# Multiresolution Analysis of Radiative Transfer through Inhomogeneous Media. Part I: Theoretical Development

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ABSTRACT

This paper derives a multiresolution formulation of the radiative transfer equation for inhomogeneous media. The multiresolution equation is separated into two sets of equations that help in its physical interpretation. The first set represents radiative transfer at some approximation scale, the second at smaller scales. These equations describe explicitly how the local-scale couplings, which occur between the fluctuations of optical properties and radiation fields at different scales, contribute to the radiation field at a prescribed scale and at a given location by introducing additional internal sourcelike functions. These functions are expressed by terms involving connection coefficients of the chosen multiresolution system and also scaling and wavelet coefficients of the inhomogeneous optical properties. This new formulation can provide new insights into the local-scale coupling governing radiative transfer in inhomogeneous media.

## 1. Introduction

Since the early 1990s, there have been many publications devoted to the study of radiative transfer in inhomogeneous clouds. The subject of some of these studies, which is germane to the present work, is the accounting of the effects of cloud inhomogeneity, within the framework of plane-parallel radiative transfer. When inhomogeneous clouds are assumed homogenous over a given scale within a plane-parallel framework, a bias in the radiances and radiative fluxes results, known as the plane-parallel bias (Cahalan 1994; Loeb et al. 1997; Oreopoulos and Barker 1999; Szczap et al. 2000a), that varies significantly with the averaging scale (Davis et al. 1997). It has been shown that inhomogeneous clouds can be treated approximately in a plane-parallel fashion by defining effective optical properties that are functions of first- and second-order statistical moments of the medium at a given scale (Szczap et al. 2000a,b).

The effects exerted by cloud spatial inhomogeneities on the radiative transfer have been studied by analyzing the relationship between the power spectrum of cloud optical properties and their radiative response (Davis et al. 1996, 1997; Marshak et al. 1995). Horizontal photon transport between adjacent cloudy columns has been also studied (Faure et al. 2001; Marshak et al. 1999). These studies showed that cloud inhomogeneities tend to (globally) smooth as well as (locally) roughen the radiation fields.

Significant progress has been seen in the development of radiative transfer codes to calculate the radiative transfer through 3D inhomogeneous clouds [for a review see Gabriel et al. (1993)]. The performance of several codes was compared and discussed in Cahalan et al. (2005). In performing such calculations, such as the Spherical Harmonic Discrete Ordinate Method (SHDOM; Evans 1998), the optical properties within the grid volume are assumed uniform. However, the effects of subgrid-scale fluctuations of optical properties on the radiative transfer are analogous to those of the problem of turbulence closures in turbulent flow simulations. In the latter, the subgrid-scale turbulence is parameterized in some way to take account of its contribution to motion on the grid scale. The notion of scale coupling arises here, defined as the effects of small scales of variability on some larger scale. Semianalytical approaches to dealing with the subgrid variability issue were formulated by Stephens (1988b) and Gabriel and Evans (1996) who also performed calculations on hypothetical clouds. We note that other numerical meth-

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ods, such as Monte Carlo simulations, can certainly be used to study the effects of cloud inhomogeneities, but do not lead to a theoretical framework from which scale coupling can be understood and ultimately parameterized in the hope of developing efficient 3D radiative transfer models. We anticipate that such developments will become increasingly important in climate or large-eddy simulation (LES) models, as the model spatial resolution increases with computer power. Already, efforts are under way to account for 3D effects in global circulation models using closures (Wood et al. 2005).

Our present knowledge on how cloud inhomogeneities affect the radiative transport at a given scale remains mostly empirical: we cannot quantify theoretically this scale coupling. The only theoretical attempts known to the authors that have grappled with the problem of scale dependence in radiative transfer, are made by Diner and Martonchik (1984a,b), Stephens (1988a,b), and Gabriel et al. (1993), all of whom used Fourier analysis. This study extends the spectral formulation of the aforementioned studies using a wavelet-based multiresolution (MR) analysis applied to a 1D horizontal inhomogeneous media. Although both Fourier and MR analysis distinguish the different scales of variability, the advantage of the latter is that it retains information about its localization. Wavelets and MR analysis are seeing increasing usage in many geophysical and engineering applications (Foufoula-Georgiou and Kumar 1994; Van Den Berg 1999), particularly in the atmospheric sciences (Fournier 2000; Harris and Foufoula-Georgiou 2001). They give an economical and interesting representation of data (Davis et al. 1994, 1999) and can yield insights into processes that involve multiple scales (Yano et al. 2001a,b; Fournier 2002, 2003). A full wavelet decomposition of the radiative transfer equation (RTE) would require the use of the spherical MR system (Schröder and Sweldens 1995; Freeden and Windheuser 1996) to expand the radiance and scattering phase functions. However, we prefer an approach in which the Fourier amplitude functions and optical thickness are developed into a scaling function with accompanying wavelets after the radiance and scattering phase functions are first expanded in a Fourier series as was done in Stephens (1986).

This paper is divided as follows: section 2 presents the RTE and its solution using successive orders of scattering and azimuthal Fourier decomposition of the radiance field. The MR formulation of wavelet systems is then developed in section 3, which also introduces notation and defines the connection coefficients. Multiscale analysis of these connection coefficients provides the means of describing and identifying local-scale couplings. The MR radiative transfer equations are derived in section 4. We show that these equations can be separated formally into two sets of equations: one at the scale of the scaling function, the other, at the scale of wavelets, with the contributions of the local-scale coupling acting as sources of additional internal radiation. The additional sources are directly related to connection coefficients through what we term effective coupling operators, or ECOs.

In Ferlay et al. (2006, hereafter Part II), the MR RTEs are solved numerically for inhomogeneous clouds. The computed radiance and flux fields are compared with those calculated using SHDOM and Monte Carlo radiative transfer codes. Finally, we present and discuss a preliminary analysis of the local-scale coupling between cloud inhomogeneities and the radiative transfer.

## 2. The radiative transfer equation

In this section, we follow closely the development of the RTE given by Diner and Martonchik (1984a) or Stephens (1986), except that the RTE is transformed into a set of equations corresponding to successive orders of scattering. While other approaches are possible, successive orders of scattering allows us to explore the importance of higher order scattering in inhomogeneous media, and may provide further insights into scale coupling.

While the mathematical notation presented hereafter does not follow standard AMS formatting and style, it does follow the above-mentioned references and others cited at appropriate steps. The appendices are also presented to aid the reader in avoiding confusion with the notation.

### a. General equation

The RTE describing the transfer of a monochromatic radiation is given by

$$\mathbf{\Omega} \cdot \boldsymbol{\nabla} N(r, \mathbf{\Omega}) = -\sigma(r)[N(r, \mathbf{\Omega}) - J(r, \mathbf{\Omega})], \qquad (1)$$

where $N(r, \mathbf{\Omega})$ and $J(r, \mathbf{\Omega})$ are the radiance and source functions, respectively, along a directional unit vector $\mathbf{\Omega}$ at a point $r$. Equation (1) is the general 3D RTE for a turbid medium. The extinction coefficient $\sigma(r)$ at point $r$ is assumed independent of the incident beam direction. The unit vector $\mathbf{\Omega}$ is expressed as $\mathbf{\Omega} = \eta \cos\varphi \mathbf{i} + \eta \sin\varphi \mathbf{j} + \mu \mathbf{k}$ with $\eta = \sin\theta$, $\mu = \cos\theta$, where $\theta$ and $\varphi$ are the zenith and azimuth angles, respectively (angular conventions are defined in Fig. 1).

The source function $J(r, \mathbf{\Omega})$ is

$$J(r, \mathbf{\Omega}) = \frac{\varpi(r)}{4\pi} \int_{4\pi} P(r, \mathbf{\Omega}, \mathbf{\Omega}')N(r, \mathbf{\Omega}')\, d\omega\, (\mathbf{\Omega}') + [1 - \varpi(r)]B(r), \qquad (2)$$

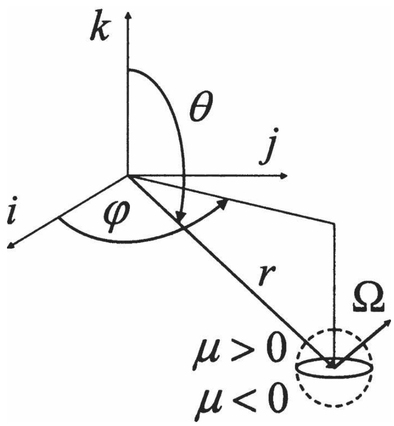


FIG. 1. Angular conventions.

where $P(r, \mathbf{\Omega}, \Omega')$ is the volume scattering phase function representing the scattering of light from an incident beam direction $\mathbf{\Omega}'$ to a new beam direction $\mathbf{\Omega}$ at point $r$ and $d\omega(\mathbf{\Omega}')$ an element of solid angle defined with respect to the directional unit vector $\mathbf{\Omega}'$; $\varpi(r)$ and $B(r)$ are the single scattering albedo and Planck function at point $r$, respectively, both independent of direction. For the sake of simplification, we will exclude the thermal emission term from the source function, though its inclusion would not pose any difficulty.

*b. Successive orders of scattering applied to two-dimensional radiation transfer*

Let us decompose the radiance function into the quasi-parallel solar component $N_{\text{dir}}(r, \mathbf{\Omega}_\odot)$ and diffuse component $N_{\text{dif}}(r, \mathbf{\Omega})$:

$$N(r, \mathbf{\Omega}) = N_{\text{dir}}(r, \mathbf{\Omega}_\odot) + N_{\text{dif}}(r, \mathbf{\Omega}). \tag{3}$$

If we combine Eq. (3) with Eq. (1) and express the radiance as the sum of successively scattered light, $N_{\text{dif}}(r, \mathbf{\Omega}) = \Sigma_{k=1}^{\infty} N_{\text{dif}}^k(r, \mathbf{\Omega})$, we obtain the RTE for the direct solar beam $f(r)$ (Goody 1964)

$$\mathbf{\Omega}_\odot \cdot \mathbf{\nabla}[f(r)] = -\sigma(r)f(r) \quad \text{with}$$

$$f(r) = \lim_{\delta\omega_\odot \to 0}\left[\int_{\delta\omega_\odot} N_{\text{dir}}(r, \mathbf{\Omega})\, d\omega\right], \tag{4}$$

and the RTE for the $k$th order of scattering

$$\mathbf{\Omega} \cdot \mathbf{\nabla} N_{\text{dif}}^k = -\sigma(r)N_{\text{dif}}^k(r, \mathbf{\Omega}) + (1 - \delta_{k,1})\frac{\varpi(r)\sigma(r)}{4\pi} \times \int_{4\pi} P(r, \mathbf{\Omega}, \mathbf{\Omega}')N_{\text{dif}}^{k-1}(r, \mathbf{\Omega}')\, d\omega\,(\mathbf{\Omega}') + \delta_{k,1}\frac{\varpi(r)\sigma(r)}{4\pi} P(r, \mathbf{\Omega}, \mathbf{\Omega}_\odot)f(r), \tag{5}$$

where $\delta_{k,1}$ is the Kronecker symbol $\delta_{k,1} = \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases}$. The last term on the right-hand side (rhs) is the source function for the diffuse radiance function caused by the scattering of collimated solar component. In what follows, we will drop the dif subscript.

