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# Stability inequality for the scalar potential in an infinite quantum waveguide

<sup>1</sup>Quang Sang Phan, <sup>2</sup>Eric Soccorsi

## Abstract

We prove Lipschitz stability in the determination of the scalar potential in the dynamic Schrödinger equation in an infinite waveguide from one boundary observation and one (arbitrarily small with respect to the infinite direction of the waveguide) internal measurement of the solution.

**AMS 2010 Mathematics Subject Classification:** 35R30.

**Keywords:** Inverse problem, Schrödinger equation, scalar potential, Carleman estimate, infinite waveguide.

## 1 Introduction

### 1.1 What we are aiming for

In the present paper we consider an infinite waveguide  $\Omega = \omega \times \mathbb{R}$ , where  $\omega$  is a connected bounded open subset of  $\mathbb{R}^{n-1}$ ,  $n \geq 2$ , with  $C^2$ -boundary  $\partial\omega$ . Given  $T > 0$  we consider the following initial boundary value problem

$$\begin{cases} -iu' - \Delta u + q(x)u = 0, & \text{in } Q = (0, T) \times \Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \\ u(t, x) = g(t, x), & (t, x) \in \Sigma = (0, T) \times \Gamma, \end{cases} \quad (1.1)$$

where  $\Gamma := \partial\omega \times \mathbb{R}$  and the sign ' stands for  $\frac{\partial}{\partial t}$ . Here  $u_0$  (resp.  $g$ ) is the initial (resp. boundary) condition associated to (1.1) and  $q$  is a function of  $x \in \Omega$  only.

Since  $\Gamma$  is unbounded it is worth making the boundary condition in the last line of (1.1) more precise. Writing  $x = (x', x_n)$  with  $x' = (x_1, \dots, x_{n-1}) \in \omega$  for every  $x \in \Omega$  we extend the mapping

$$\begin{aligned} C_0^\infty((0, T) \times \mathbb{R}; H^2(\omega)) &\longrightarrow L^2((0, T) \times \mathbb{R}; H^{3/2}(\partial\omega)) \\ v &\mapsto [(t, x_n) \in (0, T) \times \mathbb{R} \mapsto v(t, \cdot, x_n)|_{\partial\omega}], \end{aligned} \quad (1.2)$$

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to a bounded operator from  $L^2((0, T) \times \mathbb{R}; H^2(\omega))$  into  $L^2((0, T) \times \mathbb{R}; H^{3/2}(\partial\omega))$ , denoted by  $\gamma_0$ . Then for every  $u \in C^0([0, T]; H^2(\Omega))$  the above mentioned boundary condition reads  $\gamma_0 u = g$ .

The main purpose of this paper is to prove stability in the determination of the scalar potential  $q$  from both boundary and arbitrarily small (with respect to the infinite direction  $x_n$  of the waveguide) internal measurements of the solution  $u$  to (1.1).

## 1.2 Published papers

The problem of stability in determining the time-independent electric potential in a Schrödinger equation from a single boundary measurement was treated by Baudouin and Puel in [1]. This result was improved by Mercado, Osses and Rosier in [8]. In these two papers, the main assumption is that the part of the boundary where the measurement is made satisfies a geometric condition related to geometric optics condition insuring observability. This geometric condition was relaxed in [2] under the assumption that the potential is known near the boundary. In all the above mentioned papers the Schrödinger equation is defined in a bounded domain.

In this paper we investigate the problem of determining the scalar potential in the Schrödinger equation defined in a closed waveguide, which is an unbounded domain. It turns out that there is a very small number of articles dealing with inverse boundary value problems in an unbounded domain in the mathematical literature. In [7] Li and Uhlmann prove uniqueness in the determination of the scalar potential in an infinite slab from partial Dirichlet-to-Neumann map. In [4] Cardoulis, Cristofol and Gaitan obtain Lipschitz stability from a single lateral measurement performed on one side of an unbounded strip. For an inverse boundary value problem stated in a waveguide geometry we refer to [5, 6] where stability is claimed for various coefficients of the Schrödinger equation from the associated Dirichlet-to-Neumann map. Here we investigate the same type of problems in absence of any information given by the Dirichlet-to-Neumann map.

## 1.3 Main results and contents

In this section we state the main result of this article and briefly comment on it.

