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# Semilinear fractional elliptic equations involving measures

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## Abstract

We study the existence of weak solutions of (E)  $(-\Delta)^\alpha u + g(u) = \nu$  in a bounded regular domain  $\Omega$  in  $\mathbb{R}^N$  ( $N \geq 2$ ) which vanish on  $\mathbb{R}^N \setminus \Omega$ , where  $(-\Delta)^\alpha$  denotes the fractional Laplacian with  $\alpha \in (0, 1)$ ,  $\nu$  is a Radon measure and  $g$  is a nondecreasing function satisfying some extra hypothesis. When  $g$  satisfies a subcritical integrability condition, we prove the existence and uniqueness of a weak solution for problem (E) for any measure. In the case where  $\nu$  is Dirac measure, we characterize the asymptotic behavior of the solution. When  $g(r) = |r|^{k-1}r$  with  $k$  supercritical, we show that a condition of absolute continuity of the measure with respect to some Bessel capacity is a necessary and sufficient condition in order (E) to be solved.

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**Key words:** Fractional Laplacian, Radon measure, Dirac measure, Green kernel, Bessel capacities.

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# 1 Introduction

Let  $\Omega \subset \mathbb{R}^N$  be a domain and  $g : \mathbb{R} \mapsto \mathbb{R}$  a continuous function. We are concerned with the existence of weak solutions to the semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \tag{1.1}$$

where  $\alpha \in (0, 1)$ ,  $\nu$  is a Radon measure such that  $\int_\Omega \rho^\beta d|\nu| < \infty$  for some  $\beta \in [0, \alpha]$  and  $\rho(x) = \text{dist}(x, \Omega^c)$ . The fractional Laplacian  $(-\Delta)^\alpha$  is defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \rightarrow 0^+} (-\Delta)_\epsilon^\alpha u(x),$$

where for  $\epsilon > 0$ ,

$$(-\Delta)_\epsilon^\alpha u(x) = - \int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz \tag{1.2}$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \epsilon], \\ 1, & \text{if } t > \epsilon. \end{cases}$$

When  $\alpha = 1$ , the semilinear elliptic problem

$$\begin{aligned} -\Delta u + g(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \partial\Omega, \end{aligned} \tag{1.3}$$

has been extensively studied by numerous authors in the last 30 years. A fundamental contribution is due to Brezis [7], Benilan and Brezis [2], where  $\nu$  is a bounded measure in  $\Omega$  and the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing, positive on  $(0, +\infty)$  and satisfies the subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s)) s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

They proved the existence and uniqueness of the solution for problem (1.3). Baras and Pierre [1] studied (1.3) when  $g(u) = |u|^{p-1}u$  for  $p > 1$  and  $\nu$  is absolutely continuous with respect to the Bessel capacity  $C_{2, \frac{p}{p-1}}$ , to obtain a solution. In [20] Véron extended Benilan and Brezis results in replacing the Laplacian by a general uniformly elliptic second order differential operator with Lipschitz continuous coefficients; he obtained existence and uniqueness results for solutions, when  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  with  $\beta \in [0, 1]$  where  $\mathfrak{M}(\Omega, \rho^\beta)$  denotes the space of Radon measures in  $\Omega$  satisfying

$$\int_\Omega \rho^\beta d|\nu| < \infty, \tag{1.4}$$

$\mathfrak{M}(\Omega, \rho^0) = \mathfrak{M}^b(\Omega)$  is the set of bounded measures and  $g$  is nondecreasing and satisfies the  $\beta$ -subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s))s^{-2\frac{N+\beta-1}{N+\beta-2}} ds < +\infty.$$

The study of general semilinear elliptic equations with measure data have been investigated, such as the equations involving measures boundary data which was initiated by Gmira and Véron [20] who adapted the method introduced by Benilan and Brezis to obtain the existence and uniqueness of solution. This subject has been vastly expanded in recent years, see the papers of Marcus and Véron [22, 23, 24, 25], Bidaut-Véron and Vivier [5], Bidaut-Véron, Hung and Véron [4].

Recently, great attention has been devoted to non-linear equations involving fractional Laplacian or more general integro-differential operators. We mention the works by Caffarelli and Silvestre [9, 10], Cabré and Tan [8], Chen, Felmer and Quaas [14], Chen, Li and Ou [15], Li [21], Ros-oton and Serra [27], Silvestre [28], Sire and Valdinoci [29]. In particular, in [14] the authors obtained the existence of large solutions to equation

$$(-\Delta)^\alpha u + g(u) = f \quad \text{in } \Omega, \quad (1.5)$$

where  $\Omega$  is a bounded regular domain. In [13] the authors considered the properties of possibly singular solutions of (1.5) in punctured domain . It is a well-known fact [33] that for  $\alpha = 1$  the weak singular solutions of (1.5) in punctured domain are classified according the type of singularities they admits: either weak singularities with Dirac mass, or strong singularities which are the upper limit of solutions with weak singularities. One of our interests is to extend these properties to any  $\alpha \in (0, 1)$  and furthermore to consider general Radon measures.

In this paper we study the existence and uniqueness of solutions of (1.1) in a measure framework. Before stating our main theorem we make precise the notion of weak solution used in this article.

**Definition 1.1** *We say that  $u$  is a weak solution of (1.1), if  $u \in L^1(\Omega)$ ,  $g(u) \in L^1(\Omega, \rho^\alpha dx)$  and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + g(u)\xi] dx = \int_\Omega \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha, \quad (1.6)$$

where  $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$  is the space of functions  $\xi$  satisfying:

- (i)  $\text{supp}(\xi) \subset \bar{\Omega}$ ,
- (ii)  $(-\Delta)^\alpha \xi(x)$  exists for all  $x \in \Omega$  and  $|(-\Delta)^\alpha \xi(x)| \leq C$  for some  $C > 0$ ,
- (iii) there exist  $\varphi \in L^1(\Omega, \rho^\alpha dx)$  and  $\epsilon_0 > 0$  such that  $|(-\Delta)_\epsilon^\alpha \xi| \leq \varphi$  a.e. in  $\Omega$ , for all  $\epsilon \in (0, \epsilon_0]$ .

We notice that for  $\alpha = 1$ , the test space  $\mathbb{X}_\alpha$  is used as  $C_0^{1,L}(\Omega)$ , which has similar properties like (i) and (ii). The counter part for the Laplacian of assumption (iii) would be that the difference quotient  $\nabla_{x_j,h}[u](\cdot) := h^{-1}\partial_{x_j}u(\cdot + h\mathbf{e}_j) + \partial_{x_j}u(\cdot)$  is bounded by an  $L^1$ -function, which is true since

$$\nabla_{x_j,h}[u](x) = h^{-1} \int_0^h \partial_{x_j,x_j}^2 u(x + s\mathbf{e}_j) ds.$$

We denote by  $G$  the Green kernel of  $(-\Delta)^\alpha$  in  $\Omega$  and by  $\mathbb{G}[\cdot]$  the Green operator defined by

$$\mathbb{G}[f](x) = \int_\Omega G(x,y)f(y)dy \quad \forall f \in L^1(\Omega, \rho^\alpha dx). \quad (1.7)$$

For  $N \geq 2$ ,  $0 < \alpha < 1$  and  $\beta \in [0, \alpha]$ , we define the critical exponent

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if } \beta \in [0, \frac{N-2\alpha}{N}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if } \beta \in (\frac{N-2\alpha}{N}\alpha, \alpha]. \end{cases} \quad (1.8)$$

Our main result is the following:

**Theorem 1.1** *Assume  $\Omega \subset \mathbb{R}^N$  ( $N \geq 2$ ) is a bounded  $C^2$  domain,  $\alpha \in (0, 1)$ ,  $\beta \in [0, \alpha]$  and  $k_{\alpha,\beta}$  is defined by (1.8). Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous, nondecreasing function, satisfying*

$$g(r)r \geq 0, \quad \forall r \in \mathbb{R} \quad \text{and} \quad \int_1^\infty (g(s) - g(-s))s^{-1-k_{\alpha,\beta}} ds < \infty. \quad (1.9)$$

*Then for any  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  problem (1.1) admits a unique weak solution  $u$ . Furthermore*

$$-\mathbb{G}(\nu_-) \leq u \leq \mathbb{G}(\nu_+) \quad \text{a.e. in } \Omega \quad (1.10)$$

*where  $\nu_+$  and  $\nu_-$  are respectively the positive and negative part in the Jordan decomposition of  $\nu$ .*

We note that for  $\alpha = 1$  and  $\beta \in [0, 1)$ , we have

$$k > \frac{N + \beta}{N - 2 + \beta}, \quad (1.11)$$

where  $k$  is given in (1.8) and the number in right hand side of (1.11) is from Theorem 3.7 in [34]. Inspired by [20, 34], the existence of solution could be extended in assuming that  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and satisfies the  $(N, \alpha, \beta)$ -weak-singularity assumption, that is, there exists  $r_0 > 0$  such that

$$g(x, r)r \geq 0, \quad \forall (x, r) \in \Omega \times (\mathbb{R} \setminus (-r_0, r_0)),$$

and

$$|g(x, r)| \leq \tilde{g}(|r|), \quad \forall (x, r) \in \Omega \times \mathbb{R},$$

where  $\tilde{g} : [0, \infty) \rightarrow [0, \infty)$  is continuous, nondecreasing and satisfies

$$\int_1^\infty \tilde{g}(s)s^{-1-k}ds < \infty.$$

We also give a stability result which shows that problem (1.1) is weakly closed in the space of measures  $\mathfrak{M}(\Omega, \rho^\beta)$ . In the last section we characterize the behaviour of the solution  $u$  of (1.1) when  $\nu = \delta_a$  for some  $a \in \Omega$ . We also study the case where  $g(r) = |r|^{k-1}r$  when  $k \geq k_{\alpha, \beta}$ , which implies that (1.9). We show that a necessary and sufficient condition in order a weak solution to problem

$$\begin{aligned} (-\Delta)^\alpha u + |u|^{k-1}u &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \tag{1.12}$$

to exist where  $\nu$  is a positive bounded measure is that  $\nu$  vanishes on compact subsets  $K$  of  $\Omega$  with zero  $C_{2\alpha, k'}$  Bessel-capacity.

