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$W_0^{1,1}$ SOLUTIONS IN SOME BORDERLINE CASES OF ELLIPTIC EQUATIONS WITH DEGENERATE COERCIVITY

LUCIO BOCCARDO, GISELLA CROCE

ABSTRACT. We study a degenerate elliptic equation, proving existence results of distributional solutions in some borderline cases.

Bernardo, come vide li occhi miei ...
(Dante, Paradiso XXXI)

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1. INTRODUCTION

The Sobolev space $W_0^{1,2}(\Omega)$ is the natural functional framework (see [10], [12]) to find weak solutions of nonlinear elliptic problems of the following type

$$(1) \quad \begin{cases} -\operatorname{div} \left(\frac{a(x)\nabla u}{(1+|u|)^\theta} \right) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where the function f belongs to the dual space of $W_0^{1,2}(\Omega)$, Ω is a bounded, open subset of \mathbb{R}^N , with $N > 2$, θ is a real number such that

$$(2) \quad 0 \leq \theta \leq 1,$$

and $a : \Omega \rightarrow \mathbb{R}$ is a measurable function satisfying the following conditions:

$$(3) \quad \alpha \leq a(x) \leq \beta,$$

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for almost every $x \in \Omega$, where α and β are positive constants. The main difficulty to use the general results of [10], [12] is the fact that

$$A(v) = -\operatorname{div} \left(\frac{a(x)\nabla v}{(1+|v|)^\theta} \right),$$

is not coercive. Papers [7], [4] and [3] deal with the existence and summability of solutions to problem (1) if $f \in L^m(\Omega)$, for some $m \geq 1$.

Despite the lack of coercivity of the differential operator $A(v)$ appearing in problem (1), in the papers [7], [4] and [1], the authors prove the following existence results of solutions of problem (1), under assumption (3):

- A) a weak solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, if $m > \frac{N}{2}$ and (2) holds true;
- B) a weak solution $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}(1-\theta)}(\Omega)$, where $m^{**} = (m^*)^* = \frac{mN}{N-2m}$, if

$$0 < \theta < 1, \quad \frac{2N}{N+2-\theta(N-2)} \leq m < \frac{N}{2};$$

- C) a distributional solution u in $W_0^{1,q}(\Omega)$, $q = \frac{Nm(1-\theta)}{N-m(1+\theta)} < 2$,
if

$$\frac{1}{N-1} \leq \theta < 1, \quad \frac{N}{N+1-\theta(N-1)} < m < \frac{2N}{N+2-\theta(N-2)}.$$

- D) an entropy solution $u \in M^{m^{**}(1-\theta)}$, with $|\nabla u| \in M^q(\Omega)$, for $1 \leq m \leq \max \left\{ 1, \frac{N}{N+1-\theta(N-1)} \right\}$.

The borderline case $\theta = 1$ was studied in [3], proving the existence of a solution $u \in W_0^{1,2}(\Omega) \cap L^p(\Omega)$ for every $p < \infty$. The case where the source is $\frac{A}{|x|^2}$ was analyzed too.

About the different notions of solutions mentioned above, we recall that the notion of entropy solution was introduced in [2]. Let

$$(4) \quad T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

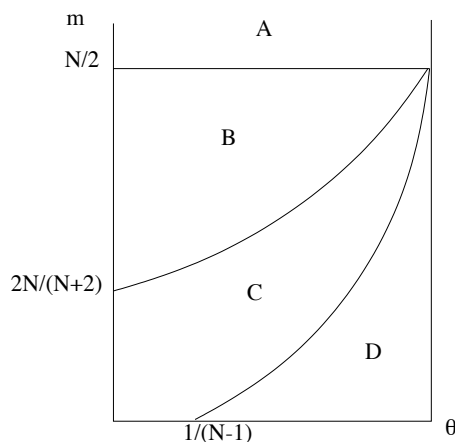
Then u is an entropy solution to problem (1) if $T_k(u) \in W_0^{1,2}(\Omega)$ for every $k > 0$ and

$$\int_{\Omega} \frac{a(x)\nabla u}{(1+|u|)^\theta} \cdot \nabla T_k(u-\varphi) \leq \int_{\Omega} f T_k(u-\varphi), \quad \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

Moreover, we say that u is a distributional solution of (1) if

$$(5) \quad \int_{\Omega} \frac{a(x) \nabla u}{(1 + |u|)^{\theta}} \cdot \nabla \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

The figure below can help to summarize the previous results, where the name of a given region corresponds to the results that we have just cited.