In plane-parallel radiative transfer, the radiance and scattering phase function are conventionally expanded into a cosine series of the azimuth angle (see, e.g., Liou 2002). The radiance function is decomposed into $2(N_F + 1)$ Fourier amplitude functions ($N_{c/s}^m$, $0 \le m \le N_F$); where the *c/s* subscript represents cosine and sine terms). This allows azimuthal decoupling of the RTE. For 2D and 3D radiative transfer problem, the RTE is transformed into a system of $2(N_F + 1)$ coupled transfer equations. By introducing the $k$-times scattered radiance function $N^k(r, \mathbf{\Omega})$ into $N(r, \mathbf{\Omega})$, we obtain the Fourier amplitude functions of the $k$-times scattered radiance function $N_{c/s}^{k,m}(r, \mu)$.

If the two-dimensional advection operator $\mathbf{\Omega} \cdot \mathbf{\nabla}$ is separated into vertical and horizontal gradient terms, we obtain the RTE for the collimated solar irradiance

$$\mu_\odot \frac{\partial}{\partial z} f(r) = -\sigma(r)f(r) - \eta_\odot \cos\varphi_\odot \frac{\partial}{\partial x} f(r) \tag{6}$$

and can derive the equations for the Fourier amplitudes of the $k$-times scattered radiance functions $N_c^{k,m}$ and $N_s^{k,m}$, to obtain a unique equation in matrix form:

$$\mu \frac{\partial}{\partial z} \tilde{N}^k(r, \mu) = -\sigma(r)\tilde{N}^k(r, \mu) - \eta\tilde{Q}\frac{\partial \tilde{N}^k(r, \mu)}{\partial x} + (1 - \delta_{k,1})\frac{\varpi(r)\sigma(r)}{4} \times \int_{-1}^{1} \tilde{P}(r, \mu, \mu')\tilde{N}^{k-1}(r, \mu')\, d\mu' + \delta_{k,1}\frac{\varpi(r)\sigma(r)}{4\pi}\tilde{P}^{\,dir} f(r), \tag{7}$$

with $\tilde{N}^k(r, \mu) = \begin{pmatrix} \tilde{N}_c^k \\ \tilde{N}_s^k \end{pmatrix}$, $\tilde{N}_c^k = (N_c^{k,0} \cdots N_c^{k,N_F})^t$ and $\tilde{N}_s^k = (N_s^{k,0} \cdots N_s^{k,N_F})^t$.

The presence of the matrix $\tilde{Q}$ prevents the separation of azimuth order in the above transfer equation. For detailed descriptions of $\tilde{Q}$, $\tilde{P}$, and $\tilde{P}^{\text{dir}}$, see Stephens (1986).

*c. Boundary conditions*

In what follows, we consider that a collimated solar irradiance illuminates the top of the medium and that the boundary condition there is

$$N_{\text{dir}}(x, a, \mathbf{\Omega}_\odot) = A_{\text{direct}}(x). \tag{8}$$

The boundary condition for the upwelling radiance, at the lower boundary $z = b$, is characterized by a bidirectional reflectance function that, by assumption, is of the form $R(r, \Omega.\Omega') = R(r, \mu, \mu', \varphi - \varphi')$:

$$R(x, b, \mu, \mu', \varphi - \varphi') = \sum_{m=0}^{N_F} R^m(x, b, \mu, \mu') \times \cos[m(\varphi - \varphi')]. \quad (9)$$

The $k$-times scattered upwelling irradiance from the underlying surface, which constitutes the boundary condition for the $k$th upwelling radiance equation, is given by

$$N^k(x, b, \mu, \varphi) = \int_{-1}^{0} \int_{0}^{2\pi} R(x, b, \mu, \mu', \varphi - \varphi') N^k(x, b, \mu', \varphi')\mu' \, d\varphi' \, d\mu' \quad (10)$$

for $0 < \mu \leq 1$. Since Eqs. (7) define a system of integro-differential equations with respect to the Fourier amplitude functions, it is also necessary to develop the boundary conditions in azimuth. Replacing $R$, $N^k$, and by their corresponding Fourier expansions, we obtain the components

$$\tilde{N}^k(x, b, \mu) = \pi C_\delta \int_{-1}^{0} \tilde{R}(x, b, \mu, \mu')\tilde{N}^k(x, b, \mu')\mu' \, d\mu' \quad (11)$$

with the $2(N_F + 1)$-component column vector $C_\delta = (2 \;\; 1 \;\; \cdots \;\; 1)'$. The boundary condition for the upwelling radiance of first-order scattering is a special case of Eq. (11) and expressed as

$$N_{c/s}^{1,m}(x, b, \mu^+) = \mu_\odot R^m(x, b, \mu^+, \mu_\odot) f(x, b) \begin{cases} \cos m\varphi_\odot \\ \sin m\varphi_\odot \end{cases}. \quad (12)$$

The two last equations may be written in vector form as in Eq. (7).

## 3. The MR formulation of wavelet systems

The plethora of references on MR analysis, for example Mallat (1998), Resnikoff and Wells (1998), Burrus et al. (1998), and Walter and Shen (2001), simplifies the developments to follow. We will only introduce some necessary definitions and properties of the MR system.

We consider here 1D MR analysis and orthogonal systems. A function $f(x) \in L^2(\mathbb{R})$ can often be better analyzed if it is decomposed linearly as $f(x) = \Sigma_m a_m\phi_m(x)$, where $m$ is an integer index for the finite or infinite sum, $a_m$ real or complex valued expansion coefficients, and $\phi_m(x)$ an orthogonal basis of real or complex valued functions of $x$. The expansion coefficients can be calculated by taking the inner product $a_m = \langle f, \phi_m\rangle = \int_{\mathbb{R}} f(x)\overline{\phi}_m(x)\, dx$, where $\overline{\phi}_m$ is the complex conjugate of $\phi_m$. The Fourier expansion is one example of such a decomposition whose $\phi_m(x)$ are trigonometric functions. The MR expansion is just another way of decomposing $f(x)$:

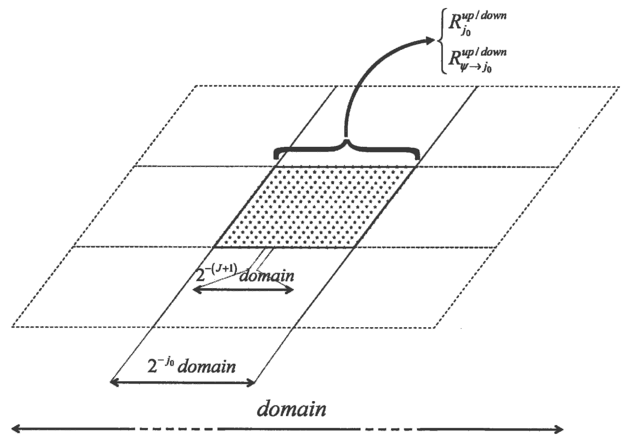


FIG. 2. Schematic representation of the cloud field. For this illustration, a 2D grid is represented even if we use here a 1D multiresolution analysis. The analysis of the scale couplings introduces three scales: that of the entire cloud field (the domain), the approximation pixels or approximation scale $2^{-j_0}$ domain, and its sampling $2^{-(J+1)}$ domain or resolution scale (the sampling points are represented with * in one of the approximation pixel). In the numerical applications in Part II, $j_0 = 3$ and $J + 1 = 7$, and the defined sampling is 50 m, so that these scales are respectively 6.4 km, 800 m, and 50 m. In each of the approximation pixels, we will compute the pixel-scale radiation fields $R_{j_0}^{\text{up/down}}$ and $R_{\psi\to j_0}^{\text{up/down}}$: the former is a consequence of the variations at pixel scale (800 m) of the cloud optical properties, and the latter is an additional field at the pixel scale, a consequence of the effects of the subpixel inhomogeneities of the cloud optical properties (between 50 and 800 m).

$$f(x) = \sum_{k_0} c_{k_0}\varphi_{j_0,k_0}(x) + \sum_{j=j_0}^{\infty}\sum_k d_{j,k}\psi_{j,k}(x), \quad (13)$$

where $\varphi_{j_0,k_0}(x)$ and $\psi_{j,k}(x)$ are the scaling functions and wavelets, respectively, defined below. The first term on the rhs represents the approximation of the function $f(x)$ in each approximation pixel indexed by $k_0$ and of size's order $2^{-j_0}$. The second term on the rhs represents the subpixel variability of the function with increasingly fine structure as $j$ increases and at different locations indexed by $k$. Figure 2, to be discussed in the next section, illustrates the notion of approximation pixel.

### a. MR expansion of a function $f(x) \in L2(\mathbb{R})$

Let us assume a function $\varphi(x)$ called scaling function and define $\varphi_{j,k}$, a two-dimensional set of functions generated from it by dilatation ($j$) and integer translation ($k$):

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k). \tag{14}$$

The scaling function has the property $\int_{-\infty}^{+\infty} \varphi(x)\, dx = 1$ [as $\varphi_{j,k}(x)$ by definition]. Further, $\varphi(x)$ is assumed to be orthogonal with respect to integer translation, that is, $\langle\varphi(x - k_1), \varphi(x - k_2)\rangle = \delta_{k_1 - k_2}$. The set $\{\varphi_{j,k}(x), k \in \mathbb{Z}\}$ spans a subspace $V_j$, called the approximation space. The approximation of a function $f(x)$ at a resolution $2^{-j}$ is given by $f_j(x)$, its orthogonal projection on $V_j$:

$$f_j(x) = \sum_k c_k \varphi_{j,k}(x). \tag{15}$$

Orthogonal projections of $f$ on $V_j$ and $V_{j+1}$ define two approximations at the scales $2^{-j}$ and $2^{-(j+1)}$, respectively. An essential condition on the scaling function is that the approximation spaces $V_j$ and $V_{j+1}$ are nested: $V_j \subset V_{j+1}$. Let $W_j$ be the orthogonal complement of $V_j$ in $V_{j+1}$: $V_{j+1} = V_j \oplus W_j$ with $U \oplus V$ the direct sum of two vector spaces. The subspace $W_j$ is spanned by a set of functions $\{\psi_{j,k}(x); k \in \mathbb{Z}\}$ called wavelets, which have the property $\int_{-\infty}^{+\infty} \psi_{j,k}(x)\, dx = 0$. Similar to $\varphi_{j,k}$ [Eq. (14)], $\psi_{j,k}$ are generated from a mother wavelet $\psi$. The wavelets carry the details necessary to increase the resolution of the function approximation. Because $V_0 \subset V_1$, and $W_0 \subset V_1$, two fundamental scaling relations exist (Mallat 1998).