The paper is organized as follows. In Section 2 we give some properties of Marcinkiewicz spaces and obtain the optimal index  $k$  for which there holds

$$\|\mathbb{G}(\nu)\|_{M^k(\Omega, \rho^\gamma dx)} \leq C\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}. \tag{1.13}$$

We also gives some integration by parts formulas and prove a Kato's type inequalities. In Section 3, we prove Theorem 1.1. In Section 4 we give applications the cases where the measure is a Dirac mass and where the nonlinearity is a power function.

## 2 Linear estimates

### 2.1 The Marcinkiewicz spaces

We recall the definition and basic properties of the Marcinkiewicz spaces.

**Definition 2.1** Let  $\Omega \subset \mathbb{R}^N$  be an open domain and  $\mu$  a positive Borel measure in  $\Omega$ . For  $\kappa > 1$ ,  $\kappa' = \kappa/(\kappa - 1)$  and  $u \in L^1_{loc}(\Omega, d\mu)$ , we set

$$\|u\|_{M^\kappa(\Omega, d\mu)} = \inf\{c \in [0, \infty) : \int_E |u|d\mu \leq c \left( \int_E d\mu \right)^{\frac{1}{\kappa'}}, \forall E \text{ Borel subset of } \Omega\} \tag{2.1}$$

and

$$M^\kappa(\Omega, d\mu) = \{u \in L^1_{loc}(\Omega, d\mu) : \|u\|_{M^\kappa(\Omega, d\mu)} < \infty\}. \tag{2.2}$$

$M^\kappa(\Omega, d\mu)$  is called the Marcinkiewicz space of exponent  $\kappa$  or weak  $L^\kappa$  space and  $\|\cdot\|_{M^\kappa(\Omega, d\mu)}$  is a quasi-norm. The following property holds.

**Proposition 2.1** [3, 16] Assume  $1 \leq q < \kappa < \infty$  and  $u \in L^1_{loc}(\Omega)$ . Then there exists  $C(q, \kappa) > 0$  such that

$$\int_E |u|^q d\mu \leq C(q, \kappa) \|u\|_{M^\kappa(\Omega, d\mu)} \left( \int_E d\mu \right)^{1-q/\kappa},$$

for any Borel set  $E$  of  $\Omega$ .

For  $\alpha \in (0, 1)$  and  $\beta, \gamma \in [0, \alpha]$  we set

$$k_1(t) = \frac{\gamma}{\alpha} + \frac{N - (N - 2\alpha)\frac{\gamma}{\alpha}}{N - 2\alpha + t}, \quad k_2(t) = \gamma + \frac{N - (N - 2\alpha)\frac{\gamma}{\alpha}}{N - 2\alpha + t} \quad (2.3)$$

and

$$t_{\alpha, \beta, \gamma} = \min\{t \in [0, \alpha] : \frac{k_2(t)}{k_1(t)} \geq \beta\}. \quad (2.4)$$

**Remark 2.1** The quantity  $t_{\alpha, \beta, \gamma}$  is well defined, since

$$\frac{k_2(\alpha)}{k_1(\alpha)} = \frac{\gamma + \alpha \frac{N - (N - 2\alpha)\frac{\gamma}{\alpha}}{N - \alpha}}{\frac{\gamma}{\alpha} + \frac{N - (N - 2\alpha)\frac{\gamma}{\alpha}}{N - \alpha}} = \alpha \geq \beta.$$

**Remark 2.2** The function  $t \mapsto k_1(t)$  is decreasing in  $[0, \alpha]$  with the following bounds

$$k_1(0) = \frac{N}{N - 2\alpha} \quad \text{and} \quad k_1(\alpha) = \frac{N + \gamma}{N - \alpha} > 1.$$

**Remark 2.3** The function  $t \mapsto \frac{k_2(t)}{k_1(t)}$  is increasing in  $[0, \alpha]$ , since

$$\left( \frac{k_2(t)}{k_1(t)} \right)' = \frac{[N - (N - 2\alpha)\frac{\gamma}{\alpha}](N + \gamma)}{k_1^2(t)} > 0.$$

As a consequence (2.4) is equivalent to

$$t_{\alpha, \beta, \gamma} = \max\{0, t_\beta\}, \quad (2.5)$$

where

$$t_\beta = \frac{\beta N - (N - 2\alpha)\gamma}{N - (N - 3\alpha + \beta)\frac{\gamma}{\alpha}}. \quad (2.6)$$

is the solution of  $\frac{k_2(t)}{k_1(t)} = \beta$ .

**Proposition 2.2** Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 2$ ) be a bounded  $C^2$  domain and  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  with  $\beta \in [0, \alpha]$ . Then

$$\|\mathbb{G}[\nu]\|_{M^{\kappa, \alpha, \beta, \gamma}(\Omega, \rho^\gamma dx)} \leq C \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}, \quad (2.7)$$

where  $\gamma \in [0, \alpha]$ ,  $\mathbb{G}[\nu](x) = \int_\Omega G(x, y) d\nu(y)$  where  $G$  is Green's kernel of  $(-\Delta)^\alpha$  and

$$k_{\alpha, \beta, \gamma} = \begin{cases} \frac{N + \gamma}{N - 2\alpha + \beta}, & \text{if } \gamma \leq \frac{N\beta}{N - 2\alpha}, \\ \frac{N}{N - 2\alpha}, & \text{if not.} \end{cases} \quad (2.8)$$

**Proof.** For  $\lambda > 0$  and  $y \in \Omega$ , we denote

$$A_\lambda(y) = \{x \in \Omega \setminus \{y\} : G(x, y) > \lambda\} \text{ and } m_\lambda(y) = \int_{A_\lambda(y)} \rho^\gamma(x) dx.$$

From [11], there exists  $C > 0$  such that for any  $(x, y) \in \Omega \times \Omega$ ,  $x \neq y$ ,

$$G(x, y) \leq C \min \left\{ \frac{1}{|x-y|^{N-2\alpha}}, \frac{\rho^\alpha(x)}{|x-y|^{N-\alpha}}, \frac{\rho^\alpha(y)}{|x-y|^{N-\alpha}} \right\} \quad (2.9)$$

and

$$G(x, y) \leq C \frac{\rho^\alpha(y)}{\rho^\alpha(x)|x-y|^{N-2\alpha}}. \quad (2.10)$$

Therefore, if  $\gamma \in [0, \alpha]$  and  $x \in A_\lambda(y)$ , there holds

$$\rho^\gamma(x) \leq \frac{C\rho^\gamma(y)}{\lambda^{\frac{\gamma}{\alpha}}|x-y|^{(N-2\alpha)\frac{\gamma}{\alpha}}}. \quad (2.11)$$

Let  $t \in [0, \alpha]$  be such that  $\frac{k_2(t)}{k_1(t)} \geq \beta$ , where  $k_1(t)$  and  $k_2(t)$  are given in (2.3), then

$$G(x, y) \leq \left( \frac{C}{|x-y|^{N-2\alpha}} \right)^{1-\frac{t}{\alpha}} \left( \frac{C\rho^\alpha(y)}{|x-y|^{N-\alpha}} \right)^{\frac{t}{\alpha}} = \frac{C\rho^t(y)}{|x-y|^{N-2\alpha+t}}.$$

We observe that

$$A_\lambda(y) \subset \left\{ x \in \Omega \setminus \{y\} : \frac{C\rho(y)^t}{|x-y|^{N-2\alpha+t}} > \lambda \right\} \subset D_\lambda(y)$$

where  $D_\lambda(y) := \left\{ x \in \Omega : |x-y| < \left( \frac{C\rho^t(y)}{\lambda} \right)^{\frac{1}{N-2\alpha+t}} \right\}$ ; together with (2.11), this implies

$$m_\lambda(y) \leq \int_{D_\lambda(y)} \frac{C\rho^\gamma(y)}{\lambda^{\frac{\gamma}{\alpha}}|x-y|^{(N-2\alpha)\frac{\gamma}{\alpha}}} dx \leq C\rho(y)^{k_2(t)}\lambda^{-k_1(t)}.$$

For any Borel set  $E$  of  $\Omega$ , we have

$$\int_E G(x, y)\rho^\gamma(x) dx \leq \int_{A_\lambda(y)} G(x, y)\rho^\gamma(x) dx + \lambda \int_E \rho^\gamma(x) dx$$

and

$$\begin{aligned} \int_{A_\lambda(y)} G(x, y)\rho^\gamma(x) dx &= - \int_\lambda^\infty sdm_s(y) \\ &= \lambda m_\lambda(y) + \int_\lambda^\infty m_s(y) ds \\ &\leq C\rho(y)^{k_2(t)}\lambda^{1-k_1(t)}. \end{aligned}$$

Thus,

$$\int_E G(x, y) \rho^\gamma(x) dx \leq C \rho(y)^{k_2(t)} \lambda^{1-k_1(t)} + \lambda \int_E \rho^\gamma(x) dx.$$

By choosing  $\lambda = [\rho(y)^{-k_2(t)} \int_E \rho^\gamma(x) dx]^{-\frac{1}{k_1(t)}}$ , we have

$$\int_E G(x, y) \rho^\gamma(x) dx \leq C \rho(y)^{\frac{k_2(t)}{k_1(t)}} \left( \int_E \rho^\gamma(x) dx \right)^{\frac{k_1(t)-1}{k_1(t)}}.$$

Therefore,

$$\begin{aligned} \int_E \mathbb{G}(|\nu|)(x) \rho^\gamma(x) dx &= \int_\Omega \int_E G(x, y) \rho^\gamma(x) dx d|\nu(y)| \\ &\leq C \int_\Omega \rho(y)^{\frac{k_2(t)}{k_1(t)}} d|\nu(y)| \left( \int_E \rho^\gamma(x) dx \right)^{\frac{k_1(t)-1}{k_1(t)}} \\ &\leq C \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} \left( \int_E \rho^\gamma(x) dx \right)^{\frac{k_1(t)-1}{k_1(t)}}, \end{aligned}$$

since by our choice of  $t$ ,  $\frac{k_2(t)}{k_1(t)} \geq \beta$ , which guarantees that

$$\int_\Omega \rho(y)^{\frac{k_2(t)}{k_1(t)}} d|\nu(y)| \leq \max_\Omega \rho^{\frac{k_2(t)}{k_1(t)} - \beta} \int_\Omega \rho(y)^\beta d|\nu(y)|.$$

As a consequence,

$$\|\mathbb{G}(\nu)\|_{M^{k_1(t)}(\Omega, \rho^\gamma dx)} \leq C \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}.$$

Therefore

$$k_{\alpha, \beta, \gamma} := \max\{k_1(t) : t \in [0, \alpha]\} = k_1(t_{\alpha, \beta, \gamma})$$

where  $t_{\alpha, \beta, \gamma}$  is defined by (2.4), we notice that  $k_{\alpha, \beta, \gamma}$  is given by (2.8), which completes the proof.  $\square$

We choose the parameter  $\gamma$  in order to make  $k_{\alpha, \beta, \gamma}$  the largest possible, and denote

$$k_{\alpha, \beta} = \max_{\gamma \in [0, \alpha]} k_{\alpha, \beta, \gamma}. \quad (2.12)$$

Since  $\gamma \mapsto k_{\alpha, \beta, \gamma}$  is increasing, the following statement holds.