**Theorem 1.1.** *For  $M > 0$ ,  $\ell > 0$  and  $\alpha > 0$  fixed, let  $u_0 \in H^4(\Omega; \mathbb{R})$  obey*

$$u_0(x) \geq \alpha > 0, \quad x \in \Omega, \quad (1.3)$$

let  $q_j \in \mathcal{Q}_M := \{q \in W^{2,\infty}(\Omega; \mathbb{R}), \|q\|_{W^{2,\infty}(\Omega)} \leq M\}$ ,  $j = 1, 2$ , fulfill

$$q_1(x) = q_2(x), \quad x \in \omega \times (\mathbb{R} \setminus (-\ell, \ell)), \quad (1.4)$$

and let  $u_j$  denote the  $C^1([0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^2([0, T]; L^2(\Omega))$ -solution to (1.1) associated to  $u_0$ ,  $g = \gamma_0 G$  and  $q_j$ , where

$$G(t, x) := u_0(x) + it(\Delta - q_2)u_0(x), \quad (t, x) \in Q. \quad (1.5)$$

Then for every  $L > \ell$ , there exist a subboundary  $\Gamma_* \subset \partial\omega \times (-L, L)$  and a constant  $C > 0$  depending only on  $L, T, M, \omega$  and  $\Gamma_*$ , such that we have

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C \left( \|\partial_\nu(u_1 - u_2)\|_{H^1(0, T; L^2(\Gamma_*))} + \|u_1 - u_2\|_{H^1(0, T; H^1(\omega \times (\ell, L)))} \right). \quad (1.6)$$

Under the prescribed conditions (1.3)–(1.5), Theorem 1.1 claims Lipschitz stability in the determination of the scalar potential appearing in the dynamic Schrödinger equation in  $\Omega$  from two different observations of the solution  $u$  to (1.1). The first one is a lateral measurement of  $\partial_\nu u$  on some subboundary of  $\partial\omega \times (-L, L)$ . The second observation is an internal measurement of  $u$  which is performed in each of the two “slices”  $S_-(L) := \omega \times (-L, -\ell)$  and  $S_+(L) := \omega \times (\ell, L)$  of  $\Omega$ . Although the  $\mathbb{R}^n$ -Lebesgue measure of  $S_\pm(L)$  can be made arbitrarily small by taking  $L$  sufficiently close to  $\ell$ , this observation cannot be removed from the rhs of (1.6) since the prefactor (ie the constant  $C$ ) tends to infinity as  $L$  goes to  $\ell$ . The occurrence of this internal observation of  $u$  in (1.6) arises from the unbounded geometry of  $\Omega$ . More precisely, the derivation of the stability inequality (1.6) being by means of a global Carleman estimate for the Schrödinger equation in a bounded domain, the strategy used in this paper involves a cut off function with first derivative supported in  $(-L, -\ell) \cup (\ell, L)$ , which gives rise to the measurement of  $u$  in  $S_\pm(L)$ .

The paper is organized as follows. Section 2 deals with the direct problem associated to (1.1). Namely §2.1 gathers existence and uniqueness results for the solution to the dynamic Schrödinger equation in the infinite domain  $\Omega$  and §2.2 is devoted to the study of the direct problem for the linearized system associated to (1.1). In §2.3 the corresponding solution is suitably extended to a function of  $[-T, T] \times \Omega$  which is continuous wrt the time variable  $t$ . This is required by the method used in the proof of the stability inequality (1.6), which is the purpose of Section 3. It is by means of a Carleman estimate for the Schrödinger equation stated in §3.1. Finally §3.2 contains the completion of the proof of Theorem 1.1.

## 2 Analysis of the direct problem

### 2.1 Existence and uniqueness results

This subsection gathers two existence and uniqueness results needed for the analysis of the direct problem. We start by recalling from [6][Proposition 2.1] the following:

**Proposition 2.1.** *Let  $M > 0$ . Then for all  $q \in \mathcal{Q}_M$ ,  $v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$  and  $f \in W^{1,1}(0, T; L^2(\Omega))$ , there is a unique solution  $v \in Z_0 = C([0, T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  to the boundary value problem*

$$\begin{cases} -iv' - \Delta v + qv = f, & \text{in } Q, \\ v(0, x) = v_0, & x \in \Omega, \\ v(t, x) = 0, & (t, x) \in \Sigma. \end{cases}$$

Moreover we have

$$\|v\|_{Z_0} \leq C \left( \|v_0\|_{\mathbf{H}^2(\Omega)} + \|f\|_{W^{1,1}(0,T;L^2(\Omega))} \right),$$

for some constant  $C > 0$  depending only on  $\omega$ ,  $T$  and  $M$ .