In general, we can write

$$L^2(\mathbb{R}) = V_\infty = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \oplus W_{j_0+2} \oplus \ldots \tag{16}$$

for an arbitrary scaling function's level $j_0$. Thus, a function $f(x) \in L^2$ can be expanded in the form of Eq. (13). Since the scaling functions and wavelets form an orthonormal basis, we can estimate the coefficients $c_{k_0}$ and $d_{j,k}$ of Eq. (13), called scaling and wavelet coefficients, respectively, from the inner products:

$$c_{k_0} = \langle f, \varphi_{j_0, k_0}\rangle = \int_{\mathbb{R}} f(x)\overline{\varphi}_{j_0,k_0}(x)\, dx$$

$$d_{j,k} = \langle f, \psi_{j,k}\rangle = \int_{\mathbb{R}} f(x)\overline{\psi}_{j,k}(x)\, dx. \tag{17}$$

It follows that the partition of the variance in the wavelet domain is given by $\langle f, f\rangle = \Sigma_{k_0} c_{k_0}^2 + \Sigma_{j=j_0}^{\infty} \Sigma_k d_{j,k}^2$, which is equivalent to the power spectral density in Fourier decomposition as computed using Parseval's formula.

The first sum on the rhs of Eq. (13) represents an approximation of the function $f(x)$ at the scale of scaling function. The second sum represents the wavelet expansion of the difference $f(x) - \Sigma_{k_0} c_{k_0}\varphi_{j_0,k_0}(x)$ between the function and its approximation. The integer translation $k$ retains the positional information of the scaling and wavelet coefficients. This is a definitive advantage of multiresolution expansions: Fourier expansion coefficients do not keep such localized information, because they are defined for the entire domain. From the physical point of view, a MR basis with a compact or quasi-compact support that rapidly decreases in physical space may be considered more appropriate to represent physical processes of finite extent.

In practice, functions are sampled at $2^{J+1}$ uniformly spaced data points, that is, approximated by their expansions in $V_{J+1}$. Defining $W_{j_0}^J = W_{j_0} \oplus \ldots \oplus W_J$ as the subspace of all the details that are represented, we can write

$$V_{J+1} = V_{J_0} \oplus W_{j_0}^J.$$

Functions are decomposed first into scaling functions, and second into all the wavelets down to the subspace $W_J$. As illustrated in Fig. 2, choosing the sampling interval, $J$ and $j_0$, introduces three scales into our analysis: the scale of the domain, the approximation scale $2^{-j_0}$ domain, and the resolution scale $2^{-(J+1)}$ domain. What we will compute in each approximation pixel follows from the separation between scaling functions (in $V_{j_0}$) and wavelets (in $W_{j_0}^J$). Note that this separation is not unique, and that we could have kept the distinction between each scale of wavelets.

To simplify the notation and reassemble all the wavelets, we first denote the basis for $V_{j_0}$ by $\varphi_k = \varphi_{j_0,k}$, $k = 0, 1, \ldots, n_\varphi$ where $n_\varphi = 2^{j_0} - 1$. To create a basis for $W_{j_0}^J$ requires an indexing algorithm. Given $(j, k)$, with $j$ in the range from $j_0$ to $J$ and $k$ in the range $0, 1, \ldots, (2^j - 1)$, we define $\alpha = 2^j - 2^{j_0} + k$. Thus, $\alpha$ lies in the range $0, 1, \ldots, n_\psi$, where $n_\psi = 2^{J+1} - 2^{j_0} - 1$. Given $f$ in $V^{J+1}$, let $f^\varphi$ and $f^\psi$ denote the projections of $f$ onto $V_{j_0}$ and $W_{j_0}^J$. The components of $f$ are defined by

$$f(x) = \sum_{k=0}^{n_\varphi} f^{\varphi k}\varphi_k(x) + \sum_{\alpha=0}^{n_\psi} f^{\psi\alpha}\psi_\alpha(x). \tag{18}$$

Using the boldface notation for the column vectors composed from the coordinates of $f$ in the basis for $V^{J+1}$ defined above, the projections onto $V_{j_0}$ and $W_{j_0}^J$ are

$$\boldsymbol{f}^\varphi = (f^{\varphi 0}, \ldots, f^{\varphi n_\varphi})^t \quad \text{and} \quad \boldsymbol{f}^\psi = (f^{\psi 0}, \ldots, f^{\psi n_\psi})^t,$$

$$\text{while} \quad \boldsymbol{f} = \begin{pmatrix} \boldsymbol{f}^\varphi \\ \boldsymbol{f}^\psi \end{pmatrix}.$$

The notations developed above will be used repeatedly throughout the rest of the paper.

### b. Connection coefficients

The question now naturally arises about how the scaling and wavelet coefficients of a function resulting from the multiplication of two functions, or differentiation, can be expressed in terms of the scaling and wavelet coefficients of the original functions. For this purpose we introduce connection coefficients as defined by Beylkin (1992), Perrier and Wickerhauser (1999), or Resnikoff and Wells (1998). They will be used to express the 1D MR formulation of terms $\sigma(x)\tilde{N}^k$ and $(\partial/\partial x)\,\tilde{N}^k$ in Eq. (7). Connection coefficients are derived below and classified into groups.

It is also possible to obtain expressions for a product of three or more functions and define corresponding connection coefficients that are tensorlike of fourth or higher order at the cost of having to deal with quite cumbersome expressions.

#### 1) CONNECTION COEFFICIENTS FOR THE PRODUCT OF TWO FUNCTIONS

When the MR analysis is applied to $h = fg$, we can decompose in two ways $h(x)$:

$$\begin{aligned} h(x) &= \sum_m h^{\varphi m}\varphi_m(x) + \sum_\gamma h^{\psi\gamma}\psi_\gamma(x) \\ &= \left(\sum_k f^{\varphi k}\varphi_k(x) + \sum_\alpha f^{\psi\alpha}\psi_\alpha(x)\right) \\ &\quad\times\left(\sum_l g^{\varphi l}\varphi_l(x) + \sum_\beta g^{\psi\beta}\psi_\beta(x)\right). \end{aligned} \tag{19}$$

Expanding and decomposing the product of two functions using the aforementioned basis, we obtain for the coefficients $h^{\varphi m}$ and $h^{\psi\gamma}$:

$$\begin{aligned} h^{\varphi m} &= \sum_k\sum_l f^{\varphi k}g^{\varphi l}\langle\varphi_k\varphi_l,\,\varphi_m\rangle \\ &\quad+ \sum_k\sum_\beta f^{\varphi k}g^{\psi\beta}\langle\varphi_k\psi_\beta,\,\varphi_m\rangle \\ &\quad+ \sum_\alpha\sum_l f^{\psi\alpha}g^{\varphi l}\langle\psi_\alpha\varphi_l,\,\varphi_m\rangle \\ &\quad+ \sum_\alpha\sum_\beta f^{\psi\alpha}g^{\psi\beta}\langle\psi_\alpha\psi_\beta,\,\varphi_m\rangle \\ \text{and}\quad h^{\psi\gamma} &= \sum_k\sum_l f^{\varphi k}g^{\varphi l}\langle\varphi_k\varphi_l,\,\psi_\gamma\rangle \\ &\quad+ \sum_k\sum_\beta f^{\varphi k}g^{\psi\beta}\langle\varphi_k\psi_\beta,\,\psi_\gamma\rangle \\ &\quad+ \sum_\alpha\sum_l f^{\psi\alpha}g^{\varphi l}\langle\psi_\alpha\varphi_l,\,\psi_\gamma\rangle \\ &\quad+ \sum_\alpha\sum_\beta f^{\psi\alpha}g^{\psi\beta}\langle\psi_\alpha\psi_\beta,\,\psi_\gamma\rangle. \end{aligned} \tag{20}$$

Scalar products of the form $\langle\chi_{\kappa_1}\chi_{\kappa_2},\,\chi_{\kappa_3}\rangle$ (with $\chi$ representing either $\varphi$ or $\psi$, and $\kappa_i$ the index $k$ or $\alpha$ as defined in the above section) are called connection coefficients of the product (Perrier and Wickerhauser 1999), defined as

$$\langle\chi_{\kappa_1}\chi_{\kappa_2},\,\chi_{\kappa_3}\rangle = \int_{\mathbb{R}} \chi_{\kappa_1}(x)\chi_{\kappa_2}(x)\bar{\chi}_{\kappa_3}(x)\,dx\,. \tag{21}$$

By using the scaling relations and permuting the subscripts $(\kappa_1,\,\kappa_2,\,\kappa_3)$ (Resnikoff and Wells 1998), it can be shown that any scalar product $\langle\chi_{\kappa_1}\chi_{\kappa_2},\,\chi_{\kappa_3}\rangle$ can be computed from the fundamental set $\{\langle\varphi_{0,k_1}\varphi_{0,k_2},\ \varphi_{0,0}\rangle;\ k_1,\ k_2 \in \mathbb{Z}\}$.

We can write $h^{\varphi m}$ and $h^{\psi\gamma}$ in vector–matrix form: $\chi$ representing either $\varphi$ or $\psi$,

$$\begin{aligned} \langle h,\,\chi_{\kappa_3}\rangle &= \begin{pmatrix}\boldsymbol{f}^\varphi\\ \boldsymbol{f}^\psi\end{pmatrix}^t \begin{pmatrix}\mathbf{A}_{\kappa_3} & \mathbf{B}_{\kappa_3}\\ \mathbf{C}_{\kappa_3} & \mathbf{D}_{\kappa_3}\end{pmatrix} \begin{pmatrix}\boldsymbol{g}^\varphi\\ \boldsymbol{g}^\psi\end{pmatrix} \\ &= (\boldsymbol{f}^\varphi)^t\mathbf{A}_{\kappa_3}\boldsymbol{g}^\varphi + (\boldsymbol{f}^\varphi)^t\mathbf{B}_{\kappa_3}\boldsymbol{g}^\psi + (\boldsymbol{f}^\psi)^t\mathbf{C}_{\kappa_3}\boldsymbol{g}^\varphi \\ &\quad+ (\boldsymbol{f}^\psi)^t\mathbf{D}_{\kappa_3}\boldsymbol{g}^\psi, \end{aligned} \tag{22}$$

where matrices $\mathbf{A}_{\kappa_3}$, $\mathbf{B}_{\kappa_3}$, $\mathbf{C}_{\kappa_3}$, and $\mathbf{D}_{\kappa_3}$ are defined by

$$\begin{aligned} \mathbf{A}_{\kappa_3} &= (\langle\chi_{k_1}\chi_{k_2},\,\chi_{\kappa_3}\rangle,\ k_1 = 0,\ldots,n_\varphi,\ k_2 = 0,\ldots,n_\varphi), \\ \mathbf{B}_{\kappa_3} &= (\langle\chi_{k_1}\chi_{\alpha_2},\,\chi_{\kappa_3}\rangle,\ k_1 = 0,\ldots,n_\varphi,\ \alpha_2 = 0,\ldots,n_\psi), \\ \mathbf{C}_{\kappa_3} &= \mathbf{B}^t_{\kappa_3}, \\ \mathbf{D}_{\kappa_3} &= (\langle\chi_{\alpha_1}\chi_{\alpha_2},\,\chi_{\kappa_3}\rangle,\ \alpha_1 = 0,\ldots,n_\psi,\ \alpha_2 = 0,\ldots,n_\psi). \end{aligned} \tag{23}$$

The distinction made between the four groups of connection coefficients $\mathbf{A}_{\kappa_3}$, $\mathbf{B}_{\kappa_3}$, $\mathbf{C}_{\kappa_3}$, and $\mathbf{D}_{\kappa_3}$ [Eq. (23)], yields four combinations [Eq. (22)] that contribute to each coefficient $\langle h,\,\chi_{\kappa_3}\rangle$. It is this distinction in the contributions that defines the scale coupling in the framework of MR analysis. Although coupling between different scales of fluctuations can also be found using the classical Fourier expansion as consequence of a convolution product (Stephens 1988a), the Fourier expansion has no counterpart to the property of localization of scale coupling. We will study in detail the characteristics of the connection coefficient matrix for Meyer's MR system in section 2a of Part II, where we analyze how the connection coefficients vary as a function of scales and horizontal distance.