**Proposition 2.3** *Let  $N \geq 2$  and  $k_{\alpha, \beta}$  be defined by (2.12), then*

$$k_{\alpha, \beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if } \beta \in [0, \frac{N-2\alpha}{N}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if } \beta \in (\frac{N-2\alpha}{N}\alpha, \alpha]. \end{cases} \quad (2.13)$$

## 2.2 Non-homogeneous problem

In this subsection, we study some properties of the solution of the linear non-homogeneous, which will play a key role in the sequel. We assume that  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$  is a bounded domain with a  $C^2$  boundary.

**Lemma 2.1** (i) *There exists  $C > 0$  such that for any  $\xi \in \mathbb{X}_\alpha$  there holds*

$$\|\xi\|_{C^\alpha(\bar{\Omega})} \leq C \|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}, \quad (2.14)$$

and

$$\|\rho^{-\alpha} \xi\|_{C^\theta(\bar{\Omega})} \leq C \|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}. \quad (2.15)$$

where  $0 < \theta < \min\{\alpha, 1 - \alpha\}$ .

(ii) *Let  $u$  be the solution of*

$$\begin{aligned} (-\Delta)^\alpha u &= f & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \quad (2.16)$$

where  $f \in C^1(\bar{\Omega})$ . Then  $u \in \mathbb{X}_\alpha$ .

**Proof.** (i). Estimates (2.14) and (2.15) are consequences of [27, Prop 1.1] and [27, Th 1.2] respectively. Furthermore, if  $\eta_1$  is the solution of (2.16) with  $f \equiv 1$  in  $\Omega$ , then  $\eta_1 > 0$  in  $\Omega$ , there exists  $C > 0$  such that

$$C^{-1} \leq \frac{\eta_1}{\rho^\alpha} \leq C \quad \text{in } \Omega. \quad (2.17)$$

In this expression the left-hand side inequality follows from the maximum principle and [11, Th 1.2]. Since  $|\xi| \leq \|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)} \eta_1$ , it follows by the maximum principle,

$$-C \|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)} \rho^\alpha(x) \leq \xi(x) \leq C \|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)} \rho^\alpha(x). \quad (2.18)$$

(ii) For  $r > 0$ , we denote

$$\Omega_r = \{z \in \Omega : \text{dist}(z, \partial\Omega) > r\}.$$

Since  $f \in C^1(\bar{\Omega})$ , then by Theorem 1.5 and Corollary 1.6 part (i) in [27], for  $\theta \in [0, \min\{\alpha, 1 - \alpha\})$ , there exists  $C > 0$  such that for any  $r > 0$ , we have

$$\|u\|_{C^{2\alpha+\theta}(\Omega_r)} \leq Cr^{-\alpha-\theta}$$

and

$$\|u\|_{C^\alpha(\mathbb{R}^N)} \leq C.$$

Then for  $x \in \Omega$ , letting  $r = \rho(x)/2$ ,

$$|\delta(u, x, y)| \leq Cr^{-\alpha-\theta} |y|^{2\alpha+\theta}, \quad \forall y \in B_r(0) \quad (2.19)$$

and

$$|\delta(u, x, y)| \leq C|y|^\alpha, \quad \forall y \in \mathbb{R}^N,$$

where  $\delta(u, x, y) = u(x + y) + u(x - y) - 2u(x)$ . Thus,

$$\begin{aligned} |(-\Delta)_\epsilon^\alpha u(x)| &\leq \frac{1}{2} \int_{\mathbb{R}^N} \frac{|\delta(u, x, y)|}{|y|^{N+2\alpha}} \chi_\epsilon(|y|) dy \\ &\leq \frac{1}{2} \int_{B_r(0)} \frac{|\delta(u, x, y)|}{|y|^{N+2\alpha}} dy + \frac{1}{2} \int_{B_r^c(0)} \frac{|\delta(u, x, y)|}{|y|^{N+2\alpha}} dy \\ &\leq \frac{Cr^{-\alpha-\theta}}{2} \int_{B_r(0)} \frac{1}{|y|^{N-\theta}} dy + \frac{C}{2} \int_{B_r^c(0)} \frac{1}{|y|^{N+\alpha}} dy \\ &\leq C\rho(x)^{-\alpha}, \quad x \in \Omega, \end{aligned}$$

for some  $C > 0$  independent of  $\epsilon$ . Moreover,  $\rho^{-\alpha}$  is in  $L^1(\Omega)$ . Finally, we prove  $(-\Delta)_\epsilon^\alpha u \rightarrow (-\Delta)^\alpha u$  as  $\epsilon \rightarrow 0^+$  pointwise. For  $x \in \Omega$ , choosing  $\epsilon \in (0, \rho(x)/2)$ , then by (2.19),

$$\begin{aligned} |(-\Delta)^\alpha u(x) - (-\Delta)_\epsilon^\alpha u(x)| &\leq \frac{1}{2} \int_{B_\epsilon(0)} \frac{|\delta(u, x, y)|}{|y|^{N+2\alpha}} dy \\ &\leq C\rho(x)^{-\alpha-\theta} \epsilon^\theta \\ &\rightarrow 0, \quad \epsilon \rightarrow 0^+. \end{aligned}$$

The proof is complete.  $\square$

The following Proposition is the basic estimate for proving the uniqueness of the solution of (1.1).

**Proposition 2.4** *If  $\nu \in L^1(\Omega, \rho^\alpha dx)$ , there exists a unique weak solution  $u$  of the problem*

$$\begin{aligned} (-\Delta)^\alpha u &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{2.20}$$

For any  $\xi \in \mathbb{X}_\alpha$ ,  $\xi \geq 0$ , we have

$$\int_{\Omega} |u| (-\Delta)^\alpha \xi dx \leq \int_{\Omega} \xi \text{sign}(u) \nu dx, \tag{2.21}$$

and

$$\int_{\Omega} u_+ (-\Delta)^\alpha \xi dx \leq \int_{\Omega} \xi \text{sign}_+(u) \nu dx, \tag{2.22}$$

We note here that for  $\alpha = 1$ , the proof of Proposition 2.4 could be seen in [34, Th 2.4]. For  $\alpha \in (0, 1)$ , we first prove some integration by parts formula.

**Lemma 2.2** *Assume  $u, \xi \in \mathbb{X}_\alpha$ , then*

$$\int_{\Omega} u (-\Delta)^\alpha \xi dx = \int_{\Omega} \xi (-\Delta)^\alpha u dx. \tag{2.23}$$

**Proof.** We first prove that

$$\int_{\Omega} u(-\Delta)_{\epsilon}^{\alpha} \xi dx = \int_{\Omega} \xi(-\Delta)_{\epsilon}^{\alpha} u dx, \quad u, \xi \in \mathbb{X}_{\alpha}. \quad (2.24)$$

Denote

$$(-\Delta)_{\Omega, \epsilon}^{\alpha} u(x) = - \int_{\Omega} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_{\epsilon}(|x - z|) dz. \quad (2.25)$$

By the definition of  $(-\Delta)_{\epsilon}^{\alpha}$ , we have

$$\begin{aligned} (-\Delta)_{\epsilon}^{\alpha} u(x) &= - \int_{\Omega} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_{\epsilon}(|x - z|) dz + u(x) \int_{\Omega^c} \frac{\chi_{\epsilon}(|x - z|)}{|z - x|^{N+2\alpha}} dz \\ &= (-\Delta)_{\Omega, \epsilon}^{\alpha} u(x) + u(x) \int_{\Omega^c} \frac{\chi_{\epsilon}(|x - z|)}{|z - x|^{N+2\alpha}} dz. \end{aligned}$$

We claim that

$$\int_{\Omega} \xi(x) (-\Delta)_{\Omega, \epsilon}^{\alpha} u(x) dx = \int_{\Omega} u(x) (-\Delta)_{\Omega, \epsilon}^{\alpha} \xi(x) dx, \quad \text{for } u, \xi \in \mathbb{X}_{\alpha}. \quad (2.26)$$

By using the fact of

$$\int_{\Omega} \int_{\Omega} \frac{[u(z) - u(x)] \xi(x)}{|z - x|^{N+2\alpha}} \chi_{\epsilon}(|x - z|) dz dx = \int_{\Omega} \int_{\Omega} \frac{[u(x) - u(z)] \xi(z)}{|z - x|^{N+2\alpha}} \chi_{\epsilon}(|x - z|) dz dx,$$

we have

$$\begin{aligned} & \int_{\Omega} \xi(x) (-\Delta)_{\Omega, \epsilon}^{\alpha} u(x) dx \\ &= -\frac{1}{2} \int_{\Omega} \int_{\Omega} \left[ \frac{(u(z) - u(x)) \xi(x)}{|z - x|^{N+2\alpha}} + \frac{(u(x) - u(z)) \xi(z)}{|z - x|^{N+2\alpha}} \right] \chi_{\epsilon}(|x - z|) dz dx \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{[u(z) - u(x)] [\xi(z) - \xi(x)]}{|z - x|^{N+2\alpha}} \chi_{\epsilon}(|x - z|) dz dx. \end{aligned}$$

