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# Nonlocal theory of sound propagation in porous media: case of circular pores

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## Abstract

Elaborating on a Maxwellian representation of longitudinal wave propagation in a viscothermal fluid, a general nonlocal macroscopic theory of sound propagation in homogeneous porous media saturated with a viscothermal fluid has been recently proposed. The present paper validates this new nonlocal Maxwellian theory by showing that, in the case of the propagation in straight circular tubes, it is in complete agreement with the long known Kirchhoff-Langevin's full solutions.

### *Keywords:*

Spatial dispersion, Circular tubes, Porous media, Kirchhoff-Langevin's theory, Viscothermal fluid, Nonlocal theory

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## 1. Introduction

In a recent paper [1], by using Kirchhoff-Langevin's description of compressional wave propagation in a fluid and following, as a powerful heuristic guide, a deep *acoustic-electromagnetic analogy*, we introduced two new up-scaling procedures allowing to compute two 'acoustic permittivities' from microstructure of porous media. These describe in a nonlocal 'Maxwellian' manner, the phenomenon of macroscopic sound propagation in rigid-framed homogeneous porous materials permeated by a viscothermal fluid. The first is a macroscopic effective density playing the role of macroscopic electric permittivity, the second a macroscopic effective compressibility playing the

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role of macroscopic magnetic permittivity. By ‘macroscopic’ we mean that the theory is not concerned with the values of the acoustic fields at every microscopic spatial position, but only, with their ‘macroscopic’ effective values, obtained by averaging in a way to be precised. In the proposed theory, an energetic ‘Umov-Poynting’ definition of the macroscopic pressure is introduced, and the permittivities are fully nonlocal, *i.e.* they are frequency and also wavenumber dependent.

The physical motivation of the new nonlocal theory is the recognition that spatial dispersion effects are not always well described by the existing theories, based on perturbative approaches. Indeed, at the zero’th order of the asymptotic two-scale homogenization theory [2, 3], and in all existing macroscopic models such as [4] and [3, 5], spatial dispersion effects are entirely absent. In the full asymptotic two-scale homogenization theory [2, 6], or its recent variants [7, 8, 9], some spatial dispersion effects can be described, but only in a limited manner, when these are small corrections.

The limited possible use of these asymptotic perturbative approaches were in recent years highlighted by their inability to cope simultaneously with all geometries and frequencies, and in particular, to describe the whole dynamics of metamaterial structures with Helmholtz resonators [10] on one hand, and the complete Bloch mode spectrum in periodic structures on the other hand. With the new ‘Maxwellian’ nonlocal and nonperturbative theory we have proposed, these limitations disappear and no restriction on periodic geometries or frequencies subsist, even if we have so far formulated the solution only for the case of propagation along a principal axis of an unbounded material, or a symmetry axis for a periodic material. Therefore, it is important to note that the new theory is suitable to predict the exact properties of the so-called metamaterials [10], from microstructure.

As this new nonlocal theory is intended to provide the true physico-mathematical solution of the macroscopic wave propagation problem, it needs to be mathematically checked in unequivocal precise manner. Indeed, a proper check is especially desirable, since a detailed verification of the general ideas of the theory cannot be performed in electromagnetics, where the counterpart of the conjectured upscaling procedures, cannot be formulated yet. In electromagnetics, the nonlocal Maxwell macroscopic theory comparable to the proposed acoustic one, is elusive. We believe that its completion is not possible yet, because the necessary thermodynamics of electromagnetic fields in matter is missing.

In this paper we concentrate on the simplest explicitly solvable example

allowing precise checking of theory. This example concerns sound propagation in cylindrical ducts. For ducts of circular cross-section the exact solution, in the framework of near-equilibrium ordinary fluid mechanics accounting for the effects of viscous losses and thermal conduction in the fluid, is known since G. Kirchhoff [11]. Kirchhoff's investigation had been conducted using the ideal gas theory. Langevin [12, 13], later showed that Kirchhoff's solution applied more generally, to a viscothermal fluid obeying an arbitrary equation of state. It will appear on this example that the predictions of the nonlocal macroscopic Maxwellian theory are, up to the numerical accuracy of the calculations, indistinguishable from those of Kirchhoff-Langevin's direct solution.

The paper is organized as follows. In section 2 we present Kirchhoff-Langevin's treatment of the problem of small-amplitude sound propagation in a tube of axis  $x$  and circular cross section filled with a viscothermal fluid. We recall how can be computed, at given angular frequency  $\omega$ , the complex wavenumbers  $k_{(m,n)}(\omega)$  of the axisymmetric normal modes  $m = 0$ ,  $n = 0, 1, 2, \dots$ , where  $m$  and  $n$  are azimuthal and radial mode indexes: the corresponding wavenumbers  $k_l(\omega) = k_{(0,l-1)}(\omega)$ ,  $l = 1, 2, \dots$ , are the complex roots of the transcendent Kirchhoff-Langevin's dispersion equation (46).

Anticipating on the Maxwellian theory's definition of a macroscopic pressure field  $H$  by means of the following fundamental 'Umov-Poynting' thermodynamic definition

$$\langle p\mathbf{v} \rangle = H \langle \mathbf{v} \rangle \quad (1)$$

where  $p$  is excess thermodynamic pressure,  $\mathbf{v}$  velocity, and  $\langle \rangle$  is the averaging operation over a cross section, we then introduce at a given real angular frequency  $\omega$ , the following complex impedance factors  $Z_l(\omega)$ ,  $l = 1, 2, \dots$

$$H = Z_l \langle \mathbf{v} \rangle \cdot \mathbf{e}_x \quad (2)$$

where  $\mathbf{e}_x$  represents the unit vector along the  $x$ -axis. These frequency-dependent Kirchhoff-Langevin's wavenumbers  $k_l(\omega)$  and impedances  $Z_l(\omega)$  enable us to evaluate two frequency-dependent (Kirchhoff-Langevin's) permittivities, namely the densities  $\rho_l(\omega)$  and bulk moduli  $\chi_l^{-1}(\omega)$  associated with the different radial modes  $n = l - 1$ ,  $l = 1, 2, \dots$

$$\rho_l(\omega) = k_l Z_l / \omega, \quad \chi_l^{-1}(\omega) = \omega Z_l / k_l \quad (3)$$

In section 3 we recall the principles of the proposed macroscopic Maxwellian nonlocal theory. There, two (Maxwell's) permittivities are introduced, which

are fundamentally nonlocal both in time and space. They correspond to two nonlocal (density and bulk modulus) operators, also described in Fourier space by frequency-dependent and wavenumber-dependent complex amplitudes  $\rho(\omega, k)$  and  $\chi^{-1}(\omega, k)$ . These functions  $\rho(\omega, k)$  and  $\chi^{-1}(\omega, k)$  are independently computable thanks to the two conjectured upscaling procedures. It will be shown that the Kirchhoff-Langevin's complex wavenumbers and impedances  $k_l$  and  $Z_l$  still make sense in the framework of the macroscopic Maxwellian theory. The wavenumbers  $k_l$  are the solutions of the 'Maxwell' dispersion equation

$$\rho(\omega, k)\chi(\omega, k)\omega^2 = k^2 \quad (4)$$

and the impedances  $Z_l$  then can be computed by

$$Z_l = \sqrt{\rho(\omega, k_l)\chi^{-1}(\omega, k_l)} \quad (5)$$

They may be referred to as Maxwell's wavenumbers and impedances. The coincidence between Kirchhoff-Langevin's and Maxwell's complex wavenumbers and impedances serves as a test of the exactness of the two upscaling procedures of the new theory. Equivalently, this coincidence will be expressed through the coincidence of Kirchhoff-Langevin's and Maxwell's densities and bulk moduli:

$$\rho_l(\omega) = \rho(\omega, k_l) \quad \chi_l^{-1}(\omega) = \chi^{-1}(\omega, k_l) \quad (6)$$

A successful numerical check of Eqs.(6), performed in section 4, will clearly indicate that the two nonlocal upscaling procedures described in [1] are exact in cylindrical tubes having circular cross section.

Kirchhoff-Langevin's theory has been scarcely used in practice. For the least attenuated plane mode which is often the only important one, it gives results undistinguishable in practice from those of the simple approximate theory developed much later by Zwikker and Kosten [14]. Zwikker and Kosten's theory may be interpreted as the one taking into account only temporal dispersion, whereas our proposed Maxwellian theory takes into account, also, spatial dispersion, hence giving the precise solution. Extended to arbitrary geometries, the physically simplified local treatment, is nothing but that of the zero'th order homogenization theory; it captures the essential physics of the wave propagation phenomenon as long as a separation condition is satisfied and the geometry is sufficiently simple. The geometry must be sufficiently simple not to include cavities in the form of Helmholtz's resonators,

and the pores sizes must be much smaller than the wavelengths. When these conditions are no longer satisfied, it no longer describes the physical solution. The propagation models ordinarily used in acoustic studies of porous media [15, 3, 5], are all developed in this local-theory framework. In electromagnetism, this would correspond to the widely used simplification which consists in assuming that the permittivities have no dependencies on the wavenumber. These simplifications, however, are clashing with the wave nature of the problems which would imply that spatial dispersion should always be present at least to some extent. Thus, even if Zwikker and Kosten's theory works very well for the least-attenuated plane wave mode, it never provides the complete physical solution of the macroscopic (*i.e.* cross-section averaged) wave propagation. It cannot predict the existence of the higher order modes, and also the change of nature of the propagation in very wide tubes, illustrated by a fundamental mode which is no longer plane-wave like, but tends to become surface-wave like.

Zwikker and Kosten's local theory is derived in Appendix A without recourse to the classical two-scale asymptotic homogenization theory, using instead, only, the language of temporal and spatial dispersion. It will then be more clear, how the present complete Maxwellian description of sound propagation in a cylindrical circular duct is nothing but a sort of Zwikker and Kosten's treatment, now suitably generalized to include spatial dispersion.

## 2. Kirchhoff-Langevin's theory of sound propagation in a tube of circular cross-section

Kirchhoff's original investigations on the effects of viscosity and heat conduction on sound propagation in free air, and also, air inside a hollow solid tube, were conducted by treating the air as an ideal gas [11]. Langevin completed much later Kirchhoff's theory by considering air as having second viscosity and a general equation of state [12]. We commence by recalling this complete Kirchhoff-Langevin's theory, which is usually not presented without simplifications in the acoustic literature.