For the case $\chi_{\kappa_3} = \varphi_{k_0}$ with $k_0 \in \{0,\ldots,n_\varphi\}$, we can define from Eq. (22) two contributions to the scaling coefficient $\langle h,\ \varphi_{k_0}\rangle = h^{\varphi k_0}$: the first arising from the coupling between scaling functions (represented by the first term on the rhs and involving the connection coefficients $\mathbf{A}_{k_0}$), the second from couplings between scaling and wavelet functions or between wavelet functions (the three last terms on the rhs, involving $\mathbf{B}_{k_0}$, $\mathbf{C}_{k_0}$, and $\mathbf{D}_{k_0}$). In the context of radiative transfer, the function $g$ corresponds to the radiation field, and $f$ to the extinction field. The physical interpretation of the combination in Eq. (22) is that the small scales of variability of the extinction field (its wavelet coefficients) may affect at larger scale the radiation field (its scaling coefficients).

Equation (22) gives the expression for one scaling or wavelet coefficient of the product $fg$. To write the MR equations in matrix formulation, we need to manipulate the column vector $\begin{pmatrix} \boldsymbol{fg}^{\varphi} \\ \boldsymbol{fg}^{\psi} \end{pmatrix}$ and to replace the function $g$ by a $n$-component column vector $G = (G_i)$, $i = 1, \ldots, n$ whose components correspond here to each Fourier amplitude function. After some developments (details are given in appendix A), we can derive a matricial formulation for the connection coefficients of the product $f\,G$:

$$\begin{aligned} \boldsymbol{fG}^{\varphi} &= \left(\widetilde{f^{\varphi}\mathbf{AI}_n}^{\varphi}\right)\boldsymbol{G}^{\varphi} + \left(\widetilde{f^{\psi}\mathbf{CI}_n}^{\varphi}\right)\boldsymbol{G}^{\varphi} + \left(\widetilde{f^{\varphi}\mathbf{BI}_n}^{\varphi}\right)\boldsymbol{G}^{\psi} \\ &\quad + \left(\widetilde{f^{\psi}\mathbf{DI}_n}^{\varphi}\right)\boldsymbol{G}^{\psi} \\ \boldsymbol{fG}^{\psi} &= \left(\widetilde{f^{\varphi}\mathbf{AI}_n}^{\psi}\right)\boldsymbol{G}^{\varphi} + \left(\widetilde{f^{\psi}\mathbf{CI}_n}^{\psi}\right)\boldsymbol{G}^{\varphi} + \left(\widetilde{f^{\varphi}\mathbf{BI}_n}^{\psi}\right)\boldsymbol{G}^{\psi} \\ &\quad + \left(\widetilde{f^{\psi}\mathbf{DI}_n}^{\psi}\right)\boldsymbol{G}^{\psi}. \end{aligned} \tag{24}$$

### 2) Connection coefficients for first-order differentiation

Before delving into the derivation of the connection coefficients for first-order differentiation, we need to introduce the notion of translation invariant MR systems, which is important to the discussion of differentiation. Translation invariance is defined by Walter and Shen (2001) as: if $f(x) \in V_j$ then $f(x - \alpha) \in V_j^1$. Translation invariance is satisfied by Fourier expansion, and also holds for differentiation and for convolution $f * g$ as far as $g$ is periodic (with period $2\pi$) and in $L(0, 2\pi)$.[1] Shannon's MR system satisfies such a translation invariance, while Meyer's system satisfies only a weak translation invariance defined as: if $f(x) \in V_j$ then $f(x - \alpha) \in V_{j+1}$. However, no other orthogonal MR system has such invariance properties.

Let us consider a MR expansion of the function $\partial g$, where $\partial$ represents the differential operator with respect to $x$:

$$\begin{aligned} \partial g(x) &= \sum_k \partial g^{\varphi k}\varphi_k(x) + \sum_\alpha \partial g^{\psi\alpha}\psi_\alpha(x) \\ &= \frac{\partial}{\partial x}\left(\sum_l g^{\varphi l}\varphi_l(x) + \sum_\beta g^{\psi\beta}\psi_\beta(x)\right). \end{aligned} \tag{25}$$

The notation here is a bit confusing, but $\partial g^{\varphi k}$ denotes the scaling coefficients of $[\partial g(x)/\partial x]$ and not the horizontal derivative of a constant scaling coefficient $g^{\varphi k}$. Upon taking the inner product of $(\partial/\partial x)\,\varphi_l(x)$ and $(\partial/\partial x)\,\psi_\beta(x)$ with the basis functions $\varphi_k$ or $\psi_\alpha$, we obtain the formulation of the coefficients $\partial g^{\varphi k}$ and $\partial g^{\psi\alpha}$ from the scaling and wavelet coefficients of $g$:

$$\begin{aligned} \partial g^{\varphi k} &= \langle \partial g, \varphi_k\rangle = \sum_l g^{\varphi l}S_l^k + \sum_\beta g^{\psi\beta}S_\beta^k \\ \partial g^{\psi\alpha} &= \langle \partial g, \psi_\alpha\rangle = \sum_l g^{\varphi l}S_l^\alpha + \sum_\beta g^{\psi\beta}S_\beta^\alpha. \end{aligned} \tag{26}$$

Coefficients $S_l^k$, $S_\beta^k$, $S_l^\alpha$, and $S_\beta^\alpha$ are uniquely determined expansion coefficients, which are called connection coefficients for first-order differentiation, defined as:

$$\begin{aligned} S_l^k &= \langle \partial\varphi_l, \varphi_k\rangle, S_\beta^k = \langle \partial\psi_\beta, \varphi_k\rangle \\ S_l^\alpha &= \langle \partial\varphi_l, \psi_\alpha\rangle, S_\beta^\alpha = \langle \partial\psi_\beta, \psi_\alpha\rangle. \end{aligned} \tag{27}$$

Since these connection coefficients are defined as scalar product between $\partial\chi_1$ and $\chi_2$, where $\chi$ stands for either $\varphi$ or $\psi$, we obtain the identity $\langle \partial\chi_1, \chi_2\rangle = -\langle \partial\overline{\chi}_2, \chi_1\rangle$ by applying integration by parts to the definition of the differentiation connection coefficient. In doing so, we have to assume that scaling function and wavelets have compact support in physical domain (or are, at least, rapidly decreasing as $x \to \pm\infty$). We can derive some identity relations for these connection coefficients in a similar way to those for connection coefficient of the product $\langle \chi_{\kappa_1}\chi_{\kappa_2}, \chi_{\kappa_3}\rangle$ (Resnikoff and Wells 1998).

When functions $g(x)$ and $\partial g(x)$ are approximated by their expansions in $V_{J+1}$, we have to neglect the terms with $j_\psi > J$ in the second sums of Eqs. (26). Because of the lack of translation invariance, these neglected terms are theoretically required for the exact expression of differentiation. Equations (26) can be expressed as

$$\langle \partial g, \varphi_k\rangle = (S_\varphi^k \quad S_\psi^k)\begin{pmatrix} \boldsymbol{g}^{\varphi} \\ \boldsymbol{g}^{\psi} \end{pmatrix} \quad \text{and}$$

$$\langle \partial g, \psi_\alpha\rangle = (S_\varphi^\alpha \quad S_\psi^\alpha)\begin{pmatrix} \boldsymbol{g}^{\varphi} \\ \boldsymbol{g}^{\psi} \end{pmatrix}, \tag{28}$$

where $S_\varphi^k$ and $S_\varphi^\alpha$($S_\psi^k$ and $S_\psi^\alpha$) are $2^{j_0}(2^{J+1} - 2^{j_0})$ column vectors with

[1] This definition of the translation invariance differs significantly from the one commonly discussed in the wavelet literature (e.g., Mallat 1998). In the latter, when a pattern in translated, its numerical descriptors, that is, the scaling and wavelet coefficients should only be translated but not modified; such translation invariance is realized by using continuous sampling and continuous wavelet transforms.

$$S^k_\varphi = (S^k_0 \cdots S^k_{n_\varphi}),\ S^k_\psi = (S^k_0 \cdots S^k_{n_\psi})$$

$$S^\alpha_\varphi = (S^\alpha_0 \cdots S^\alpha_{n_\varphi}),\ S^\alpha_\psi = (S^\alpha_0 \cdots S^\alpha_{n_\psi}).$$

As for the connection coefficients of the product, manipulating $\begin{pmatrix} \partial\mathbf{g}^\varphi \\ \partial\mathbf{g}^\psi \end{pmatrix}$ and replacing $g$ by a $n$-component column vector $G = (G_i)$, $i = 1, \ldots, n$ leads, after some developments (details are provided in appendix B), to a matricial formulation for the connection coefficients for the differentiation $\partial G$:

$$\partial\boldsymbol{G}^\varphi = \left(\widetilde{\mathbf{S}^\varphi\mathbf{I}_n}^\varphi\right)\boldsymbol{G}^\varphi + \left(\widetilde{\mathbf{S}^\psi\mathbf{I}_n}^\varphi\right)\boldsymbol{G}^\psi$$

$$\partial\boldsymbol{G}^\psi = \left(\widetilde{\mathbf{S}^\varphi\mathbf{I}_n}^\psi\right)\boldsymbol{G}^\varphi + \left(\widetilde{\mathbf{S}^\psi\mathbf{I}_n}^\psi\right)\boldsymbol{G}^\psi. \quad (29)$$