Our results are the following.

THEOREM 1.1. *Let f be a function in $L^m(\Omega)$. Assume (3) and*

$$(6) \quad m = \frac{N}{N + 1 - \theta(N - 1)}, \quad \frac{1}{N - 1} < \theta < 1$$

Then there exists a $W_0^{1,1}(\Omega)$ distributional solution to problem (1).

Observe that this case corresponds to the curve between the regions C and D of the figure. Note that $m > 1$ if and only if $\theta > \frac{1}{N-1}$.

In the following result too, we will prove the existence of a $W_0^{1,1}(\Omega)$ solution.

THEOREM 1.2. *Let f be a function in $L^m(\Omega)$. Assume (3), $f \log(1 + |f|) \in L^1(\Omega)$ and $\theta = \frac{1}{N-1}$. Then there exists a $W_0^{1,1}(\Omega)$ distributional solution of (1).*

We end our introduction just mentioning a uniqueness result of solutions to problem (1) can be found in [13].

Moreover, in [4, 5, 11, 6] it was showed that the presence of a lower order term has a regularizing effect on the existence and regularity of the solutions.

To prove our results, we will work by approximation, using the following sequence of problems:

$$(7) \quad \begin{cases} -\operatorname{div} \left(\frac{a(x) \nabla u_n}{(1 + |u_n|)^\theta} \right) = T_n(f), & \text{in } \Omega; \\ u_n = 0, & \text{on } \partial\Omega. \end{cases}$$

The existence of weak solutions $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ to problem (7) is due to [7].

2. $W_0^{1,1}(\Omega)$ SOLUTIONS

In the sequel C will denote a constant depending on $\alpha, N, \operatorname{meas}(\Omega), \theta$ and the $L^m(\Omega)$ norm of the source f .

We are going to prove Theorem 1.1, that is, the existence of a solution to problem (1) in the case where $m = \frac{N}{N+1-\theta(N-1)}$ and $\frac{1}{N-1} < \theta < 1$. Note that $m > 1$ if and only if $\theta > \frac{1}{N-1}$.

Proof. (of Theorem 1.1) We consider $T_k(u_n)$ as a test function in (7): then

$$(8) \quad \int_{\Omega} |\nabla T_k(u_n)|^2 \leq \frac{(1+k)^\theta \|f\|_{L^1(\Omega)}}{\alpha}$$

by assumption (3) on a .

Choosing $[(1 + |u_n|)^p - 1] \operatorname{sign}(u_n)$, for $p = \theta - \frac{1}{N-1}$, as a test function in (7) we have, by Hölder's inequality on the right hand side and assumption (3) on the left one

$$(9) \quad \alpha p \int_{\Omega} \left(\frac{|\nabla u_n|}{(1 + |u_n|)^{\frac{N}{2(N-1)}}} \right)^2 \leq \int_{\Omega} |f| [(1 + |u_n|)^p - 1] \leq \|f\|_{L^m(\Omega)} \left[\int_{\Omega} [(1 + |u_n|)^p - 1]^{m'} \right]^{\frac{1}{m'}}.$$

The Sobolev embedding used on the left hand side implies

$$\left[\int_{\Omega} \left\{ (1 + |u_n|)^{\frac{N-2}{2(N-1)}} - 1 \right\}^{\frac{2N}{N-2}} \right]^{\frac{2}{2^*}} \leq C \left[\int_{\Omega} [(1 + |u_n|)^p - 1]^{m'} \right]^{\frac{1}{m'}}.$$

