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Jeremie Brioussel

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FOLNER SETS OF ALTERNATE DIRECTED GROUPS

JÉRÉMIE BRIEUSSEL

KYOTO UNIVERSITY
JAPAN

ABSTRACT. An explicit family of Folner sets is constructed for some directed groups acting on a rooted tree of sublogarithmic valency by alternate permutations. In the case of bounded valency, these groups were known to be amenable by probabilistic methods. The present construction provides a new and independent proof of amenability, using neither random walks, nor word length.

1. INTRODUCTION

By a criterion of Folner [Fol], amenable groups are those that admit finite subsets with arbitrary small boundaries. A sequence of such subsets, called a Folner sequence, is easily described for abelian groups, and well-understood for solvable groups ([PSC], [Ers]). Many non-solvable amenable groups are directed groups acting on rooted trees. This family of groups gathers many examples with "exotic" properties, such as infinite torsion groups constructed by Aleshin [Ale], groups of intermediate growth by Grigorchuk [Gri] or groups with non-uniform exponential growth by Wilson [Wil1].

Their amenability in the case of bounded valency was shown in [Bri1] by use of Kesten's probabilistic criterion [Kes]. The strategy, introduced by Bartholdi and Virag in [BV], is to show that a self-similar random walk on a Cayley graph diffuses slowly, in the sense that its return probability does not decay exponentially, or that its entropy is sublinear ([KV]). The same method permits to show that automata groups are amenable when their activity is bounded [BKN] or linear [AAV]. Though it ensures their existence, such a probabilistic proof does not exhibit Folner sets.

For the groups of [Ale] and [Gri], subexponential growth easily implies the existence of a subsequence of the family of balls (for a word length) which is a Folner sequence, but it is not known if the whole sequence of balls is Folner and the subsequence (even though it has density 1) is not explicit. Even for groups of polynomial growth, it is not elementary to show that balls form a Folner sequence, a result due to Pansu [Pan], using technics from Gromov [Gro].

The object of the present article is to exhibit explicit Folner sets for some groups in a class denoted \mathcal{DP} , containing in particular directed groups acting on a rooted tree by alternate permutations. A group Γ in this class \mathcal{DP} is defined (see section

4) by two subgroups A finite and H finitely generated, together with an action on a rooted tree with valency sequence $(d_k)_{k \in \mathbb{N}}$. The main result is:

Theorem 1.1. *Let Γ belong to the class \mathcal{DP} with H amenable and $\frac{d_k}{\log k} \rightarrow 0$, then the group Γ is amenable.*

As a corollary, this provides a new proof, using neither random walks nor word length, that directed groups acting on a rooted tree of bounded valency are amenable ([Bri1]). It also provides many new examples of amenable directed groups acting on a tree of unbounded sublogarithmic valency. Moreover it also permits to reprove amenability of automata groups with bounded activity by methods different from [BKN].

The article is structured as follows. Rooted trees and their automorphism groups are described in section 2. Section 3 is devoted to the construction of explicit Folner sets for the archetypal example of the alternate mother group G_d acting on a regular rooted tree of valency $d \geq 5$. This example, treated first for simplicity of notations, is generalized to the class \mathcal{DP} in section 4. Finally, section 5 is devoted to the construction of groups in the class \mathcal{DP} , including the saturated alternate directed groups, and some groups acting on trees with unbounded valency.

2. ROOTED TREES AND THEIR GROUPS OF AUTOMORPHISMS

Let S_d denote the group of permutations of the set $\{1, \dots, d\}$ with d elements, and $\mathcal{A}_d = \mathcal{A}_{\{1, \dots, d\}}$ denote the subgroup of alternate permutations.

Given a sequence $\bar{d} = (d_k)_{k \geq 0}$ of integers ≥ 2 , the spherically homogeneous rooted tree $T_{\bar{d}}$ is the graph with vertex set $\{t_0 t_1 \dots t_k \mid t_i \in \{1, \dots, d_i\}, k \geq -1\}$, including the empty sequence \emptyset , called the root, corresponding to $k = -1$, and edge set $\{(t_0 \dots t_k, t_0 \dots t_k t_{k+1})\}$. The vertex set restricted to a fixed k is called the $k + 1$ st level of the tree. It is the direct product $\{1, \dots, d_0\} \times \dots \times \{1, \dots, d_k\}$. When the sequence \bar{d} is constant equal to d , the tree is called d -regular, denoted T_d .

The group of automorphisms $\text{Aut}(T_{\bar{d}})$ of the rooted tree $T_{\bar{d}}$ is the group of graph automorphisms that fix the root \emptyset . It satisfies a canonical isomorphism:

$$(1) \quad \text{Aut}(T_{\bar{d}}) \simeq \text{Aut}(T_{\sigma \bar{d}}) \wr S_{d_0},$$

where $\sigma \bar{d} = (d_k)_{k \geq 1}$ is the shifted sequence obtained by deleting the first entry, and $G \wr S_d = (G \times \dots \times G) \rtimes S_d$ is the semi-direct product where S_d acts by permuting factors, called wreath product. Write $g = (g_1, \dots, g_{d_0})\sigma = (g_{t_0})\sigma$ the identification by isomorphism (1). The product rule is $gg' = (g_1 g'_{\sigma(1)}, \dots, g_{d_0} g'_{\sigma(d_0)})\sigma\sigma'$.

By iterating the wreath product isomorphism (1), a family of isomorphisms is obtained:

$$\text{Aut}(T_{\bar{d}}) \simeq \text{Aut}(T_{\sigma^k \bar{d}}) \wr S_{d_{k-1}} \wr \dots \wr S_{d_0}.$$

Identifications are denoted $g = (g_{t_0 \dots t_k})(\sigma_{t_0 \dots t_{k-1}}) \dots (\sigma_{t_0})\sigma$, where $(\sigma_{t_0 \dots t_j})$ is a sequence of permutations in S_{d_j} indexed by the $j + 1$ st level of the tree and $(g_{t_0 \dots t_k})$ is a sequence of automorphisms of $T_{\sigma^k \bar{d}}$ indexed by level $k + 1$. The automorphism g is determined by the whole sequence of permutations $(\sigma_v)_{v \in T_{\bar{d}}}$, called its portrait.

The automorphism g is said to be alternate if all the permutations σ_v of its portrait are alternate permutations. Denote $Aut^{alt}(T_{\bar{d}})$ the group of alternate automorphisms of $T_{\bar{d}}$. It also satisfies isomorphisms:

$$Aut^{alt}(T_{\bar{d}}) \simeq Aut^{alt}(T_{\sigma^k \bar{d}}) \wr \mathcal{A}_{d_{k-1}} \wr \cdots \wr \mathcal{A}_{d_0}.$$

The neutral element of a group G is denoted e_G or e .

3. FOLNER SETS OF THE ALTERNATE MOTHER GROUP

3.1. The alternate mother group. Define alternate automorphisms of the d -regular rooted tree T_d by use of the wreath product isomorphism $Aut^{alt}(T_d) \simeq Aut^{alt}(T_d) \wr \mathcal{A}_d$.

- (1) Given σ in \mathcal{A}_d , denote $A = \{a(\sigma) | \sigma \in \mathcal{A}_d\} \simeq \mathcal{A}_d$ with:

$$a(\sigma) = (e, \dots, e)\sigma.$$

Elements of A are called rooted.

- (2) Given a_2, \dots, a_d in $A = \mathcal{A}_d$ and ρ in $Fix_{\mathcal{A}_d}(1) = \mathcal{A}_{\{2, \dots, d\}} = \mathcal{A}_{d-1}$, set:

$$b(a_2, \dots, a_d, \rho) = (b(a_2, \dots, a_d, \rho), a_2, \dots, a_d)\rho.$$

Denote $B = \{b(a_2, \dots, a_d, \rho) | a_2, \dots, a_d \in \mathcal{A}_d, \rho \in Fix_{\mathcal{A}_d}(1)\}$. Elements of B are called directed. The set B forms a finite subgroup of $Aut(T_d)$. Indeed, the following is an isomorphism:

$$(2) \quad \begin{array}{l} B \rightarrow (\mathcal{A}_d \times \cdots \times \mathcal{A}_d) \rtimes \mathcal{A}_{\{2, \dots, d\}} \\ b(a_2, \dots, a_d, \rho) \mapsto (a_2, \dots, a_d)\rho \end{array} .$$

Define the group generated by the sets A, B :

$$G_d = \langle A, B \rangle < Aut(T_d).$$

By construction, the group G_d is an automata group. It is essentially the mother group of degree 0 (see [BKN], [AAV]), but the permutations involved are alternate.

