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# Braided Systems: a Unified Treatment of Algebraic Structures with Several Operations

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## Abstract

Bialgebras and Hopf (bi)modules are examples of algebraic structures involving several interacting operations. The multi-operation setting significantly increases the complexity of these structures and their homology. In the present paper we develop the machinery of braided systems, tailored for handling multi-operation structures. Our construction is general enough to include as particular cases the examples above (as well as, for instance, Poisson algebras and Yetter-Drinfel'd modules, treated in separate publications). At the same time, graphical tools allow a concrete and efficient exploration of braided systems. Gerstenhaber-Schack, Panaite-Ştefan and Ospel-Taillefer (co)homology theories for bialgebras and Hopf (bi)modules, as well as the Heisenberg double, and the algebras  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  of Cibils-Rosso and Panaite, naturally appear in our braided setting. This new interpretation offers a conceptual explication, a generalization and a simplified proof of several related algebraic phenomena.

**Keywords:** braided systems; braided homology; Hopf algebras; Hopf (bi)modules; Heisenberg double; (generalized) crossed products; bialgebra homology; distributive laws; multi-quantum shuffle algebra.

**MSC 2010:** 16T25, 16T10, 18G60, 18D10.

## 1 Introduction

In [14] we introduced a homology theory for *braided objects* in a monoidal category  $\mathcal{C}$  ( $\mathcal{C} = \mathbf{Vect}_{\mathbb{k}}$  is a typical example). Interpreting associative / Leibniz algebras and self-distributive structures as braided objects, and applying our general homology theory to them, we obtained in each case a generalization of usual homology theories for the structure in question. We thus unified many familiar theories into one, explaining their otherwise mysterious similarities.

Concretely, an object  $V$  in  $\mathcal{C}$  is called braided if it is endowed with a *braiding*, i.e., a morphism  $\sigma : V \otimes V \rightarrow V \otimes V$  satisfying the *Yang-Baxter equation (YBE)*  $\sigma^1 \sigma^2 \sigma^1 = \sigma^2 \sigma^1 \sigma^2$ , where  $\sigma^1 = \sigma \otimes \text{Id}_V$  and  $\sigma^2 = \text{Id}_V \otimes \sigma$ . For instance, in [14] we showed that a unital associative algebra in  $\mathbf{Vect}_{\mathbb{k}}$  is a braided object, with  $\sigma_{Ass} : v \otimes w \mapsto 1 \otimes v \cdot w$ . However, the one-object one-morphism setting of braided object theory is too restrictive for more complicated structures. For instance, the *bialgebra* structure involves several operations (multiplication, comultiplication, etc.), and its Gerstenhaber-Shack (co)homology theory (in the version convenient for braided interpretations) is defined on  $\text{Hom}(H^{\otimes n}, H^{\otimes m}) \simeq H^{\otimes m} \otimes (H^*)^{\otimes n}$ , thus involving two different objects ( $H$  and  $H^*$ ). This suggests working with a family of objects  $(V_1, \dots, V_r)$  in  $\mathcal{C}$  endowed with morphisms  $\sigma_{i,j} : V_i \otimes V_j \rightarrow V_j \otimes V_i$  (here we impose  $i \leq j$  for staying the most general possible), satisfying the colored version of the YBE on all the tensor products  $V_i \otimes V_j \otimes V_k$  with  $i \leq j \leq k$ . This is what we call a *rank  $r$  braided system*, a notion central to this paper. For  $r = 2$  we recover the *WXZ-systems* of L. Hlavatý and L. Šnobl ([10]), motivated by the concept of quantum doubles. They classified such systems in dimension 2 and studied their symmetries.

In Sections 2 and 3 we extend the representation and homology theories for braided objects, developed in [14], to braided systems. In particular, the multi-versions of *braided modules* and *braided differentials* are defined, the former playing the role of coefficients for the latter. Further sections are devoted to examples. In each of them, we look for a braided system encoding the given algebraic structure in the sense of Table 1. The line  $\mathbf{BrSyst}_r(\mathcal{C}) \leftrightarrow \mathbf{Structure}(\mathcal{C})$  means that we want to recover the categories of the algebraic structures we are interested in (e.g., the

category of bialgebras in  $\mathcal{C}$ ) as subcategories of the category of rank  $r$  braided systems in  $\mathcal{C}$ . Diverse properties of our algebraic structures and their homologies can then be deduced from the corresponding properties of braided systems.

braided system	$\leftrightarrow$	algebraic structure
braiding components $\sigma_{i,j}$	$\leftrightarrow$	operations
YBEs	$\Leftrightarrow$	defining relations
braided morphisms	$\simeq$	structural morphisms
<b>BrSyst<math>_r(\mathcal{C})</math></b>	$\leftrightarrow$	<b>Structure(<math>\mathcal{C}</math>)</b>
braided modules	$\simeq$	usual modules
braided differentials	$\supseteq$	usual differentials

Table 1: Multi-braided interpretation for algebraic structures

All braided systems considered in this paper are composed of *unital associative algebras* (=UAAs)  $(V_i, \mu_i, \nu_i)$ , and the diagonal braiding components  $\sigma_{i,i}$  are the ‘‘associativity braidings’’  $\sigma_{Ass} = \nu_i \otimes \mu_i$  (or their right versions  $\sigma_{Ass}^r = \mu_i \otimes \nu_i$ ). In Section 4 we study in detail such braided systems of UAAs, highlighting their connections with *braided tensor products of algebras*  $\overleftarrow{V} = V_r \otimes \cdots \otimes V_1$ . Concretely, morphisms  $\xi_{i,j}$  for  $i < j$  are shown to complete the associativity braidings  $\sigma_{i,i}$  into a braided system structure if and only if they define an associative multiplication on  $\overleftarrow{V}$  by

$$\mu_{\overleftarrow{V}} := (\mu_r \otimes \cdots \otimes \mu_1) \circ \xi_{1,2}^{2r-2} \circ (\xi_{2,3}^{2r-4} \circ \xi_{1,3}^{2r-3}) \circ \cdots \circ (\xi_{r-1,r}^2 \circ \cdots \circ \xi_{2,r}^{r-1} \circ \xi_{1,r}^r) : \overleftarrow{V} \otimes \overleftarrow{V} \rightarrow \overleftarrow{V},$$

where  $\xi_{i,j}^p$  denotes the morphism  $\xi_{i,j}$  applied at positions  $p$  and  $p+1$ .

Rank 2 braided tensor products are at the heart of *braided geometry*, introduced by S. Majid in a long series of papers ([17, 16, 18], etc.). Majid’s motivation was to develop an algebra analogue of the product of two spaces in non-commutative geometry. A pleasant consequence of his work was the construction of new examples of non-commutative non-cocommutative Hopf algebras via the bicrossproduct construction (which is a particular case of braided tensor product).

The case of general  $r$  was independently treated by several authors from different viewpoints:

1. P.J. Martínez, J.L. Peña, F. Panaite and F. van Oystaeyen introduced in [11] the notion of *iterated twisted tensor products of algebras* in  $\mathbf{Vect}_{\mathbb{k}}$ , and studied various Hopf algebraic, geometric and physical examples. Their motivation lied in the braided geometry.
2. E. Cheng introduced in [4] the notion of *iterated distributive laws*, generalizing the work of J. Beck ([1]). Categorical motivations (namely, a study of interchange laws in a strict  $n$ -category) lead her to work in the monoidal category of endofunctors of a given category.

The approaches above relate the associativity of  $\mu_{\overleftarrow{V}}$  to the YBEs for the  $\xi_{i,j}$  with  $i < j$ , combined with the naturality of the  $\xi_{i,j}$  w.r.t. multiplications  $\mu_i$  and  $\mu_j$ . Our main contribution consists in a treatment of all the conditions involved in the associativity of  $\mu_{\overleftarrow{V}}$  in terms of YBEs:

$$\left. \begin{array}{l} \text{associativity of } \mu_i \iff \text{YBE on } V_i \otimes V_i \otimes V_i \\ \text{compatibility between } \xi_{i,j} \text{ \& } \mu_i \iff \text{YBE on } V_i \otimes V_i \otimes V_j \\ \text{compatibility between } \xi_{i,j} \text{ \& } \mu_j \iff \text{YBE on } V_i \otimes V_j \otimes V_j \end{array} \right\} \text{ new}$$

$$\text{compatibilities between the } \xi \iff \text{YBE on } V_i \otimes V_j \otimes V_k \left. \right\} \text{ known}$$

This entirely braided interpretation is made possible by our associativity braiding. Among its advantages is the possibility to apply the braided homology machinery to braided tensor products of algebras, which turns out to be very fruitful in the examples from this paper.

Sections 5-7 contain a detailed study of the braided systems of UAAs encoding the structures of *generalized two-sided crossed products* (as defined by D. Bulacu, F. Panaite and F. Van Oystaeyen in [3]) and finite-dimensional  $\mathbb{k}$ -linear *bialgebras*. For the latter, we propose two braided systems, recovering Hopf modules and Hopf bimodules as corresponding braided modules. Both systems are presented in Table 2, together with two systems encoding the UAA structure. Here  $\tau$  is simply the transposition  $v \otimes w \mapsto w \otimes v$  (or the underlying braiding if one works in a symmetric category),  $\sigma_{bi} : H \otimes H^* \rightarrow H^* \otimes H$  is defined, using Sweedler’s notation, by

$$\sigma_{bi}(h \otimes l) := \langle l_{(1)}, h_{(2)} \rangle l_{(2)} \otimes h_{(1)}, \quad (1)$$

and, when writing  $\sigma_{i,j} = \sigma_{Ass}$  or  $\sigma_{bi}$ , we mean the formulas for  $\sigma_{Ass}$  or  $\sigma_{bi}$  applied to the (bi)algebra corresponding to  $V_i \otimes V_j$  (e.g.,  $\sigma_{2,4}$  in the last line is calculated according to Formula (1) for  $H^{op, cop}$ ).

The  $\sigma_{i,i}$  components are omitted when they equal  $\sigma_{Ass}$ .

structure	braided system		br. modules	br. complexes
algebra $A$	$A$	$\sigma_{1,1} = \sigma_{Ass}$ or $\sigma_{Ass}^r$	algebra modules	bar, Hochschild
	$A, A^{op}$	$\sigma_{1,2} = \tau$	algebra bimodules	
bialgebra $H$	$H, H^*$	$\sigma_{1,1} = \sigma_{Ass}^r,$ $\sigma_{1,2} = \sigma_{bi}$	Hopf modules	Gerstenhaber-Schack, Panaite-Ştefan [7, 27]
	$H, H^{op}$ $H^*, (H^*)^{op}$	$\sigma_{1,2} = \tau, \sigma_{3,4} = \tau,$ other $\sigma_{i,j} = \sigma_{bi}$	Hopf bimodules	Ospel-Taillefer [25, 31]

Table 2: Braided interpretation of the algebra and the bialgebra structures

Note the the braiding components in the systems above are not necessarily invertible. For instance,  $\sigma_{bi}$  has an inverse if and only if  $H$  is a Hopf algebra. This yields a braided interpretation of the existence of an *antipode*.

The braided system from the third line of the table leads to an inclusion of the category of bialgebras in  $\mathbf{vect}_k$  into  $\mathbf{BrSyst}_2(\mathbf{vect}_k)$ . F.F. Nichita’s paper [23] can also be seen in the same light. To encode the associativity, he uses a generalization of the self-inverse braiding  $\widehat{\sigma}_{Ass} = \nu \otimes \mu + \mu \otimes \nu - \text{Id}_{V \otimes 2}$ , proposed by P. Nuss ([24]) in the context of the descent theory for noncommutative rings. This braiding was thoroughly studied by Nichita and his collaborators ([22, 2]). We stick however to our  $\sigma_{Ass}$  because of its homological applications (cf. the last column of Table 2).

Representation-theoretic aspects of braided systems of UAAs are of particular interest. They can be viewed through the prism of the general principle of presenting complicated structures using something well understood — here modules over a well chosen algebra. The structure complexity is now hidden in this algebra, which for some purposes can be treated as a black box. Table 3 contains relevant examples. Notation  $\underline{\otimes}$  is here to stress that braided tensor products are used.

“complicated” structure	corresponding “complicated” algebra
bimodule over an algebra $A$	enveloping algebra $A \otimes A^{op}$
Hopf module over a bialgebra $H$	Heisenberg double $\mathcal{H}(H) := H^* \underline{\otimes} H$
Hopf bimodule over a Hopf algebra $H$	algebras $\mathcal{X}(H) = (H \otimes H^{op}) \underline{\otimes} (H^* \otimes (H^*)^{op}),$ $\mathcal{Y}(H)$ and $\mathcal{Z}(H)$
YD module over a bialgebra $H$	Drinfel’d double $\mathcal{D}(H) := H^* \underline{\otimes} H^{op}$

Table 3: Algebras encoding Hopf and Yetter-Drinfel’d (bi)module structures

In this paper and in [15], we interpret the “complicated” structures from the table as braided modules over certain braided systems of UAAs (cf. Table 2), and the “complicated” algebras as the corresponding braided tensor product algebras. Now in order to interpret the former as algebra modules over the latter, we return to our general setting, and identify braided modules over a braided system of UAAs as modules over the corresponding braided tensor product algebra  $\underline{V}$ :

$$\mathbf{Mod}_{(V_1, \dots, V_r; \sigma_{i,i} = \sigma_{Ass}, \xi_{i,j})} \simeq \mathbf{Mod}_{(\underline{V}, \mu_{\underline{V}})}.$$

Our general braided system theory now applies to the “complicated” structures and algebras from the table. In particular, using our explicit *permutation rules* for components of a braided tensor product, we include the algebra  $\mathcal{X}(H)$  of Cibils-Rosso ([5]) and their versions  $\mathcal{Y}(H)$  and  $\mathcal{Z}(H)$  of F. Panaite ([26]) into a family of  $\#S_4 = 24$  algebras. We give explicit isomorphisms between these, as well as explicit equivalences between their categories of modules. This generalizes some results of [26], avoiding the technical calculations used there. Further, our *adjoint braided module* theory recovers the “complicated” structure of some braided differential complexes (e.g., the Hopf bimodule structure of the bar complex of a bialgebra with coefficients in a Hopf bimodule).

We finish with a list of other structures admitting a braided system interpretation; the work on all these points is in progress.

1. The braided system we constructed for generalized two-sided crossed products works in particular for  *$H$ -(bi)(co)module algebras*. Repeating our study of braided homology of bialgebras in this context, one recovers in particular D. Yau’s deformation bicomplex of module algebras ([34]). “Braided” tools also simplify A. Kaygun’s treatment of  $H$ -equivariant  $A$ -bimodule structures used in his Hopf-Hochschild homology of module algebras ([13]).

2. Combining  $\sigma_{Ass}$  with the braiding from [14] encoding Lie/Leibniz algebras, one gets a rank 2 braided system encoding the non-commutative *Poisson algebra* structure. Its braided homology is likely to include B. Fresse's Poisson algebra homology ([6]).
3. The braided system machinery can also be applied to the *quantum Koszul complexes* of D. Gurevich ([8]) and M. Wambst ([32]).