We observe that $\frac{N-2}{2(N-1)} \frac{2N}{N-2} = pm'$; moreover $\frac{2}{2^*} > \frac{1}{m'}$, since $m < \frac{N}{2}$. Therefore the above inequality implies that

$$(10) \quad \int_{\Omega} |u_n|^{\frac{N}{N-1}} \leq C.$$

One deduces that

$$(11) \quad \int_{\Omega} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{N}{N-1}}} \leq C$$

from (10) and (9). Let $v_n = \frac{2(N-1)}{N-2}(1 + |u_n|)^{\frac{N-2}{2(N-1)}} \text{sign}(u_n)$. Estimate (11) is equivalent to say that $\{v_n\}$ is a bounded sequence in $W_0^{1,2}(\Omega)$; therefore, up to a subsequence, there exists $v \in W_0^{1,2}(\Omega)$ such that $v_n \rightharpoonup v$ weakly in $W_0^{1,2}(\Omega)$ and a.e. in Ω . If we define the function $u = \left(\left[\frac{2(N-1)}{N-2} |v| \right]^{\frac{2(N-1)}{N-2}} - 1 \right) \text{sign}(v)$, the weak convergence of $\nabla v_n \rightharpoonup \nabla v$ means that

$$(12) \quad \frac{\nabla u_n}{(1 + |u_n|)^{\frac{N}{2(N-1)}}} \rightharpoonup \frac{\nabla u}{(1 + |u|)^{\frac{N}{2(N-1)}}} \quad \text{weakly in } L^2(\Omega).$$

Moreover, the Sobolev embedding for v_n implies that $u_n \rightarrow u$ in $L^s(\Omega)$, for every $1 \leq s < \frac{N}{N-1}$.

Holdér's inequality with exponent 2 applied to

$$\int_{\Omega} |\nabla u_n| = \int_{\Omega} \frac{|\nabla u_n|}{(1 + |u_n|)^{\frac{N}{2N-2}}} (1 + |u_n|)^{\frac{N}{2N-2}}$$

gives

$$(13) \quad \int_{\Omega} |\nabla u_n| \leq C,$$

due to (10) and (11). We are now going to estimate $\int_{\{k \leq |u_n|\}} |\nabla u_n|$. By using $[(1 + |u_n|)^p - (1 + k)^p]^+ \text{sign}(u_n)$ as a test function in (7), we have

$$\int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{N}{N-1}}} \leq C \left[\int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{m}} \left[\int_{\Omega} (1 + |u_n|)^{\frac{N}{N-1}} \right]^{\frac{1}{m'}},$$

which implies, by (10),

$$(14) \quad \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\frac{N}{N-1}}} \leq C \left[\int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{m}}.$$

Hölder's inequality, estimates (10) and (14) on

$$\int_{\{k \leq |u_n|\}} |\nabla u_n| = \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|}{(1 + |u_n|)^{\frac{N}{2N-2}}} (1 + |u_n|)^{\frac{N}{2N-2}} \leq C \left[\int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{2m}},$$

give
(15)

$$\int_{\{k \leq |u_n|\}} |\nabla u_n| = \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|}{(1 + |u_n|)^{\frac{N}{2N-2}}} (1 + |u_n|)^{\frac{N}{2N-2}} \leq C \left[\int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{2m}}.$$

Thus, for every measurable subset E , due to (8) and (15), we have

$$\begin{aligned} \int_E \left| \frac{\partial u_n}{\partial x_i} \right| &\leq \int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| + \int_{\{k \leq |u_n|\}} |\nabla u_n| \\ (16) \quad &\leq \text{meas}(E)^{\frac{1}{2}} \left[\frac{(1+k)^\theta \|f\|_{L^1(\Omega)}}{\alpha} \right]^{\frac{1}{2}} + C \left[\int_{\{k \leq |u_n|\}} |f|^m \right]^{\frac{1}{2m}}. \end{aligned}$$

Now we are going to prove that u_n weakly converges to u in $W_0^{1,1}(\Omega)$ following [5]. Estimates (16) and (10) imply that the sequence $\{\frac{\partial u_n}{\partial x_i}\}$ is equiintegrable. By Dunford-Pettis theorem, and up to subsequences, there exists Y_i in $L^1(\Omega)$ such that $\frac{\partial u_n}{\partial x_i}$ weakly converges to Y_i in $L^1(\Omega)$. Since $\frac{\partial u_n}{\partial x_i}$ is the distributional partial derivative of u_n , we have, for every n in \mathbb{N} ,

$$\int_{\Omega} \frac{\partial u_n}{\partial x_i} \varphi = - \int_{\Omega} u_n \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).$$

We now pass to the limit in the above identities, using that $\partial_i u_n$ weakly converges to Y_i in $L^1(\Omega)$, and that u_n strongly converges to u in $L^1(\Omega)$: we obtain

$$\int_{\Omega} Y_i \varphi = - \int_{\Omega} u \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).$$

This implies that $Y_i = \frac{\partial u}{\partial x_i}$, and this result is true for every i . Since Y_i belongs to $L^1(\Omega)$ for every i , u belongs to $W_0^{1,1}(\Omega)$.