Since \mathcal{A}_d is simple hence perfect for $d \geq 5$, the group G_d satisfies the:

Proposition 3.1. *If $d \geq 5$, the group G_d is isomorphic to its wreath product with \mathcal{A}_d :*

$$G_d \simeq G_d \wr \mathcal{A}_d = (G_d \times \cdots \times G_d) \rtimes \mathcal{A}_d.$$

The proposition follows from the:

Fact 3.2. *Let $d \geq 5$, then for any generator $a = a(\sigma) \in A$ and $b = b(a_2, \dots, a_d, \rho) \in B$, the elements (a, e, \dots, e) and (b, e, \dots, e) belong to G_d .*

Recall the conjugacy notation $g^a = aga^{-1}$, and observe that for $g = (g_1, \dots, g_d)\sigma$ and a in A , one has $g^a = (g_{a(1)}, \dots, g_{a(d)})\sigma^a$.

Proof of fact 3.2. Take τ in \mathcal{A}_d such that $\tau(1) = 1$ and $\tau^{-1}(2) = 3$ and observe the commutator relations:

$$\begin{aligned} b &= b(\alpha_2, e_G, \dots, e_G, e_A) = (b, \alpha_2, e_G, e_G, \dots, e_G)e_A, \\ b'^\tau &= b(\alpha'_2, e_G, \dots, e_G, e_A)^\tau = (b', 1, \alpha'_2, e_G, \dots, e_G)e_A, \\ [b, b'^\tau] &= ([b, b'], e_G, e_G, e_G, \dots, e_G). \end{aligned}$$

As $[b(\alpha_2, e_G, \dots, e_G, e_A), b(\alpha'_2, e_G, \dots, e_G, e_A)] = b([\alpha_2, \alpha'_2], e_G, \dots, e_G, e_A)$ and as the group $A = \mathcal{A}_d$ is perfect (because it is simple), any element a_2 in $A \simeq \mathcal{A}_d$ is a product of commutators. This shows that G_d contains (b_2, e_G, \dots, e_G) for any $b_2 = b(a_2, e_G, \dots, e_G, e_A)$ with a_2 in \mathcal{A}_d . Moreover for any $b_\emptyset = b(e_G, \dots, e_G, \rho)$ with ρ in $\text{Fix}_A(1) \simeq \mathcal{A}_{d-1}$, the group G_d contains $b_\emptyset a(\rho^{-1}) = (b_\emptyset, e_G, \dots, e_G)$.

Now the elements $b_2 = b(a_2, e_G, \dots, e_G, e_A)$ and $b_\emptyset = b(e_G, \dots, e_G, \rho)$ generate B by isomorphism (2), because ρ in $\mathcal{A}_{\{2, \dots, d\}}$ and (a_2, e_A, \dots, e_A) for a_2 in \mathcal{A}_d generate the finite group $(\mathcal{A}_d \times \dots \times \mathcal{A}_d) \rtimes \mathcal{A}_{\{2, \dots, d\}}$. Thus G_d contains (b, e_G, \dots, e_G) for any b in B .

Finally given a_2 in A , for $b_2 = b(a_2, e_G, \dots, e_G, e_A) = (b_2, a_2, e_G, \dots, e_G)$, the element $(b_2^{-1}, e_G, \dots, e_G)$ belongs to G_d by the above. So do $(b_2^{-1}, e_G, \dots, e_G)b_2 = (e_G, a_2, e_G, \dots, e_G)$ and $(e_G, a_2, e_G, \dots, e_G)^\tau = (a_2, e_G, \dots, e_G)$ for τ in $A = \mathcal{A}_d$ such that $\tau^{-1}(2) = 1$. \square

Proof of proposition 3.1. By definition, G_d admits an embedding into the wreath product $G_d \hookrightarrow G_d \wr \mathcal{A}_d$. The key point is that this embedding is surjective. Clearly $A \simeq \mathcal{A}_d$ is the set of rooted automorphisms. Moreover, fact 3.2 shows that $G_d \times \{1\} \times \dots \times \{1\}$ is in G_d . As \mathcal{A}_d acts transitively on $\{1, \dots, d\}$, conjugation shows that $\{1\} \times \dots \times G_d \times \dots \times \{1\}$ also belongs to G_d for any position of the non-trivial factor. Then $G_d \times \dots \times G_d$ belongs to G_d by product. This proves the wreath product isomorphism. \square

3.2. Definition of Folner sets. For a group Γ with finite generating set S , the boundary of a subset $L \subset \Gamma$ is defined as:

$$\partial L = \{\gamma \in L \mid \exists s \in S, \gamma s \notin L\}.$$

The interior of L is the set $\text{Int}(L) = L \setminus \partial L$.

A sequence L_k of subsets of Γ is a Folner sequence if $\frac{|\partial L_k|}{|L_k|} \rightarrow 0$. By [Fol], a finitely generated group Γ is amenable if and only if it admits a Folner sequence for some (equivalently for any) finite generating set S .

Let us define a sequence of subsets of G_d as follows:

$$L_0 = \{g \in G_d \mid \exists \beta \in B, \alpha_2, \dots, \alpha_d, \sigma \in A, g = (\beta, \alpha_2, \dots, \alpha_d)\sigma\}.$$

By induction on k , define:

$$L_{k+1} = \{g = (g_1, \dots, g_d)\sigma \mid \forall t, g_t \in L_k \text{ and } \exists T, g_T \in \text{Int}(L_k)\}.$$

By proposition 3.1, the sets L_k are included in G_d for $d \geq 5$, and not just in the automorphism group $\text{Aut}(T_d)$.

Theorem 3.3. *For $d \geq 5$, the sets L_k form a Folner sequence for G_d . In particular, the group G_d is amenable.*

The group G_d was known to be amenable by [Bri1] (use of Kesten criterion on return probability) or [BKN] (triviality of the Poisson boundary). However, these proofs, based on contraction in the wreath product of word length for some random walks, did not provide explicit Folner sets. The following proof uses neither random walks, nor word length.

3.3. Proof of theorem 3.3. Observe that for any a in A and $g = (\beta, \alpha_2, \dots, \alpha_d)\sigma$ in L_0 , the element $ga = (\beta, \alpha_2, \dots, \alpha_d)\sigma a$ still belongs to L_0 . Moreover, for any $b = b(a_2, \dots, a_d, \rho) = (b, a_2, \dots, a_d)\rho$ in B , one has:

$$gb = \begin{cases} (\beta b, \alpha_2 a_{\sigma(2)}, \dots, \alpha_d a_{\sigma(d)})\sigma\rho & \text{if } \sigma^{-1}(1) = 1, \\ (\beta a_{\sigma(1)}, \alpha_2 a_{\sigma(2)}, \dots, \alpha_{\sigma^{-1}(1)} b, \dots, \alpha_d a_{\sigma(d)})\sigma\rho & \text{if } \sigma^{-1}(1) \neq 1. \end{cases}$$

As the sets A and B are finite groups, this shows equivalence of (1), (2) and (3) in the:

Fact 3.4. *The following are equivalent:*

- (1) g belongs to $\text{Int}(L_0)$,
- (2) $gb \in L_0$ for all $b \in B$,
- (3) $\sigma^{-1}(1) = 1$,
- (4) $gb \in \text{Int}(L_0)$ for all $b \in B$.

In particular, $\frac{|\text{Int}(L_0)|}{|L_0|} = \frac{1}{d}$, hence $\delta_0 = \frac{|\partial L_0|}{|L_0|} = 1 - \frac{1}{d}$.