Similarly, by the fact that  $u \in \mathbb{X}_{\alpha}$ ,

$$\int_{\Omega} u(x) (-\Delta)_{\Omega, \epsilon}^{\alpha} \xi(x) dx = \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{[u(z) - u(x)] [\xi(z) - \xi(x)]}{|z - x|^{N+2\alpha}} \chi_{\epsilon}(|x - z|) dz dx.$$

Then (2.26) holds. In order to prove (2.23), we first notice that by (2.26),

$$\begin{aligned} & \int_{\Omega} \xi(x) (-\Delta)_{\epsilon}^{\alpha} u(x) dx \\ &= \int_{\Omega} (-\Delta)_{\Omega, \epsilon}^{\alpha} u(x) dx + \int_{\Omega} u(x) \xi(x) \int_{\Omega^c} \frac{\chi_{\epsilon}(|x - z|)}{|z - x|^{N+2\alpha}} dz dx \\ &= \int_{\Omega} u(x) (-\Delta)_{\Omega, \epsilon}^{\alpha} \xi(x) dx + \int_{\Omega} u(x) \xi(x) \int_{\Omega^c} \frac{\chi_{\epsilon}(|x - z|)}{|z - x|^{N+2\alpha}} dz dx \\ &= \int_{\Omega} u(x) (-\Delta)_{\epsilon}^{\alpha} \xi(x) dx. \end{aligned} \quad (2.27)$$

Since  $u$  and  $\xi$  belongs to  $\mathbb{X}_\alpha$ ,  $(-\Delta)_\epsilon^\alpha \xi \rightarrow (-\Delta)^\alpha \xi$  and  $(-\Delta)_\epsilon^\alpha u \rightarrow (-\Delta)^\alpha u$  and  $|u(-\Delta)_\epsilon^\alpha \xi| + |\xi(-\Delta)_\epsilon^\alpha u| \leq 2\varphi$ . It follows by the dominated convergence Theorem

$$\lim_{\epsilon \rightarrow 0^+} \int_{\Omega} \xi(x)(-\Delta)_\epsilon^\alpha u(x) dx = \int_{\Omega} \xi(x)(-\Delta)^\alpha u(x) dx$$

and

$$\lim_{\epsilon \rightarrow 0^+} \int_{\Omega} (-\Delta)_\epsilon^\alpha \xi(x) u(x) dx = \int_{\Omega} (-\Delta)^\alpha \xi(x) u(x) dx.$$

Letting  $\epsilon \rightarrow 0^+$  of (2.24) we conclude that (2.23) holds.  $\square$

For  $1 \leq p < \infty$  and  $0 < \beta < 1$ ,  $W^{\beta,p}(\Omega)$  is the set of  $\xi \in L^p(\Omega)$  such that

$$\int_{\Omega} \int_{\Omega} \frac{|\xi(x) - \xi(y)|^p}{|x - y|^{N+\beta p}} dy dx < \infty. \quad (2.28)$$

This space is endowed with the norm

$$\|\xi\|_{W^{\beta,p}(\Omega)} = \left( \int_{\Omega} |\xi(x)|^p dx + \int_{\Omega} \int_{\Omega} \frac{|\xi(x) - \xi(y)|^p}{|x - y|^{N+\beta p}} dy dx \right)^{\frac{1}{p}}. \quad (2.29)$$

Furthermore, if  $\Omega$  is bounded, the following Poincaré inequality holds [32, p 134].

$$\left( \int_{\Omega} |\xi(x)|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_{\Omega} \int_{\Omega} \frac{|\xi(x) - \xi(y)|^p}{|x - y|^{N+\beta p}} dy dx \right)^{\frac{1}{p}} \quad \forall \xi \in C_c^{\overline{\Omega}}. \quad (2.30)$$

**Lemma 2.3** *Let  $\gamma \in C^2(\mathbb{R})$  be a convex function such that  $\gamma(0) = 0$  and  $u \in \mathbb{X}_\alpha$ , then  $u \in W^{\alpha,2}(\Omega)$ ,  $\gamma \circ u \in \mathbb{X}_\alpha$  and for almost all  $x \in \Omega$ , there exists  $z_x \in \overline{\Omega}$  such that*

$$(-\Delta)^\alpha (\gamma \circ u)(x) = (\gamma' \circ u)(x) (-\Delta)^\alpha u(x) - \frac{\gamma'' \circ u(z_x)}{2} \int_{\Omega} \frac{(u(y) - u(x))^2}{|y - x|^{N+2\alpha}} dy. \quad (2.31)$$

**Proof.** Since  $u \in C(\overline{\Omega})$  vanishes in  $\Omega^c$ ,  $\gamma \circ u$  shares the same properties. By (2.14), for any  $x$  and  $y$  in  $\Omega$

$$(u(x) - u(y))^2 \leq C|x - y|^{2\alpha} \|(-\Delta u)^\alpha\|_{L^\infty(\Omega)}^2$$

Then  $u \in W^{\alpha,2}(\Omega)$ . Similarly  $\gamma \circ u \in W^{\alpha,2}(\Omega)$ . Furthermore

$$(\gamma \circ u)(y) - (\gamma \circ u)(x) = (\gamma' \circ u)(x) (u(y) - u(x)) + \int_{u(x)}^{u(y)} (u(y) - t) \gamma''(t) dt$$

By the mean value theorem, there exists some  $\tau \in [0, 1]$  such that

$$\int_{u(x)}^{u(y)} (u(y) - t)\gamma''(t)dt = \frac{\gamma''(\tau u(y) + (1 - \tau)u(x))}{2}(u(y) - u(x))^2$$

Since  $\gamma''$  is continuous and  $u$  is continuous in  $\bar{\Omega}$ ,

$$\left| \int_{u(x)}^{u(y)} (u(y) - t)\gamma''(t)dt \right| \leq \frac{\|\gamma'' \circ u\|_{L^\infty}}{2}(u(y) - u(x))^2$$

and since  $u \in W^{\alpha, 2}(\Omega)$ ,

$$\left| \int_{|y-x|>\epsilon} \int_{u(x)}^{u(y)} (u(y) - t)\gamma''(t)dt \frac{dy}{|y-x|^{N+2\alpha}} \right| \leq \frac{\|\gamma'' \circ u\|_{L^\infty}}{2} \int_{\Omega} (u(y) - u(x))^2 \frac{dy}{|y-x|^{N+2\alpha}}.$$

Notice also that  $\tau u(y) + (1 - \tau)u(x) \subset u(\Omega) = [u(z_0), u(z_1)] := I$ , therefore

$$\min_{t \in I} \gamma''(t) \leq \gamma''(\tau u(y) + (1 - \tau)u(x)) \leq \max_{t \in I} \gamma''(t)$$

thus

$$\begin{aligned} & \frac{\min_{t \in I} \gamma''(t)}{2} \int_{\Omega} \frac{(u(y) - u(x))^2}{|y-x|^{N+2\alpha}} dy \\ & \leq \int_{\Omega} \int_{u(x)}^{u(y)} (u(y) - t)\gamma''(t)dt \frac{dy}{|y-x|^{N+2\alpha}} \\ & \leq \frac{\max_{t \in I} \gamma''(t)}{2} \int_{\Omega} \frac{(u(y) - u(x))^2}{|y-x|^{N+2\alpha}} dy. \end{aligned}$$

Since  $\gamma''$  is continuous, there exists  $t_0 \in I$  such that

$$\int_{\Omega} \int_{u(x)}^{u(y)} (u(y) - t)\gamma''(t)dt \frac{dy}{|y-x|^{N+2\alpha}} = \gamma''(t_0) \int_{\Omega} \frac{(u(y) - u(x))^2}{|y-x|^{N+2\alpha}} dy,$$

and since  $\bar{\Omega}$  is connected there exists  $z_x \in \bar{\Omega}$  such that  $t_0 = u(z_x)$ , which ends the proof.  $\square$

**Proof of Proposition 2.4. Uniqueness.** Let  $w$  be a weak solution of

$$\begin{aligned} (-\Delta)^\alpha w &= 0 & \text{in } \Omega \\ w &= 0 & \text{in } \Omega^c. \end{aligned} \tag{2.32}$$

If  $\omega$  is a Borel subset of  $\Omega$  and  $\eta_\omega$  the solution of

$$\begin{aligned} (-\Delta)^\alpha \eta_\omega &= \chi_\omega & \text{in } \Omega \\ \eta_\omega &= 0 & \text{in } \Omega^c, \end{aligned} \tag{2.33}$$

then  $\eta_\omega \in \mathbb{X}_\alpha$  and

$$\int_\omega w dx = 0.$$

This implies  $w = 0$ .

