is a Hirsch coalgebra whose operations $E^{i,j} = 0$ for $j \geq 2$.

Let the cobar construction $(\Omega C, d, \mu, \nabla)$ to be a DG-bialgebra. We can define the antipode $S : \Omega C \rightarrow \Omega C$ by $S(1) = 1$ and for $\sigma \in \tilde{C}_n$ with $n \geq 2$ by

$$(3.1) \quad S(\sigma) := -\sigma - \mu(S \otimes 1)\overline{\nabla}(\sigma)$$

which makes sense since

$$\overline{\nabla}(\sigma) \subset \bigoplus_{\substack{i+j=n \\ 0 < i, j < n}} (\Omega C)_i \otimes (\Omega C)_j.$$

Remark 3.0.5. Equivalently, we can define S to be the geometric serie $(Id) \smile^{-1} = \sum_{n \geq 0} (\eta\epsilon - Id) \smile^{-n}$ with $(\eta\epsilon - Id) \smile^{-0} = \eta\epsilon$ and where \smile denote the convolution product from 1.0.1. Note that this sum is finite when it is evaluated on an element. This presentation is combinatorial and gives for $[\sigma] \in (s^{-1}C)_n$,

$$(3.2) \quad S([\sigma]) = \eta\epsilon([\sigma]) + (\eta\epsilon - Id)([\sigma]) + \mu\overline{\nabla}([\sigma]) - \mu^{(2)}\overline{\nabla}^{(2)}([\sigma]) + \dots + (-1)^{n-1}\mu^{(n)}\overline{\nabla}^{(n)}([\sigma]).$$

By the [iii](#) of Proposition 1.0.3 the antipode is an algebra antimorphism that is an algebra morphism from $(\Omega C, \mu_\Omega)$ to $\Omega C_{(12)} := (\Omega C, \mu_{\Omega_{(12)}} := \tau\mu_\Omega)$. Moreover, it is also a DG-map, therefore it corresponds to a twisting cochain $F \in \text{Tw}(C, \Omega C_{(12)})$. An antipode is determined by the underlying bialgebra structure and here, the latter is equivalent to a homotopy G-algebra structure $E^{k,1}$ on C . We make explicit the twisting cochain F in terms of $E^{k,1}$.

We write $F^i : \tilde{C} \rightarrow \tilde{C}^{\otimes i}$ the i -th component of F . The relation² $\mu_{\Omega C}(1 \otimes S_F)\nabla = \eta\epsilon$ gives

$$(3.3) \quad F^1 = -Id_{\tilde{C}};$$

$$(3.4) \quad F^n = (-1)^n \sum_{\substack{1 \leq s \leq n-1 \\ n_1 + \dots + n_s = n-1 \\ n_i \geq 1}} (1^{\otimes n_1 + \dots + n_{s-1}} \otimes E^{n_s, 1}) \dots (1^{\otimes n_1} \otimes E^{n_2, 1}) E^{n_1, 1}, \quad n \geq 2.$$

For example, the three first terms are $F^1 = -Id_{\tilde{C}}$, $F^2 = E^{1,1}$ and $F^3 = -(1 \otimes E^{1,1})E^{1,1} - E^{2,1}$.

²We can also consider $\mu_{\Omega C}(S_F \otimes 1)\nabla = \eta\epsilon$. It gives equivalent F^n but with a more complicated description because of apparition of permutations. For example $F^3 = -(E^{1,1} \otimes 1)E^{1,1} - (\tau \otimes 1)E^{2,1}$.

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Formulated in this terms, the n -th component of the difference $S^2 - Id$ is zero for $n = 1$ and

$$(3.5) \quad \sum_{s=1}^n \sum_{n_1+\dots+n_s=n} \mu_{\Omega(12)}^{(s)}(F^{n_1} \otimes \dots \otimes F^{n_s})F^s$$

for $n \geq 2$. For example, the two first terms we obtain are $F^2 - \tau F^2 = E^{1,1} - \tau E^{1,1}$ and

$$(1 + \tau_3)F^3 + \tau^{2,1}(F^2 \otimes 1)F^2 + \tau^{1,2}(1 \otimes F^2)F^2 \\ = (\tau^{1,2} - 1 - \tau_3)(1 \otimes E^{1,1})E^{1,1} + \tau^{2,1}(E^{1,1} \otimes 1)E^{1,1} - (1 + \tau_3)E^{2,1},$$

where the permutations are $\tau^{1,2}(a \otimes b \otimes c) = \pm b \otimes c \otimes a$, $\tau^{2,1}(a \otimes b \otimes c) = \pm c \otimes a \otimes b$ and $\tau_3(a \otimes b \otimes c) = \pm c \otimes b \otimes a$; the signs being given by the Koszul sign rule.