## Notations and conventions

All the structures in this paper live in a strict monoidal category  $(\mathcal{C}, \otimes, \mathbf{I})$ ; the reader can have in mind the category  $\mathbf{Vect}_k$  of vector spaces over a field  $k$  for simplicity. The word “strict” is often omitted for brevity, as well as the word “monoidal” in the terms “braided / symmetric monoidal category”. Given an object  $V$  in  $\mathcal{C}$ , we succinctly denote its tensor powers by  $V^n := V^{\otimes n}$ ,  $V^0 := \mathbf{I}$ . Further, given a morphism  $\varphi : V^l \rightarrow V^r$ , the following notation is repeatedly used:

$$\varphi^i := \text{Id}_V^{\otimes(i-1)} \otimes \varphi \otimes \text{Id}_V^{\otimes(k-i+1)} \quad : \quad V^{k+l} \rightarrow V^{k+r}, \quad (2)$$

and similarly for morphisms on tensor products of different objects. Working with a family of objects  $(V_1, V_2, \dots)$ , we put  $\text{Id}_i := \text{Id}_{V_i}$ .

The already classical *graphical calculus* is extensively used in this paper, with

- dots standing for objects in  $\mathcal{C}$ ,
- horizontal gluing corresponding to the tensor product,
- graph diagrams representing morphisms from the object which corresponds to the lower dots to that corresponding to the upper dots,
- vertical gluing standing for morphism composition, and vertical strands for identities.

All diagrams are to be read from bottom to top here.

Notations  $S_n, B_n, B_n^+$  stand for the symmetric groups, the braid groups and the positive braid monoids respectively. Their standard generators are denoted by  $s_i$  and  $\sigma_i$ ,  $1 \leq i \leq n-1$ .

## 2 Multi-braided vocabulary

### Braided systems

The notion of braided system generalizes the common notion of braided object in a monoidal category.

**Definition 2.1.** → A *braided system* in  $\mathcal{C}$  is an ordered finite family  $V_1, V_2, \dots, V_r \in \text{Ob}(\mathcal{C})$  endowed with a *braiding*, i.e., morphisms  $\sigma_{i,j} : V_i \otimes V_j \rightarrow V_j \otimes V_i$  for  $1 \leq \underline{i \leq j} \leq r$  satisfying the (*colored*) *Yang-Baxter equation*

$$(\sigma_{j,k} \otimes \text{Id}_i) \circ (\text{Id}_j \otimes \sigma_{i,k}) \circ (\sigma_{i,j} \otimes \text{Id}_k) = (\text{Id}_k \otimes \sigma_{i,j}) \circ (\sigma_{i,k} \otimes \text{Id}_j) \circ (\text{Id}_i \otimes \sigma_{j,k}) \quad (\text{YBE})$$

on all the tensor products  $V_i \otimes V_j \otimes V_k$  with  $1 \leq \underline{i \leq j \leq k} \leq r$ . Such a system is denoted by  $((V_i)_{1 \leq i \leq r}; (\sigma_{i,j})_{1 \leq i \leq j \leq r})$  or briefly  $(\overline{V}, \overline{\sigma})$ .

- The *rank* of a braided system is the number  $r$  of its components.
- A *braided morphism*  $\overline{f} : (\overline{V}, \overline{\sigma}) \rightarrow (\overline{W}, \overline{\xi})$  between two braided systems in  $\mathcal{C}$  of the same rank  $r$  is a collection of morphisms  $(f_i \in \text{Hom}_{\mathcal{C}}(V_i, W_i))_{1 \leq i \leq r}$  respecting the braiding, in the sense that, for all  $1 \leq \underline{i \leq j} \leq r$ , one has

$$(f_j \otimes f_i) \circ \sigma_{i,j} = \xi_{i,j} \circ (f_i \otimes f_j). \quad (3)$$

- The category of rank  $r$  braided systems and braided morphisms in  $\mathcal{C}$  is denoted by  $\mathbf{BrSyst}_r(\mathcal{C})$ .
- Rank 1 braided systems are called *braided objects* in  $\mathcal{C}$ .
- For given  $1 \leq s \leq t \leq r$ , the *braided  $(s, t)$ -subsystem* of  $(\overline{V}, \overline{\sigma})$ , denoted by  $(\overline{V}, \overline{\sigma})[s, t]$ , is the subfamily  $V_s, \dots, V_t$  with the corresponding components  $\sigma_{i,j}$  of  $\overline{\sigma}$ .

In order to emphasize the multi-component nature of our constructions, we sometimes talk about *multi-braidings*, *multi-braided morphisms*, etc.

The notion of multi-braiding thus defined is

1. **positive**, i.e., the  $\sigma_{i,j}$  are not supposed to be invertible (the term *pre-braiding*, usual in such situations, is avoided here in order not to overload the terminology);

2. **partial**, i.e., defined only on certain couples of objects (this is underlined in the definition);
3. **local**, i.e., contrary to the usual notion of braiding in a monoidal category, no naturality conditions are imposed.

Graphically, a braiding component is represented as on Fig. 1 (A); it is a braid whose strands are “colored” (= decorated) with the corresponding objects  $V_i$  (or sometimes simply their indices  $i$ ). Note that the definition allows a  $j$ -colored strand to overcross only strands colored with indices  $i \leq j$ . The diagrammatic counterpart of Equation (YBE) is now the (colored) third Reidemeister move (Fig. 1 (B)), which is at the heart of braid theory; this allows one to work with braided systems by manipulating positive braid diagrams.

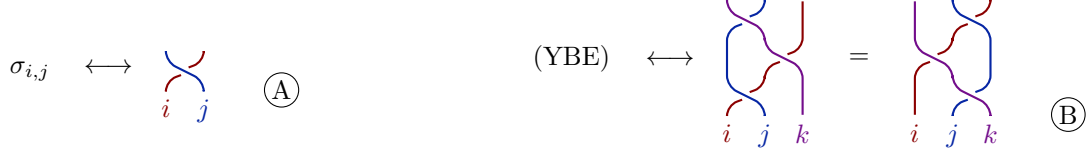


Figure 1: Braided systems versus colored braids

Each component of a braided system is a braided object. Pursuing this remark, one gets

**Proposition 2.2.** *Given a braided category  $(\mathcal{C}, \otimes, \mathbf{I}, c)$ , one has,  $\forall r \in \mathbb{N}$ , a fully faithful functor*

$$\begin{aligned}
 (\mathbf{BrSyst}_1(\mathcal{C}))^{\times r} &\hookrightarrow \mathbf{BrSyst}_r(\mathcal{C}), \\
 (V_i, \sigma_i)_{1 \leq i \leq r} &\mapsto (V_1, \dots, V_r; \sigma_{i,i} := \sigma_i, \sigma_{i,j} := c_{V_i, V_j} \text{ for } i < j), \\
 (f_i : V_i \rightarrow W_i)_{1 \leq i \leq r} &\mapsto \bar{f} := (f_i)_{1 \leq i \leq r}.
 \end{aligned} \tag{4}$$

*Proof.* There are three types of tensor products on which one should check (YBE) in order to verify that (4) defines a braided system:

1.  $V_i \otimes V_i \otimes V_i$ . Use the YBE for  $\sigma_i$  here.
2.  $V_i \otimes V_i \otimes V_j$  or  $V_i \otimes V_j \otimes V_j$  for  $i < j$ . Use the naturality of  $c$  with respect to  $\sigma_i$  or  $\sigma_j$ .
3.  $V_i \otimes V_j \otimes V_k$  for  $i < j < k$ . Use the YBE for the braiding  $c$ .

Now, for morphisms, condition (3) is automatic for  $i < j$  thanks to the naturality of  $c$ , and for  $i = j$  it is equivalent to  $f_i$  being a braided morphism. Thus our functor is well defined, full and faithful on morphisms.  $\square$

The following elementary observation will be useful in what follows:

**Observation 2.3.** *If our category  $\mathcal{C}$  is preadditive, then one has,  $\forall r \in \mathbb{N}$ , a category automorphism*

$$\begin{aligned}
 \mathbf{BrSyst}_r(\mathcal{C}) &\xrightarrow{\sim} \mathbf{BrSyst}_r(\mathcal{C}), \\
 ((V_i)_{1 \leq i \leq r}; (\sigma_{i,j})_{1 \leq i < j \leq r}) &\longleftrightarrow ((V_i)_{1 \leq i \leq r}; (-\sigma_{i,j})_{1 \leq i < j \leq r}), \\
 (f_i : V_i \rightarrow W_i)_{1 \leq i \leq r} &\longleftrightarrow (f_i : V_i \rightarrow W_i)_{1 \leq i \leq r}.
 \end{aligned}$$

## Braided modules

Actions can be defined for our new structure in a very natural way:

**Definition 2.4.**  $\rightarrow$  A *right braided module* over  $(\bar{V}, \bar{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$  is an object  $M$  equipped with morphisms  $\bar{\rho} := (\rho_i : M \otimes V_i \rightarrow M)_{1 \leq i \leq r}$  satisfying, for all  $1 \leq i \leq j \leq r$ ,

$$\rho_j \circ (\rho_i \otimes \text{Id}_j) = \rho_i \circ (\rho_j \otimes \text{Id}_i) \circ (\text{Id}_M \otimes \sigma_{i,j}) \quad : \quad M \otimes V_i \otimes V_j \rightarrow M. \tag{5}$$

- $\rightarrow$  Left braided modules and left/right braided comodules, as well as braided (co)module morphisms, are defined in the usual way.
- $\rightarrow$  The category of right braided modules and their morphisms is denoted by  $\mathbf{Mod}_{(\bar{V}, \bar{\sigma})}$ . Notation  ${}_{(\bar{V}, \bar{\sigma})}\mathbf{Mod}$  is used in the left case, and  $\mathbf{Mod}^{(\bar{V}, \bar{\sigma})}$  and  $(\bar{V}, \bar{\sigma})\mathbf{Mod}$  in the co-cases.

According to Fig. 2, braided modules can be treated by manipulating a particular type of knotted trivalent graphs (cf. [12, 33, 35] for the theory of the latter).

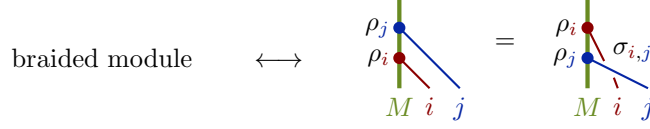


Figure 2: Right braided module

In the following sections and [15], we interpret algebra (bi)modules, Hopf (bi)modules, Yetter-Drinfel'd modules and other structures as braided modules over appropriate braided systems.

We now give some equivalent definitions of braided module structure, in particular in the important case of the unit object  $\mathbf{I}$  of  $\mathcal{C}$ .

**Observation 2.5.** *A braided  $(\overline{V}, \overline{\sigma})$ -module structure on  $M$  is the same thing as a collection of braided  $(V_i, \sigma_{i,i})$ -module structures, compatible in the sense of (5) for all  $i < j$ .*

**Observation 2.6.** *The notions of right and left braided  $(\overline{V}, \overline{\sigma})$ -modules coincide for  $\mathbf{I}$ . Condition (5) takes in this case a simpler form*

$$(\rho_j \otimes \rho_i) \circ \sigma_{i,j} = \rho_i \otimes \rho_j \quad : \quad V_i \otimes V_j \rightarrow \mathbf{I}.$$

**Definition 2.7.** A right (= left) braided  $(\overline{V}, \overline{\sigma})$ -module structure on  $\mathbf{I}$  is called a *braided character*.

**Example 2.8.** If  $\mathcal{C}$  is preadditive, then a braided character  $\varepsilon_i$  on any  $V_i$ , extended to other components by zero, trivially becomes a braided character on  $(\overline{V}, \overline{\sigma})$ .

## Invertibility questions

The invertibility of some of the  $\sigma_{i,j}$ , often encountered in practice, can be helpful in extending braided structures. It allows to interchange the corresponding components of a braided system without changing the module category:

**Proposition 2.9.** *Take  $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$  with  $\sigma_{p,p+1}$  invertible for some  $p$ . Then:*

1. *The family  $(V_1, \dots, V_{p-1}, V_{p+1}, V_p, V_{p+2}, \dots, V_r)$ , equipped with the old  $\sigma_{i,j}$  on the tensor products  $V_i \otimes V_j$  with  $(i, j) \neq (p+1, p)$  and with  $\sigma_{p,p+1}^{-1}$  on  $V_{p+1} \otimes V_p$ , is a braided system, denoted by  $s_p(\overline{V}, \overline{\sigma})$ .*
2. *The categories of braided modules for the original and the rearranged systems are equivalent:*

$$\mathbf{Mod}_{(\overline{V}, \overline{\sigma})} \simeq \mathbf{Mod}_{s_p(\overline{V}, \overline{\sigma})}. \quad (6)$$

*Proof.* 1. One has to check the following new instances of the YBE:

- (a) On  $V_i \otimes V_{p+1} \otimes V_p$  with  $i < p$ , i.e., using notation (2),

$$\sigma_{i,p+1}^2 \circ \sigma_{i,p}^1 \circ (\sigma_{p,p+1}^{-1})^2 = (\sigma_{p,p+1}^{-1})^1 \circ \sigma_{i,p}^2 \circ \sigma_{i,p+1}^1.$$

Composing both sides with  $\sigma_{p,p+1}^1$  on the left and with  $\sigma_{p,p+1}^2$  on the right, one gets

$$\sigma_{p,p+1}^1 \circ \sigma_{i,p+1}^2 \circ \sigma_{i,p}^1 = \sigma_{i,p}^2 \circ \sigma_{i,p+1}^1 \circ \sigma_{p,p+1}^2,$$

which is precisely the YBE on  $V_i \otimes V_p \otimes V_{p+1}$  for the braided system  $(\overline{V}, \overline{\sigma})$ .

- (b) On  $V_{p+1} \otimes V_p \otimes V_j$  with  $j > p+1$ . This case is similar to the previous one.  
(c) On  $V_{p+1} \otimes V_{p+1} \otimes V_p$ . Manipulations similar to case 1a iterated twice lead to the YBE on  $V_p \otimes V_{p+1} \otimes V_{p+1}$  for the braided system  $(\overline{V}, \overline{\sigma})$ .  
(d) On  $V_{p+1} \otimes V_p \otimes V_p$ . This case is similar to the previous one.
2. Given an object  $M$  equipped with morphisms  $\rho_i : M \otimes V_i \rightarrow M$ , the list of compatibility conditions (5) one has to check for  $(\overline{V}, \overline{\sigma})$  differs from the list for  $s_p(\overline{V}, \overline{\sigma})$  only in the conditions for  $i = p, j = p+1$ :

$$\begin{aligned} \rho_{p+1} \circ (\rho_p \otimes \text{Id}_{p+1}) &= \rho_p \circ (\rho_{p+1} \otimes \text{Id}_p) \circ (\text{Id}_M \otimes \sigma_{p,p+1}) \\ \text{versus} \quad \rho_p \circ (\rho_{p+1} \otimes \text{Id}_p) &= \rho_{p+1} \circ (\rho_p \otimes \text{Id}_{p+1}) \circ (\text{Id}_M \otimes \sigma_{p,p+1}^{-1}). \end{aligned}$$

The second one composed with the invertible morphism  $\text{Id}_M \otimes \sigma_{p,p+1}$  on the right yields the first one. So the identity functor of  $\mathcal{C}$  and the permutation  $\rho_p \leftrightarrow \rho_{p+1}$  of the components of  $\overline{\rho}$  give a category equivalence (6).  $\square$

*Remark 2.10.* More generally, given a permutation  $\theta \in S_r$  and a  $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$  with all the  $\sigma_{i,j}$  invertible for  $i$  and  $j$  reversed by  $\theta$ , the family  $(V_{\theta^{-1}(1)}, \dots, V_{\theta^{-1}(r)})$ , equipped with the old  $\sigma_{i,j}$  on  $V_i \otimes V_j$  with  $i < j$ ,  $\theta(i) < \theta(j)$  and with  $\sigma_{i,j}^{-1}$  on  $V_j \otimes V_i$  with  $i < j$ ,  $\theta(i) > \theta(j)$ , is a braided system, denoted by  $\theta(\overline{V}, \overline{\sigma})$ . One thus obtains a *partial  $S_r$ -action* on  $\mathbf{BrSyst}_r(\mathcal{C})$ , inducing equivalences between the corresponding categories of braided modules.