*Existence and estimate (2.21).* For  $\delta > 0$  we define an even convex function  $\phi_\delta$  by

$$\phi_\delta(t) = \begin{cases} |t| - \frac{\delta}{2}, & \text{if } |t| \geq \delta, \\ \frac{t^2}{2\delta}, & \text{if } |t| < \delta/2. \end{cases} \quad (2.34)$$

Then for any  $t, s \in \mathbb{R}$ ,  $|\phi'_\delta(t)| \leq 1$ ,  $\phi_\delta(t) \rightarrow |t|$  and  $\phi'_\delta(t) \rightarrow \text{sign}(t)$  when  $\delta \rightarrow 0^+$ . Moreover

$$\phi_\delta(s) - \phi_\delta(t) \geq \phi'_\delta(t)(s - t). \quad (2.35)$$

Let  $\{\nu_n\}$  be a sequence functions in  $C^2(\bar{\Omega})$  such that

$$\lim_{n \rightarrow \infty} \int_\Omega |\nu_n - \nu| \rho^\alpha dx = 0.$$

Let  $u_n$  be the corresponding solution to (2.20) with right-hand side  $\nu_n$ , then by Lemma 2.1,  $u_n \in \mathbb{X}_\alpha$  and by Lemmas 2.2, 2.3, for any  $\delta > 0$  and  $\xi \in \mathbb{X}_\alpha$ ,  $\xi \geq 0$ ,

$$\begin{aligned} \int_\Omega \phi_\delta(u_n)(-\Delta)^\alpha \xi dx &= \int_\Omega \xi(-\Delta)^\alpha \phi_\delta(u_n) dx \\ &\leq \int_\Omega \xi \phi'_\delta(u_n)(-\Delta)^\alpha u_n dx \\ &= \int_\Omega \xi \phi'_\delta(u_n) \nu_n dx. \end{aligned} \quad (2.36)$$

Letting  $\delta \rightarrow 0$ , we obtain

$$\int_\Omega |u_n|(-\Delta)^\alpha \xi dx \leq \int_\Omega \xi \text{sign}(u_n) \nu_n dx \leq \int_\Omega \xi |\nu_n| dx. \quad (2.37)$$

If we take  $\xi = \eta_1$  we derive from Lemma 2.1

$$\int_\Omega |u_n| dx \leq C \int_\Omega |\nu_n| \rho^\alpha dx. \quad (2.38)$$

Similarly

$$\int_\Omega |u_n - u_m| dx \leq C \int_\Omega |\nu_n - \nu_m| \rho^\alpha dx. \quad (2.39)$$

Therefore  $\{u_n\}$  is a Cauchy sequence in  $L^1$  and its limit  $u$  is a weak solution of (2.20). Letting  $n \rightarrow \infty$  in (2.37) we obtain (2.21). Inequality (2.22) is proved by replacing  $\phi_\delta$  by  $\tilde{\phi}_\delta$  which is zero on  $(-\infty, 0]$  and  $\phi_\delta$  on  $[0, \infty)$ .  $\square$

The next result is a higher order regularity result

**Proposition 2.5** *Let the assumptions of Proposition 2.2 be fulfilled and  $0 \leq \beta \leq \alpha$ . Then for  $p \in (1, \frac{N}{N+\beta-2\alpha})$  there exists  $c_p > 0$  such that for any  $\nu \in L^1(\Omega; \rho^\beta dx)$*

$$\|\mathbb{G}[\nu]\|_{W^{2\alpha-\gamma,p}(\Omega)} \leq c_p \|\nu\|_{L^1(\Omega; \rho^\beta dx)} \quad (2.40)$$

where  $\gamma = \beta + \frac{N}{p'}$  if  $\beta > 0$  and  $\gamma > \frac{N}{p'}$  if  $\beta = 0$  and

**Proof.** We use Stampacchia's duality method [30] and put  $u = \mathbb{G}[\nu]$ . If  $\psi \in C_c^\infty(\overline{\Omega})$ , then

$$\begin{aligned} \left| \int_{\Omega} \psi (-\Delta u)^\alpha dx \right| &\leq \int_{\Omega} |\nu| |\psi| dx \\ &\leq \sup_{\Omega} |\rho^{-\beta} \psi| \int_{\Omega} |\nu| \rho^\beta dx \\ &\leq \|\psi\|_{C^\beta(\overline{\Omega})} \|\nu\|_{L^1(\Omega; \rho^\beta dx)}. \end{aligned} \quad (2.41)$$

By Sobolev-Morrey inbedding type theorem (see e.g. [26, Th 8.2]), for any  $p \in ()$  and  $p' = \frac{p}{p-1}$ ,

$$\|\psi\|_{C^\beta(\overline{\Omega})} \leq C \|\psi\|_{W^{\gamma,p'}(\Omega)}$$

with  $\gamma = \beta + \frac{N}{p'}$  if  $\beta > 0$  and  $\gamma > \frac{N}{p'}$  if  $\beta = 0$ . Therefore  $\square$

$$\left| \int_{\Omega} \psi (-\Delta u)^\alpha dx \right| \leq C \|\psi\|_{W^{\gamma,p'}(\Omega)} \|\nu\|_{L^1(\Omega; \rho^\beta dx)}, \quad (2.42)$$

which implies that the mapping  $\psi \mapsto \int_{\Omega} \psi (-\Delta u)^\alpha dx$  is continuous on  $W^{\gamma,p'}(\Omega)$  and thus

$$\|(-\Delta u)^\alpha\|_{W^{-\gamma,p}(\Omega)} \leq C \|\nu\|_{L^1(\Omega; \rho^\beta dx)}. \quad (2.43)$$

Since  $(-\Delta)^{-\alpha}$  is an isomorphism from  $W^{-\gamma,p}(\Omega)$  into  $W^{2\alpha-\gamma,p}(\Omega)$ , it follows that

$$\|u\|_{W^{2\alpha-\gamma,p}(\Omega)} \leq C \|\nu\|_{L^1(\Omega; \rho^\beta dx)}. \quad (2.44)$$

$\square$

**Proposition 2.6** *Under the assumptions of Proposition 2.5 the mapping  $\nu \mapsto \mathbb{G}[\nu]$  is compact from  $L^1(\Omega)$  into  $L^q(\Omega)$  for any  $q \in [1, \frac{N}{N+\beta-2\alpha})$ .*

**Proof.** By [26, Th 7.1] the inbedding of  $W^{2\alpha-\gamma,p}(\Omega)$  into  $L^q(\Omega)$  is compact, this ends the proof.  $\square$

### 3 Proof of Theorem 1.1

Before proving the main we give a general existence result in  $L^1$ .

**Proposition 3.1** *Suppose that  $\Omega$  is a bounded  $C^2$  domain of  $\mathbb{R}^N$  ( $N \geq 2$ ),  $\alpha \in (0, 1)$  and the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, nondecreasing and  $rg(r) \geq 0$  for all  $r \in \mathbb{R}$ . Then for any  $f \in L^1(\Omega, \rho^\alpha)$  there exists a unique weak solution  $u$  of (1.1). Moreover the mapping  $f \mapsto u$  is increasing.*

**Proof.** *Step 1: Variational solutions.* Let  $j(r) = \int_0^r g(s)ds$ . We define the functional

$$J(w) = \frac{1}{2} \int_{\Omega} \left( \left( (-\Delta)^{\frac{\alpha}{2}} w \right)^2 + 2j(w) \right) dx$$

over  $L^2(\Omega)$ ;  $J$  is a strictly convex lower semi-continuous functional with domain

$$D(J) = W^{\frac{\alpha}{2}, 2}(\Omega) \cap \left\{ w \in L^2(\Omega) : \int_{\Omega} j(w) dx < \infty \right\}.$$

Furthermore  $J(w) \geq \sigma \|w\|_{L^2}^2$  for some  $\sigma > 0$  by [26, p. 134]. Therefore the subdifferential  $\partial J$  of  $J$  is a maximal monotone in the sense of Browder-Minty (see [6] and the references therein); it is onto and there exists a unique  $u \in \text{Dom}(\partial J)$  such that  $\partial J(u) = f$  for any  $f \in L^2(\Omega)$ , where  $\text{Dom}(\partial J)$  is the domain of  $\partial J$  and by [6, Cor 2.11]

$$\partial J(u) = (-\Delta)^\alpha(u) + g(u) = f,$$

with  $u \in W^{\alpha, 2}(\Omega)$  such that  $g(u) \in L^2(\Omega)$ . Furthermore

$$\int_{\Omega} \left( (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} \zeta + g(u)\zeta - f\zeta \right) dx = 0,$$

for any  $\zeta \in W^{\frac{\alpha}{2}, 2}(\Omega)$ . If  $f$  is assumed to be bounded, then  $u \in C^\alpha(\overline{\Omega})$  by [27, Prop 1.1].

*Step 2:  $L^1$  solutions.* For  $n \in \mathbb{N}_*$ , set  $f_n = \text{sgn}(f) \min\{n, |f|\}$  and denote by  $u_n$  the solution of (1.1) with right-hand side  $f_n$ . By (2.37) with  $\xi = \eta_1$

$$\int_{\Omega} (|u_n - u_m| + |g(u_n) - g(u_m)|\eta_1) dx \leq \int_{\Omega} |f_n - f_m|\eta_1 dx. \quad (3.1)$$

Since  $f_n \rightarrow f$  in  $L^1(\Omega, \rho^\alpha dx)$ ,  $\{u_n\}$  and  $\{g(u_n)\}$  are Cauchy sequences in  $L^1(\Omega)$  and  $L^1(\Omega, \rho^\alpha dx)$  respectively. Set  $u = \lim_{n \rightarrow \infty} u_n$ , we derive from the identity verified for any  $\xi \in \mathbb{X}$

$$\int_{\Omega} (u_n (-\Delta)^\alpha \xi + g(u_n)\xi) dx = \int_{\Omega} f_n \xi dx$$

that  $u$  is a solution of (1.1). Uniqueness follows from (2.37)-(3.1), since for any  $f$  and  $f'$  in  $L^1(\Omega, \rho^\alpha dx)$ , the any couple  $(u, u')$  of weak solutions with respective right-hand side  $f$  and  $f'$  satisfies

$$\int_{\Omega} (|u - u'| + |g(u) - g(u')|\eta_1) dx \leq \int_{\Omega} |f - f'|\eta_1 dx. \quad (3.2)$$

Finally, the monotonicity of the mapping  $f \mapsto u$  follows from (2.22) thanks to which (3.2) is transformed into

$$\int_{\Omega} ((u - u')_+ + (g(u) - g(u'))_+\eta_1) dx \leq \int_{\Omega} (f - f')_+\eta_1 dx. \quad (3.3)$$

□

**Proof of Theorem 1.1.** Uniqueness follows from (3.2). For existence we define

$$C_\beta(\overline{\Omega}) = \{\zeta \in C(\overline{\Omega}) : \rho^{-\beta}\zeta \in C(\overline{\Omega})\}$$

endowed with the norm

$$\|\zeta\|_{C_\beta} = \sup\{(\rho(x))^{-\beta}\zeta(x) : x \in \overline{\Omega}\}.$$