Now we are ready to establish

Proposition 3.0.6. *Let $(C, d, \nabla_C, E^{k,1})$ be a homotopy G -coalgebra.*

1. *The cobar construction ΩC is an iDGH if and only if all the obstructions $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$ defined by*

$$(3.6) \quad O_n := \sum_{s=1}^n \sum_{n_1+\dots+n_s=n} \mu_{\Omega(12)}^{(s)}(F^{n_1} \otimes \dots \otimes F^{n_s})F^s$$

for $n \geq 2$ are zero.

2. *Let C be 2-reduced.*

- (a) *If there exists an integer n such that $O_k = 0$ for $k \leq n$, then $\Omega^2(C_{\leq n})$ is a homotopy BV-algebra.*
- (b) *If all the obstructions $O_n : \tilde{C} \rightarrow \tilde{C}^{\otimes n}$ are zero, then the double cobar construction $\Omega^2 C$ is a homotopy BV-algebra given by the BV-operator defined in (2.12).*

Remark 3.0.7. The previous proposition works for a Hirsch coalgebra C instead of a homotopy G -coalgebra : the operations from (3.6) have the same definition in terms of F_n ; only the F_n 's defining the antipode differ. However, the involved technics above are quite similar for Hirsch coalgebras.

Remark 3.0.8. For a general Hirsch coalgebra $(C, E^{i,j})$ the condition of cocommutativity of the coproduct on ΩC is $E^{i,j} = \tau E^{j,i}$ for all (i, j) . Accordingly, with Proposition 1.0.3 we see (already in component three) that condition for the antipode to be involutive is weaker than condition for coproduct to be cocommutative. When the Hirsch coalgebra is an hGc, the cocommutativity of the coproduct means that the hGc is quasi trivial : $E^{1,1} = \tau E^{1,1}$ and $E^{k,1} = 0$ for $k \geq 2$.

4 The cobar construction $\Omega C_*(X)$ and the homotopy G -coalgebra $C_*(X)$

Let us fix the base ring k to be any commutative ring. Here we recall the coproduct introduced by Baues in [Bau98] and describe it via the coaction of the second filtration of the

surjection operad $F_2\chi$ on $C_*(X)$. We explicit the antipode on the cobar construction of a simplicial set.

Coproduct Let us fix X to be a 1-reduced simplicial set, that is $X_0 = X_1 = *$. We recall the augmentation

$$(4.1) \quad \begin{aligned} \epsilon : \Omega C_*(X) &\rightarrow k \\ s^{-1} a_1 \otimes \dots \otimes s^{-1} a_n &\mapsto \begin{cases} 0 & \text{for } n \geq 1 \\ id_k & \text{for } n = 0. \end{cases} \end{aligned}$$

To give an explicit formulation of the coproduct we need to introduce some notations. For $\sigma_i \in X$, $[\sigma_1|\sigma_2|\dots|\sigma_n]$ denote the tensor $s^{-1}\sigma_{i_1} \otimes s^{-1}\sigma_{i_2} \otimes \dots \otimes s^{-1}\sigma_{i_r}$ where indices i_j are such that $\sigma_{i_j} \in X_{n_{i_j}}$ with $n_{i_j} \geq 2$. For a subset $b = \{b_0 < b_1 < \dots < b_r\} \subset \{0, 1, \dots, n\}$ we denote by i_b the unique order preserving injective function $i_b : \{0, 1, \dots, r\} \rightarrow \{0, 1, \dots, n\}$ such that $\text{Im}(i_b) = b$. Let $\sigma \in X_n$ and $0 \leq b_0 < b_1 \leq n$. By $\sigma(b_0, \dots, b_1)$ we mean $i_b^* \sigma \in X_{b_1 - b_0}$ where $b = \{b_0, b_0 + 1, \dots, b_1 - 1, b_1\}$. Let $b \subset \{1, \dots, n - 1\}$. By $\sigma(0, b, n)$ we mean $i_{b'}^* \sigma$ where $b' = \{0\} \cup b \cup \{n\}$.

We define the coproduct ∇_0 on elements $\sigma \in X_n$, $n \geq 2$ by :

$$(4.2) \quad \begin{aligned} \nabla_0 : \Omega C_*(X) &\rightarrow \Omega C_*(X) \otimes \Omega C_*(X) \\ s^{-1} \sigma = [\sigma] &\mapsto \sum_{\substack{b = \{b_1 < b_2 < \dots < b_r\} \\ b \subset \{1, \dots, n-1\}}} (-1)^\zeta [\sigma(0, \dots, b_1) | \sigma(b_1, \dots, b_2) | \dots | \sigma(b_r, \dots, n)] \otimes [\sigma(0, b, n)] \end{aligned}$$

where

$$\zeta = r|\sigma(0, \dots, b_1)| + \sum_{i=2}^r (r+1-i)|\sigma(b_{i-1}, \dots, b_i)| - r(r+1)/2,$$

and extend it as an algebra morphism.