**Corollary 2.11.** *Let  $(\overline{V}, \overline{\sigma})$  be a braided system in an additive monoidal  $\mathcal{C}$ , with  $\sigma_{i,j}$  invertible for all  $s \leq i < j \leq t$ . Then one can glue the objects  $V_s, \dots, V_t$  together into  $V_{s:t} := \bigoplus_{i=s}^t V_i$  and extend the braiding onto  $(V_1, \dots, V_{s-1}, V_{s:t}, V_{t+1}, \dots, V_r)$ , putting  $\sigma|_{V_j \otimes V_i} := \sigma_{i,j}^{-1}$  for all  $s \leq i < j \leq t$ .*

Note that the invertibility of  $\sigma_{i,i}$  is not required here even for  $s \leq i \leq t$ .

*Proof.* It suffices to consider the case  $s = t - 1 =: p$  — the general case then follows by induction. The only instances of (YBE) appearing here in addition to those coming from the braided systems  $(\overline{V}, \overline{\sigma})$  and  $s_p(\overline{V}, \overline{\sigma})$  are those on  $V_p \otimes V_{p+1} \otimes V_p$  and  $V_{p+1} \otimes V_p \otimes V_{p+1}$ . They are proved by the same argument as in Point 1a (or 1b) of the proof of Proposition 2.9, taking  $i = p$  (or  $j = p + 1$ ).  $\square$

The corollary recovers in particular the *gluing procedure* for Yang-Baxter operators (or, in our terms, for braided objects) described by S. Majid and M. Markl in [19].

### 3 A homology theory for braided systems

We now generalize the *braided homology* theory, developed in [14] for braided objects in  $\mathcal{C}$ , to the braided system setting. In this section  $\mathcal{C}$  is additive monoidal. In particular, one can interpret the collection  $\overline{\sigma}$  as a *partial braiding*, still denoted by  $\overline{\sigma}$ , on

$$V := V_1 \oplus V_2 \oplus \dots \oplus V_r,$$

and the family  $\overline{\rho}$  defining a right braided  $(\overline{V}, \overline{\sigma})$ -module as a morphism  $\rho : M \otimes V \rightarrow M$ .

#### Quantum shuffles: a multi-version

We start by showing that the collection  $\overline{\sigma}$  suffices for defining a partial version of quantum (co)shuffle structures, defined in M. Rosso's pioneer papers [28, 29].

**Definition 3.1.** The permutation sets

$$Sh_{p_1, p_2, \dots, p_k} := \left\{ \theta \in S_{p_1 + p_2 + \dots + p_k} \left| \begin{array}{l} \theta(1) < \theta(2) < \dots < \theta(p_1), \\ \theta(p_1 + 1) < \dots < \theta(p_1 + p_2), \\ \dots, \\ \theta(p + 1) < \dots < \theta(p + p_k) \end{array} \right. \right\},$$

where  $p = p_1 + p_2 + \dots + p_{k-1}$ , are called *shuffle sets*.

In other words, one permutes  $p_1 + p_2 + \dots + p_k$  elements preserving the order within  $k$  consecutive blocks of size  $p_1, p_2, \dots, p_k$ , just like when shuffling cards, which explains the name.

Recall further the projection  $B_n^+ \twoheadrightarrow S_n$ , sending a generator  $\sigma_i$  to the corresponding generator  $s_i$ , and its set-theoretical (i.e. not preserving the monoid structure) *Matsumoto section*

$$\begin{aligned} S_n &\longleftarrow B_n^+, \\ \theta = s_{i_1} s_{i_2} \dots s_{i_k} &\longmapsto \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k}, \end{aligned}$$

where  $s_{i_1} s_{i_2} \dots s_{i_k}$  is any of the shortest words representing  $\theta \in S_n$ .

**Notation 3.2.** We denote by  $B_\theta$  the image of  $\theta \in S_n$  under this map.

Now let us return to the context of braided systems.

**Definition 3.3.**  $\twoheadrightarrow$  A degree  $d$  (*reversely*) *ordered tensor product* for  $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$  is a tensor product of the form  $V_{k_1} \otimes \dots \otimes V_{k_d}$  with  $k_1 \leq \dots \leq k_d$  (respectively,  $k_1 \geq \dots \geq k_d$ ).

$\twoheadrightarrow$  The direct sum of all the (reversely) ordered tensor products of degree  $d$  is denoted by  $T(\overline{V})_d^{\twoheadrightarrow}$  (respectively,  $T(\overline{V})_d^{\leftarrow}$ ).



In  $\mathbf{Vect}_k$ , the  $T(\overline{V})_d^{\rightarrow}$  sum up to  $T(\overline{V})^{\rightarrow} := T(V_1) \otimes T(V_2) \otimes \cdots \otimes T(V_r)$ , and the  $T(\overline{V})_d^{\leftarrow}$  sum up to  $T(\overline{V})^{\leftarrow} := T(V_r) \otimes T(V_{r-1}) \otimes \cdots \otimes T(V_1)$ .

The last ingredient we need is a partial  $B_d^+$ -action on  $V^d$  for  $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$ . For a generator  $\sigma_i$  of  $B_d^+$  and a summand  $V_{k_1} \otimes \cdots \otimes V_{k_d}$  of  $V^d$  with  $k_1 \leq \cdots \leq k_d$ , it is defined by

$$\sigma_i \quad \longmapsto \quad \sigma_{k_i, k_{i+1}}^i = \begin{array}{c} | \quad | \quad \text{---} \quad | \quad | \\ \text{\scriptsize } k_1 \quad \cdots \quad k_i \quad k_{i+1} \quad \cdots \quad k_d \end{array},$$

the latter morphism lying in  $\text{Hom}_{\mathcal{C}}(V_{k_1} \otimes \cdots \otimes V_{k_d}, V_{k_1} \otimes \cdots \otimes V_{k_{i+1}} \otimes V_{k_i} \otimes \cdots \otimes V_{k_d})$ . This action agrees with the common graphical depiction of elements of  $B_d^+$ .

**Notation 3.4.** The partial action described above is denoted by  $B_d^+ \ni b \mapsto b^{\overline{\sigma}}$ .

Armed with all these notations, we are ready to define the multi-versions of quantum shuffle operations:

**Definition 3.5.** Take  $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$ .

→ The family of morphisms

$$\Delta_{\overline{\sigma}}^{p,q} := \sum_{\theta \in Sh_{p,q}} (B_{\theta})^{\overline{\sigma}} \quad : \quad T(\overline{V})_p^{\leftarrow} \otimes T(\overline{V})_q^{\leftarrow} \rightarrow T(\overline{V})_{p+q}^{\leftarrow}, \quad (7)$$

where for given reversely ordered tensor products  $W$  in  $T(\overline{V})_p^{\leftarrow}$  and  $U$  in  $T(\overline{V})_q^{\leftarrow}$  the summation runs only over  $\theta$  for which the action  $(B_{\theta})^{\overline{\sigma}}$  is defined on  $W \otimes U$  and takes values in the  $T(\overline{V})_{p+q}^{\leftarrow}$  part of  $V^{p+q}$ , is called the *multi-quantum shuffle multiplication*.

→ The family of morphisms

$$\Theta_{\overline{\sigma}}^{p,q} := \sum_{\theta \in Sh_{p,q}} (B_{\theta^{-1}})^{\overline{\sigma}} \quad : \quad T(\overline{V})_{p+q}^{\rightarrow} \rightarrow T(\overline{V})_p^{\rightarrow} \otimes T(\overline{V})_q^{\rightarrow} \quad (8)$$

is called the *multi-quantum coshuffle comultiplication*.

→ More generally, replacing  $Sh_{p,q}$  with  $Sh_{p_1, \dots, p_k}$ , one gets morphisms  $\Delta_{\overline{\sigma}}^{p_1, \dots, p_k}$  and  $\Theta_{\overline{\sigma}}^{p_1, \dots, p_k}$ .

In order to better understand the condition on  $\theta$  in (7), one should think of it as a dual version of the more natural definition (8).

Note that for an ordered tensor products  $W$  in  $T(\overline{V})_{p+q}^{\rightarrow}$ , its image  $\Theta_{\overline{\sigma}}^{p,q}(W)$  lives in several summands of  $T(\overline{V})_p^{\rightarrow} \otimes T(\overline{V})_q^{\rightarrow}$ . That is why we need categories to be additive here.

**Proposition 3.6.** *The morphisms (7)-(8) are well-defined, and give an associative multiplication (respectively, a coassociative comultiplication).*

*Proof.* If (reversely) ordered tensor products are fed into these formulas, then the braiding  $\overline{\sigma}$  is applied only to components  $V_i \otimes V_j$  with  $i \leq j$ , so the morphisms are well defined. The (co)associativity is proved as in the rank 1 case (see [29] or [14]).  $\square$

## Multi-braided differentials

We now explain what we mean by a *homology theory* for a braided system  $(\overline{V}, \overline{\sigma})$  in  $\mathcal{C}$ :

**Definition 3.7.** → A *differential* for  $(\overline{V}, \overline{\sigma})$  is a family of morphisms  $\{d_n : T(\overline{V})_n^{\rightarrow} \rightarrow T(\overline{V})_{n-1}^{\rightarrow}\}_{n>0}$  satisfying  $d_{n-1} \circ d_n = 0$  for all  $n > 1$ .

→ A *bidifferential* for  $(\overline{V}, \overline{\sigma})$  consists of two families of morphisms  $\{d_n, d'_n : T(\overline{V})_n^{\rightarrow} \rightarrow T(\overline{V})_{n-1}^{\rightarrow}\}_{n>0}$  satisfying, for all  $n > 1$ ,

$$d_{n-1} \circ d_n = d'_{n-1} \circ d'_n = d'_{n-1} \circ d_n + d_{n-1} \circ d'_n = 0.$$

→ Replacing  $T(\overline{V})_n^{\rightarrow}$  with  $M \otimes T(\overline{V})_n^{\rightarrow} \otimes N$  (for some objects  $M$  and  $N$ ) above, one gets the notion of (bi)differentials with *coefficients* in  $M$  and  $N$ .

Everything is now ready for constructing a multi-version of braided differentials:

**Theorem 1.** Take a braided system  $(\overline{V}, \overline{\sigma})$  in an additive monoidal category  $\mathcal{C}$ , equipped with a right and, respectively, left braided  $(\overline{V}, \overline{\sigma})$ -modules  $(M, \overline{\rho})$  and  $(N, \overline{\lambda})$ . The families of morphisms

$$\begin{aligned} (\rho d)_n &:= (\rho \otimes \text{Id}_{T(\overline{V})_{n-1} \otimes N}) \circ (\text{Id}_M \otimes \underline{\Theta}_{1, n-1} \otimes \text{Id}_N), \\ (d^\lambda)_n &:= (-1)^{n-1} (\text{Id}_{M \otimes T(\overline{V})_{n-1}} \otimes \lambda) \circ (\text{Id}_M \otimes \underline{\Theta}_{n-1, 1} \otimes \text{Id}_N) \end{aligned}$$

from  $M \otimes T(\overline{V})_n \otimes N$  to  $M \otimes T(\overline{V})_{n-1} \otimes N$  define a bidifferential with coefficients in  $M$  and  $N$ . (Here  $-\overline{\sigma}$  is the braiding obtained from  $\overline{\sigma}$  according to Observation 2.3.)

*Proof.* The verifications use

1 the coassociativity of  $\underline{\Theta}_{-\overline{\sigma}}$ , and

2 the definition of braided modules, reformulated in a preadditive  $\mathcal{C}$  as

$$\rho \circ (\rho \otimes \text{Id}_V) \circ (\text{Id}_M \otimes \underline{\Theta}_{1, 1}) = 0, \quad \lambda \circ (\text{Id}_V \otimes \lambda) \circ (\underline{\Theta}_{1, 1} \otimes \text{Id}_N) = 0.$$

Concretely, writing  $\underline{\Theta}_{-\overline{\sigma}}$  instead of  $\text{Id}_M \otimes \underline{\Theta}_{-\overline{\sigma}} \otimes \text{Id}_N$  or  $\underline{\Theta}_{-\overline{\sigma}} \otimes \text{Id}_N$  for brevity, one calculates

$$\begin{aligned} (\rho d)_{n-1} \circ (\rho d)_n &= (\rho \otimes \text{Id} \dots) \circ \underline{\Theta}_{1, n-2} \circ (\rho \otimes \text{Id} \dots) \circ \underline{\Theta}_{1, n-1} \\ &= (\rho \otimes \text{Id} \dots) \circ (\rho \otimes \text{Id} \dots) \circ (\text{Id}_{M \otimes V} \otimes \underline{\Theta}_{1, n-2}) \circ \underline{\Theta}_{1, n-1} \\ &\stackrel{1}{=} (\rho \otimes \text{Id} \dots) \circ (\rho \otimes \text{Id} \dots) \circ (\text{Id}_M \otimes \underline{\Theta}_{1, 1} \otimes \text{Id} \dots) \circ \underline{\Theta}_{2, n-2} \\ &= ((\rho \circ (\rho \otimes \text{Id}_V) \circ (\text{Id}_M \otimes \underline{\Theta}_{1, 1})) \otimes \text{Id} \dots) \circ \underline{\Theta}_{2, n-2} \\ &\stackrel{2}{=} 0 \circ \underline{\Theta}_{2, n-2} = 0, \end{aligned}$$

and similarly for  $d^\lambda$ . Further, one has

$$(d^\lambda)_{n-1} \circ (\rho d)_n = (-1)^{n-2} (\rho \otimes \text{Id} \dots \otimes \lambda) \circ \underline{\Theta}_{1, n-2, 1} = -(\rho d)_{n-1} \circ (d^\lambda)_n. \quad \square$$

The differential  $(\rho d)_n$  — and, similarly,  $(d^\lambda)_n$  — can be viewed as a signed sum (since the negative braiding  $-\overline{\sigma}$  is used) of the form  $\sum_{i=1}^n (-1)^{i-1} (\rho d)_{n; i}$ . The term  $(\rho d)_{n; i}$  is diagrammatically presented on Fig. 3. The sign can be interpreted here as the crossing number of the diagram.

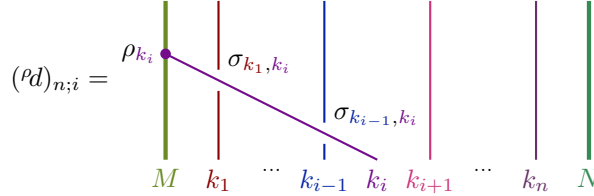


Figure 3: Multi-braided left differential

**Corollary 3.8.** Any  $\mathbb{Z}$ -linear combination of the families  $(\rho d)_n$  and  $(d^\lambda)_n$  from the theorem is a differential for  $(\overline{V}, \overline{\sigma})$  with coefficients in  $M$  and  $N$ .