## 4. The MR radiative transfer equations

We apply now the MR projection to Eqs. (6) and (7) as well as to their corresponding boundary conditions. For the sake of simplification, we assume a constant single scattering albedo and a single scattering phase function for the entire cloud. By keeping the separation between the vectors of scaling and wavelet coefficients, we obtain two equations: the first at the scale of the scaling functions, the second at the scales of the wavelets. We call these MR radiative transfer equations (MR RTEs) the RTE at the pixel scale and the RTE at subpixel scales, respectively, even if this terminology would be more appropriate for Haar's than for other MR systems. These equations involve the matricial formulations discussed in appendices A and B for the connection coefficients for the product $f\,G$ and those of first-order differentiation. Here, the functions $f$ and the vector $G$ are the extinction function $\sigma$ and the Fourier amplitude vector $\widetilde{N^k} = \tilde{N}^k(z, \mu^\pm)$, respectively. The number of Fourier amplitude functions is $n = 2(N_F + 1)$. We obtain the MR formulations

$$\widetilde{\boldsymbol{\sigma N^k}}^\varphi = \left(\widetilde{\boldsymbol{\sigma}^\varphi\mathbf{AI}_n}^\varphi + \widetilde{\boldsymbol{\sigma}^\psi\mathbf{CI}_n}^\varphi\right)\widetilde{\boldsymbol{N^k}}^\varphi + \left(\widetilde{\boldsymbol{\sigma}^\varphi\mathbf{BI}_n}^\varphi + \widetilde{\boldsymbol{\sigma}^\psi\mathbf{DI}_n}^\varphi\right)\widetilde{\boldsymbol{N^k}}^\psi$$

$$\widetilde{\boldsymbol{\sigma N^k}}^\psi = \left(\widetilde{\boldsymbol{\sigma}^\varphi\mathbf{AI}_n}^\psi + \widetilde{\boldsymbol{\sigma}^\psi\mathbf{CI}_n}^\psi\right)\widetilde{\boldsymbol{N^k}}^\varphi + \left(\widetilde{\boldsymbol{\sigma}^\varphi\mathbf{BI}_n}^\psi + \widetilde{\boldsymbol{\sigma}^\psi\mathbf{DI}_n}^\psi\right)\widetilde{\boldsymbol{N^k}}^\psi \quad (30)$$

and

$$\widetilde{\partial\boldsymbol{N^k}}^\varphi = \left(\widetilde{\mathbf{S}^\varphi\mathbf{I}_n}^\varphi\right)^t\widetilde{\boldsymbol{N^k}}^\varphi + \left(\widetilde{\mathbf{S}^\psi\mathbf{I}_n}^\varphi\right)^t\widetilde{\boldsymbol{N^k}}^\psi$$

$$\widetilde{\partial\boldsymbol{N^k}}^\psi = \left(\widetilde{\mathbf{S}^\varphi\mathbf{I}_n}^\psi\right)^t\widetilde{\boldsymbol{N^k}}^\varphi + \left(\widetilde{\mathbf{S}^\psi\mathbf{I}_n}^\psi\right)^t\widetilde{\boldsymbol{N^k}}^\psi. \quad (31)$$

By using these expressions and distinguishing every term, we can write the MR RTEs at the pixel and subpixel scales.

### a. Radiative transfer equation at the pixel scale

The RTE for the vector of scaling coefficients, or at the pixel scale, is given by

$$\mu\frac{\widetilde{\partial\boldsymbol{N^k}}^\varphi}{\partial z} = -\left(\widetilde{\boldsymbol{\sigma}^\varphi\mathbf{AI}_n}^\varphi\right)\widetilde{\boldsymbol{N^k}}^\varphi - \eta\widetilde{\mathbf{I}_\varphi Q}\left(\widetilde{\mathbf{S}^\varphi\mathbf{I}_n}^\varphi\right)^t\widetilde{\boldsymbol{N^k}}^\varphi + (1-\delta_{k,1})\frac{\varpi}{4} \times \int_{-1}^{1} P(\mu,\mu')\left(\widetilde{\boldsymbol{\sigma}^\varphi\mathbf{AI}_n}^\varphi\right)\widetilde{\boldsymbol{N^{(k-1)}}}^\varphi(\mu')\,d\mu' + \delta_{k,1}\frac{\varpi}{4\pi}P(\mu,\mu_\odot)\left(\widetilde{\boldsymbol{\sigma}^\varphi\mathbf{AI}_n}^\varphi\right)\boldsymbol{f}^\varphi + \widetilde{\mathbf{TI}}(\psi,\varphi) + \widetilde{\mathbf{TI}}(\varphi,\psi) + \widetilde{\mathbf{TI}}(\psi,\psi) + \widetilde{\mathbf{SI}}(\psi,\varphi), \quad (32)$$

where $\boldsymbol{\sigma}^\varphi$ and $\widetilde{\boldsymbol{N^k}}^\varphi$ are the vectors of scaling coefficients of extinction and radiance functions, respectively. Because we manipulate $\widetilde{\boldsymbol{N^k}}^\varphi$, $\tilde{Q}$ has to be transformed into $\widetilde{\mathbf{I}_\varphi Q} = \mathbf{I}_\varphi \otimes \tilde{Q}$ where $\otimes$ is the Kronecker product and $\mathbf{I}_\varphi$ is the $2^{j_0} \times 2^{j_0}$ identity matrix.

By solving Eq. (32) for the vector of scaling coefficients $\widetilde{\boldsymbol{N^k}}^\varphi$, the radiation field at the scale of the approximation pixels can be constructed. The first four terms on the rhs depend only on the scaling coefficients $\boldsymbol{\sigma}^\varphi$ and $\widetilde{\boldsymbol{N^{k\pm}}}^\varphi$, and represent the radiative transfer equation for a cloud approximated at the pixel scale. The additional four terms $\widetilde{\mathbf{TI}}(\psi,\varphi)$, $\widetilde{\mathbf{TI}}(\varphi,\psi)$, $\widetilde{\mathbf{TI}}(\psi,\psi)$, and $\widetilde{\mathbf{SI}}(\psi,\varphi)$ represent internal radiation sources at the scale of the approximation pixel due to couplings between the scales of scaling functions and wavelets. The quantity $\widetilde{\mathbf{TI}}(\psi,\varphi)$ represents the effect at the pixel scale of the couplings between subpixel-scale fluctuations of the extinction coefficient and pixel-scale fluctuations of the radiance function:

$$\widetilde{\mathbf{TI}}(\psi,\varphi) = -\left(\widetilde{\boldsymbol{\sigma}^\psi\mathbf{CI}_n}^\varphi\right)\widetilde{\boldsymbol{N^k}}^\varphi + (1-\delta_{k,1})\frac{\varpi}{4} \times \int_{-1}^{1} P(\mu,\mu')\left(\widetilde{\boldsymbol{\sigma}^\psi\mathbf{CI}_n}^\varphi\right)\widetilde{\boldsymbol{N^{(k-1)}}}^\varphi\,d\mu' + \delta_{k,1}\frac{\varpi}{4\pi}P(\mu,\mu_\odot)\left(\widetilde{\boldsymbol{\sigma}^\psi\mathbf{CI}_n}^\varphi\right)\boldsymbol{f}^\varphi. \quad (33)$$

The second and third additional terms, $\widetilde{\mathbf{TI}}(\varphi, \psi)$ and $\widetilde{\mathbf{TI}}(\psi, \psi)$, represent the effect at pixel scale of the couplings between pixel-scale (subpixel-scale) fluctuations of the extinction coefficient and subpixel-scale fluctuations of the radiance function. Their formulations follow that of $\widetilde{\mathbf{TI}}(\psi, \varphi)$, $\left(\widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{CI}_n}^{\varphi}\right)$ being replaced by $\left(\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{BI}_n}^{\varphi}\right)$ and $\left(\widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{DI}_n}^{\varphi}\right)$, $\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\varphi}$, respectively, and $\mathbf{f}^{\varphi}$ by $\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi}$ and $\boldsymbol{f}^{\psi}$. The fourth additional term $\widetilde{\mathbf{SI}}(\psi, \varphi)$ results from the horizontal derivatives of the wavelet functions: $\widetilde{\mathbf{SI}}(\psi, \varphi) = -\eta \widetilde{\mathbf{I}_{\varphi}Q}\left(\widetilde{\mathbf{S}^{\psi}\mathbf{I}_n}^{\varphi}\right)^t \widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi}$.

It is also necessary to express the boundary conditions [Eqs. (8), (11), and (12)] with respect to the MR expansion. With reference to the formulation for the illumination at the top of the medium at the pixel scale, we obtain

$$\widetilde{\boldsymbol{N}_{a,\mathrm{dir}}}^{\varphi} = \boldsymbol{A}^{\varphi}_{\mathrm{direct}},$$

where $\widetilde{\boldsymbol{N}_{a,\mathrm{dir}}}^{\varphi}$ represents the scaling coefficients of $N_{\mathrm{dir}}$ $(x, a, \Omega_{\odot})$, $\boldsymbol{A}_{\mathrm{direct}}{}^{\varphi}$ the scaling coefficients of $\boldsymbol{A}_{\mathrm{direct}}(x)$. In the case of a constant illumination, all the coefficients in $\boldsymbol{A}_{\mathrm{direct}}{}^{\varphi}$ are equal. The underlying surface at the lower boundary $z = b$ is characterized by a bidirectional reflectance function which may be horizontally inhomogeneous. The multiresolution formulation of the lower boundary condition at the pixel scale follows the above developments. It is given in appendix C.

### b. *Radiative transfer equation at subpixel scales*

The RTE for the vector of wavelet coefficients, or at subpixel scales is

$$\begin{aligned}\mu \frac{\partial \widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi}}{\partial z} &= -\left(\widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{DI}_n}^{\psi}\right)\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi} - \eta\widetilde{\mathbf{I}_{\psi}Q}\left(\widetilde{\mathbf{S}^{\psi}\mathbf{I}_m}^{\psi}\right)^t\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi} \\ &+ (1-\delta_{k,1})\frac{\varpi}{4} \\ &\times \int_{-1}^{1} P(\mu, \mu')\left(\widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{DI}_n}^{\psi}\right)\widetilde{\boldsymbol{N}^{(\boldsymbol{k}-1)}}^{\psi}(\mu')\, d\mu' \\ &+ \delta_{k,1}\frac{\varpi}{4\pi}P(\mu, \mu_{\odot})\left(\widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{DI}_n}^{\psi}\right)\boldsymbol{f}^{\psi} \\ &+ \widetilde{\mathbf{TI}}(\varphi, \varphi) + \widetilde{\mathbf{TI}}(\varphi, \psi) + \widetilde{\mathbf{TI}}(\psi, \varphi) + \widetilde{\mathbf{SI}}(\varphi, \psi),\end{aligned} \tag{34}$$

where $\boldsymbol{\sigma}^{\psi}$ and $\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi}$ are the vectors of wavelet coefficients of the extinction and radiance functions, respectively. Because we manipulate $\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi}$, $\tilde{Q}$ has to be transformed into $\widetilde{\mathbf{I}_{\psi}Q} = \mathbf{I}_{\psi} \otimes \tilde{Q}$ where $\otimes$ is the Kronecker product and $\mathbf{I}_{\psi}$ is the $(2^{J+1} - 2^{j_0}) \times (2^{J+1} - 2^{j_0})$ identity matrix.