Coproduct as a homotopy G-coalgebra Let X be a simplicial set. We denote by DX is the set of degenerate elements and we set $C_*(X) := \text{span}_k(X) / \text{span}_k(DX)$ to be the normalized chain complex. The chain complex $C_*(X)$ is a homotopy G-coalgebra (hGc for short) that is a coalgebra over the operad $F_2\chi$, see [Kad05, Kad03]. We establish³

Proposition 4.0.9. *Let X be a 1-reduced simplicial set. Then the homotopy G-coalgebra $C_*(X)$ given by the coproduct defined in (4.2) is the homotopy G-coalgebra $C_*(X)$ given by the coaction of $F_2\chi$.*

Proof. Let us first define succinctly the surjection operad χ and its coaction on $C_*(X)$. We refer to [BF04, Section 1.2] for the complete definition.

Let $r \geq 1$, $d \geq 0$ be two integers. Let $u : \{1, \dots, r + d\} \rightarrow \{1, \dots, r\}$ be a surjective map. We represent it by a sequence $(u(1), u(2), \dots, u(r + d))$. Such a map is said degenerate if

³ The results in the sequel work for both the sign convention of [BF04] and ours. For the sake of clarity, we do not keep track of them.

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there exists an $1 \leq i < r + d$ such that $u(i) = u(i + 1)$. We define $\chi(r)_d$ to be the module freely generated by the non-degenerate surjective maps $u : \{1, \dots, r + d\} \rightarrow \{1, \dots, r\}$. It is an S_r -module in the following sense: to a permutation $\sigma \in S_r$ and an element $u \in \chi(r)_d$ the surjective map $\sigma \cdot u$ is given by the sequence $(\sigma(u(1)), \dots, \sigma(u(r + d)))$.

The suboperad $F_2\chi \subset \chi$ is generated by $(1, 2)$ and $(1, 2, 1, 3, 1, \dots, 1, k, 1, k + 1, 1)$ for $k \geq 1$.

Let us describe shortly how the surjection operad χ acts on $C_*(X)$. Let us consider an element $u \in \chi(r)_d$ and a n -simplex σ . Integers r, d and n being fixed, we cut the interval $[0, n] = \{0, 1, \dots, n\} \subset \mathbb{N}$ into $r + d$ subintervals, $0 = n_0 \leq n_1 \leq \dots \leq n_{r+d} = n$. For $u = (u(1), \dots, u(r + d))$ we attribute to the subinterval $[n_{i-1}, n_i]$ the label $u(i)$.

$$\begin{array}{ccccccccccc} 0 = n_0 & & n_1 & & n_2 & & n_{i-1} & & n_i & & n_{r+d-1} & & n_{r+d} = n \\ \times & \dots & \times & \dots & \times & \dots & \times & \dots & \times & \dots & \times & \dots & \times \\ & & u(1) & & u(2) & & & & u(i) & & & & u(r+d) \end{array}$$

We denote by $I(j)$ the concatenation of subintervals with value j that is

$$I(j) := [n_{j_1-1}, n_{j_1}], [n_{j_2-1}, n_{j_2}], \dots, [n_{j_l-1}, n_{j_l}]$$

where $j_1 < j_2 < \dots < j_l$ are the indices such that $u(j_1) = u(j_2) = \dots = u(j_l) = j$. Finally, for such a $u = (u(1), \dots, u(r + d)) \in \chi(r)_d$, we define the operation

$$E(u) : \tilde{C}_*(X) \rightarrow \tilde{C}_*(X)^{\otimes r}$$

to be

$$E(u)(\sigma) := \sum \pm \sigma(I(1)) \otimes \dots \otimes \sigma(I(r)),$$

the sum being over all cuttings.

For example, let $(1, 2) \in \chi(2)_0$ then we obtain

$$E((1, 2))(\sigma) = \sum_{0 \leq i \leq n} \pm \sigma(0, \dots, i) \otimes \sigma(i, \dots, n),$$

that is the Alexander-Withney coproduct.