The (bi)differentials from the above theorem and corollary are called *braided*.

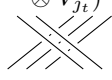
*Remark 3.9.* ✓ Braided (bi)differentials are *functorial*: for braided systems  $(\overline{V}, \overline{\sigma})$  and  $(\overline{V}', \overline{\sigma}')$  and braided modules  $(M, \overline{\rho}) \in \mathbf{Mod}_{(\overline{V}, \overline{\sigma})}$ ,  $(M', \overline{\rho}') \in \mathbf{Mod}_{(\overline{V}', \overline{\sigma}')}$ ,  $(N, \overline{\lambda}) \in {}_{(\overline{V}, \overline{\sigma})}\mathbf{Mod}$ ,  $(N', \overline{\lambda}') \in {}_{(\overline{V}', \overline{\sigma}')} \mathbf{Mod}$ , any braided morphism  $\overline{f} : (\overline{V}, \overline{\sigma}) \rightarrow (\overline{V}', \overline{\sigma}')$  and morphisms  $\varphi : M \rightarrow M'$ ,  $\psi : N \rightarrow N'$  compatible with  $\overline{f}$  (i.e.,  $\rho'_i \circ (\varphi \otimes f_i) = \varphi \circ \rho_i$  and  $\lambda'_i \circ (f_i \otimes \psi) = \psi \circ \lambda_i$  for all  $i$ ) intertwine braided differentials, in the sense that the following diagram commutes:

$$\begin{array}{ccc} M \otimes T(\overline{V})_n \otimes N & \xrightarrow{\varphi \otimes \overline{f}^{\otimes n} \otimes \psi} & M' \otimes T(\overline{V}')_n \otimes N' \\ d_n \downarrow & & d'_n \downarrow \\ M \otimes T(\overline{V})_{n-1} \otimes N & \xrightarrow{\varphi \otimes \overline{f}^{\otimes n-1} \otimes \psi} & M' \otimes T(\overline{V}')_{n-1} \otimes N' \end{array}$$

- ✓ Dually, one obtains a *cocohomology theory* for  $(\overline{V}, \overline{\sigma})$  with coefficients in braided *comodules*. Note that one works with  $T(\overline{V})_n^{\leftarrow}$  in the dual settings, since a braiding on the system  $(V_1, \dots, V_r)$  in  $\mathcal{C}$  is the same thing as a braiding on the reversed system  $(V_r, \dots, V_1)$  in  $\mathcal{C}^{\text{op}}$ .
- ✓ Braided bidifferentials can be refined to a *precubical structure*, enriched with degeneracies if the braided system is moreover endowed with a “good” comultiplication (i.e. compatible with the braiding and  $\overline{\sigma}$ -cocommutative); see [14] for details in the braided object case.
- ✓ Braided differentials  $({}^\rho d)_n$  (or  $(d^\lambda)_n$ ) can be defined with coefficients on one side only, i.e. on  $M \otimes T(\overline{V})_n^{\rightarrow}$  (or  $T(\overline{V})_n^{\rightarrow} \otimes N$ ), by analogous formulas.

## Adjoint multi-braided modules

The theory of adjoint braided modules from [14], including its homological consequences, has a multi-version as well.

**Notation 3.10.** The obvious morphism from  $(V_{i_1} \otimes \dots \otimes V_{i_s}) \otimes (V_{j_1} \otimes \dots \otimes V_{j_t})$  to  $(V_{j_1} \otimes \dots \otimes V_{j_t}) \otimes (V_{i_1} \otimes \dots \otimes V_{i_s})$ , induced by  $\overline{\sigma}$  and diagrammatically presented as , is denoted by  $\overline{\sigma}$ . (Here we suppose  $i_n \leq j_m$  for all  $n, m$ , so that  $\overline{\sigma}$  is applicable to  $V_{i_n} \otimes V_{j_m}$ .)

**Proposition 3.11.** Take  $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$  and  $(M, \overline{\rho}) \in \mathbf{Mod}_{(\overline{V}, \overline{\sigma})}$ . Fix an integer  $1 \leq t \leq r$ , and denote by  $(\overline{V}', \overline{\sigma})$  the braided  $(1, t)$ -subsystem  $(\overline{V}, \overline{\sigma})[1, t]$ .

1. For any  $n \in \mathbb{N}$ , a braided  $(\overline{V}, \overline{\sigma})[t, r]$ -module structure can be defined on  $M \otimes T(\overline{V}')_n^{\rightarrow}$  via

$${}^\rho \pi_i := (\rho_i \otimes \text{Id}_{T(\overline{V}')_n^{\rightarrow}}) \circ (\text{Id}_M \otimes \overline{\sigma}_{T(\overline{V}')_n^{\rightarrow}, V_i}) \quad : \quad M \otimes T(\overline{V}')_n^{\rightarrow} \otimes V_i \rightarrow M \otimes T(\overline{V}')_n^{\rightarrow}.$$

2. The braided differentials  ${}^\rho d$  on  $(\overline{V}, \overline{\sigma})$  with coefficients in  $M$  are braided module morphisms for the structure above.

*Proof.* Let us prove the compatibility relation (5) for  ${}^\rho \pi_i$  and  ${}^\rho \pi_j$  with  $t \leq i \leq j \leq r$ . Working on  $M \otimes T(\overline{V}')_n^{\rightarrow} \otimes V_i \otimes V_j$ , and using Notation (2), one gets

$$\begin{aligned} & {}^\rho \pi_i \circ ({}^\rho \pi_j \otimes \text{Id}_i) \circ (\text{Id}_{M \otimes T(\overline{V}')_n^{\rightarrow}} \otimes \sigma_{i,j}) \\ &= \rho_i^1 \circ \overline{\sigma}_{T(\overline{V}')_n^{\rightarrow}, V_i}^2 \circ \rho_j^1 \circ \overline{\sigma}_{T(\overline{V}')_n^{\rightarrow}, V_j}^2 \circ \sigma_{i,j}^{n+2} \\ &= \rho_i^1 \circ \rho_j^1 \circ \overline{\sigma}_{T(\overline{V}')_n^{\rightarrow}, V_j \otimes V_i}^2 \circ \sigma_{i,j}^{n+2} \\ &\stackrel{\boxed{1}}{=} \rho_i^1 \circ \rho_j^1 \circ \sigma_{i,j}^2 \circ \overline{\sigma}_{T(\overline{V}')_n^{\rightarrow}, V_i \otimes V_j}^2 \\ &\stackrel{\boxed{2}}{=} \rho_j^1 \circ \rho_i^1 \circ \overline{\sigma}_{T(\overline{V}')_n^{\rightarrow}, V_i \otimes V_j}^2 \\ &= \rho_j^1 \circ \overline{\sigma}_{T(\overline{V}')_n^{\rightarrow}, V_j}^2 \circ \rho_i^1 \circ \overline{\sigma}_{T(\overline{V}')_n^{\rightarrow}, V_i}^2 = {}^\rho \pi_j \circ ({}^\rho \pi_i \otimes \text{Id}_j), \end{aligned}$$

where  $\boxed{1}$  is a repeated application of (YBE), and  $\boxed{2}$  follows from the relation (5) for  $\rho_i$  and  $\rho_j$ . The compatibility relation for  ${}^\rho \pi_i$  and  ${}^\rho d$  with  $t \leq i \leq r$  is verified similarly.  $\square$

Applied to a braided object  $(V, \sigma)$  and a braided character on it, Proposition 3.11 endows all the tensor powers  $V^n$  with a braided  $(V, \sigma)$ -module structure. Inspired by this example, we call *adjoint* the braided modules from the proposition.

## 4 A proto-example: braided systems of associative algebras

The braided systems studied in this section have unital associative algebras (= UAAs) as components  $V_i$ , and “associativity braidings” from [14] as the diagonal parts  $\sigma_{i,i}$  of the braiding. We exhibit a bijection between such braided systems and *braided tensor products of algebras*, identifying braided modules over the former with usual modules over the latter. The “component permuting” Proposition 2.9 then yields rules for permuting the factors of braided tensor products of algebras. Concrete examples illustrating the advantages of our braided system approach follow in subsequent sections. In this section  $\mathcal{C}$  is again monoidal, not necessarily preadditive.

## A braiding encoding the associativity

The braidings we use in the associativity setting come with some additional structure:

**Definition 4.1.**  $\rightarrow$  Denote by  $\mathbf{BrSyst}_r^\downarrow(\mathcal{C})$  the category of

- ✓  $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$  endowed with distinguished morphisms  $\overline{\nu} := (\nu_i : \mathbf{I} \rightarrow V_i)_{1 \leq i \leq r}$ , called *units*, and
- ✓ morphisms from  $\mathbf{BrSyst}_r(\mathcal{C})$  preserving all the units.

Objects  $(\overline{V}, \overline{\sigma}, \overline{\nu})$  of  $\mathbf{BrSyst}_r^\downarrow(\mathcal{C})$  are called rank  $r$  *pointed braided systems*.

- $\rightarrow$  A *right braided module* over  $(\overline{V}, \overline{\sigma}, \overline{\nu}) \in \mathbf{BrSyst}_r^\downarrow(\mathcal{C})$  is a right braided  $(\overline{V}, \overline{\sigma})$ -module  $(M, \overline{\rho})$  satisfying moreover  $\rho_i \circ (\text{Id}_M \otimes \nu_i) = \text{Id}_M$  for all  $1 \leq i \leq r$  (morally, the units act by identity). The category of such modules and their morphisms is denoted by  $\mathbf{Mod}_{(\overline{V}, \overline{\sigma}, \overline{\nu})}$ . Similar definitions and notations are assumed for left modules.

**Notation 4.2.** The category of UAAs and algebra morphisms in  $\mathcal{C}$  is denoted by  $\mathbf{Alg}(\mathcal{C})$ .

We next show that different aspects of the UAA structure are captured by a braiding encoding it, called the *associativity braiding* in what follows:

**Theorem 2** ([14]). 1. One has a fully faithful functor

$$\boxed{\mathbf{Alg}(\mathcal{C}) \hookrightarrow \mathbf{BrSyst}_1^\downarrow(\mathcal{C})} \quad (9)$$

$$(V, \mu, \nu) \mapsto (V, \sigma_{Ass}, \nu),$$

$$f \mapsto f,$$

where

$$\sigma_{Ass} := \nu \otimes \mu \quad : \quad V \otimes V = \mathbf{I} \otimes V \otimes V \rightarrow V \otimes V. \quad (10)$$

2. The associativity braiding  $\sigma_{Ass}$  is idempotent:  $\sigma_{Ass} \circ \sigma_{Ass} = \sigma_{Ass}$ .
3. The YBE for  $\sigma_{Ass}$  is equivalent to the associativity for  $\mu$ , under the assumption that  $\nu$  is a unit map for  $\mu$  (i.e.,  $\mu \circ (\text{Id}_V \otimes \nu) = \mu \circ (\nu \otimes \text{Id}_V) = \text{Id}_V$ ).
4. For a UAA  $(V, \mu, \nu)$  in  $\mathcal{C}$ , one has an equivalence of right module categories

$$\boxed{\mathbf{Mod}_{(V, \mu, \nu)} \xrightarrow{\sim} \mathbf{Mod}_{(V, \sigma_{Ass}, \nu)}} \quad (M, \rho) \longleftrightarrow (M, \rho),$$

where on the left one considers usual modules over UAAs, and on the right the pointed version of braided modules.

5. For a module  $(M, \rho) \in \mathbf{Mod}_{(V, \mu, \nu)} \simeq \mathbf{Mod}_{(V, \sigma_{Ass}, \nu)}$ , the left braided differential  $\rho d$  on  $(M \otimes V^n)_{n \geq 0}$  coincides with the bar differential  $d_n = \rho^1 + \sum_{i=1}^{n-1} (-1)^i \mu^i$  with coefficients in  $M$ .

When working with several UAAs, we use notation  $\sigma_{Ass}(V)$  or  $\sigma_{Ass}(V, \mu, \nu)$  to avoid confusion.

**Remark 4.3.** ✓ A more elegant functor  $\mathbf{Alg}(\mathcal{C}) \rightarrow \mathbf{BrSyst}_1(\mathcal{C})$  is obtained by composing (9) with a forgetful functor. However, the pointed structure is needed if one wants a full functor.

- ✓ Point 2 shows that the braiding  $\sigma_{Ass}$  is *highly non-invertible* in general. This explains our choice of the positive notion of braiding.
- ✓ The equivalence in 3 holds only under a mild condition concerning units; such *normalization conditions* often appear in our “braided” study of structures.
- ✓ Point 4 applied to  $M = \mathbf{I}$  ensures that an algebra character is always a braided character.
- ✓ Dualizing, one interprets the category of *coalgebras* in  $\mathcal{C}$  as a subcategory of *co-pointed* (= endowed with a distinguished co-element) braided objects via the fully faithful functor

$$\mathbf{coAlg}(\mathcal{C}) \hookrightarrow \mathbf{BrSyst}_1^\uparrow(\mathcal{C}),$$

$$(V, \Delta, \varepsilon) \mapsto (V, \sigma_{coAss} = \varepsilon \otimes \Delta, \varepsilon),$$

$$f \mapsto f.$$

The algebra-coalgebra duality in a preadditive  $\mathcal{C}$  can now be seen inside the category of bi-pointed braided objects  $\mathbf{BrSyst}_1^\uparrow(\mathcal{C})$ , since this category is self-dual (the notion of braiding being so) and encompasses both  $\mathbf{Alg}(\mathcal{C})$  and  $\mathbf{coAlg}(\mathcal{C})$  (the missing (co)unit structure can

be taken zero):

$$\mathbf{coAlg}(\mathcal{C}) \hookrightarrow \mathbf{BrSyst}_1^\uparrow(\mathcal{C}) \hookleftarrow \mathbf{Alg}(\mathcal{C}).$$

- ✓ In the theorem, the associativity braiding can be replaced with its *right version*  $\sigma_{Ass}^r := \mu \otimes \nu$ . In this case left modules should be taken as coefficients in the last point. The diagrams of the two associativity braidings are shown on Fig. 4.



Figure 4: Associativity braidings:  $\sigma_{Ass}$  and its vertical mirror version  $\sigma_{Ass}^r$

## Multi-braided tensor products of algebras

From now on, we work in the settings of several interacting UAAs. After some general technical definitions, we study the compatibility of braidings  $\sigma_{Ass}(V_i)$  for different UAAs  $V_i$  and interpret it in terms of (a multi-version of) braided tensor products of algebras.

**Definition 4.4.** → Take an object  $V$  in  $\mathcal{C}$ . A pair of morphisms  $(\eta : \mathbf{I} \rightarrow V, \epsilon : V \rightarrow \mathbf{I})$  is called *normalized* if  $\epsilon \circ \eta = \text{Id}_{\mathbf{I}}$ .

- Take objects  $V, W$  in  $\mathcal{C}$ . A morphism  $\xi : V \otimes W \rightarrow W \otimes V$  is called *natural with respect to a morphism*  $\varphi : V^n \rightarrow V^m$  (or  $\psi : W^n \rightarrow W^m$ ) if

$$\xi^1 \circ \dots \circ \xi^m \circ (\varphi \otimes \text{Id}_W) = (\text{Id}_W \otimes \varphi) \circ \xi^1 \circ \dots \circ \xi^n$$

(where Notation (2) is used), or, respectively,

$$\xi^m \circ \dots \circ \xi^1 \circ (\text{Id}_V \otimes \psi) = (\psi \otimes \text{Id}_V) \circ \xi^m \circ \dots \circ \xi^1.$$

In the case  $V = W$  both conditions are required.