### 2.1. Linearized equations, in the Navier-Stokes-Fourier model

Following [13] we are given a homogeneous viscothermal fluid which obeys an arbitrary caloric equation of state

$$\varepsilon = \varepsilon(s, v) \tag{7}$$

with  $\varepsilon$  the specific internal energy per unit mass,  $s$  the specific entropy, and  $v = 1/\rho$  the specific volume. The thermodynamic pressure  $p$ , absolute temperature  $T$ , and specific heats at constant pressure and constant volume  $c_p$  and  $c_v$  are defined by

$$p \equiv - \left( \frac{\partial \varepsilon}{\partial v} \right)_s, \quad T \equiv \left( \frac{\partial \varepsilon}{\partial s} \right)_v, \quad c_p \equiv T \left( \frac{\partial s}{\partial T} \right)_p, \quad c_v \equiv T \left( \frac{\partial s}{\partial T} \right)_v \quad (8)$$

Elimination of  $s$  in Eqs.(7) and (8) results in a thermal equation of state  $p = p(T, v)$ . The fluid thermal expansion coefficient is defined by

$$\beta \equiv \frac{1}{v} \left( \frac{\partial v}{\partial T} \right)_p \quad (9)$$

Let us define two fixed reference sound speed values  $c_a$  and  $c_i$ , adiabatic and isothermal respectively, by

$$c_a^2 \equiv \left( \frac{\partial p}{\partial \rho} \right)_s, \quad c_i^2 \equiv \left( \frac{\partial p}{\partial \rho} \right)_T \quad (10)$$

An application of general thermodynamic methods [16] show that the quantities introduced in Eqs.(8), (9), and (10), are not independent. They are related by the following thermodynamic identities

$$\gamma - 1 = \frac{T\beta^2 c_a^2}{c_p}, \quad c_a^2 = \gamma c_i^2 \quad (11)$$

where  $\gamma \equiv c_p/c_v$  is the specific heat ratio. With  $\mathbf{v}$ , the Euler's fluid velocity,  $\sigma$  the stress tensor, and  $\mathbf{q}$  the heat flux, the conservation equations of mass, momentum, and energy are expressed by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (12a)$$

$$\frac{\partial (\rho v_i)}{\partial t} + \partial_j (\rho v_i v_j - \sigma_{ij}) = 0 \quad (12b)$$

$$\frac{\partial (\rho \varepsilon)}{\partial t} + \nabla \cdot (\rho \varepsilon \mathbf{v} + \mathbf{q}) = \sigma_{ij} \partial_j v_i \quad (12c)$$

The first law of thermodynamics is written as

$$d\varepsilon = -pd(1/\rho) + Tds \quad (13)$$

which follows from Eqs.(7) and (8). As such, Eq.(12c) may be put in the following equivalent form of a balance law for entropy

$$\frac{\partial(\rho s)}{\partial t} + \nabla \cdot (\rho s \mathbf{v} + \frac{1}{T} \mathbf{q}) = \frac{1}{T} (\sigma_{ij} + p \delta_{ij}) \partial_j v_i + \mathbf{q} \cdot \nabla \left( \frac{1}{T} \right) \quad (14)$$

In the right-hand side, one sees the density of local entropy sources in the fluid, which is required to be positive by the second law. Now substituting in Eqs.(12a), (12b) and (12c), the total fields

$$\rho(t, \mathbf{x}) = \rho_0 + \rho'(t, \mathbf{x}) \quad (15a)$$

$$\sigma_{ij}(t, \mathbf{x}) = -p \delta_{ij} + \sigma'_{ij} = -(P_0 + p'(t, \mathbf{x})) \delta_{ij} + \sigma'_{ij}(t, \mathbf{x}) \quad (15b)$$

$$s(t, \mathbf{x}) = s_0 + s'(t, \mathbf{x}) \quad (15c)$$

$$T(t, \mathbf{x}) = T_0 + \tau(t, \mathbf{x}) \quad (15d)$$

where  $\rho_0$ ,  $P_0$ ,  $s_0$ ,  $T_0$  represent the constant thermodynamic equilibrium values, and  $p'$  the thermodynamic excess pressure  $p' = p(T, 1/\rho) - p(T_0, 1/\rho_0) = p(T, 1/\rho) - P_0$ , the following linearised equations governing the small amplitudes perturbations are immediately obtained

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0 \quad (16)$$

$$\rho_0 \frac{\partial v_i}{\partial t} = -\partial_i p' + \partial_j \sigma'_{ij} \quad (17)$$

$$\rho_0 \frac{\partial s'}{\partial t} = -\frac{1}{T_0} \nabla \cdot \mathbf{q} \quad (18)$$

To close the system of equations, there remain to precise the constitutive laws which give the deviatoric stresses  $\sigma'_{ij}$  and the heat flux  $\mathbf{q}$  in terms of other variables. In the Navier-Stokes-Fourier theory used in this paper, it is assumed that the  $\sigma'_{ij}$  are purely viscous and determined by the first spatial derivatives of the velocity; the  $\mathbf{q}$  is purely thermal and determined by the first spatial derivatives of the excess temperature. The Maxwell's stress terms [17, 18, 19] appearing in nonisothermal fluids and required by the kinetic theory of gases are neglected. In addition, molecular relaxation phenomena will not be considered; if necessary they may be incorporated as it has been done in *e.g.* [20].



Within these simplifying assumptions, the constitutive equations are written in the following Newton-Stokes form [21]

$$\sigma'_{ij} = \eta \left( \partial_j v_i + \partial_i v_j - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right) + \zeta \delta_{ij} \nabla \cdot \mathbf{v} \quad (19)$$

and Fourier form

$$q_i = -\kappa \partial_i T \quad (20)$$

The coefficients of thermal conductivity  $\kappa$ , first and second viscosity  $\eta$  and  $\zeta$ , are constants to be evaluated in the ambient state  $(P_0, T_0)$ . The second law of thermodynamics results in the inequalities

$$\kappa \geq 0, \quad \eta \geq 0, \quad \zeta \geq 0 \quad (21)$$

The precise values of these fluid constitutive parameters are difficult to obtain by molecular theories; they are generally best found experimentally.

Inserting the constitutive equations (19-20) in the 5 balance equations (12a), (12b) and (14), we find for the system of 5 linearized conservation equations of mass, momentum, and energy

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0 \quad (22a)$$

$$\rho_0 \frac{\partial v_i}{\partial t} = -\partial_i p' + \partial_j \left\{ \eta \left( \partial_j v_i + \partial_i v_j - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right) + \zeta \delta_{ij} \nabla \cdot \mathbf{v} \right\} \quad (22b)$$

$$\rho_0 \frac{\partial s'}{\partial t} = \frac{\kappa}{T_0} \nabla^2 \tau \quad (22c)$$

In what follows, we keep using the 6 variables velocity, condensation  $b \equiv \rho'/\rho_0$ , excess pressure, and excess temperature. To obtain a closed system of equations on these 6 variables we use the equation of state  $\rho = \rho(p, T)$ , whose linearized version gives

$$b = \rho'/\rho_0 = \left( \frac{\gamma}{\rho c_a^2} \right)_0 p' - \beta_0 \tau \quad (23)$$

where the index 0 is used for quantities to be evaluated in the ambient state. The linearized version of the state equation  $T = T(p, s)$  gives  $\tau = \left( \frac{T\beta}{\rho c_p} \right)_0 p' + \left( \frac{T}{c_p} \right)_0 s'$ , where the coefficient  $(\partial T/\partial p)_s = \beta T/\rho c_p$  is expressed

using Maxwell's relation  $(\partial v/\partial s)_p = (\partial T/\partial p)_s$ , the identity  $(\partial v/\partial s)_p = -(\partial v/\partial p)_s (\partial p/\partial s)_v$ , and the general thermodynamic identity (11). Using this state equation to eliminate  $s'$  in (22c), and rewriting the first coefficient in (23) by introducing the adiabatic ambient bulk modulus

$$\chi_0 \equiv (1/\rho c_a^2)_0 \quad (24)$$

our starting complete system of six linearized viscothermal equations with the six variables  $\mathbf{v}$ ,  $b$ ,  $p'$ ,  $\tau$ , is finally written as

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' + \eta \nabla^2 \mathbf{v} + \left(\zeta + \frac{\eta}{3}\right) \nabla (\nabla \cdot \mathbf{v}) \quad (25a)$$

$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{v} = 0 \quad (25b)$$

$$\gamma \chi_0 p' = b + \beta_0 \tau \quad (25c)$$

$$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p'}{\partial t} + \kappa \nabla^2 \tau \quad (25d)$$

Eqs.(25a) and (25d) are the linearized Navier-Stokes and Fourier equations, respectively. For simplicity, in what follows the prime over the excess thermodynamic pressure  $p'$  is omitted.

## 2.2. Propagation in the circular tube

The solutions of the viscothermal fluid equations were given by Kirchhoff, for the case of axisymmetric wave propagation in a circular tube filled with ideal gas, taking into account viscous losses and thermal exchanges [11]. Kirchhoff assumed that the solid walls would remain at ambient temperature due to the large heat capacity and conduction coefficient of the solid compared to the fluid. This means that the excess-temperature vanishes on the solid walls. This boundary condition, well-verified in general, is used for simplicity in the following as well.

Here, we derive the axisymmetric 'Kirchhoff-Langevin's' solutions of the above equations (25), more general than Kirchhoff's as they are written for a fluid having nonzero value of the second viscosity  $\zeta$ , and, arbitrary equation of state *i.e.* a coefficient  $\beta_0$  that differs from the ideal gas value  $1/T_0$ . The reason to study only the axisymmetric solutions, is that we restrict in section 3 to consider the case where the wave motion is created by distributed source terms representing longitudinal macroscopic wave variations, directed along the tube axis  $x$  and invariant in the transverse direction.

Following Rayleigh's presentation of Kirchhoff's theory [22], we first substitute the state equation (25c) in Fourier's equation (25d) and use the thermodynamic identity (11) and definition (24) to obtain the following alternative form of (25d)

$$\frac{\partial \tau}{\partial t} = \frac{\gamma - 1}{\beta_0} \frac{\partial b}{\partial t} + \frac{\kappa}{\rho_0 c_v} \nabla^2 \tau \quad (26)$$

Using the variable

$$\tau' = \frac{\beta_0 \tau}{\gamma - 1} \quad (27)$$

Eqs.(25c) and (26) are simplified and become

$$\frac{p}{\rho_0} = c_i^2 b + (c_a^2 - c_i^2) \tau' \quad (28)$$

and

$$\frac{\partial \tau'}{\partial t} = \frac{\partial b}{\partial t} + \frac{\kappa}{\rho_0 c_v} \nabla^2 \tau' \quad (29)$$

Then, assuming that the variables  $\mathbf{v}$ ,  $b$ ,  $p$ ,  $\tau'$ , are varying with time as  $e^{-i\omega t}$ , Eqs.(25) yield

$$-\rho_0 i\omega \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} + \left(\zeta + \frac{\eta}{3}\right) i\omega \nabla b \quad (30a)$$

$$-i\omega b + \nabla \cdot \mathbf{v} = 0 \quad (30b)$$

$$\frac{p}{\rho_0} = c_i^2 b + (c_a^2 - c_i^2) \tau' \quad (30c)$$

$$-i\omega \tau' = -i\omega b + \frac{\kappa}{\rho_0 c_v} \nabla^2 \tau' \quad (30d)$$

Eliminating the pressure and condensation, give rise to the following velocity-temperature equations

$$-i\omega \mathbf{v} - \frac{\eta}{\rho_0} \nabla^2 \mathbf{v} = -\nabla X \quad (31a)$$

$$X = \left[ c_a^2 - \frac{\frac{\eta}{3} + \zeta}{\rho_0} i\omega \right] \tau' + \frac{\kappa}{\rho_0 c_v i\omega} \left[ c_i^2 - \frac{\frac{\eta}{3} + \zeta}{\rho_0} i\omega \right] \nabla^2 \tau' \quad (31b)$$

$$\nabla \cdot \mathbf{v} - i\omega \tau' - \frac{\kappa}{\rho_0 c_v} \nabla^2 \tau' = 0 \quad (31c)$$

Elimination of the velocity by taking the divergence of (31a) and using Eq.(30b) results in the temperature equation

$$\begin{aligned} \omega^2 \tau' + \left[ c_a^2 - i\omega \left( \frac{\kappa}{\rho_0 c_v} + \frac{\frac{4\eta}{3} + \zeta}{\rho_0} \right) \right] \nabla^2 \tau' \\ + \frac{\kappa}{\rho_0 c_v i\omega} \left[ c_i^2 - i\omega \frac{\frac{4\eta}{3} + \zeta}{\rho_0} \right] \nabla^2 \nabla^2 \tau' = 0 \end{aligned} \quad (32)$$

Let  $\lambda_1$  and  $\lambda_2$  be the two, small and large, solutions of the associated Kirchhoff-Langevin's characteristic equation

$$\omega^2 + \left[ c_a^2 - i\omega \left( \frac{\kappa}{\rho_0 c_v} + \frac{\frac{4\eta}{3} + \zeta}{\rho_0} \right) \right] \lambda + \frac{\kappa}{\rho_0 c_v i\omega} \left[ c_i^2 - i\omega \frac{\frac{4\eta}{3} + \zeta}{\rho_0} \right] \lambda^2 = 0 \quad (33)$$

The small solution – mainly real – describes propagating acoustic waves with small bulk absorption, the large solution – purely imaginary – highly damped diffusive entropic waves.