By solving Eq. (34) for the vector of wavelet coefficients $\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi}$, the radiation field at the wavelet scales from $j_{0_{\psi}}$ to $J$, that is, at the subpixel scales, can be constructed. The first four terms on the rhs involve only wavelets, that is, subpixel variability, whereas the last four terms represent internal additional radiation sources due to couplings between subpixel and pixel-scale fluctuations. The first term, $\widetilde{\mathbf{TI}}(\varphi, \varphi)$, appears because couplings between scaling functions can induce, in general, fluctuations at smaller scales:

$$\begin{aligned}\widetilde{\mathbf{TI}}(\varphi, \varphi) &= -\left(\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{AI}_m}^{\psi}\right)\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\varphi} + (1-\delta_{k,1})\frac{\varpi}{4} \\ &\times \int_{-1}^{1} P(\mu, \mu')\left(\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{AI}_m}^{\psi}\right)\widetilde{\boldsymbol{N}^{(\boldsymbol{k}-1)}}^{\varphi}\, d\mu' \\ &+ \delta_{k,1}\frac{\varpi}{4\pi}P(\mu, \mu_{\odot})\left(\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{AI}_m}^{\psi}\right)\boldsymbol{f}^{\varphi}.\end{aligned} \tag{35}$$

The second and third additional terms, $\widetilde{\mathbf{TI}}(\varphi, \psi)$ and $\widetilde{\mathbf{TI}}(\psi, \varphi)$, represent the effect at subpixel scale of the couplings between pixel-scale (subpixel-scale) fluctuations of the extinction coefficient and subpixel-scale (pixel-scale) fluctuations of the radiance function. Their formulations follow that of $\widetilde{\mathbf{TI}}(\varphi, \varphi)$, $\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{AI}_m}^{\psi}$ being replaced by $\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{BI}_m}^{\psi}$ and $\widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{CI}_m}^{\psi}$, respectively, $\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\varphi}$ by $\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi}$ for the case of $\widetilde{\mathbf{TI}}(\varphi, \psi)$. The fourth additional term, $\widetilde{\mathbf{SI}}(\varphi, \psi)$, results from the horizontal derivatives of the scaling functions: $\widetilde{\mathbf{SI}}(\varphi, \psi) = -\eta\widetilde{\mathbf{I}_{\varphi}Q}\left(\widetilde{\mathbf{S}^{\varphi}\mathbf{I}_m}^{\psi}\right)^t\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\varphi}$.

The formulation for the illumination at the top of the medium at subpixel scales is

$$\widetilde{\boldsymbol{N}_{a,\mathrm{dir}}}^{\psi} = \boldsymbol{A}^{\psi}_{\mathrm{direct}},$$

where $\widetilde{\boldsymbol{N}_{a,\mathrm{dir}}}^{\psi}$ represents the wavelet coefficients of $N_{\mathrm{dir}}$ $(x, a, \Omega_{\odot})$, $\boldsymbol{A}_{\mathrm{direct}}{}^{\psi}$ the wavelet coefficients of $A_{\mathrm{direct}}(x)$. In the case of a constant illumination, all the coefficients in $\boldsymbol{A}_{\mathrm{direct}}{}^{\psi}$ are zero. It is also necessary to formulate the lower boundary condition at subpixel scales. This is given in appendix C.

### c. *Effective scale coupling and additional radiation sources*

In section 3b, we stated that the local-scale couplings are represented by the connection coefficients for the product and for first-order differentiation, that is, by the matrices **A** through **D** and **S**. From Eqs. (30) and (31), it is clear, for example, that the product of a matrix times the vector $\widetilde{\boldsymbol{N}^{\boldsymbol{k}}}^{\psi}$ may provide scaling coefficients as a result, that constitutes a scale coupling. However, if in Eq. (31), this coupling is caused directly by the connection coefficients $\mathbf{S}^{\psi}$ in $\left(\widetilde{\mathbf{S}^{\psi}\mathbf{I}_n}^{\varphi}\right)^t$, in Eq. (30), it is caused indirectly by $\mathbf{B}$ and $\mathbf{D}$ through their product with the values of the scaling and wavelet coefficients associated with the cloud optical properties, $\boldsymbol{\sigma}^{\varphi}$ and $\boldsymbol{\sigma}^{\psi}$, in $\left(\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{BI}_n}^{\varphi} + \widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{DI}_n}^{\varphi}\right)$. This means that, among all the possible scale couplings described by the connection coefficients that depend only on the choice of the MR system, the medium activates and makes effective only some of these couplings. For example, if no wavelets are needed to describe the variability of the extinction coefficient, the activation of the connection coefficients contained in $\mathbf{D}$ isn't realized through $\widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{DI}_n}^{\varphi}$. We hereafter call the products

$$\begin{pmatrix} \left(\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{AI}_n}^{\varphi} + \widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{CI}_n}^{\varphi}\right) & \left(\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{BI}_n}^{\varphi} + \widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{DI}_n}^{\varphi}\right) \\ \left(\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{AI}_n}^{\psi} + \widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{CI}_n}^{\psi}\right) & \left(\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{BI}_n}^{\psi} + \widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{DI}_n}^{\psi}\right) \end{pmatrix} \tag{36}$$

effective coupling operators.

The distinction made in Eq. (36) between different types of ECOs make it possible to identify the different scales that contribute to the radiation field. For example, in Eq. (32), the four first terms on the rhs come from pixel-scale couplings, while the additional terms coming from scale couplings represent additional radiation sources. Thus, as illustrated in Fig. 2, our approach allows us to quantify in each approximation pixel the additional radiation field denoted $R_{\psi\rightarrow j_0}^{\text{up/down}}$, from the pixel-scale radiation field denoted $R_{j_0}^{\text{up/down}}$. The former is due to the subpixel variability of the cloud optical properties, while the latter is due to their variability at the scale of the approximation pixels. The pixel-scale radiation field $R_{j_0}^{\text{up/down}}$ is the result of effective pixel-scale couplings and involves $\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{A}}^{\varphi}$; the additional pixel-scale radiation field $R_{\psi\rightarrow j_0}^{\text{up/down}}$ is the result of effective scale couplings and involves $\widetilde{\boldsymbol{\sigma}^{\varphi}\mathbf{B}}^{\varphi}$, $\widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{C}}^{\varphi}$, and $\widetilde{\boldsymbol{\sigma}^{\psi}\mathbf{D}}^{\varphi}$. In Part II, we will discuss in detail the importance of this additional radiation field for two types of clouds.

## 5. Summary and conclusions

Accounting for the effects of cloud inhomogeneities over different spatial scales poses a major challenge to radiation modeling in climate studies as well as cloud and atmospheric remote sensing. The plane-parallel and independent pixel approximation (IPA) assumptions, largely used by the atmospheric science community, can address neither the radiative effects of realistic clouds on the climate processes nor the effects exerted by cloud inhomogeneities on remotely sensed cloud parameters. To improve our understanding of these problems, a theoretical framework that leads to a deeper understanding of scale coupling in radiative transfer within inhomogeneous media would be a useful development.

The aim of the present study is to develop such a theoretical framework by extending earlier studies (Stephens 1988a; Gabriel et al. 1993) and to provide a clear definition of scale couplings in radiative transfer by applying the wavelet-based multiresolution (MR) analysis. The MR analysis has enabled us to write a new set of radiative transfer equation (RTE) composed of two equations: the first at the pixel scale [Eq. (32)], the second at subpixel scales [Eq. (34)]. This leads us to introduce some concepts such as approximation pixel, local-scale coupling, effective coupling operators (ECOs), and additional radiation sources due to subpixel inhomogeneity that have no equivalents in the conventional radiative transfer formulations. These concepts can provide new insights on the scale coupling in radiative transfer theory. Figure 2 illustrates and summarizes our multiscale approach and the different scales that are relevant: the entire cloud field or the domain scale, the approximation scale, and the resolution scale at which the cloud is sampled. In each of the approximation pixels, and following the distinction between terms in Eq. (32), we can compute the pixel-scale radiation fields $R_{j_0}^{\text{up/down}}$ and $R_{\psi\rightarrow j_0}^{\text{up/down}}$, the latter being the contribution of the subpixel-scale cloud inhomogeneities to the radiation field at the pixel scale.

In Part II, we will present the results of radiative transfer simulations for some theoretical clouds, based on the Meyer's MR development of the RTE. After showing how the structure of Meyer's ECOs change with the characteristics of cloud inhomogeneity, we will compute separately the radiation fields $R_{j_0}^{\text{up/down}}$ and $R_{\psi\rightarrow j_0}^{\text{up/down}}$, then evaluate and analyze for each case the contributions of subpixel-scale couplings to the radiances and radiative fluxes at the pixel scale.

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## APPENDIX A

### Matrix Formulation for Connection Coefficients for the Product

In the main text, we have expressed one scaling or wavelet coefficient of the product $fg$ as in Eq. (22). To write the MR equations in matrix formulation, we need to replace the lhs of Eq. (22) with the column vector $\begin{pmatrix} fg^\varphi \\ fg^\psi \end{pmatrix}$ containing all the coefficient of the product $fg$. We can express $fg^\varphi$ and $fg^\psi$ as a matrix–vector product:

$$fg^\varphi = \left(\widetilde{f^\varphi\mathbf{A}}^\varphi + \widetilde{f^\psi\mathbf{C}}^\varphi\right)g^\varphi + \left(\widetilde{f^\varphi\mathbf{B}}^\varphi + \widetilde{f^\psi\mathbf{D}}^\varphi\right)g^\psi$$

$$fg^\psi = \left(\widetilde{f^\varphi\mathbf{A}}^\psi + \widetilde{f^\psi\mathbf{C}}^\psi\right)g^\varphi + \left(\widetilde{f^\varphi\mathbf{B}}^\psi + \widetilde{f^\psi\mathbf{D}}^\psi\right)g^\psi; \tag{A1}$$

$\left(\widetilde{f^\varphi\mathbf{A}}^\varphi + \widetilde{f^\psi\mathbf{C}}^\varphi\right)$ is a $2^{j_0} \times 2^{j_0}$ matrix defined by $\left(\widetilde{f^\varphi\mathbf{A}}^\varphi + \widetilde{f^\psi\mathbf{C}}^\varphi\right) = \begin{pmatrix} (f^\varphi)^t\mathbf{A}_0 + (f^\psi)^t\mathbf{C}_0 \\ \vdots \\ (f^\varphi)^t\mathbf{A}_{n_\varphi} + (f^\psi)^t\mathbf{C}_{n_\varphi} \end{pmatrix}$, while $\left(\widetilde{f^\varphi\mathbf{B}}^\varphi + \widetilde{f^\psi\mathbf{D}}^\varphi\right)$ is a $2^{j_0} \times (2^{J+1} - 2^{j_0})$ matrix defined by $\left(\widetilde{f^\varphi\mathbf{B}}^\varphi + \widetilde{f^\psi\mathbf{D}}^\varphi\right) = \begin{pmatrix} (f^\varphi)^t\mathbf{B}_0 + (f^\psi)^t\mathbf{D}_0 \\ \vdots \\ (f^\varphi)^t\mathbf{B}_{n_\varphi} + (f^\psi)^t\mathbf{D}_{n_\varphi} \end{pmatrix}$. Here, $\left(\widetilde{f^\varphi\mathbf{A}}^\psi + \widetilde{f^\psi\mathbf{C}}^\psi\right)$ and $\left(\widetilde{f^\varphi\mathbf{B}}^\psi + \widetilde{f^\psi\mathbf{D}}^\psi\right)$ are $(2^{J+1} - 2^{j_0}) \times 2^{j_0}$ and $(2^{J+1} - 2^{j_0})^2$ matrices, respectively, each row of which corresponds to $\alpha_3 = 2^{j_3} - 2^{j_0} + k_3$ with $\begin{pmatrix} j_3 = j_{0\psi}, \ldots, J \\ k_3 = 0, \ldots, 2^{j_3} - 1 \end{pmatrix}$.