The naturality conditions for  $n = 1, m = 2$  and  $V = W$  are diagrammatically presented on Fig. 5. In this example, one recovers two of the six Reidemeister moves from the theory of knotted trivalent graphs (cf. [12, 33, 35]).

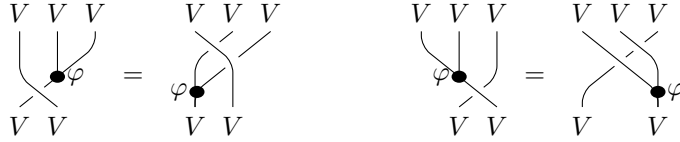


Figure 5: Naturality

**Theorem 3.** Take  $r$  UAAs  $(V_i, \mu_i, \nu_i)_{1 \leq i \leq r}$  in a monoidal category  $\mathcal{C}$ , each unit  $\nu_i$  being a part of a normalized pair  $(\nu_i, \epsilon_i)$ . For each couple of subscripts  $1 \leq i < j \leq r$ , take a morphism  $\xi_{i,j}$  natural with respect to  $\nu_i$  and  $\nu_j$ . The following statements are then equivalent:

- 1 The morphisms  $\xi_{i,i} := \sigma_{Ass}(V_i)$ ,  $1 \leq i \leq r$ , complete the  $\xi_{i,j}$  and the  $\nu_i$  into a pointed braided system structure on  $\overleftarrow{V}$ .
- 2 Each  $\xi_{i,j}$  is natural with respect to  $\mu_i$  and  $\mu_j$ , and, for each triple  $i < j < k$ , the  $\xi$  satisfy the YBE on  $V_i \otimes V_j \otimes V_k$ .
- 3 A UAA structure on

$$\overleftarrow{V} := V_r \otimes V_{r-1} \otimes \dots \otimes V_1$$

can be defined by putting

$$\mu_{\overleftarrow{V}} := (\mu_r \otimes \dots \otimes \mu_1) \circ \xi_{1,2}^{2r-2} \circ (\xi_{2,3}^{2r-4} \circ \xi_{1,3}^{2r-3}) \circ \dots \circ (\xi_{r-1,r}^2 \circ \dots \circ \xi_{2,r}^{r-1} \circ \xi_{1,r}^r), \quad (11)$$

$$\nu_{\overleftarrow{V}} := \nu_r \otimes \nu_{r-1} \otimes \dots \otimes \nu_1. \quad (12)$$

The multiplication (11) for  $r = 3$  is diagrammatically presented on Fig. 6 (A). Note the inverse component order in the definition of  $\overleftarrow{V}$ , ensuring that  $\mu_{\overleftarrow{V}}$  is well-defined.

*Proof.* We show that Points 1 and 3 are both equivalent to the (intermediate) Point 2.

Start with 1. The YBE on each  $V_i \otimes V_i \otimes V_i$  is automatic via Theorem 2. On  $V_i \otimes V_i \otimes V_j$  with  $i < j$ , the YBE becomes

$$(\xi_{i,j} \otimes \text{Id}_i) \circ (\text{Id}_i \otimes \xi_{i,j}) \circ (\nu_i \otimes \mu_i \otimes \text{Id}_j) = (\text{Id}_j \otimes \nu_i \otimes \mu_i) \circ (\xi_{i,j} \otimes \text{Id}_i) \circ (\text{Id}_i \otimes \xi_{i,j})$$

(see Fig. 6 (B) for a graphical version). But this is equivalent to  $\xi_{i,j}$  being natural with respect to  $\mu_i$  (cf. Fig. 6 (C)): compose the former with  $\text{Id}_j \otimes \mu_i$  to get the latter, and compose the latter with  $(\xi_{i,j} \otimes \text{Id}_i) \circ (\nu_i \otimes \text{Id}_j \otimes \text{Id}_i)$  to get the former (in each case, use the naturality of  $\xi_{i,j}$  with respect to the units to pull the truncated strands out of all crossings). Similarly, YBE on  $V_i \otimes V_j \otimes V_j$ ,  $i < j$ , is equivalent to  $\xi_{i,j}$  being natural with respect to  $\mu_j$ . This yields the equivalence  $1 \Leftrightarrow 2$ .

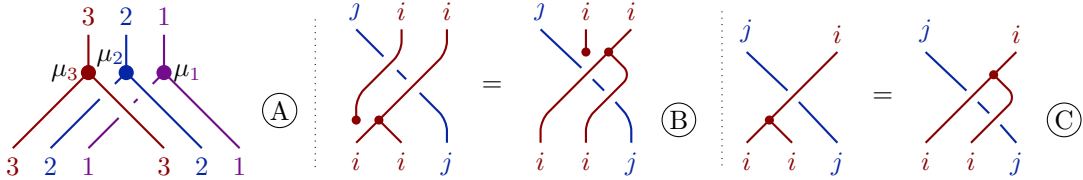


Figure 6: Braided tensor product of UAAs; YBE for  $V_i \otimes V_i \otimes V_j$ ; naturality with respect to  $\mu_i$

Let us now show the equivalence  $3 \Leftrightarrow 2$ . We use short-cut notations

$$\iota_j := \nu_r \otimes \cdots \otimes \nu_{j+1} \otimes \text{Id}_j \otimes \nu_{j-1} \otimes \cdots \otimes \nu_1 : V_j \rightarrow \overleftarrow{V}. \quad (13)$$

Given a collection  $\xi_{i,j}$  satisfying the conditions of Point 2, one checks (for instance graphically) that  $\mu_{\overleftarrow{V}}$  and  $\nu_{\overleftarrow{V}}$  from Point 3 define a UAA structure on  $\overleftarrow{V}$ . This generalizes the verifications usually made while defining tensor product of algebras in a braided category. To show that all the conditions from Point 2 are necessary, consider the associativity condition for  $\mu_{\overleftarrow{V}}$  composed with

- either  $\iota_i \otimes \iota_j \otimes \iota_k : V_i \otimes V_j \otimes V_k \rightarrow \overleftarrow{V}^{\otimes 3}$  on the right and the  $\epsilon_t$  at all the positions except for  $i, j$  and  $k$  on the left (this gives the YBE on  $V_i \otimes V_j \otimes V_k$ ,  $i < j < k$ );
- or  $\iota_i \otimes \iota_i \otimes \iota_j : V_i \otimes V_i \otimes V_j \rightarrow \overleftarrow{V}^{\otimes 3}$  on the right and the  $\epsilon_t$  at all the positions except for  $i$  and  $j$  on the left (this gives the naturality of  $\xi_{i,j}$  with respect to  $\mu_i$ );
- or  $\iota_i \otimes \iota_j \otimes \iota_j : V_i \otimes V_j \otimes V_j \rightarrow \overleftarrow{V}^{\otimes 3}$  on the right and the  $\epsilon_t$  at all the positions except for  $i, j$  on the left (this gives the naturality of  $\xi_{i,j}$  with respect to  $\mu_j$ ).

Above we used the naturality of the  $\xi$  with respect to the units and the defining property of a normalized pair.  $\square$

The theorem gives a “braided” (1), an “algebraic” (3) and a “mixed” (2) interpretations of the same phenomenon. For certain structures, the associativity verification can be considerably simplified by checking (1) or (2) instead.

**Definition 4.5.** A braided system of the type described in the theorem is called a (*pointed*) *braided system of UAAs*, and the UAA  $\overleftarrow{V}$  is called the *braided tensor product* of the UAAs  $V_1, \dots, V_r$ , denoted (abusively) by

$$\overleftarrow{V} = V_r \underset{\xi}{\otimes} V_{r-1} \underset{\xi}{\otimes} \cdots \underset{\xi}{\otimes} V_1.$$

*Remark 4.6.* The existence of the  $\epsilon_i$  was used only to prove  $3 \Rightarrow 2$ . One can replace it by demanding (3) to hold for all subsystems of  $\overleftarrow{V}$ . In this case, while proving  $3 \Rightarrow 2$ , one can work with an appropriate subsystem instead of composing with the  $\epsilon_i$  in order to get rid of the unnecessary components. In particular, the existence of the  $\epsilon_i$  is redundant for  $r = 2$ .

*Remark 4.7.* Some or all of the morphisms  $\xi_{i,i} = \sigma_{Ass}(V_i)$  can be replaced with their right versions  $\sigma_{Ass}^r(V_i)$ . The theorem still holds true, with analogous proof.

**Example 4.8.** According to Proposition 2.2, for a braided  $\mathcal{C}$  the choice  $\xi_{i,j} = c_{V_i, V_j}$  in the theorem gives a braided system. In addition, the  $c_{V_i, V_j}$  are natural with respect to everything hence in particular to the units. In this case, the UAA structure on  $\overleftarrow{V}$  predicted by the theorem is the usual tensor product of algebras in a braided category.

## Multi-braided modules as modules over algebras

The structure equivalence from Theorem 3 has an important counterpart on the level of modules:

**Proposition 4.9.** *In the settings of Theorem 3, suppose one of the three equivalent conditions satisfied for the  $\xi_{i,j}$ . Then the category of modules over the UAA  $\overleftarrow{V}$  is equivalent to the category of braided modules over the pointed braided system from the theorem:*

$$\boxed{\mathbf{Mod}_{(\overleftarrow{V}, \mu_{\overleftarrow{V}}, \nu_{\overleftarrow{V}})} \simeq \mathbf{Mod}_{(\overline{V}, \overline{\xi}, \overline{\nu})}}$$

*Proof.* According to Observation 2.5 combined with Point 4 of Theorem 2, a braided module structure over  $(\overline{V}, \overline{\xi}, \overline{\nu})$  consists of module structures  $(M, \rho_j)$  over UAAs  $(V_j, \mu_j, \nu_j)$ , compatible in the sense of (5). The map  $\rho = \rho_1 \circ (\rho_2 \otimes \text{Id}_1) \circ \dots \circ (\rho_r \otimes \text{Id}_{r-1} \otimes \dots \otimes \text{Id}_1)$  then defines a  $\overleftarrow{V}$ -module structure on  $M$ . Conversely, a module  $(M, \rho)$  over the UAA  $\overleftarrow{V}$  can also be seen as a braided  $(\overline{V}, \overline{\xi}, \overline{\nu})$ -module via  $\rho_j = \rho \circ (\text{Id}_M \otimes \iota_j)$ , where the  $\iota_j$  are defined in (13). The identity functor of  $\mathcal{C}$  and this structure correspondence give the desired category equivalence.  $\square$

We now discuss the possibility to permute the factors of braided tensor products of UAAs:

**Proposition 4.10.** *In the settings of Theorem 3, suppose one of the  $\xi_{i,i+1}$  invertible. Then*

1. UAAs  $V_1, \dots, V_{i-1}, V_{i+1}, V_i, V_{i+2}, \dots, V_r$  endowed with the  $\xi$  one had for the system  $\overline{V}$ , completed with  $\xi_{i,i+1}^{-1}$  on  $V_{i+1} \otimes V_i$ , still form a braided system of UAAs.
2. The UAA braided tensor products  $\overleftarrow{V}$  and  $s_i \cdot \overleftarrow{V} := V_r \otimes_{\xi} \dots \otimes_{\xi} V_{i+2} \otimes_{\xi} V_i \otimes_{\xi^{-1}} V_{i+1} \otimes_{\xi} V_{i-1} \otimes_{\xi} \dots \otimes_{\xi} V_1$  are related by the algebra isomorphism (abusively denoted by  $s_i$ )

$$\text{Id}_r \otimes \dots \otimes \text{Id}_{i+2} \otimes \xi_{i,i+1}^{-1} \otimes \text{Id}_{i-1} \otimes \dots \otimes \text{Id}_1 \quad : \quad \overleftarrow{V} \longrightarrow s_i \cdot \overleftarrow{V}.$$

3. The algebra isomorphism above induces an equivalence of categories of modules:

$$\begin{aligned} \mathbf{Mod}_{\overleftarrow{V}} &\simeq \mathbf{Mod}_{s_i \cdot \overleftarrow{V}}, \\ (M, \rho_{\overleftarrow{V}}) &\leftrightarrow (M, \rho_{s_i \cdot \overleftarrow{V}} \circ (\text{Id}_M \otimes s_i^{-1})). \end{aligned}$$

*Proof.* 1. Proposition 2.9 allows to interchange components  $V_i$  and  $V_{i+1}$  of the pointed braided system  $(\overline{V}, \overline{\xi}, \overline{\nu})$  from Point [1](#) of Theorem 3. The new pointed braided system  $s_i(\overline{V}, \overline{\xi}, \overline{\nu})$  then satisfies again the conditions from Point [1](#) of Theorem 3 and is thus a braided system of UAAs (note that the naturality of  $\xi_{i,i+1}^{-1}$  with respect to the units follows from that of  $\xi_{i,i+1}$ ).

2. Theorem 3 (Point [3](#)) then gives a UAA structure on  $s_i(\overleftarrow{V})$ . Applying the YBE several times, one sees that, in order to check that  $\text{Id}_r \otimes \dots \otimes \xi_{i,i+1}^{-1} \otimes \dots \otimes \text{Id}_1$  is an algebra morphism, it is sufficient to work with  $V_i$  and  $V_{i+1}$  only. Namely, one has to prove the identity

$$\xi_{i,i+1}^{-1} \circ (\nu_{i+1} \otimes \nu_i) = \nu_i \otimes \nu_{i+1},$$

which follows from the naturality of  $\xi_{i,i+1}^{-1}$  with respect to the units, and the equality

$$\begin{aligned} (\mu_i \otimes \mu_{i+1}) \circ (\text{Id}_i \otimes \xi_{i,i+1}^{-1} \otimes \text{Id}_{i+1}) \circ (\xi_{i,i+1}^{-1} \otimes \xi_{i,i+1}^{-1}) = \\ \xi_{i,i+1}^{-1} \circ (\mu_{i+1} \otimes \mu_i) \circ (\text{Id}_{i+1} \otimes \xi_{i,i+1} \otimes \text{Id}_i) \end{aligned}$$

of morphisms from  $(V_{i+1} \otimes V_i)^{\otimes 2}$  to  $V_i \otimes V_{i+1}$  (cf. Fig. 7). This relation follows from the naturality of  $\xi_{i,i+1}$  (and hence  $\xi_{i,i+1}^{-1}$ ) with respect to  $\mu_i$  and  $\mu_{i+1}$  (Point [2](#) of Theorem 3).