The field  $\tau'$  solution to (32) will be in the form

$$\tau' = A_1 \varphi_1 + A_2 \varphi_2 \quad (34)$$

with functions  $\varphi_1$  and  $\varphi_2$  verifying

$$\nabla^2 \varphi_1 = \lambda_1 \varphi_1, \quad \nabla^2 \varphi_2 = \lambda_2 \varphi_2 \quad (35)$$

The velocity  $\mathbf{v}$  will write

$$\mathbf{v} = \mathbf{v}' + B_1 \nabla \varphi_1 + B_2 \nabla \varphi_2 \quad (36)$$

with  $\mathbf{v}'$  the vortical part, such that

$$\nabla^2 \mathbf{v}' = \frac{-i\omega \rho_0}{\eta} \mathbf{v}', \quad \nabla \cdot \mathbf{v}' = 0 \quad (37)$$

The relation between coefficients  $B$  and  $A$  follows from (31c)

$$B_{1,2} = \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_{1,2}} \right) A_{1,2} \quad (38)$$

In summary, the viscothermal fields are generally decomposed as

$$\mathbf{v} = \mathbf{v}' + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \nabla \varphi_1 + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \nabla \varphi_2 \quad (39a)$$

$$b = \left( 1 + \frac{\kappa}{\rho_0 c_v i\omega} \lambda_1 \right) A_1 \varphi_1 + \left( 1 + \frac{\kappa}{\rho_0 c_v i\omega} \lambda_2 \right) A_2 \varphi_2 \quad (39b)$$

$$\frac{p}{\rho_0} = \left( c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_1 \right) A_1 \varphi_1 + \left( c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_2 \right) A_2 \varphi_2 \quad (39c)$$

$$\tau' = A_1 \varphi_1 + A_2 \varphi_2 \quad (39d)$$

Concerning the application to axisymmetric fields propagating in the right-going  $x$  direction in a tube of circular cross-section, we wish to determine normal modes as functions of  $x$ , proportional to  $e^{+ik_l x}$ , where the  $k_l$ 's,  $\Im(k_l) > 0$ ,  $l = 1, 2, \dots$ , are complex constants to be specified. In what follows, for convenience, the index  $l$  labelling the different axisymmetric modes solutions will be omitted.

For these modes, the operators  $\nabla$  and  $\nabla^2$  can be replaced by  $ik\mathbf{e}_x + \mathbf{e}_r \frac{\partial}{\partial r}$  ( $\mathbf{e}_r$  representing the radial unit vector) and  $\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} - k^2$ , and the different fields  $a(t, \mathbf{x})$  can be replaced by their amplitudes  $a$  such that  $a(t, \mathbf{x}) = a(r)e^{-i\omega t + ikx}$ . By elementary calculations there follows that the corresponding  $\varphi_1$  and  $\varphi_2$  will be described by Bessel functions

$$\varphi_{1,2} = J_0 \left( r \sqrt{-\lambda_{1,2} - k^2} \right) \quad (40)$$

Writing the vortical velocity  $\mathbf{v}'$  in the form  $\mathbf{v}' = u'\mathbf{e}_x + q'\mathbf{e}_r$ , with axial and radial amplitudes  $u'$  and  $q'$  independent of azimuthal angle  $\vartheta$ , it is easy to see that Eqs.(37) imply  $u'$  is the solution to

$$\left( \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} \right) u' = \left( \frac{-i\omega}{\nu} + k^2 \right) u' \quad (41)$$

where  $\nu \equiv \eta/\rho_0$  is the kinematic viscosity and  $q'$  is determined by the relation

$$q' = \frac{-ik}{\frac{-i\omega}{\nu} + k^2} \frac{\partial u'}{\partial r} \quad (42)$$

As a result,  $u'$  and  $q'$  will be written as

$$u' = A\varphi, \quad q' = A \frac{-ik}{\frac{-i\omega}{\nu} + k^2} \frac{\partial \varphi}{\partial r} \quad (43)$$

where  $\varphi$  is the Bessel function

$$\varphi = J_0 \left( r \sqrt{\frac{i\omega}{\nu} - k^2} \right) \quad (44)$$

Finally, writing the total velocity in the form of  $\mathbf{v} = u\mathbf{e}_x + q\mathbf{e}_r$ ,  $u$  and  $q$  are obtained as

$$u = A\varphi + ik \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \varphi_1 + ik \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \varphi_2 \quad (45a)$$

$$q = A \frac{-ik}{\frac{-i\omega}{\nu} + k^2} \frac{\partial \varphi}{\partial r} + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \frac{\partial \varphi_1}{\partial r} + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \frac{\partial \varphi_2}{\partial r} \quad (45b)$$

The tube being assumed sufficiently inert thermally to remain at ambient temperature, and no temperature-jump occurring on the tube wall, the excess temperature vanishes on the latter. On the other hand, no-slip condition is applied. Thus, (45a-45b) and (39d) vanish at  $r_w = R$ . These three homogeneous equations have non vanishing solutions only if their determinant is zero, which consequently yields the following Kirchhoff's dispersion equation, hereafter referred to as Kirchhoff-Langevin's dispersion equation to remind its validity for a general viscothermal fluid

$$\begin{aligned} & \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) \frac{1}{\varphi_{1w}} \frac{\partial \varphi_1}{\partial r_w} - \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) \frac{1}{\varphi_{2w}} \frac{\partial \varphi_2}{\partial r_w} \\ & - \frac{k^2}{\frac{-i\omega}{\nu} + k^2} \left( \frac{i\omega}{\lambda_1} - \frac{i\omega}{\lambda_2} \right) \frac{1}{\varphi_w} \frac{\partial \varphi}{\partial r_w} = 0 \end{aligned} \quad (46)$$

The index  $w$  indicates that the functions and derivatives are evaluated at the tube wall  $r_w = R$ . Kirchhoff-Langevin's dispersion equation (46) has a series of discrete complex solutions  $k_l$ ,  $\Im k_l > 0$ ,  $l = 1, 2, \dots$ , which can be sorted by convention in ascending order according to the values of  $\Im k_l$ :  $\Im k_1 < \Im k_2 < \dots$ . In order to determine the solutions  $k_l$ , the Newton-Raphson root-finding method can be used. In the lossless limit  $\eta, \zeta, \kappa \rightarrow 0$ , (46) simplifies and tends to the equation  $J_1 \left( R \sqrt{\frac{\omega^2}{c_a^2} - k^2} \right) = 0$ , yielding the following  $k_{0l}$  starting values,  $k_{0l}^2 = \frac{\omega^2}{c_a^2} - \frac{x_l^2}{R^2}$ , where  $x_l \geq 0$  are the successive zeros of the function  $J_1(x)$ . As  $J_1(0) = 0$ , *i.e.*  $x_1 = 0$ , the first solution is  $k_{01} = \omega/c_a$ , which corresponds to the plane wave mode. At a given frequency, there are

one (plane wave mode  $k_{01}$ ) or more real positive solutions describing right-going propagating waves, and an infinite discrete set of purely imaginary solutions with  $\Im(k_{0l}) > 0$ , describing evanescent waves which attenuate as  $e^{-\Im k_{0l}x}$  along positive  $x$ -axis.

In general, a few Newton-Raphson iterations suffice to make these starting lossless purely real or purely imaginary solutions  $k_{0l}$  (to which may be added either a small imaginary or small real part), converge towards the complete complex solutions  $k_l$ . The solution with positive imaginary part is retained, as we consider waves propagating in the direction  $+x$ , which can be created by a source placed on the left. The condition that the imaginary part is positive, automatically fixes the sign of the real part.

We note that, for a given solution  $l$  and on account of the two independent conditions expressing the vanishing of  $\tau'$  and  $u$  at the tube wall  $r_w = R$

$$A_1\varphi_{1w} + A_2\varphi_{2w} = 0 \quad (47a)$$

$$A\varphi_w + ik\left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1}\right)A_1\varphi_{1w} + ik\left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2}\right)A_2\varphi_{2w} = 0 \quad (47b)$$

the solution may be fixed in terms of only one arbitrary complex amplitude (related to the arbitrary choice of the sound pressure level and phase). We denote it  $\mathcal{A}$  and write

$$A = \mathcal{A}ik\left(\frac{i\omega}{\lambda_1} - \frac{i\omega}{\lambda_2}\right)\varphi_{1w}\varphi_{2w}, \quad A_1 = -\mathcal{A}\varphi_w\varphi_{2w}, \quad A_2 = \mathcal{A}\varphi_w\varphi_{1w} \quad (48)$$

In summary, with this notation the different fields write as follows

$$\begin{aligned} \frac{u}{ik\mathcal{A}} = & \left( \frac{i\omega}{\lambda_1} - \frac{i\omega}{\lambda_2} \right) \varphi_{1w}\varphi_{2w}\varphi - \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) \varphi_w\varphi_{2w}\varphi_1 \\ & + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) \varphi_w\varphi_{1w}\varphi_2 \end{aligned} \quad (49a)$$

$$\begin{aligned} \frac{q}{\mathcal{A}} = & \left( \frac{i\omega}{\lambda_1} - \frac{i\omega}{\lambda_2} \right) \varphi_{1w}\varphi_{2w} \frac{k^2}{\frac{-i\omega}{\nu} + k^2} \frac{\partial \varphi}{\partial r} - \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) \varphi_w\varphi_{2w} \frac{\partial \varphi_1}{\partial r} \\ & + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) \varphi_w\varphi_{1w} \frac{\partial \varphi_2}{\partial r} \end{aligned} \quad (49b)$$

$$\frac{b}{\mathcal{A}} = - \left( 1 + \frac{\kappa}{\rho_0 c_v i\omega} \lambda_1 \right) \varphi_w\varphi_{2w}\varphi_1 + \left( 1 + \frac{\kappa}{\rho_0 c_v i\omega} \lambda_2 \right) \varphi_w\varphi_{1w}\varphi_2 \quad (49c)$$

$$\begin{aligned} \frac{p}{-\mathcal{A}\rho_0} = & \left( c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_1 \right) \varphi_w\varphi_{2w}\varphi_1 \\ & + \left( c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_2 \right) \varphi_w\varphi_{1w}\varphi_2 \end{aligned} \quad (49d)$$

$$\frac{\tau'}{\mathcal{A}} = -\varphi_w\varphi_{2w}\varphi_1 + \varphi_w\varphi_{1w}\varphi_2 \quad (49e)$$

Definite and different field patterns are obtained by substituting for the wavenumber  $k$  in the above expressions, one of its possible values  $k_l$ . Suitable averaging of these fields patterns will then yield the quantities making sense in the forthcoming macroscopic theory, and in particular, the wavenumbers  $k_l$  and impedances  $Z_l$ .