We are now going to pass to the limit in problems (7). For the limit of the left hand side, it is sufficient to observe that $\frac{\nabla u_n}{(1+|u_n|)^{\frac{N}{2(N-1)}}} \rightharpoonup \frac{\nabla u}{(1+|u|)^{\frac{N}{2(N-1)}}}$ weakly in $L^2(\Omega)$ due to (12) and that $|a(x)\nabla\varphi|$ is bounded. \square

We prove Theorem 1.2, that is, the existence of a $W_0^{1,1}(\Omega)$ solution in the case where $\theta = \frac{1}{N-1}$ and $f \log(1 + |f|) \in L^1(\Omega)$.

Proof. (of Theorem 1.2) Let $k \geq 0$ and take $[\log(1 + |u_n|) - \log(1 + k)]^+ \text{sign}(u_n)$, as a test function in problems (7). By assumption (3) on

a one has

$$\alpha \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\theta+1}} \leq \int_{\{k \leq |u_n|\}} |f| \log(1 + |u_n|).$$

We now use the following inequality on the left hand side:

$$(17) \quad a \log(1 + b) \leq \frac{a}{\rho} \log \left(1 + \frac{a}{\rho} \right) + (1 + b)^\rho$$

where a, b are positive real numbers and $0 < \rho < \frac{N-2}{N-1}$. This gives, for any $k \geq 0$

$$(18) \quad \alpha \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{\theta+1}} \leq \int_{\{k \leq |u_n|\}} \frac{|f|}{\rho} \log \left(1 + \frac{|f|}{\rho} \right) + \int_{\{k \leq |u_n|\}} (1 + |u_n|)^\rho.$$

In particular, for $k \geq 1$ we have

$$(19) \quad \frac{\alpha}{2^{\theta+1}} \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|^2}{|u_n|^{\theta+1}} \leq \int_{\{k \leq |u_n|\}} \frac{|f|}{\rho} \log \left(1 + \frac{|f|}{\rho} \right) + 2^\rho \int_{\{k \leq |u_n|\}} |u_n|^\rho.$$

Writing the above inequality for $k = 1$ and using the Sobolev inequality on the left hand side, one has

$$\left[\int_{\Omega} (|u_n|^{\frac{1-\theta}{2}} - 1)_+^{2^*} \right]^{\frac{2}{2^*}} \leq C \int_{\{1 \leq |u_n|\}} \frac{|f|}{\rho} \log \left(1 + \frac{|f|}{\rho} \right) + C \int_{\{1 \leq |u_n|\}} |u_n|^\rho,$$

which implies that

$$\left[\int_{\Omega} |u_n|^{\frac{(1-\theta)2^*}{2}} \right]^{\frac{1}{2^*}} \leq C + C \sqrt{\int_{\{1 \leq |u_n|\}} \frac{|f|}{\rho} \log \left(1 + \frac{|f|}{\rho} \right)} + C \sqrt{\int_{\Omega} |u_n|^\rho}.$$

By using Hölder's inequality with exponent $\frac{(1-\theta)2^*}{2\rho}$ on the last term of the right hand side, we get

$$\left[\int_{\Omega} |u_n|^{\frac{(1-\theta)2^*}{2}} \right]^{\frac{1}{2^*}} \leq C + C \sqrt{\int_{\{1 \leq |u_n|\}} \frac{|f|}{\rho} \log \left(1 + \frac{|f|}{\rho} \right)} + C \left[\int_{\Omega} |u_n|^{\frac{(1-\theta)2^*}{2}} \right]^{\frac{\rho}{(1-\theta)2^*}}.$$