Set

$$\mathcal{X}_0 := \gamma_0(W^{2,2}(0, T; \mathbf{H}^2(\Omega))). \quad (2.7)$$

The space  $\mathcal{X}_0$  is equipped with the following quotient norm

$$\|g\|_{\mathcal{X}_0} = \inf\{\|G\|_{W^{2,2}(0,T;\mathbf{H}^2(\Omega))}; G \in W^{2,2}(0, T; \mathbf{H}^2(\Omega)) \text{ satisfies } \gamma_0 G = g\},$$

in such a way that every  $g \in \mathcal{X}_0$  admits an extension  $G \in W^{2,2}(0, T; \mathbf{H}^2(\Omega))$  obeying

$$\|G\|_{W^{2,2}(0,T;\mathbf{H}^2(\Omega))} \leq 2\|g\|_{\mathcal{X}_0}.$$

Further, put

$$\mathcal{L} := \{(u_0, g) \in \mathbf{H}^2(\Omega) \times \mathcal{X}_0; u_0 = g(0, \cdot) \text{ on } \Gamma\}. \quad (2.8)$$

Then, setting  $u = v + G$ , where  $v$  is defined by Proposition 2.1 for  $f = i\partial_t G + \Delta G - VG$  and  $v_0 = u_0 - G(0, \cdot)$ , we obtain the following existence and uniqueness result, which is similar to [6][Corollary 2.1].

**Theorem 2.2.** *Let  $M > 0$ . Then for every  $q \in \mathcal{Q}_M$  and  $(u_0, g) \in \mathcal{L}$  there is a unique solution  $u \in Z = C([0, T]; \mathbf{H}^2(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  to the boundary value problem (1.1). Moreover the estimate*

$$\|u\|_Z \leq C(\|u_0\|_{\mathbf{H}^2(\Omega)} + \|g\|_{\mathcal{X}_0}), \quad (2.9)$$

holds for some some positive constant  $C$  depending only on  $\omega$ ,  $T$  and  $M$ .

## 2.2 The linearized problem

Let  $M$ ,  $q_j$ ,  $j = 1, 2$ ,  $u_0$ ,  $g$  and  $G$  be the same as in Theorem 1.1. Since  $G \in W^{2,2}(0, T; \mathbf{H}^2(\Omega))$  and  $g(0, \cdot) = u_0$  on  $\Gamma$ , we have  $(u_0, g) \in \mathcal{L}$ . Applying Theorem 2.2 there is thus a unique solution  $u_j \in Z, j = 1, 2$ , to the system

$$\begin{cases} -iu'_j - \Delta u_j + q_j(x)u_j = 0, & \text{in } Q, \\ u_j(0, x) = u_0(x), & x \in \Omega, \\ u_j(t, x) = g(t, x), & (t, x) \in \Sigma. \end{cases} \quad (2.10)$$

Further, differentiating (2.10) with respect to  $t$  for  $j = 2$ , we obtain that  $u'_2$  is solution to the following boundary value problem

$$\begin{cases} -iu''_2 - \Delta u'_2 + q_2(x)u'_2 = 0, & \text{in } Q, \\ u'_2(0, x) = \nu_0(x), & x \in \Omega, \\ u'_2(t, x) = g'(t, x), & (t, x) \in \Sigma, \end{cases} \quad (2.11)$$

where  $\nu_0 = i(\Delta - q_2)u_0$ . Since  $G'(t, x) = \nu_0(x)$  for a.e.  $(t, x) \in Q$  by (1.5) then it is true that  $G' \in W^{2,2}(0, T; H^2(\Omega))$  and consequently  $(\nu_0, g') \in \mathcal{L}$ . Therefore  $u'_2 \in Z$  from (2.11) and Theorem 2.2. As a consequence we have

$$u_2 \in C^1([0, T]; H^2(\Omega)) \cap C^2([0, T]; L^2(\Omega)).$$

Moreover, it follows from (2.10) that  $u = u_1 - u_2$  is solution to the linearized system

$$\begin{cases} -iu' - \Delta u + q_1 u = f, & \text{in } Q, \\ u(0, x) = 0, & x \in \Omega, \\ u(t, x) = 0, & (t, x) \in \Sigma, \end{cases} \quad (2.12)$$

with  $f = (q_2 - q_1)u_2$ . Since  $u_2 \in Z$  and  $q_1 - q_2 \in L^\infty(\Omega)$  then  $f \in W^{1,1}(0, T; L^2(\Omega))$  so that we have  $u \in Z_0$  by Proposition 2.1.