Let us denote by a bracket  $\langle f \rangle$  the average of a field  $f$ , performed over the cross-section of the tube

$$\langle f \rangle = \frac{1}{\pi R^2} \int_0^R dr \int_0^{2\pi} r d\vartheta f \quad (50)$$

For later use, a macroscopic mean pressure  $H$  is introduced through setting up the general definition

$$\langle p\mathbf{v} \rangle = H \langle \mathbf{v} \rangle \quad (51)$$

Thermodynamic-relating reasons to define in such a way the macroscopic pressure were given in [1] with an ensemble-averaging conception. As explained in the next section, for the single tube considered here, this average



is interpreted as a cross-section average. The notation  $H$  comes from the affinity of this concept with that of Maxwell's magnetic density field  $\mathbf{H}$ . In accordance with this general definition, a characteristic complex impedance factor  $Z_l$  can be defined for a given mode, by setting

$$H = Z_l \langle u \rangle, \quad i.e. \quad Z_l = \frac{\langle pu \rangle}{\langle u \rangle^2} \quad (52)$$

where the averages can be computed analytically knowing the wavenumber  $k_l$ .

By knowing the complex wavenumber  $k_l(\omega)$  and impedance  $Z_l(\omega)$  for a given radial mode solution  $l$ , an equivalent-fluid complex density  $\rho_l(\omega)$  and bulk modulus  $\chi_l^{-1}(\omega)$  might be defined through setting relations having the usual form (see e.g. [14])  $k_l = \omega/c_l = \omega\sqrt{\rho_l\chi_l}$  and  $Z_l = \rho_l c_l = \sqrt{\rho_l\chi_l^{-1}}$ , *i.e.*

$$\rho_l(\omega) = \frac{k_l}{\omega} Z_l = \frac{k_l}{\omega} \frac{\langle pu \rangle}{\langle u \rangle^2}, \quad \chi_l^{-1}(\omega) = \frac{\omega}{k_l} Z_l = \frac{\omega}{k_l} \frac{\langle pu \rangle}{\langle u \rangle^2} \quad (53)$$

For a given mode  $l$  propagating in the  $+x$  direction, macroscopic equivalent-fluid equations of motion of the following form

$$-\rho_l(\omega)i\omega\langle u \rangle = -ik_l H, \quad -\chi_l(\omega)i\omega H = -ik_l \langle u \rangle \quad (54)$$

are then satisfied.

There are known formulae giving the cross-section average  $\langle \rangle$  of Bessel functions  $\varphi$ ,  $\varphi_1$  and  $\varphi_2$  and their products, in terms of other Bessel functions. Thus, the above Kirchhoff-Langevin's impedance factors  $Z_l(\omega)$  (52) and densities and bulk moduli  $\rho_l(\omega)$  and  $\chi_l^{-1}(\omega)$  (53) are the expression of closed-form functions  $F(\omega, k_l)$ , directly computable once  $k_l$  is evaluated.

### 3. General nonlocal theory applied to sound propagation in a tube with circular cross-section

The general nonlocal theory of sound propagation presented in [1] is intended to be exact – within the considered Navier-Stokes-Fourier physics – for the description of sound propagation in rigid-framed fluid-saturated porous materials whose macroscopic properties are defined in an ensemble-averaged sense. To justify the fact that this theory also exactly describes the cross-section-averaged propagation in one single tube, let us imagine that we

are given an ensemble  $\Omega$  of samples  $\omega$ , each of which is obtained by random translation in the axial directions, of one reference sample  $\omega_0$  permeated by identical parallel pores of radius  $R$  directed in the axial direction  $x$ . Then, apart from a constant porosity normalization factor of which we can make abstraction, the ensemble-average operation in the sense of [1], over the space  $\Omega$ , becomes the above cross-section average  $\langle \rangle$ , in a single tube. In fact, the fields relating to the harmonic action-response problems considered in [1] have the form  $a(\omega, k, r)e^{-i\omega t + ikx}$ , with amplitudes  $a(\omega, k, r)$  independent of  $x$ , and thereby, needed to be averaged only over a cross-section.

### 3.1. Linearized macroscopic equations in Maxwellian nonlocal theory

Considering, as explained above, that the operation  $\langle \rangle$  is the cross-section average in one single tube, and introducing the macroscopic variables  $V = \mathbf{V} \cdot \mathbf{e}_x = \langle u \rangle$  and  $B = \langle b \rangle$ , where  $u$  is the axial velocity, the nonlocal theory [1] predicts that the wave propagation of the averaged quantities  $V$  and  $B$  is described by the following field equations (see Eqs.(60), (61), (64a) and (64b) in [1])

$$\frac{\partial B}{\partial t} + \frac{\partial V}{\partial x} = 0 \quad (55)$$

$$\frac{\partial D}{\partial t} = -\frac{\partial H}{\partial x} \quad (56)$$

and constitutive relations

$$D(t, x) = \int_{-\infty}^t dt' \int dx' \rho(t - t', x - x') V(t', x') \quad (57)$$

$$H(t, x) = \int_{-\infty}^t dt' \int dx' \chi^{-1}(t - t', x - x') B(t', x') \quad (58)$$

where  $\rho(t, x)$  and  $\chi^{-1}(t, x)$  are constitutive kernel functions independent of temporal and spatial variations of the fields. They are determined by the fluid constants and the microgeometry of the porous medium, *i.e.* here, the tube radius  $R$ . The integrations over  $t'$  determine the so-called temporal dispersion effects and the integrations over  $x'$  determine the so-called spatial dispersion effects [23, 24]. The upscaling recipes [1] lead to specify the Fourier coefficients  $\rho(\omega, k)$  and  $\chi^{-1}(\omega, k)$  of these constitutive functions, and will be described in the next two sections.

### 3.2. Determination of the nonlocal density $\rho(\omega, k)$

To compute  $\rho(\omega, k)$ , we first consider the response of the fluid subjected to the action of an external driving bulk force  $\mathbf{f}$  (per unit volume), which derives from a fictitious harmonic pressure waveform inserted in Navier-Stokes equation (see section 4 in [1]). Thus we consider solving the action-response problem

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho_0} \nabla p = \nu \nabla^2 \mathbf{v} - \frac{(\frac{\eta}{3} + \zeta)}{\rho_0} \nabla \frac{\partial b}{\partial t} + \frac{1}{\rho_0} \mathbf{f} \quad (59a)$$

$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{v} = 0 \quad (59b)$$

$$\gamma \chi_0 p = b + \beta_0 \tau \quad (59c)$$

$$\frac{\partial \tau}{\partial t} = \frac{\beta_0 T_0}{\rho_0 c_P} \frac{\partial p}{\partial t} + \frac{\kappa}{\rho_0 c_P} \nabla^2 \tau \quad (59d)$$

for  $r < R$ , and

$$\mathbf{v} = 0 \quad (60a)$$

$$\tau = 0 \quad (60b)$$

at  $r = R$ , with driving force given by

$$\mathbf{f} = -\nabla \mathcal{P} = -ik \mathbf{e}_x \mathcal{P}_0 e^{-i\omega t + ikx} \quad (61)$$

With calculations entirely similar to those which have been done before, Eqs.(31) become

$$-i\omega \mathbf{v} - \nu \nabla^2 \mathbf{v} = -\nabla X + \frac{1}{\rho_0} \mathbf{f} \quad (62a)$$

$$X = \left[ c_a^2 - \frac{\frac{\eta}{3} + \zeta}{\rho_0} i\omega \right] \tau' + \frac{\kappa}{\rho_0 c_v i\omega} \left[ c_i^2 - \frac{\frac{\eta}{3} + \zeta}{\rho_0} i\omega \right] \nabla^2 \tau' \quad (62b)$$

$$\nabla \cdot \mathbf{v} - i\omega \tau' - \frac{\kappa}{\rho_0 c_v} \nabla^2 \tau' = 0 \quad (62c)$$

and the temperature Eq.(32) becomes

$$\begin{aligned} -\omega^2 \tau' - \left[ c_a^2 - i\omega \left( \frac{\kappa}{\rho_0 c_v} + \frac{\frac{4\eta}{3} - \zeta}{\rho_0} \right) \right] \nabla^2 \tau' \\ - \frac{\kappa}{\rho_0 c_v i\omega} \left[ c_i^2 - i\omega \frac{\frac{4\eta}{3} + \zeta}{\rho_0} \right] \nabla^2 \nabla^2 \tau' + \frac{k^2}{\rho_0} \mathcal{P} = 0 \end{aligned} \quad (63)$$

A particular solution of the above equation is

$$\tau'_p = C \frac{k^2}{\rho_0} \mathcal{P} \quad (64)$$

with

$$C = \left\{ \omega^2 - \left[ c_a^2 - i\omega \left( \frac{\kappa}{\rho_0 c_v} + \frac{\frac{4\eta}{3} + \zeta}{\rho_0} \right) \right] k^2 + \frac{\kappa}{\rho_0 c_v i\omega} \left[ c_i^2 - i\omega \frac{\frac{4\eta}{3} + \zeta}{\rho_0} \right] k^4 \right\}^{-1} \quad (65)$$

The complete solution of Eq.(63) is this particular solution added to the general solution (34) of the homogeneous equation (32)

$$\tau' = \tau'_0 + \tau'_p = A_1 \varphi_1 + A_2 \varphi_2 + C \frac{k^2}{\rho_0} \mathcal{P} \quad (66)$$

Similarly, the complete expression of the velocity will be

$$\mathbf{v} = u \mathbf{e}_x + q \mathbf{e}_r = \mathbf{v}_0 + \mathbf{v}_p \quad (67)$$

where  $\mathbf{v}_0$  is written as in (39a), and  $\mathbf{v}_p = u_p \mathbf{e}_x$  is the particular solution with  $u_p$  determined by (62c), and  $\tau' = \tau'_p$ , *i.e.*,  $iku_p = (i\omega - \kappa k^2 / \rho_0 c_v) \tau'_p$ . Only the  $x$  component  $u$  will be required to compute  $\rho(\omega, k)$ . However, the radial component  $q$  needs also to be written as it is involved in the boundary conditions, by means of which the amplitudes  $A, A_1, A_2$  are finally fixed. Both components write accordingly

$$u = A\varphi + ik \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \varphi_1 + ik \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \varphi_2 + \left( -i\omega + \frac{\kappa}{\rho_0 c_v} k^2 \right) C \frac{ik}{\rho_0} \mathcal{P} \quad (68a)$$

$$q = A \frac{-ik}{\frac{-i\omega}{\nu} + k^2} \frac{\partial \varphi}{\partial r} + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \frac{\partial \varphi_1}{\partial r} + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \frac{\partial \varphi_2}{\partial r} \quad (68b)$$