When $g$ is replaced by a $n$-component column vector $\mathbf{G}$, scaling and wavelet coefficients $\langle f\mathbf{G}, \chi_{\kappa_3}\rangle$ can be represented in an expression similar to Eq. (22), except that we have to repeat the same operation for all the components $G_i$ of the vector $\mathbf{G}$. This means that we have to replace $\begin{pmatrix} f^\varphi \\ f^\psi \end{pmatrix}$ and $\begin{pmatrix} \mathbf{A}_{\kappa_3} & \mathbf{B}_{\kappa_3} \\ \mathbf{C}_{\kappa_3} & \mathbf{D}_{\kappa_3} \end{pmatrix}$ by the tensor (or Kronecker) products $\left[\begin{pmatrix} f^\varphi \\ f^\psi \end{pmatrix} \otimes \mathbf{I}_n\right]$ and $\left[\begin{pmatrix} \mathbf{A}_{\kappa_3} & \mathbf{B}_{\kappa_3} \\ \mathbf{C}_{\kappa_3} & \mathbf{D}_{\kappa_3} \end{pmatrix} \otimes \mathbf{I}_n\right]$ where $\mathbf{I}_n$ is the $n \times n$ identify matrix. Let us define a $n \times 2^{j_0}$ column vector $\mathbf{G}^\varphi = (G^{\varphi 0}, \ldots, G^{\varphi n_\varphi})^t$ with $G^{\varphi k} = (G_1^{\varphi k} \ldots G_n^{\varphi k})^t$ and similarly $\mathbf{G}^\psi$ a $n \times (2^{J+1} - 2^{j_0})$ column vector $\mathbf{G}^\psi = (G^{\psi 0}, \ldots, G^{\psi n_\psi})^t$ with $G^{\psi k} = (G_1^{\psi k} \ldots G_n^{\psi k})^t$. Thus, we can write another version of Eq. (A1) as

$$f\mathbf{G}^\varphi = \left(\widetilde{f^\varphi\mathbf{A}\mathbf{I}_n}^\varphi\right)\mathbf{G}^\varphi + \left(\widetilde{f^\psi\mathbf{C}\mathbf{I}_n}^\varphi\right)\mathbf{G}^\varphi + \left(\widetilde{f^\varphi\mathbf{B}\mathbf{I}_n}^\varphi\right)\mathbf{G}^\psi + \left(\widetilde{f^\psi\mathbf{D}\mathbf{I}_n}^\varphi\right)\mathbf{G}^\psi$$

$$f\mathbf{G}^\psi = \left(\widetilde{f^\varphi\mathbf{A}\mathbf{I}_n}^\psi\right)\mathbf{G}^\varphi + \left(\widetilde{f^\psi\mathbf{C}\mathbf{I}_n}^\psi\right)\mathbf{G}^\varphi + \left(\widetilde{f^\varphi\mathbf{B}\mathbf{I}_n}^\psi\right)\mathbf{G}^\psi + \left(\widetilde{f^\psi\mathbf{D}\mathbf{I}_n}^\psi\right)\mathbf{G}^\psi,$$

where the matrix $\left(\widetilde{f^\varphi\mathbf{A}\mathbf{I}_n}^\varphi\right)$ and other similar products are defined by $\left(\widetilde{f^\varphi\mathbf{A}\mathbf{I}_n}^\varphi\right) = \widetilde{f^\varphi\mathbf{A}}^\varphi \otimes \mathbf{I}_n$.

# APPENDIX B

## Matrix Formulation for Connection Coefficients for First-Order Differentiation

In the main text, we have expressed a scaling or wavelet coefficient of $\partial g(x)$ as in Eq. (28). As in appendix A, we need first to express the lhs of this equation as a column vector $\begin{pmatrix} \partial g^\varphi \\ \partial g^\psi \end{pmatrix}$ and second to replace $g$ by a $n$-component column vector $\mathbf{G} = (G_i)$, $i = 1, \ldots, n$. We can write

$$\partial g^\varphi = \widetilde{\mathbf{S}^\varphi}^\varphi g^\varphi + \widetilde{\mathbf{S}^\psi}^\varphi g^\psi$$

$$\partial g^\psi = \widetilde{\mathbf{S}^\varphi}^\psi g^\varphi + \widetilde{\mathbf{S}^\psi}^\psi g^\psi$$

$$\text{or} \quad \begin{pmatrix} \partial g^\varphi \\ \partial g^\psi \end{pmatrix} = \mathbf{S}^{\varphi\psi}\begin{pmatrix} g^\varphi \\ g^\psi \end{pmatrix}, \tag{B1}$$

where $\widetilde{\mathbf{S}^\varphi}^\varphi$ and $\widetilde{\mathbf{S}^\psi}^\varphi$ are $2^{j_0} \times 2^{j_0}$ and $(2^{J+1} - 2^{j_0}) \times 2^{j_0}$ matrices, respectively, defined by

$$\widetilde{\mathbf{S}^\varphi}^\varphi = \begin{pmatrix} S_\varphi^0 \\ \vdots \\ S_\varphi^{n_\varphi} \end{pmatrix}, \widetilde{\mathbf{S}^\psi}^\varphi = \begin{pmatrix} S_\psi^0 \\ \vdots \\ S_\psi^{n_\varphi} \end{pmatrix},$$

where $\widetilde{\mathbf{S}^\varphi}^\psi$ and $\widetilde{\mathbf{S}^\psi}^\psi$ are similarly defined as $2^{j_0} \times (2^{J+1} - 2^{j_0})$ and $(2^{J+1} - 2^{j_0})^2$ matrices, respectively, each column of which corresponds to a particular value of $\alpha = 2^j - 2^{j_0} + k$.

When $g$ is replaced by a $n$-component column vector $\mathbf{G}$ in Eqs. (28) or (B1), the same mathematical operations occur as in appendix A, and manipulating $\mathbf{G}^\varphi$ and $\mathbf{G}^\psi$ as defined above, we can write another version of Eq. (B1):

$$\partial\mathbf{G}^\varphi = \left(\widetilde{\mathbf{S}^\varphi\mathbf{I}_n}^\varphi\right)\mathbf{G}^\varphi + \left(\widetilde{\mathbf{S}^\psi\mathbf{I}_n}^\varphi\right)\mathbf{G}^\psi$$

$$\partial\mathbf{G}^\psi = \left(\widetilde{\mathbf{S}^\varphi\mathbf{I}_n}^\psi\right)\mathbf{G}^\varphi + \left(\widetilde{\mathbf{S}^\psi\mathbf{I}_n}^\psi\right)\mathbf{G}^\psi, \tag{B2}$$

where the matrix $\widetilde{\mathbf{S}^\varphi\mathbf{I}_n}^\varphi$ and the other similar products are defined by $\widetilde{\mathbf{S}^\varphi\mathbf{I}_n}^\varphi = \widetilde{\mathbf{S}^\varphi}^\varphi \otimes \mathbf{I}_n$. Vectors $\partial\mathbf{G}^\varphi$ and $\partial\mathbf{G}^\psi$ have the same structure as $\mathbf{G}^\varphi$ and $\mathbf{G}^\psi$.

# APPENDIX C

## Multiresolution Formulation of the Lower Boundary Conditions

### a. At the pixel scale

The underlying surface at the lower boundary $z = b$ is characterized by a bidirectional reflectance function that may be horizontally inhomogeneous. The MR formulation of the lower boundary condition at the pixel scale follows the developments in section 4a. We obtain

$$\widetilde{N_b^{k+}}^{\varphi} = \pi\widetilde{\mathbf{I}_\varphi C_\delta} \int_{-1}^{0} \left(\widetilde{\boldsymbol{R}_+^{\varphi}\mathbf{AI}_n}^{\varphi}\right)\widetilde{N_b^{k-1}}^{\varphi}\mu'\,d\mu' + \widetilde{\mathbf{TI}}(\psi,\varphi)(\mu) + \widetilde{\mathbf{TI}}(\varphi,\psi)(\mu) + \widetilde{\mathbf{TI}}(\psi,\psi)(\mu), \tag{C1}$$

where $\widetilde{N_b^{k+}}^{\varphi}$ represents the scaling coefficients of $\tilde{N}^k(x, b, \mu)$, $\widetilde{N_b^{k-1}}^{\varphi}$ the scaling coefficients of $\tilde{N}^{k-1}(x, b, \mu')$, $\boldsymbol{R}_+^{\varphi}$ the scaling coefficients of $\tilde{R}(b, \mu, \mu')$, $0 < \mu \le 1$ and $\widetilde{\mathbf{I}_\varphi C_\delta} = \mathbf{I}_\varphi \otimes C_\delta$. The formulation for the upward boundary condition for the first-order scattering is

$$\widetilde{N_{b,c/s}^{1+}}^{\varphi} = \mu_\odot\left(\widetilde{\boldsymbol{R}_{\mu_\odot}^{\varphi}\mathbf{AI}_n}^{\varphi}\right)\boldsymbol{f}_b^{\varphi}\begin{cases}\cos m\varphi_\odot\\ \sin m\varphi_\odot\end{cases} + \widetilde{\mathbf{TI}}(\psi,\varphi)(\mu) + \widetilde{\mathbf{TI}}(\varphi,\psi)(\mu) + \widetilde{\mathbf{TI}}(\psi,\psi)(\mu), \tag{C2}$$

where $\widetilde{N_{b,c/s}^{1+}}^{\varphi}$ represents the scaling coefficients of $\boldsymbol{N}_{c/s}^{1,m}(b, \mu)$, $\boldsymbol{f}_b^{\varphi}$ the scaling coefficients of $f(x, b)$, $\boldsymbol{R}_{\mu_\odot}^{\varphi}$ the scaling coefficients of $\tilde{R}(b, \mu, \mu_\odot)$. The first term on the rhs of the two last equations depend only on scaling coefficients and represent the lower boundary conditions approximated at the scale of scaling functions. Additional terms of the form $\widetilde{\mathbf{TI}}$ represent contributions of scale couplings between the scaling function and wavelets scales of both downward radiances and bidirectional reflectance functions to the upward pixel-scale radiances at $b$ level. These terms (not explicitly given here) can be derived easily in same ways as in Eqs. (32).