Proof. Point (4) is equivalent to (3) due to the fixed point assumption $\rho(1) = 1$ in the definition of B , which guarantees that $(\sigma\rho)^{-1}(1) = (\rho^{-1}\sigma^{-1})(1) = \sigma^{-1}(\rho^{-1}(1)) = 1$ when $\sigma^{-1}(1) = 1$.

The evaluation of δ_0 is done by counting $|L_0| = |B||A|^d$ as g is described by $\beta, \alpha_2, \dots, \alpha_d, \sigma$, and condition $\sigma^{-1}(1) = 1$ occurs with probability $\frac{1}{d}$. \square

Lemma 3.5. *The following are equivalent:*

- (1) g belongs to $\text{Int}(L_k)$,
- (2) $gb \in L_k$ for all $b \in B$,
- (3) $\sigma^{-1}(1) \in I(g) = \{T | g_T \in \text{Int}(L_{k-1})\}$,
- (4) $gb \in \text{Int}(L_k)$ for all $b \in B$.

Proof of lemma 3.5. The case $k = 0$ is treated by fact 3.4 with convention that $I(g) = \{1\}$ if $g \in L_0$. Assume by induction that the result is true for $k - 1$, and prove it for k .

Again $ga = (g_1, \dots, g_d)\sigma a$ belongs to L_k for any value of a in A , g in L_k . Moreover:

$$gb = (g_1 a_{\sigma(1)}, \dots, g_{\sigma^{-1}(1)} b, \dots, g_d a_{\sigma(d)})\sigma\rho.$$

Suppose (3) that $g_{\sigma^{-1}(1)} \in \text{Int}(L_{k-1})$, then as (1) implies (4) for $k - 1$, the element $g_{\sigma^{-1}(1)} b$ belongs to $\text{Int}(L_{k-1})$ for any b in B , so that gb belongs to L_k for any b in

B , proving (2). Then (1) follows because ga also belongs to L_k for a in A , hence g is an interior point of L_k .

Suppose (3) does not hold, so $g_{\sigma^{-1}(1)} \in \partial L_{k-1}$. By equivalence of (1) and (2) for $k-1$, there exists b in B such that $g_{\sigma^{-1}(1)}b \notin L_{k-1}$, so that gb is not in L_k , disclaiming (1) and (2) for g . This proves equivalence of (1), (2) and (3) for k .

Now gb belongs to $\text{Int}(L_k)$ if and only if $(\sigma\rho)^{-1}(1) \in I(g)$ by equivalence of (1) and (3). But $(\sigma\rho)^{-1}(1) = \sigma^{-1}(\rho^{-1}(1)) = \sigma^{-1}(1)$ because $\rho(1) = 1$. So (3) implies (4). Obviously, (4) implies (2), closing step k of induction. \square

There remains to evaluate the sizes of the interior and boundary of L_k . Set:

$$\delta_k = \frac{|\partial L_k|}{|L_k|}, \quad 1 - \delta_k = \frac{|\text{Int}(L_k)|}{|L_k|}.$$

Lemma 3.6. *The sequence (δ_k) satisfies:*

$$1 - \delta_{k+1} = \frac{1 - \delta_k}{1 - \delta_k^d}.$$

Proof of lemma 3.6. Given a subset $I \subset \{1, \dots, d\}$, denote:

$$J_I = \{g = (g_1, \dots, g_d) \sigma \mid \forall T \in I, g_T \in \text{Int}(L_k) \text{ and } \forall t \notin I, g_t \in \partial L_k\}.$$

By definition, L_{k+1} is the disjoint union $L_{k+1} = \sqcup_{|I| \geq 1} J_I$.

For $i = |I|$, the size of J_I and its intersection with $\text{Int}(L_{k+1})$ are evaluated as:

$$\begin{aligned} |J_I| &= |\mathcal{A}_d| |\text{Int}(L_k)|^i |\partial L_k|^{d-i} = |\mathcal{A}_d| |L_k|^d (1 - \delta_k)^i \delta_k^{d-i}, \\ |J_I \cap \text{Int}(L_{k+1})| &= \frac{|I|}{d} |\mathcal{A}_d| |\text{Int}(L_k)|^i |\partial L_k|^{d-i} = \frac{i}{d} |J_I|, \end{aligned}$$

where the factor $\frac{i}{d}$ comes from (3) of lemma 3.5. Denote C_d^i the number of subsets of size i in $\{1, \dots, d\}$, and use the mean of binomial distribution to get:

$$\begin{aligned} |\text{Int}(L_{k+1})| &= \sum_{i=1}^d C_d^i (1 - \delta_k)^i \delta_k^{d-i} \frac{i}{d} |L_k|^d |\mathcal{A}_d| = (1 - \delta_k) |L_k|^d |\mathcal{A}_d|, \\ |L_{k+1}| &= \sum_{i=1}^d C_d^i (1 - \delta_k)^i \delta_k^{d-i} |L_k|^d |\mathcal{A}_d| = (1 - \delta_k^d) |L_k|^d |\mathcal{A}_d|. \end{aligned}$$

This shows that:

$$1 - \delta_{k+1} = \frac{|\text{Int}(L_{k+1})|}{|L_{k+1}|} = \frac{1 - \delta_k}{1 - \delta_k^d}.$$

\square

Proof of theorem 3.3. As $\delta_k > 0$, lemma 3.6 implies $1 - \delta_{k+1} > 1 - \delta_k$, so the sequence (δ_k) is decreasing, tending to a limit δ satisfying $1 - \delta = \frac{1 - \delta}{1 - \delta^d}$, hence δ is 0 (or 1, ruled out by $\delta_0 < 1$). \square

More precisely, lemma 3.6 implies that for any $\eta < \frac{1}{d-1}$, one has $\delta_k = O(k^{-\eta})$, as shown below in lemma 4.12. On the other hand:

$$|L_k| = |B|^{d^k} |A|^{(d-1)d^k + (d^k + \dots + d + 1)} \geq 2^{2^k}.$$

Remark 3.7. Lemma 3.5 provides a complete combinatorial description of L_k . An element g of G_d has the form $g = (g_{t_0 \dots t_k})(\sigma_{t_0 \dots t_{k-1}}) \dots (\sigma_{t_0})\sigma$ in the k th iteration of the wreath product. Such an element g belongs to L_k if and only if it satisfies the three following conditions:

- (1) $\forall t_0 \dots t_{k-1}$, the element $g_{t_0 \dots t_{k-1}}$ is in B and $g_{t_0 \dots t_{k-1}2}, \dots, g_{t_0 \dots t_{k-1}d}$ are in A ,
- (2) $\forall t_0 \dots t_{k-2}$, the set $I(t_0 \dots t_{k-2}) = \{T_{k-1} | \sigma_{t_0 \dots t_{k-2}T_{k-1}}^{-1}(1) = 1\}$ is non-empty.
- (3) $\forall 3 \leq l \leq k+1, \forall t_0 \dots t_{k-l}$, the set

$$I(t_0 \dots t_{k-l}) = \{T_{k-l+1} | \sigma_{t_1 \dots t_{k-l}T_{k-l+1}}^{-1}(1) \in I(t_1 \dots t_{k-l}T_{k-l+1})\},$$

defined by induction on l , is non-empty (for $l = k+1$, consider $I(\emptyset)$ where \emptyset is the root vertex of T_d).

The element g belongs to $\text{Int}(L_k)$ if and only if it satisfies (1), (2), (3) and moreover:

- (4) $\sigma^{-1}(1) \in I(\emptyset) = \{T | \sigma_T \in I(T)\}$.

Note that condition (2) is a specific case of condition (3) where $I(t_0 \dots t_{k-1}) = \{1\}$ for all $t_0 \dots t_k$. As an interpretation, say a vertex $v = t_0 \dots t_l$ with $l \leq k-1$ is open if $\sigma_v^{-1}(1) \in I(v)$. Conditions (1), (2), (3) ensure that g belongs to L_k if and only if each vertex v has at least one neighbour of next level vT which is open. Condition (4) ensures that g is in the interior $\text{Int}(L_k)$ if and only if the root itself is open.