Now, we seek the excess pressure solution as the last required quantity. It has the complete form

$$p = p_0 + p_p \quad (69)$$

where  $p_0$  is given by (39c), and the particular solution  $p_p$  is determined by (25c) with  $\tau' = \tau'_p$ ,  $b = b_p$ , and  $i\omega b_p = iku_p$ . Thus

$$\begin{aligned} \frac{p}{\rho_0} = & \left( c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i \omega} \lambda_1 \right) A_1 \varphi_1 + \left( c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i \omega} \lambda_2 \right) A_2 \varphi_2 \\ & + \left( c_a^2 - c_i^2 \frac{\kappa}{\rho_0 c_v i \omega} k^2 \right) C \frac{k^2}{\rho_0} \mathcal{P} \end{aligned} \quad (70)$$

The boundary conditions imply that the three quantities, excess temperature and the two components of velocity (66), (68a) and (68b), should vanish at the tube wall  $r = r_w = R$ . This yields a linear system whose solution uniquely determines the three response amplitudes  $A$ ,  $A_1$  and  $A_2$ , in terms of the arbitrary driving pressure amplitude  $\mathcal{P}_0$ . Knowing the response fields (68a) and (70) as functions of  $\omega, k$  and  $r$ , is all we need to compute the density  $\rho(\omega, k)$ . According to the conjectured upscaling procedure we assume that in the action-response problem (59), the role of the macroscopic field  $H$  in Eqs.(55-58) is played by the field  $\mathbf{P} + \mathcal{P}$  where  $\mathbf{P}$  is the macroscopic part of the response pressure field  $p$ , which is defined by (see section 4 in [1])

$$\langle p \mathbf{v} \rangle = \mathbf{P} \langle \mathbf{v} \rangle, \quad i.e. \quad \langle pu \rangle = \mathbf{P} \langle u \rangle \quad (71)$$

This assumption leads to Eqs.(68) and (69) in [1], which give here

$$\rho(\omega, k) = \frac{k(\mathbf{P} + \mathcal{P})}{\omega \langle u \rangle} \quad (72)$$

A direct verification of this conjectured upscaling procedure would be to see whether or not, when the amplitudes  $\mathcal{A}$  and  $\mathcal{P}$  are adjusted so that the gradient  $-\nabla H = -ik_l \mathbf{e}_x H$  in section 2.2 is the same as the gradient  $-\nabla(\mathbf{P} + \mathcal{P}) = -ik_l \mathbf{e}_x (\mathbf{P} + \mathcal{P})$  in this section, the averages of the velocities  $\mathbf{v}$  appearing respectively in Eqs.(25) and (59) turn out to be exactly the same. This macroscopic coincidence of the two mean velocities obtained in two different problems, is highly nontrivial even in the present simplest case of cylindrical duct geometry. In principle, the two problems have fundamentally different nature; one is an eigenvalue (source-free) problem and the other is an action-response (source-driven) problem. In the two problems, the corresponding sets of microscopic field patterns are not the same; but after averaging, the two mean velocities divided by the corresponding two gradients are conjectured to be exactly the same.

A comparison between the first Eq.(54) and (72) shows that this matching is expressed in explicit equivalent form by the following equation

$$\rho_l(\omega) = \rho(\omega, k_l) \quad (73)$$

It is in this last convenient form, that the validity of the upscaling procedure for  $\rho(\omega, k)$  will be directly checked. We may name ‘Maxwell’s’ the nonlocal density function (72). There are known formulae giving the average of Bessel functions  $\varphi$ ,  $\varphi_1$  and  $\varphi_2$  and their products, in terms of other Bessel functions, which allow to write the nonlocal density function (72) in closed form. This function is a complicated ratio of sums containing many terms, each of which involving the product of 6 Bessel functions, multiplied by factors involving  $\omega$ ,  $k^2$ ,  $\lambda_1$ ,  $\lambda_2$ . To save time, instead of seeking its most compact final expression, we have made a direct Matlab programming of the function  $\rho(\omega, k)$ , with  $\omega$  and  $k$  as input arguments.

### 3.3. Determination of the nonlocal bulk modulus $\chi^{-1}(\omega, k)$

The same type of calculations as seen in the above to obtain  $\rho(\omega, k)$ , can be performed to compute  $\chi^{-1}(\omega, k)$ . Here, we need first to consider the response of the fluid subjected to the action of an external driving bulk rate of heat supply  $\dot{Q}$  (per unit volume and unit time), which derives from a fictitious harmonic pressure waveform inserted in Fourier equation (see section 4 in [1]). Thus we consider solving the action-response problem

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho_0} \nabla p = \nu \nabla^2 \mathbf{v} - \frac{(\frac{\eta}{3} + \zeta)}{\rho_0} \nabla \frac{\partial b}{\partial t} \quad (74a)$$

$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{v} = 0 \quad (74b)$$

$$\gamma \chi_0 p = b + \beta_0 \tau \quad (74c)$$

$$\frac{\partial \tau}{\partial t} = \frac{\beta_0 T_0}{\rho_0 c_p} \frac{\partial p}{\partial t} + \frac{\kappa}{\rho_0 c_p} \nabla^2 \tau + \frac{1}{\rho_0 c_p} \dot{Q} \quad (74d)$$

for  $r < R$ , and

$$\mathbf{v} = 0 \quad (75a)$$

$$\tau = 0 \quad (75b)$$

at  $r = R$ , with the driving rate of heat supply given by

$$\dot{Q} = \beta_0 T_0 \frac{\partial \mathcal{P}}{\partial t} = -i\omega \beta_0 T_0 \mathcal{P}_0 e^{-i\omega t + ikx} \quad (76)$$

Through similar calculations as seen before, Eqs.(31) now become

$$-i\omega \mathbf{v} - \nu \nabla^2 \mathbf{v} = -\nabla X \quad (77a)$$

$$X = \left[ c_a^2 - \frac{\frac{\eta}{3} + \zeta}{\rho_0} i\omega \right] \tau' + \frac{\kappa}{\rho_0 c_v i\omega} \left[ c_i^2 - \frac{\frac{\eta}{3} + \zeta}{\rho_0} i\omega \right] \nabla^2 \tau' \quad (77b)$$

$$\nabla \cdot \mathbf{v} - i\omega \tau' - \frac{\kappa}{\rho_0 c_v} \nabla^2 \tau' = \gamma \chi_0 \frac{\partial \mathcal{P}}{\partial t} \quad (77c)$$

and the temperature Eq.(32) becomes

$$\begin{aligned} & -\omega^2 \tau' - \left[ c_a^2 - i\omega \left( \frac{\kappa}{\rho_0 c_v} + \frac{\frac{4\eta}{3} - \zeta}{\rho_0} \right) \right] \nabla^2 \tau' \\ & - \frac{\kappa}{\rho_0 c_v i\omega} \left[ c_i^2 - i\omega \frac{\frac{4\eta}{3} + \zeta}{\rho_0} \right] \nabla^2 \nabla^2 \tau' - (i\omega - \nu k^2) \gamma \chi_0 i\omega \mathcal{P} = 0 \end{aligned} \quad (78)$$

A particular solution to the above equation will be

$$\tau'_p = -C(i\omega - \nu k^2) \gamma \chi_0 i\omega \mathcal{P} \quad (79)$$

with the same constant  $C$  as before. The complete excess temperature solution is this particular solution added to the general solution (34) of the homogeneous equation (32):

$$\tau' = \tau'_0 + \tau'_p = A_1 \varphi_1 + A_2 \varphi_2 - C(i\omega - \nu k^2) \gamma \chi_0 i\omega \mathcal{P} \quad (80)$$

Likewise, the complete velocity solution is the form

$$\mathbf{v} = u \mathbf{e}_x + q \mathbf{e}_r = \mathbf{v}_0 + \mathbf{v}_p \quad (81)$$

where  $\mathbf{v}_0$  writes as in (40), and  $\mathbf{v}_p = u_p \mathbf{e}_x$  is the particular solution with  $u_p$  determined by (77c) and  $\tau' = \tau'_p$ , i.e.,  $iku_p = (i\omega - \kappa k^2 / \rho_0 c_v) \tau'_p - i\omega \gamma \chi_0 \mathcal{P}$ . We obtain for the two components of the velocity

$$\begin{aligned} u = A\varphi + ik \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \varphi_1 + ik \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \varphi_2 \\ + \left[ \left( -i\omega + \frac{\kappa}{\rho_0 c_v} k^2 \right) C(i\omega - \nu k^2) - 1 \right] \gamma \chi_0 \frac{\omega}{k} \mathcal{P} \end{aligned} \quad (82a)$$

$$q = A \frac{-ik}{\frac{-i\omega}{\nu} + k^2} \frac{\partial \varphi}{\partial r} + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \frac{\partial \varphi_1}{\partial r} + \left( \frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \frac{\partial \varphi_2}{\partial r} \quad (82b)$$

The complete pressure solution, similarly, is written as

$$p = p_0 + p_p \quad (83)$$

where  $p_0$  writes as in (39c) and  $p_p$  is determined by (25c) with  $\tau' = \tau'_p$  and  $b = b_p$ ,  $i\omega b_p = iku_p$ . We will have

$$\begin{aligned} \frac{p}{\rho_0} = & \left( c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_1 \right) A_1 \varphi_1 + \left( c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_2 \right) A_2 \varphi_2 \\ & - C(i\omega - \nu k^2) i\omega \left( c_a^2 - c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} k^2 \right) \gamma \chi_0 \mathcal{P} - c_i^2 \gamma \chi_0 \mathcal{P} \end{aligned} \quad (84)$$

The complete condensation solution will now also be required. It may be written from (30c) and the expressions (80) and (84), which yield

$$\begin{aligned} b = & \left( 1 + \frac{\kappa}{\rho_0 c_v i\omega} \lambda_1 \right) A_1 \varphi_1 + \left( 1 + \frac{\kappa}{\rho_0 c_v i\omega} \lambda_2 \right) A_2 \varphi_2 \\ & - C(i\omega - \nu k^2) i\omega \left( 1 - \frac{\kappa}{\rho_0 c_v i\omega} k^2 \right) \gamma \chi_0 \mathcal{P} - \gamma \chi_0 \mathcal{P} \end{aligned} \quad (85)$$

As before, the boundary conditions imply that the three quantities, excess temperature and the two components of velocity (80), (82a) and (82b), should vanish at the tube wall  $r = r_w = R$ . This yields a linear system whose solution uniquely determines the three response amplitudes  $A$ ,  $A_1$  and  $A_2$ , in terms of the arbitrary driving amplitude  $\mathcal{P}_0$ . Knowing the response fields (82a), (84) and (85) as functions of  $\omega, k$  and  $r$ , is what we need to compute the bulk modulus  $\chi^{-1}(\omega, k)$ . According to the conjectured upscaling procedure [1] we assume that in the action-response problem (74), the role of the macroscopic field  $H$  in Eqs.(55-58) is played by the field  $\mathbf{P} + \mathcal{P}$  where  $\mathbf{P}$  is the macroscopic part of the response pressure field  $p$ , which is defined by

$$\langle p\mathbf{v} \rangle = \mathbf{P}\langle \mathbf{v} \rangle, \quad i.e. \quad \langle pu \rangle = \mathbf{P}\langle u \rangle \quad (86)$$

and, at the same time, the role of the macroscopic field  $B$  is played by the averaged field  $\langle b + \gamma \chi_0 \mathcal{P} \rangle$ .