### b. *At subpixel scales*

The MR formulation of the lower boundary condition at subpixel scales is

$$\widetilde{N_b^{k+}}^{\psi} = \pi\widetilde{\mathbf{I}_\psi C_\delta} \int_{-1}^{0} \left(\widetilde{\boldsymbol{R}_+^{\psi}\mathbf{AI}_m}^{\psi}\right)\widetilde{N_b^{k-1}}^{\psi}\mu'\,d\mu' + \widetilde{\mathbf{TI}}(\varphi,\varphi)(\mu) + \widetilde{\mathbf{TI}}(\psi,\varphi)(\mu) + \widetilde{\mathbf{TI}}(\varphi,\psi)(\mu), \tag{C3}$$

where $\widetilde{N_b^{k+}}^{\psi}$ represents the wavelet coefficients of $\tilde{N}^k(x, b, \mu)$, $\widetilde{N_b^{k-1}}^{\psi}$ the wavelet coefficients of $\tilde{N}^{k-1}(x, b, \mu')$, $\boldsymbol{R}_+^{\psi}$ the wavelet coefficients of $\tilde{R}(b, \mu^+, \mu')$, $0 < \mu \le 1$, and $\widetilde{\mathbf{I}_\psi C_\delta} = \mathbf{I}_\psi \otimes C_\delta$. For the first-order scattering,

$$\widetilde{N_{b,c/s}^{1+}}^{\psi} = \mu_\odot\left(\widetilde{\boldsymbol{R}_{\mu_\odot}^{\psi}\mathbf{AI}_m}^{\psi}\right)\boldsymbol{f}_b^{\psi}\begin{cases}\cos m\varphi_\odot\\ \sin m\varphi_\odot\end{cases} + \widetilde{\mathbf{TI}}(\varphi,\varphi)(\mu) + \widetilde{\mathbf{TI}}(\psi,\varphi)(\mu) + \widetilde{\mathbf{TI}}(\varphi,\psi)(\mu), \tag{C4}$$

where $\widetilde{N_{b,c/s}^{1+}}^{\psi}$ represents the wavelet coefficients of $\boldsymbol{N}_{c/s}^{1,m}(b, \mu)$, $\boldsymbol{f}_b^{\psi}$ the wavelet coefficients of $f(x, b)$, $\boldsymbol{R}_{\mu_\odot}^{\psi}$ the wavelet coefficients of $\tilde{R}(b, \mu, \mu_\odot)$. The first term on the rhs of the two last equations depend only on wavelet coefficients, so involves only the subpixel variability of $N^{k,m}(b, \mu')$ and $\tilde{R}(b, \mu, \mu')$. Additional terms of the form $\widetilde{\mathbf{TI}}$ represent contributions of scale couplings between the scaling function and wavelets scales of both downward radiances and bidirectional reflectance functions to the upward subpixel-scale radiances at $b$ level. They may be also derived easily in same ways as in Eqs. (34).

## REFERENCES


Beylkin, G., 1992: On the representation of operators in bases of compactly supported wavelets. *SIAM J. Numer. Anal.,* **108,** 1716–1740.

Burrus, C. S., R. A. Gopinath, and H. Guo, 1998: *Introduction to Wavelet and Wavelet Transforms.* Prentice-Hall, 268 pp.

Cahalan, R. F., 1994: Bounded cascade clouds: Albedo and effective thickness. *Nonlinear Processes Geophys.,* **1,** 156–167.

——, and Coauthors, 2005: The International Intercomparison of 3D Radiation Codes (I3RC): Bringing together the most advanced radiative transfer tools for cloudy atmospheres. *Bull. Amer. Meteor. Soc.,* **86,** 1275–1293.

Davis, A., A. Marshak, and W. Wiscombe, 1994: Wavelet-based multifractal analysis of nonstationary and/or intermittent geophysical signals. *Wavelets in Geophysics,* E. Foufoula-Georgiou and P. Kumar, Eds., Academic Press, 249–298.

——, ——, ——, and R. Cahalan, 1996: Scale invariance of liquid water distributions in marine stratocumulus. Part I: Spectral properties and stationarity issues. *J. Atmos. Sci.,* **53,** 1538–1558.

——, ——, R. Cahalan, and W. Wiscombe, 1997: The Landsat scale break in stratocumulus as a three-dimensional radiative transfer effect: Implications for cloud remote sensing. *J. Atmos. Sci.,* **54,** 241–260.

——, ——, and E. Clothiaux, 1999: Anisotropic multi-resolution analysis in 2D, application to long-range correlations in cloud mm-radar fields. *Proc. SPIE,* **3723,** 194–207.

Diner, D. J., and J. V. Martonchik, 1984a: Atmospheric transfer of radiation above an inhomogeneous non-Lambertian reflective ground—I. Theory. *J. Quant. Spectrosc. Radiat. Transfer,* **31** (2), 97–125.

——, and ——, 1984b: Atmospheric transfer of radiation above an inhomogeneous non-Lambertian reflective ground—II. Computational considerations and results. *J. Quant. Sspectrosc. Radiat. Transfer,* **31** (4), 279–304.

Evans, K. F., 1998: The spherical harmonics discrete ordinate method for three-dimensional atmospheric radiative transfer. *J. Atmos. Sci.,* **55,** 429–446.

Faure, T., H. Isaka, and B. Guillemet, 2001: Neural network analysis of the radiative interaction between neighboring cloud pixel of inhomogeneous clouds. *J. Geophys. Res.,* **106,** 14 465–14 484.

Ferlay, F., H. Isaka, P. Gabriel, and A. Benassi, 2006: Multiresolution analysis of radiative transfer through inhomogeneous media. Part II: Validation and new insights. *J. Atmos. Sci.,* **63,** 1213–1230.

Foufoula-Georgiou, E., and P. Kumar, Eds., 1994: *Wavelets in Geophysics.* Academic Press, 373 pp.

Fournier, A., 2000: Introduction to orthonormal wavelet analysis with shift invariance: Application to observed atmospheric blocking spatial structure. *J. Atmos. Sci.,* **57,** 3856–3880.

——, 2002: Atmospheric energetics in the wavelet domain. Part I: Governing equations and interpretation for idealized flows. *J. Atmos. Sci.,* **59,** 1182–1197.

——, 2003: Atmospheric energetics in the wavelet domain. Part II: Time-averaged observed atmospheric blocking. *J. Atmos. Sci.,* **60,** 319–338.

Freeden, W., and U. Windheuser, 1996: Spherical wavelet transform and its discretization. *Adv. Comput. Math.,* **5,** 51–94.

Gabriel, P. M., and K. F. Evans, 1996: Simple radiative transfer methods for calculating domain-averaged solar fluxes in inhomogeneous clouds. *J. Atmos. Sci.,* **53,** 858–877.

——, S.-C. Tsay, and G. L. Stephens, 1993: A Fourier–Ricatti approach to radiative transfer. Part I: Foundations. *J. Atmos. Sci.,* **50,** 3125–3147.

Goody, R. M., 1964: *Atmospheric Radiation.* Oxford University Press, 59 pp.

Harris, D., and E. Foufoula-Georgiou, 2001: Subgrid variability and stochastic downscaling of modeled clouds: Effects on radiative computations for rainfall retrieval. *J. Geophys. Res.,* **106D,** 10 349–10 362.

Liou, K. N., 2002: *An Introduction to Atmospheric Radiation*. 3d ed., International Geophysics Series, Vol. 84, Academic Press, 583 pp.

Loeb, N. G., T. Várnai, and R. Davies, 1997: Effects of clouds inhomogeneities on the solar zenith angle dependence of nadir reflectance. *J. Geophys. Res.,* **102,** 9387–9395.

Mallat, S., 1998: *A Wavelet Tour of Signal Processing*. Academic Press, 637 pp.

Marshak, A., A. Davis, W. Wiscombe, and R. Cahalan, 1995: Radiative smoothing in fractal clouds. *J. Geophys. Res.,* **100,** 26 247–26 261.

——, L. Oreopoulos, A. Davis, W. Wiscombe, and R. Cahalan, 1999: Horizontal radiative fluxes in clouds and accuracy of the independent pixel approximation nonlocal independent pixel approximation at absorbing wavelengths. *Geophys. Res. Lett.,* **26,** 1585–1588.

Oreopoulos, L., and H. W. Barker, 1999: Accounting for subgrid-scale cloud variability in a multilayer 1d solar radiative transfer algorithm. *Quart. J. Roy. Meteor. Soc.,* **125,** 301–330.

Perrier, V., and M. V. Wickerhauser, 1999: Multiplication of short wavelet series using wavelet coefficients. *Advances in Wavelets,* K.-S. Lau, Ed., Springer-Verlag, 77–101.

Resnikoff, H. L., and R. O. Wells, 1998: *Wavelet Analysis: The Scalable Structure of Information.* Springer-Verlag, 435 pp.

Schröder, P., and W. Sweldens, 1995: Spherical wavelets: Efficiently representing functions on the sphere. *Proc. Siggraph 95,* Los Angeles, CA, ACM. [Available online at http://www.multires.caltech.edu/pubs/sphwave.pdf.]

Stephens, G. L., 1986: Radiative transfer in spatially heterogeneous, two-dimensional, anisotropically scattering media. *J. Quant. Spectrosc. Radiat. Transfer,* **36,** 51–67.

——, 1988a: Radiative transfer through arbitrarily shaped optical media. Part I: A general method of solution. *J. Atmos. Sci.,* **45,** 1818–1836.

——, 1988b: Radiative transfer through arbitrarily shaped optical media. Part II: Group theory and simple closures. *J. Atmos. Sci.,* **45,** 1837–1848.

Szczap, F., H. Isaka, M. Saute, B. Guillemet, and I. Ioltukhovski, 2000a: Effective radiative properties of bounded cascade nonabsorbing clouds: Definition of the equivalent homogeneous cloud approximation. *J. Geophys. Res.,* **105,** 20 617–20 634.

——, ——, ——, ——, and ——, 2000b: Effective radiative properties of bounded cascade absorbing clouds: Definition of an effective single-scattering albedo. *J. Geophys. Res.,* **105,** 20 635–20 648.

Van Den Berg, J. C., Ed., 1999: *Wavelets in Physics.* Cambridge University Press, 453 pp.

Walter, G. G., and X. Shen, 2001: *Wavelets and Others Orthogonal Systems*. 2d ed., Studies in Advanced Mathematics, Chapman and Hall/CRC, 370 pp.

Wood, N. B., P. M. Gabriel, and G. L. Stephens, 2005: An assessment of the parameterization of subgrid-scale cloud effects on radiative transfer. Part II: Horizontal inhomogeneity. *J. Atmos. Sci.,* **62**, 2895–2909.

Yano, J. I., M. W. Moncrieff, and X. Wu, 2001a: Wavelet analysis of simulated tropical convective cloud systems. Part II: Decomposition of convective and mesoscales. *J. Atmos. Sci.,* **58,** 868–876.

——, ——, ——, and M. Yamada, 2001b: Wavelet analysis of simulated tropical convective cloud systems. Part I: Basic analysis. *J. Atmos. Sci.,* **58,** 850–867.