Last, we deduce from (2.12) that  $v = u'$  satisfies

$$\begin{cases} -iv' - \Delta v + q_1 v = f', & \text{in } Q, \\ v(0, x) = i(q_2 - q_1)(x)u_0(x), & x \in \Omega. \\ v(t, x) = 0, & (t, x) \in \Sigma, \end{cases} \quad (2.13)$$

with  $f' = (q_2 - q_1)u'_2 \in W^{1,1}(0, T; L^2(\Omega))$ . From the identity  $q_1 = q_2$  on  $\Gamma$  we get that  $i(q_2 - q_1)u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ . Therefore  $v = u'_1 - u'_2 \in Z_0$  by Proposition 2.1. Bearing in mind that  $u'_2 \in Z$  we deduce from this that  $u'_1 \in Z$ . Summing up, we have obtained that

$$u_j \in C^1([0, T]; H^2(\Omega)) \cap C^2([0, T]; L^2(\Omega)), \quad j = 1, 2.$$

### 2.3 Time symmetrization

In this subsection we prove that  $v$  may be extended to a solution of (2.13) over the time span  $[-T, T]$ .

We extend  $v$  (resp.  $f'$ ) on  $(-T, 0) \times \Omega$  by setting  $v(t, x) = \overline{-v(-t, x)}$  (resp.  $f'(t, x) = -\overline{f'(-t, x)}$ ) for  $(t, x) \in (-T, 0) \times \Omega$ . Since the initial conditions  $v(0, \cdot)$  and  $f'(0, \cdot)$  are purely complex valued according to (2.13), the mappings  $t \mapsto v(t, x)$  and  $t \mapsto f'(t, x)$  are thus continuous at  $t = 0$  for a.e.  $x \in \Omega$ . Therefore, we have  $v \in C^0([-T, T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([-T, T]; L^2(\Omega))$  and  $f' \in W^{1,1}(-T, T; L^2(\Omega))$ . Moreover  $v$  is solution to (2.13) over the whole time span  $(-T, T)$ :

$$\begin{cases} -iv' - \Delta v + q_1 v = f', & \text{in } (-T, T) \times \Omega, \\ v(0, x) = i(q_2 - q_1)(x)u_0(x), & x \in \Omega. \\ v(t, x) = 0, & (t, x) \in (-T, T) \times \partial\Omega, \end{cases} \quad (2.14)$$

### 3 Stability inequality

#### 3.1 Global Carleman estimate

In this subsection we recall a global Carleman estimate for the Schrödinger operator, which is borrowed from [1][proposition 3].

Given an arbitrary bounded domain  $\mathcal{O} \subset \mathbb{R}^n$  with  $C^2$  boundary, we consider the Schrödinger operator  $L$  acting in  $(C_0^\infty)'(( -T, T) \times \mathcal{O})$ ,

$$L := -i\partial_t - \Delta + q, \quad (3.15)$$

with scalar potential  $q \in L^\infty(\mathcal{O})$ .

Next we introduce an open subset  $\Gamma_0$  of  $\partial\mathcal{O}$ , and a function  $\tilde{\beta} \in C^4(\overline{\mathcal{O}}; \mathbb{R}_+)$  satisfying the following conditions:

**Assumption 3.1.**

- (i)  $\exists C_0 > 0$  such that the estimate  $|\nabla\tilde{\beta}(x)| \geq C_0$  holds for all  $x \in \mathcal{O}$ ;
- (ii)  $\partial_\nu\tilde{\beta}(x) := \nabla\tilde{\beta}(x) \cdot \nu(x) < 0$  for all  $x \in \partial\mathcal{O} \setminus \Gamma_0$ ;
- (iii)  $\exists \Lambda_1 > 0, \exists \epsilon > 0$  such that we have  $\lambda|\nabla\tilde{\beta}(x) \cdot \zeta|^2 + D^2\tilde{\beta}(\zeta, \bar{\zeta}) \geq \epsilon|\zeta|^2$  for all  $\zeta \in \mathbb{R}^n$  and  $\lambda > \Lambda_1$ .