To the elements $u = (1, 2, 1, 3, 1, \dots, 1, k, 1, k + 1, 1)$ with $k \geq 1$ corresponds the operations $E^{1,k}$ defining the hGc-structure on $C_*(X)$. For an n -simplex σ we have

$$\begin{aligned} E^{1,k}(\sigma) = & \sum_{0 \leq j_1 < j_2 < \dots < j_{2k} \leq n} \\ & \pm \sigma(0, \dots, j_1, j_2, \dots, j_3, j_4, \dots, j_{2k-1}, j_{2k}, \dots, n) \otimes \sigma(j_1, \dots, j_2) \otimes \sigma(j_3, \dots, j_4) \otimes \dots \otimes \sigma(j_{2k-1}, \dots, j_{2k}). \end{aligned}$$

Now we show that the coproduct ∇_0 from (4.2) defines the same hGc-structure on $C_*(X)$ when X is 1-reduced. We remark that since X is 1-reduced, the elements $\sigma(j_l, \dots, j_{l+1})$ where $j_{l+1} - j_l = 1$ belong to $X_1 = * = s_0(*)$ and are degenerate. We denote by $E_0^{k,1}$ the operations defined by the coproduct ∇_0 . To show the correspondance it is sufficient to show that each $(k, 1)$ -component are equal i.e $E_0^{k,1} = \tau^{1,k} E^{1,k}$. Let σ be an n -simplex, this coproduct is a sum over all subsets $b = \{b_1 < b_2 < \dots < b_r\} \subset \{1, \dots, n - 1\}$. Let $b_0 := 0, \beta_l \in b \cup \emptyset$ defined as follow. For $l = 1$ we set

$$\beta_1 := \min_{1 \leq i \leq r} \{b_i \mid b_i - b_{i-1} \geq 2\},$$

and η_1 the index such that $b_{\eta_1} = \beta_1$; for $l \geq 2$, we set

$$\beta_l := \min_{\eta_{l-1}+1 \leq i \leq r} \{b_i \mid b_i - b_{i-1} \geq 2\}$$

where η_l is the index such that $b_{\eta_l} = \beta_l$. Additionally, we set $\alpha_l := b_{\eta_{l-1}}$. We obtain that

$$E_0^{k,1}(\sigma) = \sum \pm \sigma(\alpha_1, \dots, \beta_1) \otimes \sigma(\alpha_2, \dots, \beta_2) \otimes \dots \otimes \sigma(\alpha_k, \dots, \beta_k) \otimes \sigma(0, b, n)$$

To establish the complete correspondance it remains to compare $\sigma(0, b, n) = \sigma(0, b_1, b_2, \dots, b_r, n)$ with $\sigma(0, \dots, \alpha_1, \beta_1, \dots, \alpha_2, \dots, \alpha_k, \beta_k, \dots, n)$. But they are equal by minimality of β_l . ■

5 Obstruction to the involutivity of the antipode of $\Omega C_*(\Sigma X)$

Let ΣX be a simplicial suspension of a simplicial set X . We show that the family of obstructions $O_n : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes n}$ defined in (3.6) can be reduced to $O_2 : \tilde{C}_*(\Sigma X) \rightarrow \tilde{C}_*(\Sigma X)^{\otimes 2}$; the latter being governed by the (lack of) cocommutativity of the operation $E^{1,1}$.

Definition 5.0.10. A **simplicial suspension** ΣX of a simplicial set X with $X_0 = *$ is the simplicial set given by $(\Sigma X)_0 = a_0$ and $(\Sigma X)_n = \{(i, x) \in \mathbb{N}^{\geq 1} \times X_{n-i}\} / \left((i, s_0^n(*)) = s_{n+i}(a_0) = a_{n+i} \right)$. The face and degeneracy operators are generated by :

- $d_0(1, x) = a_n$ for all $x \in X_n$;
- $d_1(1, x) = a_0$ for all $x \in X_0$;
- $d_{i+1}(1, x) = (1, d_i(x))$ for all $x \in X_n, n > 0$;
- $s_0(i, x) = (i+1, x)$;
- $s_{i+1}(1, x) = (1, s_i(x))$.