4. GENERALIZATION

4.1. **The class \mathcal{DP} .** Theorem 3.3 can be generalized to a wider setting.

Definition 4.1. A group Γ_0 belongs to the class \mathcal{DP} if and only if it satisfies the two following conditions:

- (1) the group Γ_0 contains two subgroups A_0 and H_0 such that:
 - (a) the set $A_0 \cup H_0$ generates the group Γ_0 ,
 - (b) the group A_0 is finite, acting transitively on a finite set $\{1, \dots, d_0\}$ of size $d_0 \geq 2$,
 - (c) the group H_0 is finitely generated (denote B_0 some finite generating set of H_0 , so $\Gamma_0 = \langle A_0 \cup B_0 \rangle$),
- (2) there is a group Γ_1 in the class \mathcal{DP} with an isomorphism:

$$\varphi_0 : \Gamma_0 \longrightarrow \Gamma_1 \wr A_0 = (\Gamma_1 \times \dots \times \Gamma_1) \rtimes A_0,$$

with d_0 factors in the direct product, on which A_0 is acting by permutation of coordinates, according to its transitive action on $\{1, \dots, d_0\}$. Moreover, this isomorphism φ_0 satisfies:

- (a) $\forall s \in A_0, \varphi_0(s) = (e_{\Gamma_1}, \dots, e_{\Gamma_1})s$,
- (b) $\forall h \in H_0, \exists h_1 \in H_1, \exists a_2, \dots, a_{d_0} \in A_1, \exists \rho \in A_0$, with $\rho(1) = 1$ and:

$$\varphi_0(h) = (h_1, a_2, \dots, a_{d_0})\rho.$$

Note that in (2)(b), the groups A_1 and H_1 are the subgroups of Γ_1 satisfying condition (1) for Γ_1 , which belongs to \mathcal{DP} .

Consider a group Γ_0 in the class \mathcal{DP} , together with a finite generating set $A_0 \cup B_0$. There exists a sequence of groups Γ_i with finite generating sets $A_i \cup B_i$, such that A_i is a finite group acting transitively on $\{1, \dots, d_i\}$, and an isomorphism:

$$\varphi_i : \Gamma_i \longrightarrow \Gamma_{i+1} \wr A_i,$$

such that for any $b_i \in B_i$, there exists $b_{i+1} \in B_{i+1}$, $a_{i,2}, \dots, a_{i,d_i} \in A_{i+1}$, $\rho_i \in A_i$ with $\rho_i(1) = 1$ and:

$$\varphi_i(b_i) = (b_{i+1}, a_{i,2}, \dots, a_{i,d_i})\rho_i.$$

Moreover, one may assume that $|B_i| = |B_0|$ for all i (possible because in (2)(b) above, h_1 is unique, hence from a given b_0 , there is a unique associated sequence b_i). This sequence of groups Γ_i is obtained inductively, applying definition 4.1 to Γ_i , which belongs to \mathcal{DP} , to define Γ_{i+1} . Set $H_i = \langle B_i \rangle < \Gamma_i$.

The groups of the class \mathcal{DP} are related to the groups of non-uniform growth constructed by Wilson (see [Wil1],[Wil2],[Bri1]). In particular, if all the groups Γ_i associated to a group Γ_0 are generated by a finite number (independent of i) of involutions, and if all the groups A_i involved are alternate groups \mathcal{A}_{d_i} acting on sets of size $d_i \geq 29$, then they have non-uniform growth by [Wil2]. This is the case of the examples in proposition 5.3 below.

Fact 4.2. *If Γ_0 belongs to \mathcal{DP} , there exists a sequence $\bar{d} = (d_i)_i$ of integers $d_i \geq 2$, and the group Γ_0 is acting by automorphisms on the spherically homogeneous rooted tree $T_{\bar{d}}$. This action is transitive on each level.*

Note that this action on the tree is not necessarily faithful (for instance, the subgroup F of the group $\Gamma = \Gamma(\mathcal{A}_{d_0}, A_{\bar{d}}, F)$ of section 2.4 of [Bri3] has a trivial action on the tree $T_{\bar{d}}$, even though Γ belongs to the class \mathcal{DP}).

Proof. By iteration of definition 4.1, there is an isomorphism $\Gamma_0 \simeq \Gamma_{i+1} \wr A_i \wr \dots \wr A_0$. As A_i is acting transitively on $\{1, \dots, d_i\}$, the group $A_i \wr \dots \wr A_0$ is acting transitively on $\{1, \dots, d_0\} \times \dots \times \{1, \dots, d_i\}$, which is the $i + 1$ st level of $T_{\bar{d}}$. Taking the limit with i , this provides the action on the tree $T_{\bar{d}}$. \square

Fact 4.3. *The group H_0 is amenable if and only if the groups H_i are amenable for all i .*

Proof. By (2)(b), the restriction of φ_0 to H_0 provides an embedding:

$$\varphi_0|_{H_0} : H_0 \hookrightarrow H_1 \times (A_1 \wr \text{Fix}_{A_0}(1)).$$

As the second factor is a finite group, amenability of H_1 implies that of H_0 .

Conversely assume that H_0 is amenable. Denote p_1 the projection on the factor H_1 and p_2 the projection on the factor $A_1 \wr \text{Fix}_{A_0}(1)$. Then $\ker(p_2) \cap \varphi_0(H_0)$ has finite index in $\varphi_0(H_0) \simeq H_0$, hence is amenable. Now $p_1 \circ \varphi_0$ surjects onto H_1 (by choice that $|B_1| = |B_0|$) and $p_1|_{\ker(p_2)}$ is an isomorphism, thus H_1 is a finite extension of $\ker(p_2) \cap \varphi_0(H_0)$, hence is amenable.

The same proof shows that amenability of H_{i+1} is equivalent to that of H_i . \square

Question 4.4. If a group $\Gamma_0 = \langle A_0, H_0 \rangle$ belongs to the class \mathcal{DP} with H_0 amenable, is the group Γ_0 amenable?

The following theorem provides a partial answer, with a condition on the sequence of integers $\bar{d} = (d_i)_i$.

Theorem 4.5. *Let Γ_0 belong to the class \mathcal{DP} with H_0 amenable and \bar{d} growing sufficiently slowly (for instance $\frac{d_k}{\log k} \rightarrow 0$), then Γ_0 is amenable.*

This theorem generalizes theorem 3.3. The proof is similar, though slightly more technical.

4.2. Proof of theorem 4.5. Given $\Gamma_0 = \langle A_0 \cup B_0 \rangle$ in the class \mathcal{DP} , consider the sequence of finitely generated groups $\Gamma_K = \langle A_K \cup B_K \rangle$ obtained by iterating definition 4.1, where B_K is the canonical generating set of the group H_K .

To ease notations, write g instead of $\varphi_K(g)$. For $\Omega \subset H_{K+1}$, set:

$$L_0^K(\Omega) = \{g \in \Gamma_K \mid \exists h \in \Omega, \alpha_2, \dots, \alpha_{d_K} \in A_{K+1}, \sigma \in A_K, g = (h, \alpha_2, \dots, \alpha_{d_K})\sigma\},$$

$$\iota L_0^K(\Omega) = \{g \in L_0^K(\Omega) \mid \sigma^{-1}(1) = 1\},$$

and by induction for $1 \leq k \leq K$, set:

$$L_k^K(\Omega) = \{g = (g_1, \dots, g_{d_{K-k}})\sigma \in \Gamma_{K-k} \mid \forall t, g_t \in L_{k-1}^K(\Omega), \exists T, g_T \in \iota L_{k-1}^K(\Omega)\},$$

$$\iota L_k^K(\Omega) = \{g \in L_k^K(\Omega) \mid g_{\sigma^{-1}(1)} \in \iota L_{k-1}^K(\Omega)\}.$$

The sets $\iota L_k^K(\Omega)$ should be considered as ‘‘combinatorial interiors’’ of $L_k^K(\Omega)$. They satisfy a combinatorial description as remark 3.7, but slightly differ from the actual interior of $L_k^K(\Omega)$, unless the set Ω has empty boundary (see remark 4.9 below). Fact 3.4 generalizes as:

Fact 4.6. *The three following are equivalent:*

- (1) $g \in \text{Int}(L_0^K(\Omega))$,
- (2) $gb_K \in L_0^K(\Omega)$ for all $b_K \in B_K$,
- (3) $\sigma^{-1}(1) = 1$ and $h \in \text{Int}(\Omega) \subset \Omega \subset H_{K+1}$.