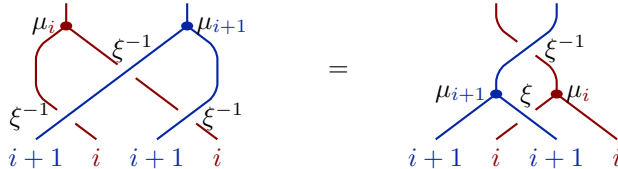


Figure 7: Checking that  $\xi_{i,i+1}^{-1}$  is an algebra morphism

3. (The proofs of) Propositions 2.9 and 4.9 yield the following chain of category equivalences:

$$\begin{array}{ccc} \mathbf{Mod}_{\overleftarrow{V}} & \simeq & \mathbf{Mod}_{(\overline{V}, \overline{\xi}, \overline{\nu})} & \simeq & \mathbf{Mod}_{s_i(\overline{V}, \overline{\xi}, \overline{\nu})} & \simeq & \mathbf{Mod}_{s_i(\overleftarrow{V})} & \square \\ (M, \rho_{\overleftarrow{V}}) & & \longleftarrow & & & & (M, \rho_{\overleftarrow{V}} \circ (\mathrm{Id}_M \otimes s_i^{-1})) \end{array}$$

*Remark 4.11.* Like in Remark 2.10, one obtains *partial*  $S_r$ -actions on braided systems of UAAs and UAA braided tensor products of rank  $r$ . Concretely, a permutation  $\theta \in S_r$  with a minimal decomposition  $\theta = s_{i_1} \cdots s_{i_k}$  sends  $(\overline{V}, \overline{\xi}, \overline{\nu})$  to  $s_{i_1}(\cdots (s_{i_k}(\overline{V}, \overline{\xi}, \overline{\nu})) \cdots)$  and acts on UAA braided tensor products by the algebra morphism  $s_{i_1} \circ \cdots \circ s_{i_k}$  (still denoted by  $\theta$ ), provided that the braiding components are invertible when necessary. These actions are mutually compatible and induce equivalences  $(M, \rho_{\overleftarrow{V}}) \leftrightarrow (M, \rho_{\overleftarrow{V}} \circ (\mathrm{Id}_M \otimes \theta^{-1}))$  of module categories.

## A toy example: algebra bimodules

As a first illustration of the braided system theory, we now upgrade Theorem 2 to the rank 2 level. A braided category  $(\mathcal{C}, \otimes, \mathbf{I}, c)$  is needed here.

First note that, for  $(V, \mu, \nu) \in \mathbf{Alg}(\mathcal{C})$ , the data  $(\mu \circ c, \nu)$  define another UAA structure on  $V$ . Notation  $V^{op}$  is used for  $V$  endowed with this *twisted* multiplication. The associativity braiding becomes here  $\sigma_{Ass}(V^{op}) = \nu \otimes (\mu \circ c)$ . The twisted structure provides a useful transition between left and right modules:

**Lemma 4.12.** *For  $(V, \mu, \nu) \in \mathbf{Alg}(\mathcal{C})$ , the functors*

$$\begin{array}{ccc} \mathbf{Mod}_{V^{op}} & \xrightarrow{\sim} & {}_V\mathbf{Mod}, \\ (M, \rho) & \mapsto & (M, \mathcal{L}(\rho) := \rho \circ c_{M,V}^{-1}), \end{array} \quad (14)$$

$$(M, \mathcal{R}(\lambda) := \lambda \circ c_{M,V}) \longleftarrow (M, \lambda), \quad (15)$$

*extended on morphisms by identities, give a category equivalence.*

Take now two UAAs  $(V, \mu, \nu)$  and  $(V', \mu', \nu')$  in  $\mathcal{C}$ . Returning to Example 4.8, one gets

**Lemma 4.13.** *The data  $(V_1 = V, V_2 = V'; \sigma_{1,1} = \sigma_{Ass}(V), \sigma_{2,2} = \sigma_{Ass}(V'^{op}), \sigma_{1,2} = c_{V,V'})$  define a braided system of UAAs, denoted by  $\mathcal{BM}(V, V')$ .*

The proofs of the above lemmas are straightforward.

The module category equivalence from Proposition 4.9 and permutation rules from Proposition 4.10 can now be applied to  $\mathcal{BM}(V, V')$ . Keeping in mind Observation 2.5 and Lemma 4.12, one interprets braided modules over this system as familiar *algebra bimodules*:

**Proposition 4.14.** *Take two UAAs  $(V, \mu, \nu)$  and  $(V', \mu', \nu')$  in a braided category  $\mathcal{C}$ . Denote by  ${}_V\mathbf{Mod}_V$  the category of  $(V', V)$ -bimodules. The following categories of modules are equivalent:*

$$\mathbf{Mod}_{V'^{op} \otimes_c V} \simeq \mathbf{Mod}_{\mathcal{BM}(V, V')} \simeq {}_{V'}\mathbf{Mod}_V \simeq \mathbf{Mod}_{s_2(\mathcal{BM}(V, V'))} \simeq \mathbf{Mod}_{V \otimes_{c^{-1}} V'^{op}}.$$

In the case  $V' = V$ , one identifies  $V'^{op} \otimes_c V$  as the *enveloping algebra* of the UAA  $V$ .

We finish by applying the adjoint module theory to our bimodule context. Recall Notation (2).

**Proposition 4.15.** *Take a bimodule  $(M, \rho : M \otimes V \rightarrow M, \lambda : V' \otimes M \rightarrow M)$  over UAAs  $V$  and  $V'$  in a braided category  $\mathcal{C}$ . The bar complex  $(M \otimes T(V), d_{bar})$  for  $V$  with coefficients in  $M$  is a complex in  ${}_V\mathbf{Mod}_V$ . In other words, the differentials  $(d_{bar})_n$  are bimodule morphisms, where a bimodule structure on  $M \otimes V^n$  is given by*

$$\begin{array}{ll} \rho_{bar} := \mu^{n+1} & : \quad M \otimes V^n \otimes V \rightarrow M \otimes V^n, \\ \lambda_{bar} := \lambda^1 & : \quad V' \otimes M \otimes V^n \rightarrow M \otimes V^n. \end{array}$$

*Proof.* Plugging into Proposition 3.11 the braided system  $\mathcal{BM}(V, V')$ , the bimodule  $(M, \rho, \lambda)$ , interpreted as a braided  $\mathcal{BM}(V, V')$ -module as in Proposition 4.14, and choosing  $t = 2$ , one obtains the compatibility of the the braided differential  $\rho d$  (which Theorem 2.5 shows to coincide with  $d_{bar}$ ) with the braided  $\mathcal{BM}(V, V')$ -module structures on the  $M \otimes V^n$ . Using Proposition 4.14 again, one interprets these braided modules as  $(V', V)$ -bimodules, with the structure explicitly written using



Lemma 4.12:

$$\begin{aligned}
\rho\pi_1 &= \rho_1 \circ (\text{Id}_M \otimes \overline{\sigma}_{V^n, V}) = \mu^{n+1}, \\
\lambda(\mathcal{R}(\lambda)\pi_2) &= \mathcal{R}(\lambda)\pi_2 \circ c_{M \otimes V^n, V'}^{-1} \\
&= (\lambda \circ c_{M, V'})^1 \circ (\text{Id}_M \otimes \overline{\sigma}_{V^n, V'}) \circ c_{M \otimes V^n, V'}^{-1} \\
&= (\lambda \circ c_{M, V'})^1 \circ (\text{Id}_M \otimes c_{V^n, V'}) \circ c_{M \otimes V^n, V'}^{-1} = \lambda^1. \quad \square
\end{aligned}$$

This bimodule structure on the bar complex is important for one of the constructions of the *Hochschild (co)homology*; namely, one considers the differential induced on the coinvariants by  $d_{\text{bar}}$ .

## 5 A braided interpretation of two-sided crossed products

We now present a rank 3 braided system of UAAs, which recovers F. Panaite's braided treatment of *two-sided crossed products* from [26] and its extension to *generalized two-sided crossed products*  $A \blacktriangleleft C \blacktriangleright B$  of Bulacu-Panaite-Van Oystaeyen ([3]) from [11]. Proposition 4.10 then yields six equivalent versions of the algebra  $A \blacktriangleleft C \blacktriangleright B$  via component permutations, generalizing and simplifying arguments from [26, 11]. Further, our adjoint braided module concept endows  $C^n$  with a  $(B, A)$ -bimodule structure, used in Section 6 to construct a bialgebra homology theory.

### Categorical bialgebras and module algebras

We need the categorical versions of some familiar algebraic notions:

**Definition 5.1.**  $\rightarrow$  A *bialgebra* in a braided category  $(\mathcal{C}, \otimes, \mathbf{I}, c)$  is an object  $H$  endowed with a UAA structure  $(\mu, \nu)$  and a counital coassociative coalgebra ( $=$  *coUAA*) structure  $(\Delta, \varepsilon)$ , compatible in the following sense:

$$\Delta \circ \mu = (\mu \otimes \mu) \circ c^2 \circ (\Delta \otimes \Delta), \quad \Delta \circ \nu = \nu \otimes \nu, \quad (16)$$

$$\varepsilon \circ \mu = \varepsilon \otimes \varepsilon, \quad \varepsilon \circ \nu = \text{Id}_{\mathbf{I}}. \quad (17)$$

$\rightarrow$  A *Hopf algebra* in  $\mathcal{C}$  is a bialgebra  $H$  with an *antipode*, i.e. an endomorphism  $s$  satisfying

$$\mu \circ (s \otimes \text{Id}_H) \circ \Delta = \mu \circ (\text{Id}_H \otimes s) \circ \Delta = \nu \circ \varepsilon. \quad (18)$$

$\rightarrow$  For a bialgebra  $H$  in  $\mathcal{C}$ , a *left  $H$ -module algebra* is a UAA  $(M, \mu_M, \nu_M)$  endowed with a left  $H$ -module structure  $\lambda : H \otimes M \rightarrow M$ , such that  $\mu_M$  and  $\nu_M$  are  $H$ -module morphisms:

$$\lambda \circ (\text{Id}_H \otimes \mu_M) = \mu_M \circ (\lambda \otimes \lambda) \circ c^2 \circ (\Delta \otimes \text{Id}_M^{\otimes 2}), \quad (19)$$

$$\lambda \circ (\text{Id}_H \otimes \nu_M) = \nu_M \circ \varepsilon. \quad (20)$$

Right  $H$ -module algebras and  $H$ -(bi)(co)module algebras are defined similarly.

$\rightarrow$  The categories of bialgebras / Hopf algebras /  $H$ -(co)module algebras and their morphisms in  $\mathcal{C}$  are denoted by, respectively,  $\mathbf{Bialg}(\mathcal{C})$ ,  $\mathbf{HAlg}(\mathcal{C})$ ,  ${}^H\mathbf{ModAlg}$ ,  $\mathbf{ModAlg}_H$ ,  ${}^H\mathbf{ModAlg}$ , etc.

Compatibility conditions (16), (19) and (20) are presented on Fig. 8.

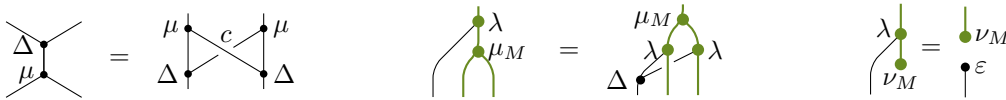


Figure 8: Bialgebra and module algebra relations

Note that a bialgebra  $H$  is an  $H$ -bicomodule, but in general not an  $H$ -bimodule over itself.

### Two-sided crossed products as braided tensor products

**Proposition 5.2.** *Take a bialgebra  $H$ , a left  $H$ -module algebra  $(A, \lambda)$ , a right  $H$ -module algebra  $(B, \rho)$  and an  $H$ -bicomodule algebra  $(C, \delta_l : C \rightarrow H \otimes C, \delta_r : C \rightarrow C \otimes H)$  in a symmetric category  $(\mathcal{C}, \otimes, \mathbf{I}, c)$ . Then*

1. The UAAs  $(B, C, A)$  form a braided system of UAAs when endowed with morphisms

$$\begin{aligned}\xi_{1,2} &= (\text{Id}_C \otimes \rho) \circ (c_{B,C} \otimes \text{Id}_H) \circ (\text{Id}_B \otimes \delta_r), \\ \xi_{2,3} &= (\lambda \otimes \text{Id}_C) \circ (\text{Id}_H \otimes c_{C,A}) \circ (\delta_l \otimes \text{Id}_A), \\ \xi_{1,3} &= c_{B,A}.\end{aligned}$$

2. Formulas (11)-(12) for the  $\xi_{i,j}$  above define a UAA structure on  $A \otimes C \otimes B$ .

3. One has a module category equivalence

$$\mathbf{Mod}_{(B,C,A;\bar{\xi})} \simeq \mathbf{Mod}_{A \otimes_{\xi} C \otimes_{\xi} B}. \quad (21)$$

The braiding from the proposition is shown on Fig. 9. Here and afterwards the underlying braiding of a symmetric category is depicted by a solid crossing, in order to distinguish it from “structural” braiding  $\xi$ .

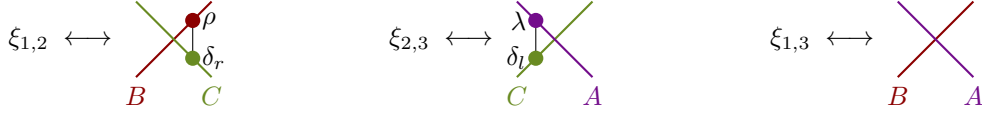


Figure 9: A braided system for a two-sided crossed product

*Proof.* The key point is to verify that the  $\xi$  satisfy the conditions of Theorem 3.2:

- ✓ the YBE on  $B \otimes C \otimes A$  follows from the compatibility of left and the right  $H$ -coactions for  $C$ ;
- ✓ the naturality of the  $\xi$  with respect to  $\mu_C$  is a consequence of the defining properties of  $H$ -bicomodule algebras for  $C$ ;
- ✓ the naturality of the  $\xi$  with respect to  $\mu_A$  and  $\mu_B$  can be deduced from the defining properties of  $H$ -module algebras for  $A$  and  $B$ .

Here we show that  $\xi_{1,2}$  is natural with respect to  $\mu_B$ , the other proofs being of the same spirit:

$$\begin{aligned}\xi_{1,2} \circ (\mu_B \otimes \text{Id}_C) &\stackrel{1}{=} (\text{Id}_C \otimes \rho) \circ (c_{B,C} \otimes \text{Id}_H) \circ (\mu_B \otimes \delta_r) \\ &\stackrel{2}{=} (\text{Id}_C \otimes \rho) \circ (\text{Id}_C \otimes \mu_B \otimes \text{Id}_H) \circ (c_{B \otimes B, C} \otimes \text{Id}_H) \circ (\text{Id}_B^{\otimes 2} \otimes \delta_r) \\ &\stackrel{3}{=} (\text{Id}_C \otimes \mu_B) \circ (\text{Id}_C \otimes \rho^{\otimes 2}) \circ (\text{Id}_{C \otimes B} \otimes c_{B,H} \otimes \text{Id}_H) \circ (c_{B \otimes B, C} \otimes \Delta_H) \circ (\text{Id}_B^{\otimes 2} \otimes \delta_r) \\ &\stackrel{4}{=} (\text{Id}_C \otimes \mu_B) \circ (\text{Id}_C \otimes \rho \otimes \text{Id}_B) \circ (c_{B,C} \otimes \text{Id}_{H \otimes B}) \circ \\ &\quad (\text{Id}_B \otimes \delta_r \otimes \rho) \circ (\text{Id}_B \otimes c_{B,C} \otimes \text{Id}_H) \circ (\text{Id}_B^{\otimes 2} \otimes \delta_r) \\ &\stackrel{5}{=} (\text{Id}_C \otimes \mu_B) \circ (\xi_{1,2} \otimes \text{Id}_B) \circ (\text{Id}_B \otimes \xi_{1,2}),\end{aligned}$$

where we use the definition of  $\xi_{1,2}$  (equalities 1 and 5), the naturality of  $c$  (equality 2), the definition of right  $H$ -module algebra for  $B$  (equality 3) and that of right  $H$ -comodule for  $C$  (equality 4). The reader is advised to draw diagrams in order to better follow the verifications.