We consider a sequence  $\{\nu_n\} \subset C(\overline{\Omega})$  such that  $\nu_{n\pm} \rightarrow \nu_\pm$  in the duality sense with  $C_\beta(\overline{\Omega})$ , which means

$$\lim_{n \rightarrow \infty} \int_{\overline{\Omega}} \zeta d\nu_{n\pm} = \int_{\overline{\Omega}} \zeta d\nu_\pm$$

for all  $\zeta \in C_\beta(\overline{\Omega})$ . It follows from the Banach-Steinhaus theorem that  $\|\nu_n\|_{\mathfrak{M}(\Omega, \rho^\beta)}$  is bounded independently of  $n$ , therefore

$$\int_{\Omega} (|u_n| + |g(u_n)|\eta_1) dx \leq \int_{\Omega} |\nu_n|\eta_1 dx \leq C. \quad (3.4)$$

Therefore  $\|g(u_n)\|_{\mathfrak{M}(\Omega, \rho^\alpha)}$  is bounded independently of  $n$ . For  $\epsilon > 0$ , set  $\xi_\epsilon = (\eta_1 + \epsilon)^{\frac{\beta}{\alpha}} - \epsilon^{\frac{\beta}{\alpha}}$ . Then, by Lemma 2.3,

$$\begin{aligned} (-\Delta)^\alpha \xi_\epsilon &= \frac{\beta}{\alpha}(\eta_1 + \epsilon)^{\frac{\beta-\alpha}{\alpha}} (-\Delta)^\alpha \eta_1 + \frac{\beta(\beta-\alpha)}{\alpha^2}(\eta_1 + \epsilon)^{\frac{\beta-2\alpha}{\alpha}} \int_{\Omega} \frac{(\eta_1(y) - \eta_1(x))^2}{|y-x|^{N+2\alpha}} dy \\ &\leq \frac{\beta}{\alpha}(\eta_1 + \epsilon)^{\frac{\beta-\alpha}{\alpha}}, \end{aligned}$$

and  $\xi_\epsilon \in \mathbb{X}_\alpha$ . Since

$$\int_{\Omega} (|u_n|(-\Delta)^\alpha \xi_\epsilon + |g(u_n)|\xi_\epsilon) dx \leq \int_{\Omega} \xi_\epsilon d|\nu_n|$$

we obtain

$$\int_{\Omega} \left( |u_n| \frac{\beta}{\alpha} (\eta_1 + \epsilon)^{\frac{\beta-\alpha}{\alpha}} + |g(u_n)| \xi_{\epsilon} \right) dx \leq \int_{\Omega} \xi_{\epsilon} d|\nu_n|.$$

If we let  $\epsilon \rightarrow 0$ , we obtain

$$\int_{\Omega} \left( |u_n| \frac{\beta}{\alpha} \eta_1^{\frac{\beta-\alpha}{\alpha}} + |g(u_n)| \eta_1^{\frac{\beta}{\alpha}} \right) dx \leq \int_{\Omega} \eta_1^{\frac{\beta}{\alpha}} d|\nu_n|.$$

By Lemma 2.3, we derive the estimate

$$\int_{\Omega} \left( |u_n| \rho^{\beta-\alpha} + |g(u_n)| \rho^{\beta} \right) dx \leq C \|\nu_n\|_{\mathfrak{M}(\Omega, \rho^{\beta})} \leq C' \quad (3.5)$$

Since  $u_n = \mathbb{G}[\nu_n - g(u_n)]$ , it follows by (2.7), that

$$\|u_n\|_{M^{k_{\alpha, \beta}}(\Omega, \rho^{\beta})} \leq \|\nu_n - g(u_n)\|_{\mathfrak{M}(\Omega, \rho^{\beta})}, \quad (3.6)$$

where  $k_{\alpha, \beta}$  is defined by (2.13). By Corollary 2.6 the sequence  $\{u_n\}$  is relatively compact in the  $L^q(\Omega)$  for  $1 \leq q < \frac{N}{N+\beta-2\alpha}$ . Therefore there exist a sub-sequence  $\{u_{n_k}\}$  and some  $u \in L^1(\Omega) \cap L^q(\Omega)$  such that  $u_{n_k} \rightarrow u$  in  $L^q(\Omega)$  and almost every where in  $\Omega$ . Furthermore  $g(u_{n_k}) \rightarrow g(u)$  almost every where. Put  $\tilde{g}(r) = g(|r|) - g(-|r|)$ . For  $\lambda > 0$ , we set  $S_{\lambda} = \{x \in \Omega : |u_{n_k}(x)| > \lambda\}$  and  $\omega(\lambda) = \int_{S_{\lambda}} \rho^{\beta} dx$ . Then for any Borel set  $E \subset \Omega$ , we have

$$\begin{aligned} \int_E |g(u_{n_k})| \rho^{\beta} dx &= \int_{E \cap S_{\lambda}^c} |g(u_{n_k})| \rho^{\beta} dx + \int_{E \cap S_{\lambda}} |g(u_{n_k})| \rho^{\beta} dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^{\beta} dx + \int_{S_{\lambda}} \tilde{g}(|u_{n_k}|) \rho^{\beta} dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^{\beta} dx - \int_{\lambda}^{\infty} \tilde{g}(s) d\omega(s). \end{aligned}$$

But

$$\int_{\lambda}^{\infty} \tilde{g}(s) d\omega(s) = \lim_{T \rightarrow \infty} \int_{\lambda}^T \tilde{g}(s) d\omega(s).$$

Since  $u_{n_k} \in M^{k_{\alpha, \beta}}(\Omega, \rho^{\beta})$ ,  $\omega(s) \leq cs^{-k_{\alpha, \beta}}$  and

$$\begin{aligned} \int_{\lambda}^T \tilde{g}(s) d\omega(s) &= - \left[ \tilde{g}(s) \omega(s) \right]_{s=\lambda}^{s=T} + \int_{\lambda}^T \omega(s) d\tilde{g}(s) \\ &\leq \tilde{g}(\lambda) \omega(\lambda) - \tilde{g}(T) \omega(T) + c \int_{\lambda}^T s^{-k_{\alpha, \beta}} d\tilde{g}(s) \\ &\leq \tilde{g}(\lambda) \omega(\lambda) - \tilde{g}(T) \omega(T) + c \left( T^{-k_{\alpha, \beta}} \tilde{g}(T) - \lambda^{-k_{\alpha, \beta}} \tilde{g}(\lambda) \right) \\ &\quad + \frac{c}{k_{\alpha, \beta} + 1} \int_{\lambda}^T s^{-1-k_{\alpha, \beta}} \tilde{g}(s) ds \end{aligned}$$

By assumption (1.9) there exists  $\{T_n\} \rightarrow \infty$  such that  $T_n^{-k_{\alpha,\beta}} \tilde{g}(T_n) \rightarrow 0$  when  $n \rightarrow \infty$ . Furthermore  $\tilde{g}(\lambda)\omega(\lambda) \leq c\lambda^{-k_{\alpha,\beta}} \tilde{g}(\lambda)$ , therefore

$$-\int_{\lambda}^{\infty} \tilde{g}(s) d\omega(s) \leq \frac{c}{k_{\alpha,\beta} + 1} \int_{\lambda}^{\infty} s^{-1-k_{\alpha,\beta}} \tilde{g}(s) ds.$$

Notice that the above quantity on the right-hand side tends to 0 when  $\lambda \rightarrow \infty$ . The conclusion follows: for any  $\epsilon > 0$  there exists  $\lambda > 0$  such that

$$\frac{c}{k_{\alpha,\beta} + 1} \int_{\lambda}^{\infty} s^{-1-k_{\alpha,\beta}} \tilde{g}(s) ds \leq \frac{\epsilon}{2}$$

and  $\delta > 0$  such that

$$\int_E \rho^{\beta} dx \leq \delta \implies \tilde{g}(\lambda) \int_E \rho^{\beta} dx \leq \frac{\epsilon}{2}.$$

This proves that  $\{g \circ u_n\}$  is uniformly integrable in  $L^1(\Omega, \rho^{\beta} dx)$ . Then  $g \circ u_{n_k} \rightarrow g \circ u$  in  $L^1(\Omega, \rho^{\beta} dx)$  by Vitali convergence theorem. Letting  $n_k \rightarrow \infty$  in the identity

$$\int_{\Omega} (u_{n_k} (-\Delta)^{\alpha} \xi + \xi g \circ u_{n_k}) dx = \int_{\Omega} \nu_{n_k} \xi dx$$

where  $\xi \in \mathbb{X}_{\alpha}$ , it infers that  $u$  is a weak solution of (1.1).