Moreover they also imply:

- (4) $gb_K \in \iota L_0^K(\Omega)$ for all $b_K \in B_K$.

In particular, $\frac{|\text{Int}(L_0^K(\Omega))|}{|L_0^K(\Omega)|} = \frac{|\text{Int}(\Omega)|}{d_K|\Omega|}$, and $\delta_0^K(\Omega) = \frac{|\partial L_0^K(\Omega)|}{|L_0^K(\Omega)|} = 1 - \frac{|\text{Int}(\Omega)|}{d_K|\Omega|}$.

Proof. Let $g = (h, \alpha_2, \dots, \alpha_{d_K})\sigma$ belong to $L_0^K(\Omega)$. By (2)(a) of definition 4.1 for A_K , the element ga_K still belongs to $L_0^K(\Omega)$ for a_K in A_K . This proves equivalence of (1) and (2).

Now take $b_K = (b_{K+1}, a_2, \dots, a_{d_K})\rho$ in B_K , then:

$$gb_K = \begin{cases} (hb_{K+1}, \alpha_2 a_{\sigma(2)}, \dots, \alpha_d a_{\sigma(d)})\sigma\rho & \text{if } \sigma^{-1}(1) = 1, \\ (ha_{\sigma(1)}, \alpha_2 a_{\sigma(2)}, \dots, \alpha_{\sigma^{-1}(1)} b_{K+1}, \dots, \alpha_d a_{\sigma(d)})\sigma\rho & \text{if } \sigma^{-1}(1) \neq 1. \end{cases}$$

This shows that gb_K belongs to $L_0^K(\Omega)$ for all b_K if and only if $\sigma^{-1}(1) = 1$ and h belongs to $\text{Int}(\Omega)$, i.e. equivalence of (2) and (3).

This implies (4) because then $(\sigma\rho)^{-1}(1) = 1$. Computing the sizes follows from (3). \square

Notation 4.7. Let $g = (g_1, \dots, g_{d_i})\sigma = (g_{t_i})\sigma$ in Γ_i , with σ in A_i , g_{t_i} in Γ_{i+1} for $t_i \in \{1, \dots, d_i\}$ by identification of g with $\varphi_i(g)$. More generally, identify $g_{t_i \dots t_j}$ with $\varphi_{j+1}(g_{t_i \dots t_j})$ for $i \leq j \leq K$ and denote:

$$g = (g_{t_i \dots t_K})(\sigma_{t_i \dots t_{K-1}}) \cdots (\sigma_{t_i})\sigma,$$

where $\sigma_{t_i \dots t_j}$ belongs to A_{j+1} and $g_{t_i \dots t_K}$ to Γ_K . Set $\tau_i = \sigma^{-1}(1) \in \{1, \dots, d_i\}$, and by induction $\tau_{j+1} = (\sigma_{\tau_i \dots \tau_j})^{-1}(1) \in \{1, \dots, d_{j+1}\}$, which guarantees $g(\tau_i \tau_{i+1} \dots \tau_j) = 11 \dots 1$ for the action on the tree of fact 4.2.

The following generalizes lemma 3.5.

Lemma 4.8. *For $0 \leq k \leq K$, the three following are equivalent:*

- (1) $g \in \text{Int}(L_k^K(\Omega))$,
- (2) $gb_{K-k} \in L_k^K(\Omega)$ for all $b_{K-k} \in B_{K-k}$,
- (3) $g \in \iota L_k^K(\Omega)$ (i.e. $\sigma^{-1}(1) \in I(g) = \{T | g_T \in \iota L_{k-1}^K(\Omega)\}$) and $g_{\tau_{K-k} \dots \tau_K} \in \text{Int}(\Omega)$.

Moreover, they also imply:

- (4) $gb_{K-k} \in \iota L_k^K(\Omega)$ for all $b_{K-k} \in B_{K-k}$.

Observe that if $g \in \iota L_k^K(\Omega)$, then $g_{\tau_{K-k} \dots \tau_K} \in \Omega$, by definitions of $\iota L_k^K(\Omega)$ and $\tau_{K-k} \dots \tau_K$.

Proof. Let $g = (g_1, \dots, g_{d_{K-k}})\sigma$ belong to $L_k^K(\Omega)$. For a in A_{K-k} , ga still belongs to $L_k^K(\Omega)$ (no condition on σ). Thus (1) is equivalent to (2). To prove equivalence with (3) and implication of (4), proceed by induction on $0 \leq k \leq K$. The case $k = 0$ was treated as fact 4.6 (where $h = g_1 = g_{\sigma^{-1}(1)} = g_{\tau_K}$), now assume the lemma is known for $k - 1$.

For $b_{K-k} = (b_{K-k+1}, a_2, \dots, a_{d_{K-k}})\rho$, one has:

$$gb_{K-k} = (g_1 a_{\sigma(1)}, \dots, g_{\sigma^{-1}(1)} b_{K-k+1}, \dots, g_{d_{K-k}} a_{\sigma(d_{K-k})})\sigma\rho.$$

Assume (2) for g , then $g_{\sigma^{-1}(1)} b_{K-k+1} \in L_{k-1}^K(\Omega)$ for all $b_{K-k+1} \in B_{K-k+1}$, which means (2) for $k - 1$ applied to $g_{\sigma^{-1}(1)}$. By induction hypothesis, $g_{\sigma^{-1}(1)}$ satisfies (3), which means that it belongs to $\iota L_{k-1}^K(\Omega)$, so $g \in \iota L_k^K(\Omega)$, and $g_{\sigma^{-1}(1)\tau_{K-k+1} \dots \tau_K} = g_{\tau_{K-k}\tau_{K-k+1} \dots \tau_K} \in \text{Int}(\Omega)$, proving (3) for g .

Moreover, (2) applied to $g_{\sigma^{-1}(1)}$ implies, by induction, (4) that $g_{\sigma^{-1}(1)} b_{K-k+1} \in \iota L_{k-1}^K(\Omega)$ for all $b_{K-k+1} \in B_{K-k+1}$. As $(\sigma\rho)^{-1}(1) = \sigma^{-1}(\rho^{-1}(1)) = \sigma^{-1}(1)$, this shows $gb_{K-k} \in \iota L_{K-k}^K(\Omega)$, which is (4) for g .

Conversely, assume (3) for g , then $g_{\sigma^{-1}(1)} \in \iota L_{k-1}^K(\Omega)$, and $g_{\tau_{K-k}\tau_{K-k+1} \dots \tau_K} = g_{\sigma^{-1}(1)\tau_{K-k+1} \dots \tau_K} \in \text{Int}(\Omega)$, i.e. (3) for $g_{\sigma^{-1}(1)}$. As (3) implies (4) for $k - 1$, one has $g_{\sigma^{-1}(1)} b_{K-k+1} \in \iota L_{k-1}^K(\Omega)$ for all $b_{K-k+1} \in B_{K-k+1}$, so $gb_{K-k} \in L_k^K(\Omega)$ for all $b_{K-k} \in B_{K-k}$, which means (2) for g . \square

Remark 4.9. The combinatorial description of remark 3.7 still applies to an element $g \in \Gamma_{K-k}$ of the form:

$$g = (g_{t_{K-k} \dots t_K})(\sigma_{t_{K-k} \dots t_{K-1}}) \cdots (\sigma_{t_{K-k}})\sigma,$$

with $t_{K-k+l} \in \{1, \dots, d_{K-k+l}\}$, $\sigma_{t_{K-k} \dots t_{K-k+l}} \in A_{K-k+l+1}$ and $g_{t_{K-k} \dots t_K} \in \Gamma_{K+1}$. Such an element g belongs to $L_k^K(\Omega)$ if and only if it satisfies the three following conditions:

- (1) $\forall t_{K-k} \dots t_{K-1}$, the element $g_{t_{K-k} \dots t_{K-1}1}$ is in $\Omega \subset H_{K+1}$ and the elements $g_{t_{K-k} \dots t_{K-1}2}, \dots, g_{t_{K-k} \dots t_{K-1}d_K}$ are in A_{K+1} ,
- (2) $\forall t_{K-k} \dots t_{K-2}$, the set:

$$\begin{aligned} I(t_{K-k} \dots t_{K-2}) &= \{T_{K-1} \in \{1, \dots, d_{K-1}\} | \sigma_{t_{K-k} \dots t_{K-2}T_{K-1}}^{-1}(1) = 1\} \\ &= \{T_{K-1} \in \{1, \dots, d_{K-1}\} | g_{t_{K-k} \dots t_{K-2}T_{K-1}} \in \iota L_0^K(\Omega) \subset \Gamma_K\} \end{aligned}$$

is non-empty.