This leads to Eqs.(68) and (73) in [1], which write here

$$\chi^{-1}(\omega, k) = \frac{\mathbf{P}(\omega, k) + \mathcal{P}}{\langle b(\omega, k, r) \rangle + \gamma \chi_0 \mathcal{P}} \quad (87)$$



A direct verification of this, would be to see whether or not, when the amplitudes  $\mathcal{A}$  and  $\mathcal{P}$  are adjusted so that the amplitude  $H$  in section 2.2 is the same as the amplitude  $\mathcal{P} + \mathcal{P}$  in this section, then, the average of the condensation  $b$  appearing in Eqs.(25) and the average of the condensation  $b + \gamma\chi_0\mathcal{P}$ , where  $b$  is the quantity appearing in (74), turn out to be also exactly the same. Again, this macroscopic accordance is by no means trivial, even in the present simplest case of cylindrical duct geometry.

A comparison between the second Eq.(54) and (87) shows that this matching is equivalently expressed by the following equation

$$\chi_l^{-1}(\omega) = \chi^{-1}(\omega, k_l) \quad (88)$$

It is in this last convenient form, that the validity of the upscaling procedure for  $\chi^{-1}(\omega, k)$  will be directly checked. As for the nonlocal density, we may name ‘Maxwell’s’ the nonlocal bulk modulus function (87). As before, this Maxwell’s nonlocal bulk modulus is known in closed form as a complicated ratio of sums containing many terms, each of which involving the product of 6 Bessels. Again to save time, a direct programming of the function  $\chi(\omega, k)$  was made, with  $\omega$  and  $k$  as input arguments.

#### 3.4. Dispersion equation, wavenumbers, and impedances

The above nonlocal theory predicts that, at a given frequency, normal mode solutions with averaged fields varying as  $e^{-i\omega t + ikx}$  will exist, for wavenumber  $k$  solution to the dispersion equation

$$\rho(\omega, k)\chi(\omega, k)\omega^2 = k^2 \quad (89)$$

Indeed, this equation comes from Eqs.(55-58), making use of macroscopic fields varying like  $e^{-i\omega t + ikx}$ . Hereafter, we refer to this dispersion equation as to the nonlocal Maxwell’s dispersion equation.

For the proposed nonlocal theory to be correct, Maxwell’s dispersion equation (89) must be mathematically equivalent to the original Kirchhoff-Langevin’s dispersion equation (46). In particular, both equations must have the same set of solutions  $k_l$  at given  $\omega$ . In addition, the macroscopic impedances of the corresponding modes must also be the same. Recall that for any field freely propagating in the tube, the nonlocal theory definition of the macroscopic field  $H(t, x)$  originates from the fundamental ‘Umov-Poynting’s’ identification (Eq.(65) in [1])

$$\langle p(t, x, r)u(t, x, r) \rangle = H(t, x)\langle u(t, x, r) \rangle \quad (90)$$

Since we have used the same expression to define ‘Kirchhoff-Langevin’s’ macroscopic impedance factors  $Z_l$  (52), Kirchhoff-Langevin’s and Maxwell’s wavenumbers and macroscopic impedances match automatically, provided the following aforementioned relations are satisfied

$$\rho_l(\omega) = \frac{k_l}{\omega} Z_l = \rho(\omega, k_l), \quad \chi_l^{-1}(\omega) = \frac{\omega}{k_l} Z_l = \chi^{-1}(\omega, k_l) \quad (91)$$

The calculations to be performed in order to show the mathematical equivalence between (46) and (89) appear very tedious, because of the large number of terms to be collected and rearranged in order to express the mean term  $\langle pu \rangle$ . Moreover, it is even not evident that the mathematical identity between the two forms of the dispersion equation would directly appear, once the calculations are performed. In what follows, in order to check this equivalence and thus the validity of the theory, the easier way is to make a direct numerical check of the relations such as (91). The finite precision of the Matlab computations would limit the number of modes that can be valuably checked; the results to be presented next, however, clearly validate the theory.

#### 4. A check on the nonlocal theory

We describe in what follows three significant different cases representative of the three main types of duct wave frequency regimes referred to as ‘narrow’ tube, ‘wide’ tube, and ‘very wide’ tube, in acoustic literature [25].

In the section 4.1 we consider the case of low frequencies or ‘narrow’ tubes. In this low frequency range, the viscous skin depth  $\delta = (2\nu/\omega)^{1/2}$  and thermal skin depth – having the same order for air – are greater than  $R$ . Calculations are performed at frequency  $f = 100Hz$  for a tube of radius  $R = 10^{-4}m$ . This is the narrow tube configuration considered in [26]. The value  $R = 10^{-4}m$  is typical for the pore size dimensions found in ordinary porous materials used in noise control applications [15] – such as pore-size parameter  $\Lambda$  of dynamically connected pores in [4]. With a viscous skin depth equal to two times the radius, the fundamental plane-wave like mode is mostly diffusive and the higher order modes are highly attenuated.

Certainly due to insufficient accuracy of Matlab Bessel’s functions, only the first mode appears to be numerically very well characterized. However, it provides a first check of the theory: Kirchhoff-Langevin’s and Maxwell’s

Table 1: Fluid properties used in all computations.

$\rho_0$ ( $kg/m^3$ )	$T_0$ ( $K$ )	$c_0$ ( $m/s$ )	$\eta$ ( $kg\ ms^{-1}$ )	$\zeta$ ( $kg\ ms^{-1}$ )	$\kappa$ ( $Wm^{-1}K^{-1}$ )	$\chi_0$ ( $Pa^{-1}$ )	$c_p$ ( $J\ kg^{-1}K^{-1}$ )	$\gamma$
1.205	293.5	340.1391	$1.8369 \times 10^{-5}$	$0.6\ \eta$	$2.57 \times 10^{-2}$	$7.173 \times 10^{-6}$	997.5422	1.4

results for this mode are found to be identically the same, up to the numerical accuracy.

In the section 4.2 we consider the case of high frequencies or ‘wide’ tubes. In this frequency range, the viscous skin depth and thermal skin depth become significantly smaller than  $R$ . The calculations are done at frequency  $f = 10kHz$  for a tube of radius  $R = 10^{-3}m$ . This is the ‘wide’ tube configuration considered in [26]. With the radius now being more than 50 times the viscous skin depth, the fundamental plane-wave-like mode is a well-propagating mode, whose macroscopic characteristics may be followed with great numerical accuracy by the Matlab nonlocal theory computations. Several higher order modes are also successfully described, whether they are below or above the cutoff frequency. Again, the results provide unequivocal validation of the proposed nonlocal theory.

Finally, in subsection 4.3, taking a tube radius of  $1cm$ , and a frequency  $f = 500kHz$ , we consider the case of ‘very wide’ tubes. There, the fundamental least attenuated mode is no longer plane-like as predicted by Zwikker and Kosten theory. Sound energy tends to concentrate near the walls. Once again Kirchhoff-Langevin’s results are exactly reproduced by the nonlocal Maxwellian theory and provide unequivocal remarkable validation of the theory. In all foregoing calculations the parameters of the air are set to the values shown in Table 1.

Given the radius  $R$  and the frequency  $f$ , we proceed as follows to evaluate different quantities. To evaluate the Zwikker and Kosten density and bulk modulus  $\rho_Z$  and  $\chi_Z^{-1}$ , and then the corresponding wavenumber and impedance  $k_Z$  and  $Z_Z$ , the formulae reported in Appendix A are used.

To evaluate the same set of quantities for a given mode using Kirchhoff-Langevin’s theory – not necessarily the least-attenuated mode considered in Zwikker and Kosten’s theory – we have first to determine the Kirchhoff-Langevin mode wavenumber  $k_K$  by solving, via a Newton-Raphson scheme, the Kirchhoff-Langevin’s dispersion equation (46). In order to do this, we dispose of the explicit analytical expression of the function  $F(\omega, k)$ . Thus we also have an explicit analytical expression for its derivative with respect

to  $k$ . Once a wavenumber  $k = k_K$  solution to (46) is determined, the field patterns in the Kirchhoff-Langevin's source-free propagation problem can be explicitly written. The impedance  $Z_K$  can then be analytically computed by using the explicit expression (52). The density  $\rho_K$  and bulk modulus  $\chi_K^{-1}$  are obtained using the relations (53).

In order to evaluate anew a complete set of quantities, for a given mode, using the nonlocal Maxwellian theory, we now have first to determine the Maxwell mode wavenumber  $k_M$  by solving the Maxwell's dispersion equation (89). Here in the Newton scheme, to save time we compute numerically the  $k$  derivative of function  $F(\omega, k)$ . This is carefully done by evaluating the function  $F$  at two close values of the wavenumber  $k(1 \pm \epsilon_d/2 e^{i\theta})$ , with  $\epsilon_d$  a very small and adjustable parameter (e.g.  $\epsilon_d = 10^{-9}$ ), and then averaging over several random orientations  $\theta$  between 0 and  $2\pi$ . The results which are reported below are insensitive to the variation of  $\epsilon_d$  and to the angles  $\theta$ . Our stopping condition, in the Newton scheme, is that the relative error between two successive evaluations of  $k_K$  or  $k_M$  should be less than a very small fixed value  $\epsilon_s$  (e.g.  $\epsilon_s = 10^{-12}$ ). The Newton scheme is more stable and converges in fewer iterations for Kirchhoff-Langevin's dispersion equation (46) than for Maxwell's dispersion equation (89). Once a mode wavenumber  $k = k_M$  is determined, the field patterns in the Maxwell's source-driven action-response problems, can be explicitly written. The Maxwell's density  $\rho_M = \rho(\omega, k_M)$  and bulk modulus  $\chi_M^{-1} = \chi^{-1}(\omega, k_M)$  can then be computed in analytical way, using the explicit expressions (71) and (72), and (71) and (87). Then, the impedance  $Z_M$  is obtained using the general relation (5).

As the two wavenumbers  $k_K$  and  $k_M$  may not coincide exactly due to the finite precision of calculations and especially, the small inaccuracies in the Matlab computation of Bessel functions, all other quantities ' $K$ ' and ' $M$ ' may also not coincide exactly; nevertheless, if the theory is correct, we expect to see consistency between the different calculations, when no numerical problems arise – what will be shown to be the case, below. The precise matching of the values of densities and bulk moduli  $\rho_K$ ,  $\rho_M$ , and  $\chi_K^{-1}$ ,  $\chi_M^{-1}$ , will express the coincidences (91).

#### 4.1. Narrow tubes: $R = 10^{-4}m$ , $f = 100Hz$

For the least attenuated plane wave mode, the values obtained of the wavenumbers, impedances, densities and bulk moduli are given in Tables 2-5.

Table 2: Narrow tubes – least attenuated plane mode. Wavenumber.

$k_Z$	$7.01099685499484 + 6.61504658906530i$
$k_K$	$7.01099585405403 + 6.61504764250774i$
$k_M$	$7.01099585405408 + 6.61504764250779i$
$\Re(\Delta k/k), \Im(\Delta k/k)$	$< 10^{-14}$

Table 3: Narrow tubes – least attenuated plane mode. Impedance.