There are actual functions  $\tilde{\beta}$  fulfilling Assumption 3.1, such as  $\Omega \ni x \mapsto |x - x_0|^2$  where  $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$  is fixed.

Further we put

$$\beta := \tilde{\beta} + K, \text{ where } K := m\|\tilde{\beta}\|_\infty \text{ for some } m > 1, \quad (3.16)$$

and define the two following weight functions for  $\lambda > 0$ :

$$\varphi(t, x) = \frac{e^{\lambda\beta(x)}}{(T+t)(T-t)} \text{ and } \eta(t, x) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T+t)(T-t)}, \quad (t, x) \in (-T, T) \times \mathcal{O}. \quad (3.17)$$

Finally, for all  $s > 0$  we introduce two operators acting in  $(C_0^\infty)'(( -T, T) \times \mathcal{O})$ ,

$$M_1 := i\partial_t + \Delta + s^2|\nabla\eta|^2 \text{ and } M_2 := is\eta' + 2s\nabla\eta \cdot \nabla + s(\Delta\eta). \quad (3.18)$$

$M_1$  (resp.  $M_2$ ) is the adjoint (resp. skew-adjoint) part of the operator  $e^{-s\eta}Le^{s\eta}$ , where  $L$  is given by (3.15).

**Proposition 3.2.** *Let  $q \in L^\infty(\mathcal{O})$ , let  $\beta$  be given by (3.16), where  $\tilde{\beta} \in C^4(\overline{\mathcal{O}}; \mathbb{R}_+)$  fulfills Assumption 3.1, let  $\varphi$  and  $\eta$  be as in (3.17), and let  $L, M_1$  and  $M_2$  be defined by (3.15)-(3.18). Then there exist  $\lambda > 0, s_0 > 0$ , and a constant  $C > 0$  depending only on  $T, \|q\|_{L^\infty(\mathcal{O})}, \mathcal{O}$  and  $\Gamma_0$ , such that the estimate*

$$\begin{aligned} & s\|e^{-s\eta}\nabla w\|_{L^2((-T, T) \times \mathcal{O})}^2 + s^3\|e^{-s\eta}w\|_{L^2((-T, T) \times \mathcal{O})}^2 + \sum_{j=1,2} \|M_j e^{-s\eta}w\|_{L^2((-T, T) \times \mathcal{O})}^2 \\ & \leq C \left( s\|e^{-s\eta}\varphi^{1/2}(\partial_\nu\beta)^{1/2}\partial_\nu w\|_{L^2((-T, T) \times \Gamma_0)}^2 + \|e^{-s\eta}Lw\|_{L^2((-T, T) \times \mathcal{O})}^2 \right), \end{aligned} \quad (3.19)$$

holds for all  $s \geq s_0$  and all  $w \in L^2(-T, T; H_0^1(\mathcal{O}))$  satisfying  $Lw \in L^2((-T, T) \times \mathcal{O})$  and  $\partial_\nu w \in L^2(-T, T; L^2(\Gamma_0))$ .

## 3.2 Proof of Theorem 1.1

In this section we apply the Bugkheim-Klibanov method defined in [3] to prove the stability inequality (1.6). To this purpose we first introduce the following useful notation used throughout the section. For all  $d > 0$  we define  $\Omega_d := \omega \times (-d, d)$ .

Fix  $L > \ell$ , set  $r = (L + \ell)/2$  and consider a domain  $\mathcal{O}$  in  $\mathbb{R}^n$ , with  $C^2$  boundary  $\partial\mathcal{O}$ , obeying

$$\Omega_\ell \subset \Omega_r \subset \mathcal{O} \subset \Omega_L. \quad (3.20)$$

Next choose  $\chi \in C_0^\infty(\mathbb{R}_{x_n}, [0, 1])$  such that

$$\chi(x_n) = \begin{cases} 1 & \text{if } |x_n| \leq \ell \\ 0 & \text{if } |x_n| \geq r. \end{cases}$$