The right-hand side of estimate (1.9) follows from the fact that  $v_{n+} := \mathbb{G}[\nu_{n+}]$  satisfies

$$(-\Delta)^{\alpha} v_{n+} + g(v_{n+}) = \nu_{n+} + g(v_{n+}) \geq \nu_n$$

Therefore  $v_{n+} \geq u_n$  by Proposition 3.1. Letting  $n \rightarrow \infty$  yields to (1.1). The left-hand side is proved similarly.  $\square$

**Corollary 3.1** *Let  $0 \leq \beta \leq \alpha$ ,  $\Omega$  be a bounded  $C^2$  domain,  $\{\nu_n\}$  a sequence of measures in  $\mathfrak{M}(\Omega, \rho^{\beta})$  and  $\nu \in \mathfrak{M}(\Omega, \rho^{\beta})$ . Assume that for any  $\xi \in C_{\beta}(\overline{\Omega})$*

$$\int_{\Omega} \xi d\nu_n \rightarrow \int_{\Omega} \xi d\nu.$$

*Then the sequence  $\{u_n\}$  of weak solutions of*

$$\begin{aligned} (-\Delta)^{\alpha} u_n + g \circ u_n &= \nu_n & \text{in } \Omega, \\ u_n &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{3.7}$$

*converges to the solution  $u$  of (1.1) in  $L^q(\Omega)$  for  $1 \leq q < \frac{N}{N+\beta-2\alpha}$  and  $\{g \circ u_n\}$  converges to  $g \circ u$  in  $L^1(\Omega)$ .*

**Proof.** The method is an adaptation of [35]. Since  $\nu_n \rightarrow \nu$  in the duality sense of  $C_\beta(\overline{\Omega})$ , there exists  $M > 0$  such that

$$\|\nu_n\|_{\mathfrak{M}(\Omega, \rho^\beta)} \leq M \quad \forall n \in \mathbb{N}.$$

Therefore (3.5), (3.6) hold (but with  $u_n$  solution of (3.7)). The above proof shows that  $\{g \circ u_n\}$  is uniformly integrable in  $L^1(\Omega, \rho^\beta dx)$  and  $\{u_n\}$  relatively compact in  $L^q(\Omega)$  for  $1 \leq q < \frac{N}{N+\beta-2\alpha}$ . Thus, up to a subsequence  $\{u_{n_k}\} \subset \{u_n\}$ ,  $u_{n_k} \rightarrow u$ , and  $u$  is the weak solution of (1.1). Since  $u$  is unique,  $u_n \rightarrow u$ .  $\square$

**Remark 3.1** *Under the hypotheses of Theorem 1.1, we assume  $\nu \geq 0$ , then*

$$\mathbb{G}(\nu) - \mathbb{G}(g(\mathbb{G}(\nu))) \leq u \leq \mathbb{G}(\nu). \quad (3.8)$$

*Indeed, since  $g$  is nondecreasing and  $u \leq \mathbb{G}(\nu)$ , then*

$$\begin{aligned} u &= \mathbb{G}(\nu) - \mathbb{G}(g(u)) \\ &\geq \mathbb{G}(\nu) - \mathbb{G}(g(\mathbb{G}(\nu))). \end{aligned}$$

## 4 Applications

### 4.1 The case of a Dirac mass

In this section we characterize the asymptotic behavior of a solution near a singularity created by a Dirac mass.

**Theorem 4.1** *Assume that  $\Omega$  is an open, bounded and  $C^2$  domain of  $\mathbb{R}^N$  ( $N \geq 2$ ) with  $0 \in \Omega$ ,  $\alpha \in (0, 1)$ ,  $\nu = \delta_0$  and the function  $g : [0, \infty) \rightarrow [0, \infty)$  is continuous, nondecreasing and (1.9) holds for*

$$k_{\alpha,0} = \frac{N}{N-2\alpha}. \quad (4.1)$$

*Then problem (1.1) admits a unique positive weak solution  $u$  such that*

$$\lim_{x \rightarrow 0} u(x)|x|^{N-2\alpha} = C, \quad (4.2)$$

*for some  $C > 0$ .*

**Remark 4.1** *We note here that a weak solution  $u$  of (1.1) with  $\nu = \delta_0$  satisfies*

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= 0 \quad \text{in } \Omega \setminus \{0\}, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned} \quad (4.3)$$

*The asymptotic behavior (4.2) is one of the possible singular behaviors of solutions of (4.3) given in [13].*

Before proving Theorem 4.1, we give an auxiliary lemma.

**Lemma 4.1** *Assume that  $g : [0, \infty) \rightarrow [0, \infty)$  is continuous, nondecreasing and (1.9) holds with  $k_{\alpha, \beta} = k > 1$ . Then*

$$\lim_{s \rightarrow \infty} g(s)s^{-k} = 0.$$

**Proof.** Since

$$\int_s^{2s} g(t)t^{-1-k} dt \geq g(s)(2s)^{-1-k} \int_s^{2s} dt = 2^{-1-k} g(s)s^{-k}$$

and by (1.9),

$$\lim_{s \rightarrow \infty} \int_s^{2s} g(t)t^{-1-k} dt = 0.$$

Then

$$\lim_{s \rightarrow \infty} g(s)s^{-k} = 0.$$

The proof is complete.  $\square$

**Proof of Theorem 4.1.** Existence follows from Theorem 1.1 with  $\beta = 0$ . For (4.2), we shall use (1.10). From [12] there holds,

$$0 < \frac{C(N, \alpha)}{|x|^{N-2\alpha}} - G(x, 0) < \frac{C(N, \alpha)}{\rho(0)^{N-2\alpha}}, \quad x \in \Omega \setminus \{0\}. \quad (4.4)$$

for some  $C(N, \alpha) > 0$ . Since

$$\mathbb{G}(\delta_0)(x) = G(x, 0) < \frac{C(N, \alpha)}{|x|^{N-2\alpha}}, \quad x \in \Omega \setminus \{0\}$$

and

$$\begin{aligned} \mathbb{G}(g(\mathbb{G}(\delta_0)))(x)|x|^{N-2\alpha} &\leq \int_{\Omega} \frac{1}{|x-y|^{N-2\alpha}} g\left(\frac{C}{|y|^{N-2\alpha}}\right) dy |x|^{N-2\alpha} \\ &\leq \int_{\Omega} \frac{1}{|e_x-y|^{N-2\alpha}} g\left(\frac{C}{(|x||z|)^{N-2\alpha}}\right) dz |x|^N \\ &= |x|^N \int_{\Omega \cap B_{1/2}(e_x)} \frac{1}{|e_x-y|^{N-2\alpha}} g\left(\frac{C}{(|x||z|)^{N-2\alpha}}\right) dz \\ &\quad + |x|^N \int_{\Omega \cap B_{1/2}^c(e_x)} \frac{1}{|e_x-y|^{N-2\alpha}} g\left(\frac{C}{(|x||z|)^{N-2\alpha}}\right) dz \\ &:= A_1(x) + A_2(x), \end{aligned}$$

where  $e_x = x/|x|$ . By Lemma 4.1,

$$\begin{aligned} A_1(x) &\leq |x|^N g\left(\frac{C2^{N-2\alpha}}{|x|^{N-2\alpha}}\right) \int_{B_{1/2}(e_x)} \frac{1}{|e_x-y|^{N-2\alpha}} dz \\ &\rightarrow 0 \quad \text{as } |x| \rightarrow 0 \end{aligned}$$

and by (1.9),

$$\begin{aligned}
A_2(x) &\leq C|x|^N \int_{B_R(0)} g\left(\frac{C}{(|x||z|)^{N-2\alpha}} dz\right) \\
&\leq C \int_{\frac{R^{1/(N-2\alpha)}}{|x|}}^{\infty} g(Cs) s^{-1-\frac{N}{N-2\alpha}} ds \\
&\rightarrow 0 \quad \text{as } |x| \rightarrow 0,
\end{aligned}$$

where  $R > 0$  such that  $B_R(0) \supset \Omega$ . That is

$$\lim_{|x| \rightarrow 0} \mathbb{G}(g(\mathbb{G}(\delta_0)))(x) |x|^{N-2\alpha} = 0. \quad (4.5)$$

We plug (4.4) and (4.5) into (3.8), then (4.2) holds.  $\square$

## 4.2 The power case

If  $g(s) = |s|^{k-1}s$  with  $k \geq 1$ , then (1.9) is satisfied if  $1 \leq q < k_{\alpha,\beta}$  where  $k_{\alpha,\beta}$  defined by (2.13) is called the *critical exponent* with limit values  $k_{\alpha,0} = \frac{N}{N-2\alpha}$  and  $k_{\alpha,\alpha} = \frac{N+\alpha}{N-\alpha}$ . If we consider the problem

$$\begin{aligned}
(-\Delta)^\alpha u + |u|^{k-1}u &= \nu && \text{in } \Omega \\
u &= 0 && \text{in } \Omega^c
\end{aligned} \quad (4.6)$$

then if  $1 < k < k_{\alpha,\beta}$  it is solvable for any  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ , but it may not be the case if  $k \geq k_{\alpha,\beta}$ . As in the case  $\alpha = 1$ , the sharp solvability of (4.6) is associated to a concentration property of the measure  $\nu$  and this concentration is expressed by the mean of Bessel capacities. If  $k > 1$  and  $k' = \frac{k}{k-1}$ , we define for any  $K \subset \mathbb{R}^N$ ,  $K$  compact,

$$C_{2\alpha,k'}^\Omega(K) = \inf\{\|\phi\|_{W^{2\alpha,k'}(\Omega)}^{k'} : \phi \in C_c^\infty(\Omega), 0 \leq \phi \leq 1, \phi \equiv 1 \text{ on } K\}. \quad (4.7)$$

Then  $C_{2\alpha,k'}^\Omega$  is an outer measure or capacity in  $\Omega$  extended to Borel sets by standard processes. Our result is the following in the case of bounded measures

**Theorem 4.2** *Assume  $\Omega$  is a bounded  $C^2$  domain in  $\mathbb{R}^N$  and  $k > 1$ . Then problem (4.6) can be solved with a nonnegative bounded measure  $\nu$  if and only if  $\nu$  satisfies on compact subsets  $K \subset \Omega$*

$$C_{2\alpha,k'}^\Omega(K) = 0 \implies \nu(K) = 0. \quad (4.8)$$

**Proof.** *1-The condition is necessary.* Assume  $u$  is a weak solution and let  $K \subset \Omega$  be compact. Let  $\phi \in C_c^\infty(\Omega)$  such that  $0 \leq \phi \leq 1$  and  $\phi(x) = 1$  for all  $x \in K$  and set  $\xi = \phi^{k'}$ , then  $\xi \in \mathbb{X}_\alpha$  and

$$\int_\Omega \left( u(-\Delta)^\alpha \xi + u^k \xi \right) dx = \int_\Omega \xi d\nu.$$

Since  $\xi \geq \chi_K$  it follows from (2.31) that

$$\int_{\Omega} \left( k' \phi^{k'-1} u (-\Delta)^{\alpha} \phi + \phi^{k'} u^k \right) dx \geq \nu(K). \quad (4.9)$$

By Hölder's inequality

$$\left| \int_{\Omega} \phi^{k'-1} u (-\Delta)^{\alpha} \phi dx \right| \leq k' \left( \int_{\Omega} \phi^{k'} u^k dx \right)^{\frac{1}{k}} \left( \int_{\Omega} |(-\Delta)^{\alpha} \phi|^{k'} dx \right)^{\frac{1}{k'}} \quad (4.10)$$

By regularity theory for fractional Laplacian and Poincaré inequality [32, p 134]

$$\int_{\Omega} |(-\Delta)^{\alpha} \phi|^{k'} dx \leq C \|\phi\|_{W^{2\alpha, k'}(\Omega)}^{k'}. \quad (4.11)$$

Therefore (4.8) yields to

$$C \|\phi\|_{W^{2\alpha, k'}(\Omega)}^{k'} \left( \int_{\Omega} \phi^{k'} u^k dx \right)^{\frac{1}{k}} + \int_{\Omega} \phi^{k'} u^k dx \geq \nu(K). \quad (4.12)$$

If  $C_{2\alpha, k'}^{\Omega}(K) = 0$ , there exists a sequence  $\{\phi_n\} \subset C_c^{\infty}(\Omega)$  such that  $0 \leq \phi_n \leq 1$  and  $\phi_n = 1$  on  $K$  and  $\|\phi_n\|_{W^{2\alpha, k'}(\Omega)}^{k'} \rightarrow 0$  as  $n \rightarrow \infty$ . Furthermore  $K$  has zero Lebesgue measure and  $\phi_n \rightarrow 0$  almost everywhere. If we replace  $\phi$  by  $\phi_n$  in (4.12) and let  $n \rightarrow \infty$  we obtain  $\nu(K) = 0$ .