$Z_Z$	$1.122582910810147 \times 10^3 + 1.037174340699598 \times 10^3 i$
$Z_K$	$1.122582790953336 \times 10^3 + 1.037174463077600 \times 10^3 i$
$Z_M$	$1.122582790953347 \times 10^3 + 1.037174463077563 \times 10^3 i$
$\Re(\Delta Z/Z), \Im(\Delta Z/Z)$	$< 10^{-14}$

Table 4: Narrow tubes – least attenuated plane mode. Density.

$\rho_Z$	$1.60661929116825 + 23.39190009091327i$
$\rho_K$	$1.60661313808787 + 23.39190042443754i$
$\rho_M$	$1.60661313808843 + 23.39190042443734i$
$\Re(\Delta \rho/\rho), \Im(\Delta \rho/\rho)$	$< 10^{-14}$

Table 5: Narrow tubes – least attenuated plane mode. Bulk modulus.

$\chi_Z^{-1}$	$9.962016440824576 \times 10^4 - 1.043527914142315 \times 10^3 i$
$\chi_K^{-1}$	$9.962016409534546 \times 10^4 - 1.043531769001552 \times 10^3 i$
$\chi_M^{-1}$	$9.962016409534391 \times 10^4 - 1.043531769003666 \times 10^3 i$
$\Re(\Delta \chi^{-1}/\chi^{-1}), \Im(\Delta \chi^{-1}/\chi^{-1})$	$< 10^{-14}$

Table 6: Wide tubes – least attenuated plane mode. Wavenumber.

$k_Z$	$1.877218171102030 \times 10^2 + 3.047328259173055ei$
$k_K$	$1.877217761268940 \times 10^2 + 3.050105788888088i$
$k_M$	$1.877217761268940 \times 10^2 + 3.050105788888080i$
$\Re(\Delta k/k), \Im(\Delta k/k)$	$< 10^{-17}$

Table 7: Wide tubes – least attenuated plane mode. Impedance.

$Z_Z$	$4.122429513133025 \times 10^2 + 2.490000257701453i$
$Z_K$	$4.122428467151478 \times 10^2 + 2.490594992301288i$
$Z_M$	$4.122428467151476 \times 10^2 + 2.490594992301310i$
$\Re(\Delta Z/Z), \Im(\Delta Z/Z)$	$< 10^{-16}$

Zwikker and Kosten’s wavenumber differs from the exact wavenumber on the 6th decimal. Kirchhoff-Langevin’s and Maxwell’s values of the wavenumber are the same: the difference expresses on the 14th decimal, which is not meaningful numerically. Kirchhoff-Langevin’s value  $k_K$  is relatively insensitive to the starting value. Convergence to the given solution is obtained by starting from the lossless case solution  $k = \omega/c_a$ , the Zwikker-Kosten solution, or the value  $k = 6 + 2i$  taken in [26]. On the contrary, Maxwell’s value  $k_M$  is sensitive to the starting value. A meaningless unattenuated value  $k_M$  is found, when using as starting value the lossless-relating solution  $k = \omega/c_a$  which is not reported here.

The errors  $\Delta f/f$  indicate the relative differences computed between Maxwell’s values and Kirchhoff-Langevin’s reference values. Their small magnitudes show that the discrepancies are numerically insignificant. There is complete matching between Kirchhoff-Langevin’s and Maxwell’s wavenumbers, impedances, densities and bulk moduli, for the first least attenuated mode. For the first higher order mode, the imaginary part of the wavenumber is already in the order of  $5. \times 10^4 i$ . This leads to a large imaginary part in the complex arguments of the Bessel functions  $\varphi$  and  $\varphi_{1,2}$ . The resulting loss of precision prevents making precise checks with Matlab.

#### 4.2. Wide tubes: $R = 10^{-3}m$ , $f = 10kHz$

For the least attenuated plane wave mode, the values obtained of the wavenumbers, impedances, densities and bulk moduli are given in Tables 66–9.

Table 8: Wide tubes – least attenuated plane mode. Density.

$\rho_Z$	$1.23153152867920 + 0.02743301182273i$
$\rho_K$	$1.23153080833621 + 0.02745300551283i$
$\rho_M$	$1.23153080833621 + 0.02745300551283i$
$Re(\Delta\rho/\rho), (\Im\Delta\rho/\rho)$	$< 10^{-15}$

Table 9: Wide tubes – least attenuated plane mode. Bulk modulus.

$\chi_Z^{-1}$	$1.379578791782648 \times 10^5 - 1.406078512972857 \times 10^3i$
$\chi_K^{-1}$	$1.379578235612869 \times 10^5 - 1.407920077798555 \times 10^3i$
$\chi_M^{-1}$	$1.379578235612869 \times 10^5 - 1.407920077798562 \times 10^3i$
$\Re(\Delta\chi^{-1}/\chi^{-1}), \Im(\Delta\chi^{-1}/\chi^{-1})$	$< 10^{-15}$

The deviations are not significant owing to the calculation precision. This is again a clear validation of the nonlocal upscaling procedures. The cutoff frequency of the first higher-order axisymmetric mode is a little above  $10kHz$ . While this mode is still very significantly attenuated, its macroscopic characteristics  $k$ ,  $Z$ ,  $\rho$  and  $\chi^{-1}$  shown in Tables 10–13, are nevertheless once again obtained with a precision showing the exactness of the upscaling performed by the proposed nonlocal Maxwellian theory.

It may be noted that the negative real part of the bulk modulus is the type of behavior described for metamaterials [10], the negative real part of wavenumber also being present and associated with the phenomenon of negative group velocity. Spatial nonlocality necessarily plays an essential role in this metamaterial-type behavior. Without spatial nonlocality the local Zwikker and Kosten description is obtained and no higher-order mode exists.

#### 4.3. Very wide tubes: $R = 10^{-2}m$ , $f = 500kHz$

In this new regime of wave propagation, the least attenuated mode is no longer a plane mode. It tends to concentrate near the walls [25]. The

Table 10: Wide tubes – first higher-order axisymmetric mode. Wavenumber.

$k_K$	$-4.306909087685141 \times 10 + 3.869321168683033 \times 10^3i$
$k_M$	$-4.306909087685137 \times 10 + 3.869321168683033 \times 10^3i$
$\Re(\Delta k/k), \Im(\Delta k/k)$	$< 10^{-15}$

Table 11: Wide tubes – first higher-order axisymmetric mode. Impedance.

$Z_K$	$1.776687018193479 \times 10^9 + 3.970076285107318 \times 10^7 i$
$Z_M$	$1.776687018142111 \times 10^9 + 3.970076267272126 \times 10^7 i$
$\Re(\Delta Z/Z), \Im(\Delta Z/Z)$	$< 10^{-16}$

Table 12: Wide tubes – first higher-order axisymmetric mode. Density.

$\rho_K$	$3.662717005908636 \times 10^6 - 1.093850090017855 \times 10^8 i$
$\rho_M$	$3.662716993171710 \times 10^6 - 1.093850089982904 \times 10^8 i$
$\Re(\Delta \rho/\rho), \Im(\Delta \rho/\rho)$	$< 10^{-10}$

obtained values of the wavenumbers, impedances, densities and bulk moduli are indicated in Tables 14–17.

They show the exactness of the upscaling procedure in this regime, as well. Since there are now considerable differences between Zwikker and Kosten’s values and the exact ones, spatial nonlocality plays an essential role here already for the least attenuated mode. Recall that the local Zwikker and Kosten theory assimilates the field  $H$  with mean pressure  $\langle p \rangle$  and the mean pressure with the pressure itself, which is very nearly a cross-sectional constant. But now, this is no longer the case. The pressure is no longer a constant over the section, and thus, the local approach is largely in error. The proposed theory, with its fundamental Umov-Poynting definition (1) of the  $H$  field, properly takes into account the nonlocal behavior, and thus, all different physical regimes of wave propagation.

## 5. Conclusion

The perfect matching between the macroscopic translation of the long-known Kirchhoff-Langevin’s results and the results obtained on the basis of the nonlocal macroscopic theory recently proposed in [1], provides a clear validation of this theory, whose nonlocal-relating upscaling procedures are

Table 13: Wide tubes – first higher-order axisymmetric mode. Bulk modulus.

$\chi_K^{-1}$	$-3.235050540472611 \times 10^8 + 2.885427853699511 \times 10^{10} i$
$\chi_M^{-1}$	$-3.235050516110349 \times 10^8 + 2.885427853625859 \times 10^{10} i$
$\Re(\Delta \chi^{-1}/\chi^{-1}), \Im(\Delta \chi^{-1}/\chi^{-1})$	$< 10^{-10}$



Table 14: Very wide tubes – least attenuated mode. Wavenumber.

$k_Z$	$9.238319493530025 \times 10^3 + 2.120643432935189i$
$k_K$	$9.230724176891270 \times 10^3 + 6.352252888390387i$
$k_M$	$9.230724176891270 \times 10^3 + 6.352252888390393i$
$\Re(\Delta k/k), \Im(\Delta k/k)$	$< 10^{-18}$

Table 15: Very wide tubes – least attenuated mode. Impedance.

$Z_Z$	$4.099012309061263 \times 10^2 + 3.362191197362033 \times 10^{-2}i$
$Z_K$	$2.443313123663708 \times 10^2 - 1.548257724791978 \times 10^3i$
$Z_M$	$2.443313123674066 \times 10^2 - 1.548257724791642 \times 10^3i$
$\Re(\Delta Z/Z), \Im(\Delta Z/Z)$	$< 10^{-12}$

Table 16: Very wide tubes – least attenuated mode. Density.

$\rho_Z$	$1.20537538699515 + 0.00037556247686i$
$\rho_K$	$0.72103233188038 - 4.54864444048231i$
$\rho_M$	$0.72103233188340 - 4.54864444048128i$
$\Re(\Delta \rho/\rho), \Im(\Delta \rho/\rho)$	$< 10^{-12}$

Table 17: Very wide tubes – least attenuated mode. Bulk modulus.

$\chi_Z^{-1}$	$1.393914394285537 \times 10^5 - 2.056360890188126 \times 10i$
$\chi_K^{-1}$	$8.279327305799672 \times 10^4 - 5.269923491004282 \times 10^5i$
$\chi_M^{-1}$	$8.279327305835144 \times 10^4 - 5.269923491003189 \times 10^5i$
$\Re(\Delta \chi^{-1}/\chi^{-1}), \Im(\Delta \chi^{-1}/\chi^{-1})$	$< 10^{-12}$

not based on conventional homogenization.