Proposition 5.0.11. [HPS07] *The differential of $(\Omega C_*(\Sigma X), d_0 + d_1)$ is reduced to its linear part d_0 .*

Proof. The only non degenerate elements in ΣX are $(1, x)$ and we have that $d_0(1, x) = a_n$ is degenerate. Therefore the Alexander-Whitney coproduct on $C_*(\Sigma X)$ is primitive (we recall that C_* are the normalized chains). Then, the reduced coalgebra $\tilde{C}_*(\Sigma X)$ is a trivial coalgebra; so the quadratic part d_1 of the differential $d_\Omega = d_0 + d_1$ on the cobar construction is trivial. ■

A natural coproduct to define is the shuffle coproduct (primitive on cogenerators) which gives a cocommutative DG-Hopf structure to $\Omega C_*(\Sigma X)$; applying Proposition 2.2.3 to X 1-reduced, we obtain on $\Omega^2 C_*(\Sigma X)$ an hBVa-structure. However, this coproduct does not correspond to the one defined in (4.2). Indeed, the hGa-structure on $C_*(\Sigma X)$ is not completely trivial. Because of cocommutativity of the Alexander-Whitney coproduct, the operation $E^{1,1}$ must be a chain map but not necessary the zero map. We obtain,

Proposition 5.0.12. *The hGc-structure $E^{k,1}$ on $C_*(\Sigma X)$ defined in Proposition 4.0.9 is*

$$E^{1,1}(\sigma) = \sum_{1 < l < n} \pm \sigma(0, \dots, l) \otimes \sigma(0, l, l+1, \dots, n);$$

$$E^{k,1} = 0 \text{ for } k \geq 2.$$

Proof. For a non degenerate $\sigma \in (\Sigma X)_n$, we have $d_0\sigma = \sigma(1, \dots, n)$ is degenerate. Consequently, the operation defined in Proposition 4.0.9

$$E^{1,1}(\sigma) = \sum_{k < l} \pm \sigma(k, \dots, l) \otimes \sigma(0, \dots, k, l, l+1, \dots, n)$$

is reduced to

$$E^{1,1}(\sigma) = \sum_{1 < l < n} \pm \sigma(0, \dots, l) \otimes \sigma(0, l, l+1, \dots, n).$$

The higher operations $E^{k,1}$ for $k \geq 2$ given by

$$E^{1,k}(\sigma) = \sum_{0 \leq j_1 < j_2 < \dots < j_{2k} \leq n} \pm \sigma(0, \dots, j_1, j_2, \dots, j_3, j_4, \dots, j_{2k-1}, j_{2k}, \dots, n) \otimes \sigma(j_1, \dots, j_2) \otimes \sigma(j_3, \dots, j_4) \otimes \dots \otimes \sigma(j_{2k-1}, \dots, j_{2k})$$

are zero since the terms $\sigma(j_l, \dots, j_{l+1})$ for $l \leq 3$ are degenerate. \blacksquare

The triviality of higher operations $E^{k,1}$ for $k \geq 2$ does not imply the vanishing of higher obstructions O_n since the $E^{1,1}$ operation appears in all the O_n 's. However, we can reduce the family of obstructions O_n to only O_2 in this case.

Proposition 5.0.13. *Let $(C, d, \nabla_C, E^{k,1})$ be an hGc with $E^{k,1} = 0$ for $k \geq 2$. If $O_2 = E^{1,1} - \tau E^{1,1}$ is zero, then so is O_n for $n \geq 2$.*

Proof. By making explicit the coassociativity of coproduct $\nabla : \Omega C \rightarrow \Omega C$ we obtain in particular the equation

$$(1 \otimes E^{1,1})E^{1,1} - (E^{1,1} \otimes 1)E^{1,1} = (\tau \otimes 1)E^{2,1} - (1 \otimes 1)E^{2,1}.$$

Therefore the triviality of the higher operation $E^{2,1}$ implies that $E^{1,1}$ is coassociative. Together with the vanishing of O_2 we obtain a coassociative and cocommutative operation $E^{1,1}$. We use this fact to vanish the O_n : by (3.4), the terms F^i , $i \leq n$ involved in O_n are

$$F^n = (-1)^n (1^{\otimes n-2} \otimes E^{1,1}) \dots (1 \otimes E^{1,1}) E^{1,1}.$$

Now because of the coassociativity of $E^{1,1}$ we can write each term $\mu_{\Omega(12)}^{(s)} (F^{n_1} \otimes \dots \otimes F^{n_s}) F^s$ of the sum

$$O_n = \sum_{s=1}^n \sum_{n_1 + \dots + n_s = n} \mu_{\Omega(12)}^{(s)} (F^{n_1} \otimes \dots \otimes F^{n_s}) F^s$$

as $\pm \sigma \circ F^n$ where σ is a permutation (depending of the term we consider). Now using the cocommutativity and again the coassociativity of $E^{1,1}$ we can remove all transpositions to obtain $\pm F^n$. A direct counting shows that positive and negative terms are equal in number: the sign of $(F^{n_1} \otimes \dots \otimes F^{n_s}) F^s$ is $(-1)^{n_1 + \dots + n_s + s} = (-1)^{n+s}$ and the number of partitions of n into s integers ≥ 1 is $\binom{n-1}{s-1}$. \blacksquare

Therefore we can consider O_2 as the only obstruction to the involutivity of the antipode on the cobar construction.