*2-The condition is sufficient.* We first assume that  $\nu \in W^{-2\alpha, k}(\Omega) \cap \mathfrak{M}_+^b(\Omega)$ ; for  $n \in \mathbb{N}_*$ , we denote by  $u_n$  the solution of

$$\begin{aligned} (-\Delta)^{\alpha} u + |T_n(u)|^{k-1} T_n(u) &= \nu & \text{in } \Omega \\ u &= 0 & \text{in } \Omega^c \end{aligned} \quad (4.13)$$

where  $T_n(r) = \text{sign}(r) \min\{n, |r|\}$ . Such a solution exists by Theorem 1.1, is nonnegative and the sequence  $\{u_n\}$  is decreasing and converges to some nonnegative  $u$  since  $\{T_n(r)\}$  is increasing on  $\mathbb{R}_+$ . Furthermore

$$0 \leq u_n \leq \mathbb{G}[\nu],$$

by (1.1). This implies that the convergence holds in  $L^1(\Omega)$ . Since  $\nu \in W^{-2\alpha, k}(\Omega)$ ,  $\mathbb{G}[\nu] \in L^k(\Omega)$ , it infers that

$$|T_n(u_n)|^{k-1} T_n(u_n) = (T_n(u_n))^k \leq (\mathbb{G}[\nu])^k.$$

Since for any  $\xi \in \mathbb{X}_{\alpha}$  there holds

$$\int_{\Omega} \left( u_n (-\Delta)^{\alpha} \xi + (T_n(u_n))^k \xi \right) dx = \int_{\Omega} \xi d\nu \quad (4.14)$$

we can let  $n \rightarrow \infty$  and conclude that  $u$  is a solution of (4.6), unique by (3.2). Next we assume that (4.8) holds. By a result of Feyel and de la

Pradelle [19] (see also [17]), there exists an increasing sequence  $\{\nu_n\} \subset W^{-2\alpha,k}(\Omega) \cap \mathfrak{M}_+(\Omega)$  which converges to  $\nu$  in the weak sense of measures. This implies that the sequence  $\{u_n\}$  of weak solutions of

$$\begin{aligned} (-\Delta)^\alpha u_n + u_n^k &= \nu_n && \text{in } \Omega \\ u_n &= 0 && \text{in } \Omega^c \end{aligned} \quad (4.15)$$

is increasing with limit  $u$ . Taking  $\eta_1 := \mathbb{G}[1]$  as a test function in the weak formulation, we have

$$\int_{\Omega} (u_n + u_n^k \eta_1) dx = \int_{\Omega} \eta_1 d\nu_n \leq \int_{\Omega} \eta_1 d\nu.$$

Therefore  $u_n \rightarrow u$  in  $L^1(\Omega) \cap L^k(\Omega, \rho^\alpha dx)$ . Letting  $n \rightarrow \infty$  we deduce that  $u$  satisfies (4.6).  $\square$

**Remark 4.2** *If  $\nu$  is a signed bounded measure a sufficient condition for solving (4.6) is*

$$C_{2\alpha,k'}^\Omega(K) = 0 \implies |\nu|(K) = 0. \quad (4.16)$$

*This can be obtained by using the fact that the solutions of (4.6) with right-hand side  $\nu_+$  and  $-\nu_-$  are respectively a supersolution and a subsolution of (4.6). It is not clear whether it is also a necessary condition.*

## References

- [1] P. Baras and M. Pierre, Singularité séliminables pour des équations semi linéaires, *Ann. Inst. Fourier Grenoble* 34 , 185-206 (1984).
- [2] Ph. Bénéilan and H. Brezis, Nonlinear problems related to the Thomas-Fermi equation, *J. Evolution Eq.* 3, 673-770, (2003).
- [3] Ph. Bénéilan, H. Brezis and M. Crandall, A semilinear elliptic equation in  $L^1(\mathbb{R}^N)$ , *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* 2,523555 (1975).
- [4] M. F. Bidaut-Véron, N. Hung and L. Véron, Quasilinear Lane-Emden equations with absorption and measure data, *arXiv:1212.6314 [math.AP]*, 15 (Jan 2013).
- [5] M. F. Bidaut-Véron and L. Vivier, An elliptic semilinear equation with source term involving boundary measures: the subcritical case, *Rev. Mat. Iberoamericana* 16, 477-513 (2000).
- [6] H. Brezis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, *North-Holland Mathematics Studies. 5. Notas de matematica 50*, North-Holland, Amsterdam (1973).

- [7] H. Brezis, Some variational problems of the Thomas-Fermi type. Variational inequalities and complementarity problems, *Proc. Internat. School, Erice, Wiley, Chichester*, 53-73 (1980).
- [8] X. Cabré and J. Tan, Positive solutions of non-linear problems involving the square root of the Laplacian, *Advances in Mathematics* 224, 2052-2093 (2010).
- [9] L. Caffarelli and L. Silvestre, An extension problem related to the fractional laplacian, *Comm. Partial Differential Equations* 32, 1245-1260 (2007).
- [10] L. Caffarelli and L. Silvestre, Regularity theory for fully non-linear integrodifferential equations, *Communications on Pure and Applied Mathematics* 62, 597-638.(2009)
- [11] Z. Chen, and R. Song, Estimates on Green functions and poisson kernels for symmetric stable process, *Math. Ann.* 312, 465-501 (1998).
- [12] H. Chen and L. Véron, Solutions of fractional equations involving sources and Radon measures, *preprint*.
- [13] H. Chen and L. Véron, Singular solutions of fractional elliptic equations with absorption, *arXiv:1302.1427v1*, [math.AP] 6 Feb 2013.
- [14] H. Chen, P. Felmer and A. Quaas, Large solution to elliptic equations involving fractional Laplacian, *preprint*.
- [15] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.* 59, 330-343 (2006).
- [16] R. Cignoli and M. Cottlar, An Introduction to Functional Analysis, *North-Holland, Amsterdam*, 1974.
- [17] G. Dal Maso, On the integral representation of certain local functionals, *Ricerche Mat.* 32, 85-113 (1983).
- [18] P. Felmer and E. Topp, Uniform equicontinuity for a family of zero order operators approaching the fractional Laplacian, *preprint*.
- [19] D. Feyel and A. de la Pradelle, Topologies fines et compactifications associées à certains espaces de Dirichlet, *Ann. Inst. Fourier (Grenoble)* 27, 121-146 (1977).
- [20] A. Gmira and L. Véron , Boundary singularities of solutions of some nonlinear elliptic equations, *Duke Math. J.* 64, 271-324 (1991).
- [21] Y.Y. Li, Remark on some conformally invariant integral equations: the method for moving spheres, *J. Eur. Math. Soc.* 6, 153-180 (2004).

- [22] M. Marcus and L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case, *Arch. Rat. Mech. Anal.* 144, 201-231 (1998).
- [23] M. Marcus and L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case, *J. Math. Pures Appl.* 77, 481-524 (1998).
- [24] M. Marcus and L. Véron, Removable singularities and boundary traces, *J. Math. Pures Appl.* 80, 879-900 (2001).
- [25] M. Marcus and L. Véron, The boundary trace and generalized B.V.P. for semilinear elliptic equations with coercive absorption, *Comm. Pure Appl. Math.* 56, 689-731(2003).
- [26] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *arXiv:1104.4345v3*, [math.FA] 19 Nov 2011.
- [27] X. Ros-oton and J. Serra, The Dirichlet problem for the fractional laplacian: regularity up to the boundary, *arXiv:1207.5985v1* [math.AP], 25 Jul 2012.
- [28] L. Silvestre, Regularity of the obstacle problem for a fractional power of the laplace operator, *Comm. Pure Appl. Math.* 60, 67-112 (2007).
- [29] Y. Sire and E. Valdinoci, Fractional laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result, *J. Funct. Anal.* 256, 1842-1864 (2009).
- [30] G. Stampacchia, Some limit cases of  $L^p$ -estimates for solutions of second order elliptic equations, *Comm. Pure Appl. Math.* 16, 505-510 (1963).
- [31] E. Stein, Singular Integrals and Differentiability Properties of Functions, *Princeton University Press* (1970).
- [32] L. Tartar, Sur un lemme d'équivalence utilisé en analyse numérique, *Calcolo* 24, 129-140 (1987).
- [33] L. Véron, Singular solutions of some nonlinear elliptic equations, *Non-linear Anal. T., M. & A.* 5, 225-242, 1981.
- [34] L. Véron, Elliptic equations involving Measures, Stationary Partial Differential equations, Vol. I, 593-712, *Handb. Differ. Equ., North-Holland, Amsterdam* (2004).
- [35] L. Véron, Existence and Stability of Solutions of General Semilinear Elliptic Equations with Measure Data, *Advanced Nonlinear Studies* 13, 447-460 (2013).