The important concept, which leads us in [1] to these exact new homogenization upscaling procedures, is the ‘Umov-Poynting-Heaviside-Schoch’ physical concept of ‘acoustic part of energy current density’. Using an analogy with electromagnetics, we had to identify exactly this acoustic part of the energy current for wave propagation in the fluid, with the quantity  $\mathbf{s} = p\mathbf{v}$ , where  $p$  is the thermodynamic pressure. This served in turn as a basis to define a macroscopic pressure field in a fluid-saturated material, through the macroscopic relation-definition  $\langle p\mathbf{v} \rangle = H\langle \mathbf{v} \rangle$ . This thermodynamic definition appeared to be the key to make use of the solutions of two simple action-response problems in the appropriate way, leading finally to the independent computation of the two nonlocal acoustical susceptibilities  $\rho$  and  $\chi$ . In forthcoming papers it will be shown that the proposed nonlocal Maxwellian theory, providing exact homogenization procedures, is valid also in the case of nontrivial geometries, thus including metamaterials. Indeed, while in the present case of cylindrical ducts, the inclusion of spatial dispersion effects leads only to relatively marginal extensions of the local treatment, it will be seen that in the case of nontrivial geometries, the inclusion of spatial dispersion plays central role to describe the new metamaterial behaviors.

## Appendix A. Zwikker and Kosten’s simplified local theory

The complete equations of the Navier-Stokes-Fourier linear wave propagation problem in the circular tube are

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \eta \nabla^2 \mathbf{v} + \left(\zeta + \frac{\eta}{3}\right) \nabla(\nabla \cdot \mathbf{v}) \quad (\text{A.1a})$$

$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{v} = 0 \quad (\text{A.1b})$$

$$\gamma \chi_0 p = b + \beta_0 \tau \quad (\text{A.1c})$$

$$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \nabla^2 \tau \quad (\text{A.1d})$$

for  $r < R$ , and

$$\mathbf{v} = 0 \quad (\text{A.1e})$$

$$\tau = 0 \quad (\text{A.1f})$$

at  $r = R$ . A simplified Zwikker and Kosten’s solution to these equations can be found in a long-wavelength limit. The aim of Zwikker and Kosten’s

theory is to find  $\rho_Z(\omega)$  and  $\chi_Z(\omega)$ , such that in harmonic regime

$$\rho_Z(\omega) \frac{\partial V}{\partial t} = - \frac{\partial P}{\partial x} \quad (\text{A.2a})$$

$$\chi_Z(\omega) \frac{\partial P}{\partial t} = - \frac{\partial V}{\partial x} \quad (\text{A.2b})$$

where  $V$  and  $P$  are the cross-section averages of velocity and pressure.

Knowing  $\rho_Z(\omega)$  and  $\chi_Z(\omega)$  would entirely specifies the tube propagation characteristics. It would give the propagation constant

$$k_Z = \omega \sqrt{\rho_Z(\omega) \chi_Z(\omega)} \quad (\text{A.3})$$

and the characteristic impedance of the progressive plane wave

$$Z_Z = \sqrt{\rho_Z(\omega) \chi_Z^{-1}(\omega)} \quad (\text{A.4})$$

where the index  $Z$  stands for ‘Zwikker and Kosten’.

We note first that, in addition to Eqs.(A.2), it must also be assumed a relation between the cross-section average of excess temperature and the pressure. This relation, between  $\langle \tau \rangle$  and  $\langle p \rangle$ , would play the role of (A.2a) between  $V$  and  $P$ . Using the similarity between (A.1a) and (A.1d) we write it

$$\rho'_Z(\omega) \frac{\partial \langle \tau \rangle}{\partial t} = \beta_0 T_0 \frac{\partial P}{\partial t} \quad (\text{A.5})$$

We note next, that, combining (A.1b) and (A.1c) and averaging over a cross-section, a general relation between  $\partial P / \partial t$ ,  $\partial \langle \tau \rangle / \partial t$ , and  $\partial V / \partial x$  can be obtained

$$\gamma \chi_0 \frac{\partial P}{\partial t} = - \frac{\partial V}{\partial x} + \beta_0 \frac{\partial \langle \tau \rangle}{\partial t} \quad (\text{A.6})$$

Thus, putting in this equation the relation (A.5) and using the general thermodynamic identity (11), it is easy to verify that, as soon as the functions  $\chi_Z$  and  $\rho'_Z$  exist, they must be such that

$$\chi_Z(\omega) = \chi_0 \left[ \gamma - (\gamma - 1) \frac{\rho_0 c_p}{\rho'_Z(\omega)} \right] \quad (\text{A.7})$$

Now we observe that the equations (A.2) have exactly the same form as the Maxwellian equations (55-58) with, however, the crucial difference that they are written excluding spatial dispersion:

$$\frac{\partial B}{\partial t} + \frac{\partial V}{\partial x} = 0 \quad (\text{A.8a})$$

$$\frac{\partial D}{\partial t} = -\frac{\partial H}{\partial x} \quad (\text{A.8b})$$

$$D(t, x) = \int_{-\infty}^t dt' \rho_Z(t - t') V(t', x) \quad (\text{A.8c})$$

$$H(t, x) = \int_{-\infty}^t dt' \chi_Z^{-1}(t - t') B(t', x), \quad H \equiv P \quad (\text{A.8d})$$

(without confusion, the previous  $\rho_Z(\omega)$  and  $\chi_Z^{-1}(\omega)$  are now the complex Fourier amplitudes of the present real kernel functions  $\rho_Z(t)$  and  $\chi_Z^{-1}(t)$ ).

We explain later on in conclusion, why, in this context, the field  $H$  coincides with the mean pressure. For the moment, we note that Eqs.(A.2-A.8), local in space, are incompatible with Eqs.(A.1). Indeed, as we have seen, Eqs.(A.1) consistently lead to Maxwellian acoustic equations which are – contrary to the above – nonlocal in space as well as in time.

Thus, what Zwikker and Kosten's theory is doing to arrive at equations having the local form (A.2), is not to solve the complete Eqs.(A.1), expressed in the complete action-response problems (59-61) and (74-76), but, by introducing various simplifications and idealizations, to solve only some truncated simplified versions of these equations.

The simplifications to be made will be justified to capture the characteristics of the plane-wave component fields, in the limit where the wavelengths are large compared to the duct transverse dimensions. It is only in this limit that the functions  $\rho_Z$  and  $\chi_Z^{-1}$  of Zwikker and Kosten's theory allow to describe with high precision the propagation of the least-attenuated, plane wave mode. The simplified versions of the equations, and action-response problems determining  $\rho_Z$  and  $\chi_Z^{-1}$ , can be directly guessed on the basis that they have to neglect spatial dispersion.

To compute the density we consider that, because the wavelengths are very large compared to the duct transverse dimensions, the spatial variation of the pressure gradient term in Eq.(A.1a) can be neglected for the purpose of determining the fluid velocity pattern accross a section. We thus look at the response of the fluid subjected to the action of an external driving force-per-unit-volume  $\mathbf{f}$ , which is, while harmonic in time, a pure spatial constant

(in any real physical wave propagation problem, the temporal variation of the pressure gradient would mean that this gradient is also, to some extent, spatially variable). In this circumstance we have not only to replace the driving  $\mathbf{f}$  (61) by a spatial constant  $\mathbf{f}_0 e^{-i\omega t}$ , but also to drop the response pressure gradient term in (59). Indeed, in the cylindrical duct geometry and with constant driving force, no response pressure is generated and no compression-dilatation of the fluid occurs. Thus we also have to drop the two other fields  $b$  and  $\tau$ . The resulting fictitious problem reads:

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \nabla^2 \mathbf{v} + \frac{1}{\rho_0} \mathbf{f} \quad (\text{A.9a})$$

$$\nabla \cdot \mathbf{v} = 0 \quad (\text{A.9b})$$

for  $r < R$ , with

$$\mathbf{v} = 0 \quad (\text{A.10a})$$

at  $r = R$ , and driving force given by

$$\mathbf{f} = -\nabla \mathcal{P} = \mathbf{e}_x f_0 e^{-i\omega t} \quad (\text{A.11})$$

The corresponding density  $\rho_Z(\omega)$ , such that

$$-i\omega \rho_Z(\omega) \langle \mathbf{v} \rangle = \mathbf{f} \quad (\text{A.12})$$

or, to compare with (72)

$$\rho_Z(\omega) = \frac{f_0}{-i\omega \langle u_0 \rangle} \quad (\text{A.13})$$

is the wanted Zwikker and Kosten density. It is found by elementary calculations to be given by

$$\frac{1}{\rho_Z(\omega)} = \frac{1}{\rho_0} [1 - \xi_Z(\omega)] \quad (\text{A.14})$$

where  $\xi_Z(\omega)$  is the following Zwikker and Kosten's relaxation function

$$\xi_Z(\omega) = \frac{2J_1\left(\left(\frac{i\omega}{\nu} R^2\right)^{1/2}\right)}{\left(\frac{i\omega}{\nu} R^2\right)^{1/2} J_0\left(\left(\frac{i\omega}{\nu} R^2\right)^{1/2}\right)} \quad (\text{A.15})$$

In a similar manner, to compute the compressibility, we now consider that in the long-wavelength limit, the pressure driving term in Eq.(A.1d) may be

viewed as a pure *spatial constant* for the purpose of determining the excess temperature response pattern, across a section. Thus we consider that this driving term acts as a spatial constant  $\beta_0 T_0 \frac{\partial p}{\partial t} \equiv \dot{Q}_0 e^{-i\omega t}$  with  $\dot{Q}_0$  a constant. The resulting fictitious heat conduction problem reads

$$\frac{\partial \tau}{\partial t} = \frac{\kappa}{\rho_0 c_p} \nabla^2 \tau + \frac{1}{\rho_0 c_p} \dot{Q} \quad (\text{A.16})$$

for  $r < R$ , and

$$\tau = 0 \quad (\text{A.17})$$

at  $r = R$ , with

$$\dot{Q} = \beta_0 T_0 \frac{\partial \mathcal{P}}{\partial t} = \dot{Q}_0 e^{-i\omega t} \quad (\text{A.18})$$

The corresponding function  $\rho'_Z(\omega)$  such that

$$-i\omega \rho'(\omega) \langle \tau \rangle = \dot{Q} \quad (\text{A.19})$$

is found by elementary calculations as

$$\frac{1}{\rho'_Z(\omega)} = \frac{1}{\rho_0 c_p} [1 - \xi_Z(\omega \text{Pr})] \quad (\text{A.20})$$

where  $\xi_Z(\omega)$  is the previous relaxation function (A.15) and  $\text{Pr} = \eta c_p / \kappa$  is the Prandtl number. Finally, as  $\rho'_Z$  is related to the compressibility  $\chi_Z$  by the relation (A.7), the end Zwikker and Kosten's result for  $\chi_Z(\omega)$  is

$$\chi_Z(\omega) = \chi_0 [1 + (\gamma - 1) \xi_Z(\omega \text{Pr})] \quad (\text{A.21})$$

In conclusion we note that, within the simplifications made in Zwikker and Kosten's local theory, as the excess pressure gradient term in Eq.(A.1a) is represented by the constant term  $\mathbf{f}$ , and as the excess pressure time derivative term in Eq.(A.1a) is represented by the constant term  $\dot{Q}$ , the excess pressure is replaced by a constant over the cross-section. Thus, when writing the definition  $\langle up \rangle = H \langle u \rangle$  of the  $H$  field, the pressure can be extracted from the averaging symbol and it turns out that  $H = p$ , which also yields  $H = P$  since the pressure is constant. We now see why, in the framework of Zwikker and Kosten's local theory, no distinction is to be made between the mean pressure  $P$  and the effective macroscopic pressure  $H$ . This remarks extends more generally to the case of the usual local description [3, Appendix A], which generalizes Zwikker and Kosten's local solution to the case of arbitrary geometries.

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