By the choice of ρ , this inequality implies that

$$(20) \quad \int_{\Omega} |u_n|^{\frac{(1-\theta)2^*}{2}} \leq C.$$

Inequalities (20) and (18) written for $k = 0$ imply that $\{v_n\} = \left\{ \frac{2}{1-\theta} (1 + |u_n|)^{\frac{1-\theta}{2}} \text{sign}(u_n) \right\}$ is a bounded sequence in $W_0^{1,2}(\Omega)$, as in the proof

of Theorem 1.1. Therefore, up to a subsequence there exists $v \in W_0^{1,2}(\Omega)$ such that $v_n \rightharpoonup v$ weakly in $W_0^{1,2}(\Omega)$ and a.e. in Ω . Let $u = \left\{ \left[\frac{1-\theta}{2} v \right]^{\frac{2}{1-\theta}} - 1 \right\} \text{sign}(v)$; the weak convergence of $\nabla v_n \rightharpoonup \nabla v$ means that

$$(21) \quad \frac{\nabla u_n}{(1 + |u_n|)^{\frac{\theta+1}{2}}} \rightharpoonup \frac{\nabla u}{(1 + |u|)^{\frac{\theta+1}{2}}} \quad \text{weakly in } L^2(\Omega).$$

Moreover, the Sobolev embedding for v_n implies that $u_n \rightarrow u$ in $L^s(\Omega)$, $s < \frac{N}{N-1}$.

By (8) one has

$$\int_{\Omega} |\nabla u_n| = \int_{\Omega} |\nabla T_1(u_n)| + \int_{\{1 \leq |u_n|\}} |\nabla u_n| \leq C + \int_{\{1 \leq |u_n|\}} \frac{|\nabla u_n|}{|u_n|^{\frac{\theta+1}{2}}} |u_n|^{\frac{\theta+1}{2}}.$$

Hölder's inequality on the right hand side, and estimates (19) written with $k = 1$ and (20) imply that the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$.

Moreover, due to (19)

$$\begin{aligned} \int_{\{k \leq |u_n|\}} |\nabla u_n| &= \int_{\{k \leq |u_n|\}} \frac{|\nabla u_n|}{|u_n|^{\frac{\theta+1}{2}}} |u_n|^{\frac{\theta+1}{2}} \leq \\ &\leq C \sqrt{\int_{\{k \leq |u_n|\}} \frac{|f|}{\rho} \log \left(1 + \frac{|f|}{\rho} \right) + \int_{\{k \leq |u_n|\}} |u_n|^\rho}. \end{aligned}$$

For every measurable subset E , the previous inequality and (8) imply

$$\begin{aligned} \int_E \left| \frac{\partial u_n}{\partial x_i} \right| &\leq \int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| + \int_{\{k \leq |u_n|\}} |\nabla u_n| \\ &\leq C \text{meas}(E)^{\frac{1}{2}} (1+k)^{\frac{\theta}{2}} + C \sqrt{\int_{\{k \leq |u_n|\}} \frac{|f|}{\rho} \log \left(1 + \frac{|f|}{\rho} \right) + \int_{\{k \leq |u_n|\}} |u_n|^\rho}. \end{aligned}$$

Since $\rho < \frac{(1-\theta)2^*}{2}$, by using Hölder's inequality on the last term and estimate (20), one has

$$\begin{aligned} \int_E \left| \frac{\partial u_n}{\partial x_i} \right| &\leq C \text{meas}(E)^{\frac{1}{2}} (1+k)^{\frac{\theta}{2}} + \\ &+ C \sqrt{\int_{\{k \leq |u_n|\}} \frac{|f|}{\rho} \log \left(1 + \frac{|f|}{\rho} \right) + \text{meas}(\{|u_n| \geq k\})^{1 - \frac{2\rho}{(1-\theta)2^*}}}. \end{aligned}$$

One can argue as in the proof of Theorem 1.1 to deduce that $u_n \rightarrow u$ weakly in $W_0^{1,1}(\Omega)$.

To pass to the limit in problems (7), as in the proof of Theorem 1.1, it is sufficient to observe that $\frac{\nabla u_n}{(1+|u_n|)^{\frac{N}{2(N-1)}}} \rightharpoonup \frac{\nabla u}{(1+|u|)^{\frac{N}{2(N-1)}}}$ weakly in $L^2(\Omega)$, due to (21). \square

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