The naturality with respect to the units follows from the naturality of  $c$  and from the definition of  $H$ -(co)module algebras. Point 1 of Theorem 3 then confirms that the  $\xi$  together with the  $\sigma_{Ass}$  form a braiding, while Point 3 asserts that  $A \otimes_{\xi} C \otimes_{\xi} B$  is an UAA. Finally, Proposition 4.9 gives the required module category equivalence.  $\square$

The braided tensor product algebra from the proposition is known as the *generalized two-sided crossed product*  $A \blacktriangleright_{\xi} C \blacktriangleleft_{\xi} B := A \otimes_{\xi} C \otimes_{\xi} B$  from [3]. The choice  $C = H$  (with comodule structures given by  $\Delta_H$ ) gives the *two-sided crossed product*  $A \#_{\xi} H \#_{\xi} B := A \otimes_{\xi} H \otimes_{\xi} B$  of Hausser-Nill ([9]).

*Remark 5.3.* Forgetting the  $B$  (or  $A$ ) part of the structure and taking as  $C$  a left (respectively, right)  $H$ -comodule, one obtains rank 2 braided systems. This gives a “braided” treatment of (a generalized version of) left/right *crossed (or smash) products*  $A \#_{\xi} H := A \otimes_{\xi} H$  and  $H \#_{\xi} B := H \otimes_{\xi} B$ .

We have thus obtained a conceptual proof of the associativity of  $A \blacktriangleright_{\xi} C \blacktriangleleft_{\xi} B$  and of the category equivalence (21), otherwise very technical.

*Remark 5.4.* If  $H$  is a Hopf algebra with an invertible antipode  $s$ , then all the  $\xi$  are invertible:

$$\begin{aligned}\xi_{1,2}^{-1} &= ((\rho \circ c_{H,B}) \otimes \text{Id}_C) \circ (s^{-1} \otimes c_{C,B}) \circ ((c_{C,H} \circ \delta_r) \otimes \text{Id}_B), \\ \xi_{2,3}^{-1} &= (\text{Id}_C \otimes (\lambda \circ c_{A,H})) \circ (c_{A,C} \otimes s^{-1}) \circ (\text{Id}_A \otimes (c_{H,C} \circ \delta_l)), \\ \xi_{1,3}^{-1} &= c_{A,B}.\end{aligned}$$

Proposition 4.10 then allows one to permute components of  $A \otimes_{\xi} C \otimes_{\xi} B$ , giving six pairwise isomorphic UAAs with pairwise equivalent categories of modules. In particular, one recovers the algebra isomorphism  $A \# H \# B \simeq (A \otimes B) \bowtie H$  from [9].

## Adjoint actions

Next, after a preliminary general lemma, we apply the theory of adjoint braided modules to the braided system of UAAs from Proposition 5.2, choosing trivial coefficients ( $M = \mathbf{I}$ ).

**Lemma 5.5.** *Take a rank  $r$  braided system  $(\overline{V}, \overline{\sigma})$  in a symmetric category  $(\mathcal{C}, \otimes, \mathbf{I}, c)$ , with  $\sigma_{1,r} = c_{V_1, V_r}$ . For this system, take braided characters  $\overline{\epsilon}$  and  $\overline{\zeta}$ . Then the right braided  $(V_r, \sigma_{r,r})$ -module structure  $\epsilon_{\pi_r}$  and the left braided  $(V_1, \sigma_{1,1})$ -module structure  $\pi_1^{\zeta}$  on  $T(\overline{V})_n^{\rightarrow}$  commute:*

$$\epsilon_{\pi_r} \circ (\pi_1^{\zeta} \otimes \text{Id}_r) = \pi_1^{\zeta} \circ (\text{Id}_1 \otimes \epsilon_{\pi_r}) \quad : \quad V_1 \otimes T(\overline{V})_n^{\rightarrow} \otimes V_r \rightarrow T(\overline{V})_n^{\rightarrow}.$$

*Proof.* The underlying braiding  $c$  is natural with respect to everything, in particular to the components  $\epsilon_r$  and  $\zeta_1$  of our braided characters. With the help of the YBE, both sides of the desired identity can then be brought to the form  $(\epsilon_r \otimes \text{Id}_{T(\overline{V})_n^{\rightarrow}} \otimes \zeta_1) \circ (\overline{\sigma}_{T(\overline{V})_n^{\rightarrow}, V_r} \otimes \text{Id}_1) \circ \overline{\sigma}_{V_1, T(\overline{V})_n^{\rightarrow} \otimes V_r}$ .  $\square$

Now return to two-sided crossed products. Recall notation  $\varphi^i$  from (2). Put

$$\omega_{2n} := \begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n \\ 1 & 3 & \dots & 2n-1 & 2 & 4 & \dots & 2n \end{pmatrix} \in S_{2n}. \quad (22)$$

**Proposition 5.6.** *In the settings of Proposition 5.2, choose algebra characters  $\epsilon_A$  and  $\epsilon_B$  for  $A$  and  $B$ . The tensor powers of  $C$  then become bimodules,  $C^n \in {}_B \mathbf{Mod}_A$ , via the formulas*

$$\begin{aligned}\epsilon^A \pi &= (\epsilon_A)^1 \circ \lambda^1 \circ (\text{Id}_H \otimes c_{C^n, A}) \circ (\mu^1)^{\circ(n-1)} \circ ((\omega_{2n}^{-1} \circ \delta_l^{\otimes n}) \otimes \text{Id}_A) \quad : \quad C^n \otimes A \rightarrow C^n, \\ \pi^{\epsilon_B} &= (\epsilon_B)^{n+1} \circ \rho^{n+1} \circ (c_{B, C^n} \otimes \text{Id}_H) \circ (\mu^{n+2})^{\circ(n-1)} \circ (\text{Id}_B \otimes (\omega_{2n}^{-1} \circ \delta_r^{\otimes n})) \quad : \quad B \otimes C^n \rightarrow C^n,\end{aligned}$$

where  $S_{2n}$  acts on tensor products of objects from  $\mathcal{C}$  via the symmetric braiding  $c$ .

The actions from the proposition for  $n = 3$  are depicted on Fig. 10.

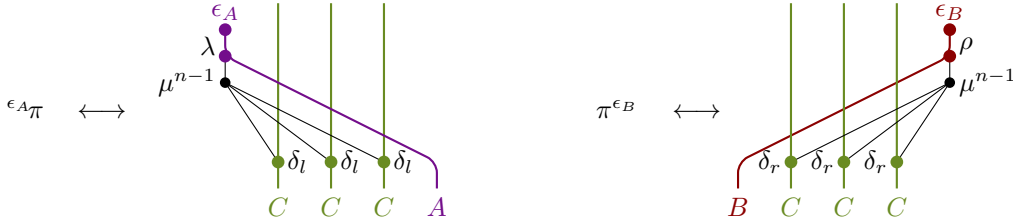


Figure 10:  ${}_B \mathbf{Mod}_A$  structure on  $C^n$

*Proof.* Note that for Point 1 of Proposition 3.11 to hold true,

- ✓ the additivity of  $\mathcal{C}$  is not necessary, and
- ✓ a braided module  $(M, \overline{\rho}) \in \mathbf{Mod}_{(\overline{V}, \overline{\sigma})[t, r]}$  (instead of  $(M, \overline{\rho}) \in \mathbf{Mod}_{(\overline{V}, \overline{\sigma})}$ ) suffices.

Apply that proposition and its mirror version to the braided system of UAAs  $(B, C, A)$  from Proposition 5.2 and to the algebra characters (hence braided characters)  $\epsilon_A$  and  $\epsilon_B$ . One gets a right braided  $(A, \sigma_{Ass}(A))$ -module structure and a left braided  $(B, \sigma_{Ass}(B))$ -module structure on all  $B^k \otimes C^n \otimes A^m$ , and hence on  $C^n$ . Further, since the  $\xi_{1,2}$  and  $\xi_{2,3}$  components of the braiding on  $(B, C, A)$  are natural with respect to the units of  $A$  and  $B$ , the latter act on  $C^n$  by identity. Theorem 2 then ensures that our braided module structures on  $C^n$  are actually module structures over UAAs  $A$  and  $B$ , which are easily checked to coincide with the desired ones. The compatibility between  $A$ - and  $B$ -actions follows from Lemma 5.5.  $\square$

## 6 A braided interpretation of bialgebras and Hopf modules

In this section we present a rank 2 braided system of UAAs  $\mathcal{B}(H)$  which encodes the *bialgebra* structure on  $H$ , in the same sense that the braiding  $\sigma_{Ass}$  encodes the UAA structure (cf. Table 1). This braided system is a particular case of the one constructed for crossed products in Proposition 5.2. In  $\mathcal{B}(H)$ , the invertibility property (trivially false for  $\sigma_{Ass}$ ) becomes algebraically significant: for the braiding component  $\sigma_{1,2}$ , it is equivalent to the existence of an *antipode*. We further identify braided  $\mathcal{B}(H)$ -modules as *Hopf modules* over  $H$ , and show the braided homology theory for  $\mathcal{B}(H)$  to include *Gerstenhaber-Schack bialgebra homology* and *Panaite-Ştefan Hopf module homology*.

Except for some general observations, we specialize here to the category  $\mathcal{C} = \mathbf{vect}_{\mathbb{k}}$  of finite-dimensional vector spaces over  $\mathbb{k}$ . Note that one could stay in the general setting of a braided category  $\mathcal{C}$  and choose a bialgebra in  $\mathcal{C}$  admitting a dual. When working in  $\mathbf{vect}_{\mathbb{k}}$ , we use Sweedler's notation for comultiplications and comodule structures. A simplified notation

$$v_1 v_2 \dots v_n := v_1 \otimes v_2 \otimes \dots \otimes v_n \in V^n$$

is preferred for pure tensors in  $V^n$ , leaving the tensor product sign for

$$v_1 v_2 \dots v_n \otimes w_1 w_2 \dots w_m \in V^n \otimes W^m.$$

The dual space of  $V \in \mathbf{vect}_{\mathbb{k}}$  is denoted by  $V^*$ . Letters  $h_i$  and  $l_j$  always stay here for elements of  $V$  and  $V^*$  respectively. The pairing  $\langle \cdot, \cdot \rangle$  is the *evaluation map*  $ev : V^* \otimes V \rightarrow \mathbb{k}$  sending  $l \otimes h$  to  $l(h)$ . Multiplications on different spaces are simply denoted by  $\cdot$  when it does not lead to confusion.

### Duality: conventions and observations

Take  $\mathbb{k}$ -vector spaces  $V, W$  and a pairing  $B : V \otimes W \rightarrow \mathbb{k}$  (e.g., the evaluation map). Table 4 presents two common ways of extending it to  $B : V^n \otimes W^n \rightarrow \mathbb{k}$ . The “arched” one is more common in literature, but we use the “rainbow” one here (like, for instance, D. Gurevich in [8]), avoiding unnecessary argument permutations in formulas and getting crossingless diagrams. Because of this non-conventional choice, some of our formulas are slightly different from those in literature.

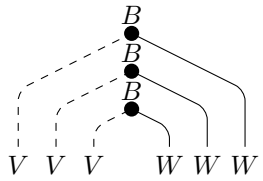
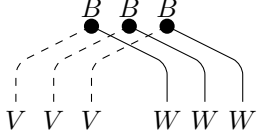
$B(v_1 v_2 \dots v_n \otimes w_1 w_2 \dots w_n) :=$	
$B(v_n \otimes w_1) \cdots B(v_1 \otimes w_n)$	$B(v_1 \otimes w_1) \cdots B(v_n \otimes w_n)$
	
<b>“rainbow”</b>	<b>“arched”</b>

Table 4: Rainbow and arched dualities

Similar conventions are used in the dual and multi-pairing situations. A linear duality in  $\mathbf{vect}_{\mathbb{k}}$  allows then to construct out of a morphism  $f : V_1 \otimes \dots \otimes V_n \rightarrow W_1 \otimes \dots \otimes W_m$  its *dual morphism*  $f^* : W_m^* \otimes \dots \otimes W_1^* \rightarrow V_n^* \otimes \dots \otimes V_1^*$  (note the inverse order of factors). Graphically, this morphism duality corresponds to the *central symmetry*, while with the arched duality one would get the *horizontal mirror symmetry*.

In what follows, we explicitly endow the dual of a coalgebra  $V$  in  $\mathbf{vect}_{\mathbb{k}}$  with the *induced algebra structure* via the pairing  $ev$ , extended to  $(V^*)^2$  and  $V^2$  using the rainbow pattern (Fig. 11 (A)):

$$\langle l_1 l_2, h \rangle = \langle l_1, h_{(2)} \rangle \langle l_2, h_{(1)} \rangle, \quad h \in V, l_1, l_2 \in V^*.$$

Multiplication and (co)units are dualized similarly. The same structure on  $V^*$  can be obtained via the dual *coevaluation map*  $coev$  or via twisted (co)pairings  $ev \circ \tau : V \otimes V^* \rightarrow \mathbb{k}$  and  $\tau \circ coev : \mathbb{k} \rightarrow V \otimes V^*$ . Here  $\tau$  is the transposition of factors (which is the underlying braiding of our category  $\mathbf{vect}_{\mathbb{k}}$ ). We shall often simplify notations, writing  $ev$  and  $coev$  even for the twisted maps.

Lemma 4.12 provides a handy transition tool between *left*  $V$ -modules and *right*  $V^{op}$ -modules for an algebra  $V$  in  $\mathbf{vect}_{\mathbb{k}}$ . We now state an analogous transition lemma for modules and comodules. A classical general observation concerning twisted bialgebra structures is first due:

**Observation 6.1.** Take a bialgebra  $(H, \mu, \nu, \Delta, \varepsilon)$  in a braided category  $(\mathcal{C}, \otimes, \mathbf{I}, c)$ . Then

1.  $H^{op} := (H, \mu^{op} := \mu \circ c^{-1}, \nu, \Delta, \varepsilon)$  and  $H^{cop} := (H, \mu, \nu, \Delta^{cop} := c^{-1} \circ \Delta, \varepsilon)$  are bialgebras in  $(\mathcal{C}, \otimes, \mathbf{I}, c^{-1})$ , while  $H^{op,cop} := (H, \mu \circ c^{-1}, \nu, c \circ \Delta, \varepsilon)$  and  $H^{cop,op} := (H, \mu \circ c, \nu, c^{-1} \circ \Delta, \varepsilon)$  are bialgebras in  $(\mathcal{C}, \otimes, \mathbf{I}, c)$ .
2. If  $H$  is a Hopf algebra with an antipode  $s$ , then so are  $H^{op,cop}$  and  $H^{cop,op}$ , with the same antipode. If  $s$  is invertible, then  $s^{-1}$  becomes an antipode for  $H^{op}$  and  $H^{cop}$ .
3. Moreover, one has the following bialgebra or Hopf algebra isomorphisms:

$$(H^{op})^* \simeq (H^*)^{cop}, \quad (H^{cop})^* \simeq (H^*)^{op}, \quad (H^{op,cop})^* \simeq (H^*)^{cop,op}.$$

Depending on the context, notations  $H, H^*, H^{op}$ , etc. will stay for the bialgebra / Hopf algebra structure, or for the corresponding (co)algebra structure, or simply the underlying vector space.