6 hBVa-structure on $\Omega^2 C_*(\Sigma^2 X)$

We prove here the hGc-structure on $C_*(\Sigma^2 X)$ is trivial and then that $\Omega C_*(\Sigma^2 X)$ is an involutive DG-Hopf algebra with the shuffle coproduct.

Proposition 6.0.14. *Let $\Sigma^2 X$ be a double suspension. Then :*

- the hGc-structure on $C_*(\Sigma^2 X)$, defined in Proposition 4.0.9, is trivial;
- the double cobar construction $\Omega^2 C_*(\Sigma^2 X)$ is an hBVa with the BV-operator from (2.12).

Proof. By the Proposition 5.0.12 the operations $E^{k,1}$ for $k \geq 2$ are trivial. Therefore to prove the first statement it remains to prove that $E^{1,1}$ is trivial. This follows from the fact that we have $d_1(1, (1, x)) = (1, d_0(1, x)) = (1, a_n) = s_{n+1}(1, a_0)$ is degenerate. The last operation

$$E^{1,1}(\sigma) = \sum_{1 < l < n} \pm \sigma(0, \dots, l) \otimes \sigma(0, l, l+1, \dots, n)$$

becomes trivial since $l > 1$.

For the second statement since the hGc-structure is trivial, all the obstructions O_n are zero and so, by the Proposition 3.0.6 we obtain the announced hBVa-structure on $\Omega^2 C_*(\Sigma^2 X)$. ■

We make explicit the hBVa-structure on $\Omega^2 C_*(\Sigma^2 X)$ by first observing that the coproduct ∇_0 from (4.2) is a shuffle coproduct; we write it as

$$\nabla_0([a_1 | \dots | a_n]) = \sum_{I_0 \sqcup I_1} \pm [a_{I_0}] \otimes [a_{I_1}]$$

where the sum is taken over all partitions $I_0 \sqcup I_1$ of $I = \{1, \dots, n\}$ with $I_0 = \{i_1 < i_2 < \dots < i_k\}$ and $I_1 = \{j_1 < j_2 < \dots < j_{n-k}\}$. We denote a_{I_0} to be $a_{i_1} \otimes a_{i_2} \otimes \dots \otimes a_{i_k}$. Also we denote $a_{I_0^{-1}}$ to be $a_{i_k} \otimes \dots \otimes a_{i_2} \otimes a_{i_1}$ and similarly for a_{I_1} .

We make explicit now the BV-operator on a 2 and 3-tensor. For the sake of simplicity we do not keep track of signs; also to avoid confusion we write $\underline{a} = [a_1 | a_2 | \dots | a_k]$ an element in $\Omega C_*(\Sigma^2 X)$ and $[\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n]$ an element in $\Omega^2 C_*(\Sigma^2 X)$.

Thus $[\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n]$ is $[[a_{1,1} | a_{1,2} | \dots | a_{1,k_1}], [a_{2,1} | a_{2,2} | \dots | a_{2,k_2}], \dots, [a_{n,1} | a_{n,2} | \dots | a_{n,k_n}]]$.

With these conventions, the BV-operator B is given by

$$B([\underline{a}, \underline{b}]) = B([a_1 | \dots | a_m], [b_1 | \dots | b_n]) = [[a_m | \dots | a_1 | b_1 | \dots | b_n]] + [[b_n | \dots | b_1 | a_1 | \dots | a_m]].$$

$$B([a_1 | \dots | a_m], [b_1 | \dots | b_n], [c_1 | \dots | c_r]) = [[a_{I_1^{-1}} | b_1 | \dots | b_n], [a_{I_0^{-1}} | c_1 | \dots | c_r]] + [[b_{I_1^{-1}} | c_1 | \dots | c_r], [b_{I_0^{-1}} | a_1 | \dots | a_m]] \\ + [[c_{I_1^{-1}} | a_1 | \dots | a_m], [c_{I_0^{-1}} | b_1 | \dots | b_n]].$$

Remark 6.0.15. In topological point of view, let us consider X to be a connected (countable) CW-complex with one vertex. The James/Milgram's models $J_i(X)$ are H-spaces homotopically equivalent to $\Omega^i \Sigma^i X$, see [Mil66, theorem 5.2]. Moreover, for $i \geq 1$ the cellular chain complex $C_*(J_i(\Sigma X))$ is a cocommutative, primitively generated DG-Hopf algebra, and $C_*(J_{i+1}(X))$ is isomorphic to $\Omega C_*(J_i(\Sigma X))$, [Mil66, theorem 6.1 and 6.2]. Then using Proposition 2.2.3 we get an hBVa-structure on it which is similar to the one obtained in the simplicial context. Also, we have a DGA-quasi-isomorphism

$$\Omega C_*(J_1(\Sigma X)) \longrightarrow C_*(J_2(X)) \xrightarrow{C_*(j_2)} C_*(\Omega^2 \Sigma^2 X).$$