Put  $w = \chi v$ , where  $v$  is the  $C^0([-T, T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([-T, T]; L^2(\Omega))$ -solution to (2.14), in such a way that the restriction  $w|_{\mathcal{O}}$  of  $w$  to  $\mathcal{O}$  satisfies

$$w|_{\mathcal{O}} \in C^0([-T, T]; H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})) \cap C^1([-T, T]; L^2(\mathcal{O})) \quad (3.21)$$

and is solution to the following system

$$\begin{cases} -iw' - \Delta w + q_1 w = \chi f' - K v, & \text{in } (-T, T) \times \mathcal{O}, \\ w(0, x) = i\chi(q_2 - q_1)(x)u_0(x), & x \in \mathcal{O}. \\ w(t, x) = 0, & (t, x) \in (-T, T) \times \partial\mathcal{O}, \end{cases} \quad (3.22)$$

where  $K = [\Delta, \chi] = \ddot{\chi} + 2\dot{\chi}\partial_{x_n}$ . Here  $\dot{\chi}$  (resp.  $\ddot{\chi}$ ) is a shorthand for the first (resp. second) derivative of  $\chi$ .

As a preamble to the proof of the stability inequality stated in Theorem 1.1 we first establish two elementary technical results.

### 3.2.1 Two auxiliary results

**Lemma 3.3.** *For  $s > 0$ , let  $\phi := e^{-s\eta}w$ , where  $w$  is defined by (3.21)-(3.22). Then we have*

$$J := \|e^{-s\eta(0, \cdot)}w(0, \cdot)\|_{L^2(\mathcal{O})}^2 = 2\text{Im} \left( \int_{(-T, 0) \times \mathcal{O}} M_1 \phi \bar{\phi} dt dx \right).$$

*Proof.* In light of (3.16)-(3.17) we have  $\lim_{t \downarrow (-T)} \eta(t, x) = +\infty$  for all  $x \in \mathcal{O}$  hence

$$\lim_{t \downarrow (-T)} \phi(t, x) = 0.$$



Therefore  $J = \|\phi(0, \cdot)\|_{L^2(\mathcal{O})}^2 = \int_{(-T,0) \times \mathcal{O}} \partial_t |\phi|^2 dt dx$ , from where we get that

$$J = 2\operatorname{Re} \left( \int_{(-T,0) \times \mathcal{O}} \partial_t \phi \bar{\phi} dt dx \right). \quad (3.23)$$

On the other hand, (3.18) and the Green formula yield

$$\begin{aligned} & \operatorname{Im} \left( \int_{(-T,0) \times \mathcal{O}} (M_1 \phi) \bar{\phi} dt dx \right) \\ &= \operatorname{Re} \left( \int_{(-T,0) \times \mathcal{O}} \partial_t \phi \bar{\phi} dt dx \right) + \operatorname{Im} \left( \int_{(-T,0) \times \mathcal{O}} \Delta \phi \bar{\phi} dt dx + s^2 \|\nabla \eta \phi\|_{L^2((-T,0) \times \mathcal{O})}^2 \right) \\ &= \operatorname{Re} \left( \int_{(-T,0) \times \mathcal{O}} \partial_t \phi \bar{\phi} dt dx \right) + \operatorname{Im} \left( \|\nabla \phi\|_{L^2((-T,0) \times \mathcal{O})}^2 \right) = \operatorname{Re} \left( \int_{(-T,0) \times \mathcal{O}} \partial_t \phi \bar{\phi} dt dx \right), \end{aligned}$$

so the result follows from this and (3.23).  $\square$

**Lemma 3.4.** *Let  $w$  and  $J$  be the same as in Lemma 3.3. Then we have*

$$J \leq s^{-3/2} I(w), \quad s > 0,$$

where

$$I(w) := s \|e^{-s\eta} \nabla w\|_{L^2((-T,T) \times \mathcal{O})}^2 + s^3 \|e^{-s\eta} w\|_{L^2((-T,T) \times \mathcal{O})}^2 + \sum_{j=1,2} \|M_j e^{-s\eta} w\|_{L^2((-T,T) \times \mathcal{O})}^2. \quad (3.24)$$

*Proof.* In view of Lemma 3.3 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} J &\leq 2 \|M_1 \phi\|_{L^2((-T,T) \times \mathcal{O})} \|\phi\|_{L^2((-T,T) \times \mathcal{O})} \\ &\leq s^{-3/2} \left( \|M_1(e^{-s\eta} w)\|_{L^2((-T,T) \times \mathcal{O})}^2 + s^3 \|e^{-s\eta} w\|_{L^2((-T,T) \times \mathcal{O})}^2 \right). \end{aligned}$$