- (3) $\forall 2 \leq l \leq k, \forall t_{K-k} \dots t_{K-l}$, the following subset of $\{1, \dots, d_{K-l+1}\}$:

$$\begin{aligned} I(t_{K-k} \dots t_{K-l}) &= \{T_{K-l+1} | \sigma_{t_{K-k} \dots t_{K-l}T_{K-l+1}}^{-1}(1) \in I(t_{K-k} \dots t_{K-l}T_{K-l+1})\}, \\ &= \{T_{K-l+1} | g_{t_{K-k} \dots t_{K-l}T_{K-l+1}} \in \iota L_{l-2}^K(\Omega) \subset \Gamma_{K-l+2}\}, \end{aligned}$$

defined by induction on l , is non-empty.

The element g belongs to $\iota L_k^K(\Omega)$ if and only if it satisfies (1), (2), (3) and moreover:

- (4) $\sigma^{-1}(1)$ belongs to the set:

$$I(\emptyset) = \{T_{K-k} | \sigma_{T_{K-k}}^{-1}(1) \in I(T_{K-k})\} = \{T_{K-k} | g_{T_{K-k}} \in \iota L_{k-1}^K(\Omega) \subset \Gamma_{K-k+1}\}.$$

The element g belongs to $\text{Int}(L_k^K(\Omega))$ if and only if it satisfies (1), (2), (3), (4) and moreover:

- (5) $g_{\tau_{K-k} \dots \tau_K} \in \text{Int}(\Omega)$.

This description and especially point (5) prove the:

Fact 4.10. *With respect to the generating set $A_{K-k} \cup B_{K-k}$ of the group Γ_{K-k} , and the generating set B_{K+1} of the group H_{K+1} , one has:*

$$|\text{Int}(L_k^K(\Omega))| = |\iota L_k^K(\Omega)| \frac{|\text{Int}(\Omega)|}{|\Omega|}.$$

In particular, the set $\iota L_k^K(\Omega)$ is precisely the interior $\text{Int}(L_k^K(\Omega))$ when $\text{Int}(\Omega) = \Omega$. This happens when H_{K+1} (hence H_0) is finite.

For $0 \leq k \leq K$, set $\frac{|\iota L_k^K(\Omega)|}{|L_k^K(\Omega)|} = 1 - \varepsilon_k$. The number ε_k will be denoted ε_k^K later on to emphasize the dependance on K . Lemma 3.6 generalizes as:

Lemma 4.11. *The sequence $(\varepsilon_k)_{0 \leq k \leq K}$ satisfies $\varepsilon_0 = 1 - \frac{1}{d_K}$ and:*

$$1 - \varepsilon_{k+1} = \frac{1 - \varepsilon_k}{1 - \varepsilon_k^{d_{K-k-1}}}.$$

Proof. Given a subset $I \subset \{1, \dots, d_{K-k-1}\}$, denote:

$$J_I = \{g = (g_1, \dots, g_{d_{K-k-1}}) \sigma | \forall T \in I, g_T \in \iota L_k^K(\Omega) \text{ and } \forall t \notin I, g_t \in L_k^K(\Omega) \setminus \iota L_k^K(\Omega)\}.$$

By definition, $L_{k+1}^K(\Omega)$ is the disjoint union $L_{k+1}^K(\Omega) = \sqcup_{|I| \geq 1} J_I$.

As in the proof of lemma 3.6, one has for $i = |I|$:

$$\begin{aligned} |J_I| &= |A_{K-k-1}| |L_k^K(\Omega)|^{d_{K-k-1}} (1 - \varepsilon_k)^i \varepsilon_k^{d_{K-k-1}-i}, \\ |J_I \cap \iota L_{k+1}^K(\Omega)| &= \frac{i}{d_{K-k-1}} |J_I|. \end{aligned}$$

Again by use of the mean of binomial distribution, get:

$$\begin{aligned} |\iota L_{k+1}^K(\Omega)| &= \sum_{i=1}^{d_{K-k-1}} C_{d_{K-k-1}}^i (1 - \varepsilon_k)^i \varepsilon_k^{d_{K-k-1}-i} \frac{i}{d_{K-k-1}} |L_k^K(\Omega)|^d |A_{K-k-1}| \\ &= (1 - \varepsilon_k) |L_k^K(\Omega)|^d |A_{K-k-1}|, \\ |L_{k+1}^K(\Omega)| &= \sum_{i=1}^{d_{K-k-1}} C_{d_{K-k-1}}^i (1 - \varepsilon_k)^i \varepsilon_k^{d_{K-k-1}-i} |L_k^K(\Omega)|^d |A_{K-k-1}| \\ &= (1 - \varepsilon_k^{d_{K-k-1}}) |L_k^K(\Omega)|^d |A_{K-k-1}|. \end{aligned}$$

This proves the lemma. \square

Lemma 4.12. *If $\frac{d_k}{\log k} \rightarrow 0$, then $\varepsilon_K^K \rightarrow 0$.*

If $d_k \leq D$ for all k , then $\varepsilon_K^K = O(K^{-\eta})$ for all $\eta < \frac{1}{D-1}$.

First check the elementary:

Fact 4.13. *Let $f(D, \varepsilon) = \frac{1 - \varepsilon^{D-1}}{1 - \varepsilon^D}$, for $D \geq 2$ and $\varepsilon \in (0, 1)$. Then for fixed D , the function $f(D, \varepsilon)$ is decreasing with ε , and for fixed ε , the function $f(D, \varepsilon)$ is increasing with D .*

Proof. Compute derivatives:

$$\begin{aligned} (1 - \varepsilon^D)^2 \frac{\partial f}{\partial \varepsilon}(D, \varepsilon) &= \varepsilon^{D-2} (1 - \varepsilon) (\varepsilon^{D-1} + \dots + \varepsilon^2 + \varepsilon - (D-1)) < 0, \\ (1 - \varepsilon^D)^2 \frac{\partial f}{\partial D}(D, \varepsilon) &= \varepsilon^{D-1} (\varepsilon - 1) \log \varepsilon > 0. \end{aligned}$$

\square

Proof of lemma 4.12. For a fixed K , and $0 \leq k \leq K$, set $D_k = d_{K-k}$, and $D(K) = \max_{0 \leq k \leq K} \{d_k\} = o(\log K)$. By lemma 4.11, one has:

$$\varepsilon_{k+1} = \varepsilon_k \frac{1 - \varepsilon_k^{D_{k+1}-1}}{1 - \varepsilon_k^{D_{k+1}}} = \varepsilon_k f(D_{k+1}, \varepsilon_k).$$

By fact 4.13, as long as $\varepsilon_k \geq E$, one has:

$$\varepsilon_{k+1} \leq \varepsilon_k f(D_{k+1}, E) \leq \varepsilon_k f(D(K), E),$$

so $\varepsilon_K = \varepsilon_K^K \leq \max\{E, f(D(K), E)^K\}$ for any $E \in (0, 1)$. Now consider a sequence $E_K \rightarrow 0$ so that $|D(K) \log E_K| = o(\log K)$ (it exists). One has:

$$\begin{aligned} f(D(K), E_K)^K &= \exp K (\log(1 - E_K^{D(K)-1}) - \log(1 - E_K^{D(K)})), \\ &= \exp(-K E_K^{D(K)-1} + O(K E_K^{D(K)})) \rightarrow 0, \end{aligned}$$

because $K E_K^{D(K)-1} \rightarrow +\infty$. This shows $\varepsilon_K^K \rightarrow 0$.