7 hBVa-structure on $\Omega^2 C_*(X)$ over $R \supset \mathbb{Q}$

In [Bau98], Baues gives an explicit construction of the cobar construction of a 1-reduced simplicial set X over the integer coefficient ring as a cocommutative up to homotopy DG-bialgebra. By using methods of [Ani89], he shows that over a ring R containing \mathbb{Q} as a subring, this DG-bialgebra can be deformed into a strictly cocommutative DG-bialgebra [Bau98, Theorem 4.7]. The latter is isomorphic as DG-Hopf algebra to the universal enveloping algebra of the Lie algebra $L(s^{-1} \tilde{C}_* X)$ generated by the desuspension of the reduced coalgebra of normalized chain complex, combine [Bau98, Theorem 4.8] and [Kas95, Proposition V.2.4]. Hence, over such a ring R , the cobar construction $\Omega C_*(X)$ is (can be deformed into) an involutive DG-Hopf algebra. By applying the Proposition 2.2.3 to $\mathcal{H} = \Omega C_*(X)$ we obtain

Proposition 7.0.16. *Let X be a 2-reduced simplicial set. Then the double cobar construction $(\Omega^2 C_*(X), d_{\Omega^2})$ over $R \supset \mathbb{Q}$ coefficients is a homotopy BV-algebra with BV-operator defined in (2.12).*

In fact, we can obtain this corollary in a slightly different way. Over the integer ring, the cobar construction of a 1-reduced simplicial set, $\Omega C_*(X)$ is a DG-Hopf algebra (not involutive). Using the structure of the rational ring \mathbb{Q} , we deform step by step this DG-Hopf algebra into a cocommutative (hence involutive) DG-Hopf algebra over a ring $R \supset \mathbb{Q}$.

We present first how to deform a DG-Hopf algebra with some good properties into a cocommutative one and we apply this deformation to the cobar construction.

Let \mathbf{DGA}_0 be the category of connected DG-algebras (associative). For a free module V , we denote by $L(V)$ the DG-Lie algebra which is the free graded Lie algebra on the free module V .

Definition 7.0.17. Let $f, g : A \rightarrow B$ be two maps between two DG-modules A, B . A **derivation homotopy** between f and g is a map $F : A \rightarrow B$ satisfying

$$(7.1) \quad dF + Fd = f - g$$

$$(7.2) \quad F(ab) = F(a)g(b) + (-1)^{|F||a|} f(a)F(b).$$

Definition 7.0.18. A **homotopy DG-bialgebra** (A, ψ, G_1, G_2) is an object A in \mathbf{DGA}_0 together with a coproduct ψ in \mathbf{DGA}_0 making (A, ψ) into a coalgebra in \mathbf{DGA}_0 cocommutative up to G_1 , coassociative up to G_2 , both being derivation homotopies.

Definition 7.0.19. Let (A, ψ, G_1, G_2) be a homotopy DG-bialgebra such that $A = TV$, V being a DG-module with $V_0 = 0$. We call (A, ψ, G_1, G_2, V) **n -good** if the following conditions (7.3), (7.4), (7.5) hold.

$$(7.3) \quad \psi = \tau\psi \text{ and } (\psi \otimes 1)\psi = (1 \otimes \psi)\psi \text{ on } V_{\leq n}$$

$$(7.4) \quad d(V_{\leq n+1}) \subset L(V_{\leq n}) \subset T(V) = A$$

$$(7.5) \quad V_{\leq n} \subset \ker(\bar{\psi}).$$

In [Ani89], Anick showed when $R \supset \mathbb{Q}$ that a homotopy DG-bialgebra satisfying the n -good condition is also $(n+1)$ -good. That is there exist an extension of this homotopy DG-bialgebra which is $(n+1)$ -good. We consider the case of an n -good coassociative homotopy DG-bialgebra with an antipode S and show we can deform it into a $(n+1)$ -good one.