This and (3.24) yields the desired result.  $\square$

### 3.2.2 Completion of the proof

The next step of the proof involves majorizing  $I(w)$  with the aid of the Carleman inequality of Proposition 3.2. In view of (3.21)-(3.22) and (3.24) the estimate

$$\begin{aligned} I(w) &\leq C \left( s \|e^{-s\eta} \varphi^{1/2} (\partial_\nu \beta)^{1/2} \partial_\nu w\|_{L^2((-T,T) \times \Gamma_0)}^2 + \|e^{-s\eta} (\chi f' - K v)\|_{L^2((-T,T) \times \mathcal{O})}^2 \right) \\ &\leq C \left( s \|e^{-s\eta} \varphi^{1/2} (\partial_\nu \beta)^{1/2} \partial_\nu w\|_{L^2((-T,T) \times \Gamma_0)}^2 \right. \\ &\quad \left. + \sum_{j=0,1} \|e^{-s\eta} \nabla^j v\|_{L^2((-T,T) \times (\Omega_r \setminus \Omega_\ell))}^2 + \|e^{-s\eta} \chi f'\|_{L^2((-T,T) \times \Omega_r)}^2 \right), \quad (3.25) \end{aligned}$$

holds for  $s \geq s_0$ . Here and henceforth  $C$  denotes a generic positive constant. Bearing in mind that  $\eta(0, x) = \inf_{t \in (-T, T)} n(t, x)$  for all  $x \in \Omega$  and that  $f' = (q_2 - q_1)u'_2$ ,  $u'_2$ , we have

$$\|e^{-s\eta} \chi f'\|_{L^2((-T, T) \times \Omega_r)} \leq C \|e^{-s\eta(0, \cdot)}(q_1 - q_2)\|_{L^2(\Omega_\ell)}. \quad (3.26)$$

Here we used the fact that  $\|u'_2\|_{L^\infty((-T, T) \times \Omega_\ell)} < \infty$ . On the other hand, the function  $\varphi \partial_\nu \beta$  being bounded in  $(-T, T) \times \Gamma_0$  as well, we deduce from (3.25)-(3.26) that

$$\begin{aligned} I(w) &\leq C \left( s \|e^{-s\eta(0, \cdot)} \partial_\nu w\|_{L^2((-T, T) \times \Gamma_0)}^2 + \sum_{j=0,1} \|e^{-s\eta(0, \cdot)} \nabla^j v\|_{L^2((-T, T) \times (\Omega_r \setminus \Omega_\ell))}^2 \right. \\ &\quad \left. + \|e^{-s\eta(0, \cdot)}(q_1 - q_2)\|_{L^2(\Omega_\ell)}^2 \right) \\ &\leq C \left( \mathbf{obs} + \|e^{-s\eta(0, \cdot)}(q_1 - q_2)\|_{L^2(\Omega_\ell)}^2 \right), \quad s \geq s_0, \end{aligned} \quad (3.27)$$

where

$$\mathbf{obs} := s \|e^{-s\eta(0, \cdot)} \partial_\nu w\|_{L^2((-T, T) \times \Gamma_0)}^2 + \sum_{j=0,1} \|e^{-s\eta(0, \cdot)} \nabla^j v\|_{L^2((-T, T) \times (\Omega_r \setminus \Omega_\ell))}^2. \quad (3.28)$$

In light of Lemma 3.4, (3.27)-(3.28) then yields

$$J = \|e^{-s\eta(0, \cdot)} \chi(q_1 - q_2)u_0\|_{L^2(\mathcal{O})}^2 \leq C s^{-3/2} \left( \mathbf{obs} + \|e^{-s\eta(0, \cdot)}(q_1 - q_2)\|_{L^2(\Omega_\ell)}^2 \right), \quad (3.29)$$

for all  $s \geq s_0$ . Further we have  $J \geq \alpha^2 \|e^{-s\eta(0, \cdot)}(q_1 - q_2)\|_{L^2(\Omega_\ell)}^2$  from the assumption (1.3) whence

$$(\alpha^2 - C s^{-3/2}) \|e^{-s\eta(0, \cdot)}(q_1 - q_2)\|_{L^2(\Omega_\ell)}^2 \leq C s^{-3/2} \mathbf{obs}, \quad s \geq s_0, \quad (3.30)$$