If moreover $d_k \leq D$, take $E_K = K^{-\eta}$ with $\eta < \frac{1}{D-1}$, then:

$$f(D, E_K)^K = \exp(-K^{1-\eta(D-1)} + O(K^{1-\eta D})) = o(K^{-\eta}),$$

so $\varepsilon_K^K = O(K^{-\eta})$. \square

Proof of theorem 4.5. By fact 4.10, one has:

$$\frac{|Int(L_K^K(\Omega))|}{|L_K^K(\Omega)|} = \frac{|L_K^K(\Omega)|}{|L_K^K(\Omega)|} \frac{|Int(\Omega)|}{|\Omega|} = (1 - \varepsilon_K^K) \frac{|Int(\Omega)|}{|\Omega|}.$$

As the group H_{K+1} is amenable by fact 4.3, the set Ω can be chosen with $\frac{|Int(\Omega)|}{|\Omega|}$ arbitrarily close to 1. By lemma 4.12, this shows that there exists a sequence of sets $\Omega_K \subset H_{K+1}$ so that the sets $L_K^K(\Omega_K) \subset \Gamma_0$ form a Folner sequence. \square

5. EXAMPLES OF GROUPS IN THE CLASS \mathcal{DP}

5.1. Alternate directed groups. Given a sequence $\bar{d} = (d_i)_{i \in \mathbb{N}}$ of integers $d_i \geq 2$, set:

$$AT_i = AT(d_i, d_{i+1}) = (\mathcal{A}_{d_{i+1}} \times \cdots \times \mathcal{A}_{d_{i+1}}) \rtimes \mathcal{A}_{d_i-1} = \mathcal{A}_{d_{i+1}} \wr \mathcal{A}_{d_i-1},$$

where \mathcal{A}_d is the alternate group of even permutations of the set $\{1, \dots, d\}$, there are $d_i - 1$ factors in the product (indexed by $\{2, \dots, d_i\}$), and \mathcal{A}_{d_i-1} acts by permuting these factors. Consider the countable infinite direct product:

$$H_{\bar{d}}^{alt} = \prod_{i=0}^{\infty} AT_i = \prod_{i=0}^{\infty} \mathcal{A}_{d_{i+1}} \wr \mathcal{A}_{d_i-1}.$$

Its elements are denoted as sequences $h = (h_i)_{i=0}^{\infty}$ with $h_i = (a_{i,2}, \dots, a_{i,d_i})\rho_i \in AT_i$.

The group $H_{\bar{d}}^{alt}$ acts faithfully on the spherically homogeneous rooted tree $T_{\bar{d}}$ in the direction of the ray 1^∞ , where under the canonical isomorphism $Aut(T_{\bar{d}}) \simeq Aut(T_{\sigma \bar{d}}) \wr S_{d_0}$, one has:

$$(h_i)_{i=0}^{\infty} = ((h_i)_{i=1}^{\infty}, a_{0,2}, \dots, a_{0,d_0})\rho_0,$$

where $\rho_0 \in \mathcal{A}_{d_0-1} \simeq Fix_{\mathcal{A}_{d_0}}(1)$. Inductively under isomorphism $Aut(T_{\sigma^k \bar{d}}) \simeq Aut(T_{\sigma^{k+1} \bar{d}}) \wr S_{d_k}$, one has $(h_i)_{i=k}^{\infty} = ((h_i)_{i=k+1}^{\infty}, a_{k,2}, \dots, a_{k,d_k})\rho_k$.

On the other hand, the group \mathcal{A}_{d_0} acts on $T_{\bar{d}}$ by rooted automorphisms:

$$\mathcal{A}_{d_0} \ni a = (e, \dots, e)a.$$

Definition 5.1. An alternate directed group G is a subgroup of $Aut(T_{\bar{d}})$ with generating set $A \cup H$, with $A \subset \mathcal{A}_{d_0}$ and $H \subset H_{\bar{d}}^{alt}$. Denote:

$$G(A, H) = \langle A \cup H \rangle < Aut(T_{\bar{d}}).$$

When the sequence \bar{d} is constant $d_i = d$, if $A = \mathcal{A}_d$ and $H \simeq \mathcal{A}_d \wr \mathcal{A}_{d-1}$ is diagonally embedded into the direct product $H_{\bar{d}}^{alt}$, then $G(A, H) = G_d$ is the alternate mother group of section 3. Directed groups (not necessarily alternate) satisfy the same definition without requirement that the permutations involved are even, that is with S_d instead of \mathcal{A}_d and $H_{\bar{d}} = \prod_{i=0}^{\infty} S_{d_{i+1}} \wr S_{d_i-1}$ instead of $H_{\bar{d}}^{alt}$ (see [Bri1], [Bri3]).

5.2. Case of bounded valency. In this section, assume that the sequence \bar{d} is bounded $5 \leq d_i \leq D$. Let $B \subset H_{\bar{d}}^{\text{alt}}$ be a finite subset, and denote its elements by $\beta = (\beta_i)_{i=0}^{\infty} \in H_{\bar{d}}^{\text{alt}}$. Then for each i , the set $\{\beta_i, \beta \in B\}$ is a B -indexed subset of $AT_i = AT(d_i, d_{i+1})$. As the valency sequence \bar{d} is bounded, there is a finite set of pairs:

$$\{(AT(s), \{\beta(s), \beta \in B\}), s \in J\},$$

such that for any i , there exists $s(i)$ in the finite set J with $(AT_i, \{\beta_i, \beta \in B\}) = (AT(s(i)), \{\beta(s(i)), \beta \in B\})$, as pairs of finite groups with B -indexed subsets.

This provides an isomorphism:

$$H_{\bar{d}}^{\text{alt}} \langle H = \langle \beta, \beta \in B \rangle \simeq \langle (\beta(s))_{s \in J}, \beta \in B \rangle < \prod_{s \in J} AT(s).$$

The group H is said saturated if $H = \prod_{s \in J} AT(s)$. (Mind a difference with the notion of saturation in [Bri1] and [Bri3], where it was only required that H surjects on each factor $AT(s)$. The present condition is slightly stronger.) Finiteness of J shows the:

Fact 5.2. *If \bar{d} is bounded, any finitely generated subgroup of $H_{\bar{d}}^{\text{alt}}$ is contained in a finite saturated subgroup H .*

The following proposition will permit to show amenability of all directed groups acting on a tree of bounded valency.

Proposition 5.3. *Let \bar{d} be a bounded sequence of integers $d_i \geq 5$. If $H < H_{\bar{d}}^{\text{alt}}$ is a finite saturated subgroup, then the alternate directed group $G(\mathcal{A}_{d_0}, H) < \text{Aut}(T_{\bar{d}})$ belongs to the class \mathcal{DP} with $A_0 = \mathcal{A}_{d_0}$ and $H_0 = H$.*

Proof. Set $H_1 = \{(h_i)_{i=1}^{\infty} | (h_i)_{i=0}^{\infty} \in H\}$. The only non-trivial point in order to verify definition 4.1 is surjectivity of the isomorphism:

$$\varphi_0 : G(\mathcal{A}_{d_0}, H) \longrightarrow G(\mathcal{A}_{d_1}, H_1) \wr \mathcal{A}_{d_0}.$$

Given $h = (h_i)_{i=0}^{\infty}$ in $H_{\bar{d}}$ with $h_i = (a_{i,2}, \dots, a_{i,d_i})\rho_i$, set:

$$h(2) = ((a_{i,2}, e, \dots, e)e)_{i=0}^{\infty}, \text{ and } h(\emptyset) = ((e, \dots, e)\rho_i)_{i=0}^{\infty}.$$

In each factor $AT(s) = \mathcal{A}_{d'(s)} \wr \mathcal{A}_{d(s)-1}$, the subset

$$\{(a_2, e, \dots, e) | a_2 \in \mathcal{A}_{d'(s)}\} \cup \{(e, \dots, e)\rho | \rho \in \mathcal{A}_{d(s)-1}\}$$

generates the group $AT(s)$. Thus by saturation

$$\langle h(2), h \in H \rangle \simeq \prod_{s \in J} \mathcal{A}_{d'(s)} \times \{e\} \times \dots \times \{e\}, \text{ and } \langle h(\emptyset), h \in H \rangle \simeq \prod_{s \in J} \mathcal{A}_{d'(s)}.$$

So saturation shows that the subsets $H(2) = \{h(2), h \in H\}$ and $H(\emptyset) = \{h(\emptyset), h \in H\}$ are subgroups of H , and moreover $\langle H(2) \cup H(\emptyset) \rangle = H$.