Lemma 7.0.20. *Let $\mathcal{A} := (A, \psi, G_1, 0, S, V)$ be a homotopy DG-bialgebra with antipode S . We suppose it is n -good. Then there exists a DG-Hopf algebra $(A, \psi^{n+1}, G_1^{n+1}, 0, S^{n+1}, V^{n+1})$ which both extends \mathcal{A} and is $(n+1)$ -good. Moreover, there are derivation homotopies $F^{n+1} : \psi \simeq \psi^{n+1}$ and $\Gamma^{n+1} : S \simeq S^{n+1}$ with $F^{n+1}(a) = 0$ and $\Gamma^{n+1}(a) = 0$ for $|a| < n$, $a \in A$.*

Proof. The part $(n+1)$ -good as bialgebra is already done in [Bau98, Theorem 4.5] where $\psi^{n+1}, G_1^{n+1}, G_2^{n+1}, F^{n+1}$ and V^{n+1} are defined. Then we are just interested by the construction of S^{n+1} and Γ^{n+1} .

Let $S^{n+1} = S - R^{n+1}$ where $R^{n+1} = \mu(1 \otimes S)\psi^{n+1} - \eta\epsilon$. Then $R^{n+1} = 0$ on $V_{\leq n}$. On V_{n+1} we have

$$\begin{aligned} \mu(1 \otimes S^{n+1})\psi^{n+1} &= \mu(1 \otimes S)\psi^{n+1} - \mu(1 \otimes R^{n+1})\psi^{n+1} \\ &= \mu(1 \otimes S)\psi^{n+1} - R^{n+1} \\ &= \mu(1 \otimes S)\psi^{n+1} - \mu(1 \otimes S)\psi^{n+1} + \eta\epsilon \\ &= \eta\epsilon. \end{aligned}$$

Let F^{n+1} be the homotopy between ψ and ψ^{n+1} , we have $dF^{n+1} + F^{n+1}d = \psi - \psi^{n+1}$. Setting $\Gamma^{n+1} = -\mu(1 \otimes S)F^{n+1}$ we obtain the desired homotopy. Indeed,

$$d\Gamma^{n+1} + \Gamma^{n+1}d = -\mu(1 \otimes S)(dF^{n+1} + F^{n+1}d) = \mu(1 \otimes S)(\psi^{n+1} - \psi) = R^{n+1} = S - S^{n+1}.$$

By construction F^{n+1} is such that $F^{n+1} = 0$ on $V_{<n} \cup V_{>n+1}$ then Γ^{n+1} so.

The same argument is true with $S^{n+1} = S - R^{n+1}$ with $R^{n+1} = \mu(S \otimes 1)\psi^{n+1} - \eta\epsilon$ and $\Gamma^{n+1} = -\mu(S \otimes 1)F^{n+1}$. Because of coassociativity, both S and S' are equal. ■

Lemma 7.0.21. *Let $(A = TV, \psi, G_1, S)$ be a DG-Hopf algebra cocommutative up to the homotopy G_1 and with S as antipode. Suppose that (A, ψ, G_1, S, V) is 1-good. Then there are a coproduct ψ^∞ and an antipode S^∞ such that $(A, \psi^\infty, S^\infty)$ is a cocommutative DG-Hopf algebra. Moreover there are derivation homotopies $F : \psi \simeq \psi^\infty$ and $\Gamma : S \simeq S^\infty$.*

Proof. By iterating the previous Lemma 7.0.20 we obtain a n -good deformation as DG-Hopf algebra for each $n \geq 1$. Hence we define $S^\infty = S^n$. We have $\mu(1 \otimes S^\infty)\psi^\infty = \eta\epsilon$ and $S^\infty = S'^\infty$. The homotopy Γ is $\sum_{n \geq 1} \Gamma^n$. ■

Proposition 7.0.22. *Let $\mathbb{Q} \subset R$ and let X be a 1-reduced simplicial set. Then there is both a coproduct ψ^∞ and an antipode S^∞ on the cobar construction $\Omega C_* X$ such that $(\Omega C_*(X), \psi^\infty, S^\infty)$ is a cocommutative DG-Hopf algebra. Moreover there are derivation homotopies $F : \nabla_0 \simeq \psi^\infty$ and $\Gamma : S_0 \simeq S^\infty$, where ∇_0 and S_0 are respectively the coproduct from (4.2) and the resulting antipode.*

Proof. We denote by G_1 the homotopy to the cocommutativity of the coproduct ∇_0 . It is defined in [Bau98, (4)]. We apply the Lemma 7.0.21 to $(\Omega C_*(X), \nabla_0, G_1, 0, S_0, s^{-1} \tilde{C}_* X)$ which is 1-good. ■

Finally we obtain

Proposition 7.0.16. *Let X be a 2-reduced simplicial set. Then the double cobar construction $(\Omega^2 C_*(X), d_{\Omega^2})$ over $R \supset \mathbb{Q}$ coefficients is an hBV-algebra with BV-operator defined in (2.12).*

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