by (3.29). Choosing  $s \geq s_0$  so large that  $\alpha^2 - C s^{-3/2} \geq \alpha^2/2$  and taking into account that  $\inf_{x \in \Omega_\ell} e^{-s\eta(0, x)} > 0$ , we derive from (3.28) and (3.30) that

$$\|q_1 - q_2\|_{L^2(\Omega_\ell)}^2 \leq C \left( \|\partial_\nu w\|_{L^2((-T, T) \times \Gamma_0)}^2 + \sum_{j=0,1} \|\nabla^j v\|_{L^2((-T, T) \times (\Omega_r \setminus \Omega_\ell))}^2 \right). \quad (3.31)$$

Now, recalling from §2.3 that  $\|\nabla^j v\|_{L^2((-T, T) \times (\Omega_r \setminus \Omega_\ell))} = 2\|\nabla^j v\|_{L^2((0, T) \times (\Omega_r \setminus \Omega_\ell))}$  for  $j = 0, 1$ , and from the identity  $w = \chi v$  that  $\|\partial_\nu w\|_{L^2((-T, T) \times \Gamma_0)} = 2\|\partial_\nu w\|_{L^2((0, T) \times \Gamma_0)}$ , it follows from (3.31) that

$$\|q_1 - q_2\|_{L^2(\Omega)}^2 \leq C \left( \|\partial_\nu w\|_{L^2((0, T) \times \Gamma_0)}^2 + \sum_{j=0,1} \|\nabla^j v\|_{L^2((0, T) \times (\Omega_r \setminus \Omega_\ell))}^2 \right). \quad (3.32)$$

Here we used the identity  $\|q_1 - q_2\|_{L^2(\Omega_\ell)} = \|q_1 - q_2\|_{L^2(\Omega)}$  arising from (1.4).

Set  $\Gamma_1 := \partial\omega \times (-r, r)$  and  $\Gamma_2 := \Gamma_0 \cap (\overline{\Omega_L} \setminus \Omega_r)$  in such a way that

$$\Gamma_0 = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \emptyset,$$

in virtue of (3.20). Since  $\chi(x_n) = 0$  for every  $|x_n| \geq r$  then the function  $w = \chi v$  is uniformly zero in  $(0, T) \times (\Omega_L \setminus \Omega_r)$ , entailing:

$$\partial_\nu w(t, \sigma) = 0, \quad (t, \sigma) \in (0, T) \times \Gamma_2.$$

As a consequence we have

$$\|\partial_\nu w\|_{L^2((0,T) \times \Gamma_0)} = \|\partial_\nu w\|_{L^2((0,T) \times \Gamma_1)} \leq C \|\partial_\nu v\|_{L^2((0,T) \times \Gamma_1)}.$$

Putting this together with (3.32) and the identity  $v = (u_1 - u_2)'$  we obtain (1.6).

## References

- [1] L. BAUDOIN, J.-P. PUEL, *Uniqueness and stability in an inverse problem for the Schrödinger equation*, Inverse Probl., **18** (2002), 1537-1554.
- [2] M. BELLASSOUED, M. CHOULLI, *Logarithmic stability in the dynamical inverse problem for the Schrödinger equation by arbitrary boundary observation*, J. Math. Pures Appl. **91**(3) (2009), 233255.
- [3] A.L. BUKHGEIM, M.V. KLIBANOV, *Uniqueness in the large of a class of multi-dimensional inverse problems*, Sov. Math. Dokl. **17** (1981), 244247.
- [4] L. CARDOULIS, M. CRISTOFOL, P. GAITAN, *Inverse problems for the Schrödinger operator in an unbounded strip*, J. Inverse Ill-Posed Probl. **16**(2) (2008), 127146.
- [5] M. CHOULLI, E. SOCCORSI, *Recovering the twisting function in a twisted waveguide from the DN map*, arXiv:1209.5662.
- [6] M. CHOULLI, Y. KIAN, E. SOCCORSI, *Stable determination of time dependent scalar potential from boundary measurements in a periodic quantum waveguide*, preprint.
- [7] X. LI, G. UHLMANN, *Inverse problems with partial data in a slab*, Inverse Probl. and Imaging **4**(3) (2010), 449-462.
- [8] A. MERCADO, A. OSSES, L. ROSIER, *Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights*, Inverse Probl. **24**(1) (2008), 015017.