The proofs of fact 3.2 and proposition 3.1 apply directly, replacing the generators $b_2 = b(a_2, e, \dots, e, e_A)$ and $b_{\emptyset} = b(e, \dots, e, \rho)$ by $h(2)$ and $h(\emptyset)$ respectively. \square

Let σ be a permutation of the set $\{1, \dots, d\}$. Denote σ' another copy of σ acting on the set $\{d+1, \dots, 2d\}$ by $\sigma'(t) = \sigma(t-d) + d$, and consider the embedding $a : S_d \hookrightarrow \mathcal{A}_{2d}$ given by $a(\sigma) = \sigma\sigma'$. It can be extended to furnish:

$$a : \text{Aut}(T_{\bar{d}}) \rightarrow \text{Aut}^{\text{alt}}(T_{\bar{2d}}),$$

an embedding of the group of automorphisms of the tree $T_{\bar{d}}$ into the group of alternate automorphisms of the tree $T_{\bar{2d}}$.

Indeed, let $\gamma \in \text{Aut}(T_{\bar{d}})$ be described by a family of permutations $\{\sigma_v\}_{v \in T_{\bar{d}}}$, where $\sigma_v \in S_{d_k}$ for every $v = t_1 \dots t_k$ in $T_{\bar{d}}$. The automorphism $a(\gamma)$ is described by a family of permutations $\{a(\gamma)_v\}_{v \in T_{\bar{2d}}}$ given by $a(\gamma)_v = a(\gamma_v) \in \mathcal{A}_{2d_k}$ for $v = t_1 \dots t_k$ in $T_{\bar{d}} \subset T_{\bar{2d}}$ and $a(\gamma)_v = e$ for $v \in T_{\bar{2d}} \setminus T_{\bar{d}}$.

Fact 5.4. *Directed elements have directed image under a , i.e. $a(H_{\bar{d}}) \subset H_{\bar{2d}}^{\text{alt}}$. In particular, the mother group of degree 0 acting on a d -regular tree embeds in the alternate mother group G_{2d} acting on a $2d$ -regular tree.*

Proof. As a shortcut denote 1^k for the sequence $11\dots 1$ with k ones. By definition, an automorphism γ is directed if and only if $\sigma_{1^k} \in \text{Fix}_{S_{d_k}}(1) \simeq S_{d_k-1}$ and $\sigma_v = e$ if v is not of the form $1^{k-1}t$ for some t in $\{1, \dots, d_k\}$. This is still the case for $a(\gamma)$. \square

The following result from [Bri1] can now be reproved.

Corollary 5.5. *Directed groups acting on a tree of bounded valency are amenable.*

Proof. Let Γ be a directed group, with generating set $S \cup H$ where $S \subset S_{d_0}$ and $H \subset H_{\bar{d}}$. By fact 5.4, the group $a(\Gamma) < \text{Aut}^{\text{alt}}(T_{\bar{2d}})$ is alternate and directed. By fact 5.2, it can be included in a directed, alternate and saturated subgroup of $\text{Aut}^{\text{alt}}(T_{\bar{2d}})$, which is in the class \mathcal{DP} by proposition 5.3, hence amenable by theorem 4.5, since $\bar{2d}$ is bounded and H_0 finite. The group Γ is also amenable as a subgroup. \square

Corollary 5.6 (Main theorem in [BKN]). *Automata groups with bounded activity are amenable.*

Proof. By theorem 3.3 in [BKN], an automata group Γ with bounded activity is a subgroup of the alternate mother group of degree 0 acting on a d -regular tree for d large enough. By fact 5.4, Γ is a subgroup of G_{2d} , hence is amenable by theorem 3.3. \square

5.3. Examples with unbounded valency. This section aims at constructing examples of groups in the class \mathcal{DP} for which the sequence \bar{d} of fact 4.2 is unbounded.

Let H_0 be a finitely generated, residually finite, perfect group with a sequence of normal subgroups $(N_k)_{k \geq 0}$ of finite index so that each quotient $A_k = H_0/N_k$ is perfect, acting faithfully and transitively on a finite set $\{1, \dots, d_k\}$ of size $d_k \geq 2$. For h in H_0 , denote $a_k(h) = hN_k \in A_k$.

To the group H_0 together with subgroup sequence $(N_k)_{k \geq 0}$ is associated an action on the rooted tree $T_{\bar{d}}$ of valency sequence $\bar{d} = (d_k)_{k \geq 0}$, denoted $b_0 : H_0 \rightarrow \text{Aut}(T_{\bar{d}})$, given by the portrait $(b_0(h))_{1^{k-1}2} = a_k(h)$ and $(b_0(h))_v = e$ if v is not of the form

$1^{k-1}2$ for $k \geq 1$ (notation 1^k is a shortcut for $11 \dots 1$ with k ones). In the wreath product isomorphism, one has:

$$b_0(h) = (b_1(h), a_1(h), e, \dots, e),$$

where $b_1(h)$ is the similar action of the group H_0 on the tree $T_{\sigma\bar{d}}$ associated to the shifted subgroup sequence $(N_k)_{k \geq 1}$. The group $A_0 = H_0/N_0$ also acts on $T_{\bar{d}}$ as a rooted automorphism acting on $\{1, \dots, d_0\}$, i.e. $a_0 = (e, \dots, e)a_0$.

Fact 5.7. *With the actions described above, the group $\Gamma_0 = \langle A_0 \cup H_0 \rangle < \text{Aut}(T_{\bar{d}})$ belongs to the class \mathcal{DP} .*

Proof. Let Γ_1 be the subgroup of $\text{Aut}(T_{\sigma\bar{d}})$ generated by $H_1 = H_0$ with the $b_1(h)$ action and $A_1 = H_1/N_1$ with a rooted action. Properties (1) and (2) of definition 4.1 follow from the construction above. Thus Γ_0 belongs to the class \mathcal{DP} as soon as Γ_1 does. As Γ_1 satisfies the same properties as Γ_0 , they are in the class \mathcal{DP} . \square

As an example of such a finitely generated, residually finite, perfect group H_0 , one may take the alternate mother group G_d of section 3 for $d \geq 6$ (then both finite generating subgroups A and B are perfect). This group satisfies $G_d \simeq G_d \wr \mathcal{A}_d$. Its finite index normal subgroups are:

$$St_j = \ker(G_d \rightarrow \mathcal{A}_d \wr \dots \wr \mathcal{A}_d),$$

where the j factors in the iterated wreath product are obtained by iteration of the above isomorphism. The group St_j is called stabilizer of level j of the group G_d . The quotient G_d/St_j is acting transitively on level j , which is the set $\{1, \dots, d\}^j$. By [Neu], these stabilizers St_j are the only finite index normal subgroups of G_d .

For an arbitrary function $j : \mathbb{N} \rightarrow \mathbb{N}$, take $N_k = St_{j(k)}$ as a sequence of normal subgroups. The group Γ_0 defined by G_d together with the function $j(k)$ belongs to the class \mathcal{DP} by fact 5.7. It is amenable when $d^{j(k)}$ is sublogarithmic by theorem 4.5. Note that in the construction above, one could use any group of proposition 5.3 with $d_i \geq 6$ instead of G_d .

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E-mail address: jeremie.brieussel@gmail.com