**Lemma 6.2.** For a coalgebra  $V$  in  $\mathbf{vect}_{\mathbb{k}}$ , the following functors (extended on morphisms by identities) give a category equivalence:

$$\mathbf{Mod}^V \xleftarrow{\sim} \mathbf{Mod}_{V^*},$$

$$(M, \delta) \longmapsto (M, \delta^{co} := (\mathrm{Id}_M \otimes ev) \circ (\delta \otimes \mathrm{Id}_{V^*})), \quad (23)$$

$$(M, \rho^{co} := (\rho \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_M \otimes coev)) \longleftarrow (M, \rho). \quad (24)$$

The proof of the lemma is routine and is best done graphically. A diagrammatic version of, for instance, the (23) part of the equivalence is given on Fig. 11 (B). Note that with the arched dualities, one would have to take the category  $\mathbf{Mod}_{(V^*)^{op}}$  on the right.



Figure 11: Multiplication-comultiplication and action-coaction dualities

**Convention 6.3.** Here and afterwards thin lines stand for the basic vector space  $V$  or  $H$ , dashed lines for its dual, and thick colored lines for different types of modules over it.

Similar equivalences hold for the categories of module (co)algebras:

**Lemma 6.4.** For a bialgebra  $H$  in  $\mathbf{vect}_{\mathbb{k}}$ , the functors from Lemmas 4.12 and 6.2 (combined with identities for the algebra structures) induce the following category equivalences:

$$\mathbf{ModAlg}^H \xleftarrow{\sim} \mathbf{ModAlg}_{(H^*)^{cop}}, \quad {}_H\mathbf{ModAlg} \xleftarrow{\sim} \mathbf{ModAlg}_{H^{op}}.$$

## A braiding encoding the bialgebra structure

We now show how to include the groupoid  ${}^*\mathbf{Bialg}(\mathbf{vect}_{\mathbb{k}})$  of bialgebras and bialgebra isomorphisms in  $\mathbf{vect}_{\mathbb{k}}$  into the groupoid of bipointed rank 2 braided systems in  $\mathbf{vect}_{\mathbb{k}}$ , taking inspiration from what was done in Theorem 2 for UAAs and pointed rank 1 systems.

**Definition 6.5.**  $\rightarrow$  Given a monoidal category  $\mathcal{C}$ , denote by  $\mathbf{BrSyst}_r^\dagger(\mathcal{C})$  the category of

- ✓ rank  $r$  pointed braided systems  $(\overline{V}, \overline{\sigma}, \overline{\nu})$  in  $\mathcal{C}$  endowed with distinguished morphisms  $\overline{\varepsilon} := (\varepsilon_i : V_i \rightarrow \mathbf{I})_{1 \leq i \leq r}$ , called  *counits* , such that all  $(\nu_i, \varepsilon_i)$  are normalized pairs, and
- ✓ morphisms from  $\mathbf{BrSyst}_r^\dagger(\mathcal{C})$  preserving moreover all the counits.

Objects  $(\overline{V}, \overline{\sigma}, \overline{\nu}, \overline{\varepsilon})$  of  $\mathbf{BrSyst}_r^\dagger(\mathcal{C})$  are called rank  $r$  *bipointed braided systems*.

$\rightarrow$  The groupoid of rank  $r$  bipointed braided systems and their isomorphisms is denoted by  ${}^*\mathbf{BrSyst}_r^\dagger(\mathcal{C})$ .

We also recall the notion of Hopf modules, giving their general categorical definition:

**Definition 6.6.** In a braided category  $\mathcal{C}$ , a (*right-right*) Hopf module over  $H \in \mathbf{Bialg}(\mathcal{C})$  is an object  $M$  endowed with a right module structure  $\rho : M \otimes H \rightarrow M$  and a right comodule structure  $\delta : M \rightarrow M \otimes H$ , satisfying the *Hopf compatibility condition* (cf. Fig. 12):

$$\delta \circ \rho = (\rho \circ \mu) \circ (\mathrm{Id}_M \otimes c_{H,H} \otimes \mathrm{Id}_H) \circ (\delta \otimes \Delta) \quad : \quad M \otimes H \rightarrow M \otimes H. \quad (25)$$

The category of right-right Hopf modules over  $H$  and their morphisms is denoted by  $\mathbf{Mod}_H^H$ .

An important example of  $H$ -Hopf module is given by  $H$  itself, with  $\rho = \mu_H$ ,  $\delta = \Delta_H$ .

Figure 12: Right-right Hopf compatibility condition

We now return to our category  $\mathcal{C} = \mathbf{vect}_{\mathbb{k}}$ , omitted in further notations for simplicity.

**Theorem 4.** 1. One has a fully faithful functor

$$\boxed{\mathcal{F} : {}^* \mathbf{Bialg} \hookrightarrow {}^* \mathbf{BrSyst}_2^\updownarrow} \quad (26)$$

$$(H, \mu, \nu, \Delta, \varepsilon) \mapsto \mathcal{B}(H) := (V_1 := H, V_2 := H^*;$$

$$\sigma_{1,1} := \sigma_{Ass}^r(H), \sigma_{2,2} := \sigma_{Ass}(H^*), \sigma_{1,2} = \sigma_{bi};$$

$$\nu, \varepsilon^*; \varepsilon, \nu^*);$$

$$f \mapsto (f, (f^{-1})^*),$$

where  $\sigma_{bi}(h \otimes l) = \langle l_{(1)}, h_{(2)} \rangle l_{(2)} \otimes h_{(1)}$  (cf. Fig. 13).

2. For a bialgebra  $H$ , the braiding component  $\sigma_{bi}$  is invertible if and only if  $H$  has an antipode.
3. Take an  $H \in \mathbf{vect}_{\mathbb{k}}$  endowed with a UAA and a coUAA structures  $(\mu, \nu)$  and  $(\Delta, \varepsilon)$ . Suppose the pair  $(\nu, \varepsilon)$  normalized. Then the YBE on  $H \otimes H \otimes H^*$  (or, symmetrically, on  $H \otimes H^* \otimes H$ ) for  $\mathcal{B}(H)$ , together with the naturality of  $\sigma_{bi}$  with respect to the units, are equivalent to the bialgebra compatibility conditions (16)-(17) for  $H$ .
4. For a bialgebra  $H$ , one has category equivalences

$$\boxed{\mathbf{Mod}_H^H \xrightarrow{\sim} \mathbf{Mod}_{\mathcal{B}(H)} \xrightarrow{\sim} \mathbf{Mod}_{H^* \otimes H}^{\sigma_{bi}}}$$

$$(M, \rho, \delta) \mapsto (M; \rho, \delta^{co}) \mapsto (M, \delta^{co} \otimes \rho)$$

If  $H$  is a Hopf algebra with an antipode  $s$ , then this chain of category equivalences can be continued on the left by  $\mathbf{Mod}_{H \otimes H^*} \simeq \mathbf{Mod}_{s_1 \cdot \mathcal{B}(H)} \simeq \mathbf{Mod}_H^H$ , where  $\theta = \sigma_{bi}^{-1}$ .

Figure 13: A braiding encoding the bialgebra structure

The graphical interpretation of  $\mathcal{B}(H)$  suggests that, applied to the dual bialgebra  $H^*$  instead of  $H$ , the construction from the theorem gives a vertical mirror version of the system  $\mathcal{B}(H)$ .

*Proof.* Take a bialgebra  $H$ . The data  $(H^*, \Delta^*, \varepsilon^*, \mu^*)$  define a left  $H^*$ -comodule algebra. (A left version of) Lemma 6.4 transforms it into a module algebra  $(H^*, \Delta^*, \varepsilon^*, (\mu^*)^{co}) \in {}_{H^{cop}} \mathbf{ModAlg}$ . Together with  $(H^{cop}, \mu, \nu, \Delta^{cop}, \Delta^{cop}) \in {}_{H^{cop}} \mathbf{ModAlg}^{H^{cop}}$ , the latter can be fed into Proposition 5.2 as the  $A$  and  $C$  parts (Remark 5.3 allows one to forget the  $B$  part). The  $\xi_{2,3}$  component of the braided system from that proposition coincides with  $\sigma_{bi}$ . Further,  $H^{cop}$  and  $H$  share the same UAA structure, hence our  $\sigma_{i-1, i-1}$  can be chosen as the  $\xi_{i,i}$  components (cf. Remark 4.7). Proposition 5.2 then implies that  $\mathcal{B}(H)$  is a braided system of UAAs. It is clearly bipointed. Moreover, the braiding on  $\mathcal{B}(H)$ , the units and the counits suffice to recover all ingredients of the bialgebra structure on  $H$ , hence the functor  $\mathcal{F}$  is injective on objects.

To prove Point 1, it remains to understand, for bialgebras  $H$  and  $K$ , isomorphisms of bipointed braided systems  $(f, g) : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ . By definition, the latter consist of bijections  $f : H \rightarrow K$  and  $g : H^* \rightarrow K^*$  which intertwine the braidings of  $\mathcal{B}(H)$  and  $\mathcal{B}(K)$  and respect the units/counits. Due to Theorem 2 (Point 1), this is equivalent to  $f$  and  $g$  being UAA isomorphisms compatible with counits (i.e.,  $\varepsilon_K \circ f = \varepsilon_H$  and  $\nu_K^* \circ g = \nu_H^*$ ) and satisfying

$$\sigma_{bi}(K) \circ (f \otimes g) = (g \otimes f) \circ \sigma_{bi}(H) \quad (27)$$

(cf. Fig. 14 (A)). Applying  $\nu_K^* \otimes \varepsilon_K$  to both sides of (27), using the compatibility of  $f$  and  $g$  with the counits, and playing with dualities, one deduces that  $g^* \circ f = \text{Id}_H$ , hence  $g = (f^{-1})^*$ . Since  $g$

is a UAA isomorphism, so is  $g^{-1}$ , hence  $f = (g^{-1})^*$  is a coUAA morphism, which completes its properties to show that it is a bialgebra isomorphism. Reversing the argument, one checks that the choice  $g = (f^{-1})^*$  for a bialgebra isomorphism  $f$  implies (27). Thus the bipointed braided system isomorphisms are precisely the pairs  $(f, (f^{-1})^*)$  for bialgebra isomorphisms  $f$ . This shows the functor  $\mathcal{F}$  to be well defined, full and faithful, and finishes the proof of Point 1.

In Point 3, the compatibility between  $\Delta$  and  $\nu$  follows by applying  $\nu^* \otimes \text{Id}_H$  to the naturality condition for  $\sigma_{bi}$  with respect to  $\nu$ . Symmetrically, the  $\mu$ - $\varepsilon$  compatibility follows from the naturality of  $\sigma_{bi}$  with respect to  $\varepsilon^*$ . The converse (compatibility  $\Rightarrow$  naturality) is easy. Now, according to (the proof of) Theorem 3, the YBE on  $H \otimes H \otimes H^*$  is equivalent to the naturality condition of  $\sigma_{bi}$  with respect to  $\mu$  (Fig. 14(B)), which implies the bialgebra  $\mu$ - $\Delta$  compatibility condition (apply  $\nu^* \otimes \text{Id}_H$  to both sides and use duality). Conversely, the bialgebra compatibility suffices to deduce the above naturality. By symmetry, one also gets a proof for  $H \otimes H^* \otimes H^*$ .

The “if” part of Point 2 can be proved by exhibiting an explicit formula for  $\sigma_{bi}^{-1}$ :

$$\sigma_{bi}^{-1}(l \otimes h) = \langle l_{(1)}, s(h_{(2)}) \rangle h_{(1)} \otimes l_{(2)} \quad (28)$$

(or by using Remark 5.4 and Point 2 of Observation 6.1). The “only if” part is more delicate. Suppose the existence of  $\sigma_{bi}^{-1}$  and put

$$\tilde{s} := (((\varepsilon \otimes \nu^*) \circ \sigma_{bi}^{-1}) \otimes \text{Id}_H) \circ (\text{Id}_{H^*} \otimes c_{H,H}) \circ (\text{coev} \otimes \text{Id}_H) \quad : \quad H \rightarrow H$$

(cf. Fig. 14). Let us prove that  $\tilde{s}$  is the antipode. The part

$$\mu \circ (\tilde{s} \otimes \text{Id}_H) \circ \Delta = \nu \circ \varepsilon \quad (29)$$

of the defining relation (18) follows from  $\sigma_{bi}^{-1} \circ \sigma_{bi} = \text{Id}_{H \otimes H^*}$  using duality manipulations. One would expect to deduce  $\mu \circ (\text{Id}_H \otimes \tilde{s}) \circ \Delta = \nu \circ \varepsilon$  from  $\sigma_{bi} \circ \sigma_{bi}^{-1} = \text{Id}_{H^* \otimes H}$ , but surprisingly this does not seem to work. Some algebraic tricks come into play instead. Mimicking (28), set

$$\tilde{\sigma} := (\text{Id}_H \otimes (\text{ev} \circ (\tilde{s} \otimes \text{Id}_{H^*}))) \otimes \text{Id}_{H^*} \circ (\Delta \otimes \mu^*) \circ c_{H^*,H} \quad : \quad H^* \otimes H \rightarrow H \otimes H^*.$$

Relation (29) implies  $\tilde{\sigma} \circ \sigma_{bi} = \text{Id}_{H \otimes H^*}$ . Consequently,  $\tilde{\sigma}$  coincides with the inverse  $\sigma_{bi}^{-1}$  of  $\sigma_{bi}$ , so  $\sigma_{bi} \circ \tilde{\sigma} = \text{Id}_{H^* \otimes H}$ . Applying  $\nu^* \otimes \varepsilon$  to both sides, one recovers the second part of (18) for  $\tilde{s}$ .

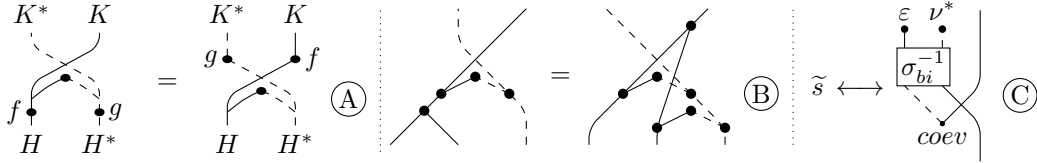


Figure 14: Naturality and invertibility issues for  $\sigma_{bi}$

We now move to Point 4. Equivalence  $\mathbf{Mod}_{\mathcal{B}(H)} \xrightarrow{\sim} \mathbf{Mod}_{H^* \otimes_{\sigma_{bi}} H}$  follows from Proposition 4.9.

Further, Observation 2.5, combined with Point 4 of Theorem 2, present a right braided  $\mathcal{B}(H)$ -module  $M$  via right module structures  $\rho_H$  and  $\rho_{H^*}$  over the UAAs  $H$  and  $H^*$  respectively, compatible in the sense of (5):

$$\rho_{H^*} \circ (\rho_H \otimes \text{Id}_{H^*}) = \rho_H \circ (\rho_{H^*} \otimes \text{Id}_H) \circ (\text{Id}_M \otimes (\tau \circ (\text{Id}_H \otimes \text{ev} \otimes \text{Id}_{H^*}) \circ (\Delta \otimes \mu^*))) \quad (30)$$

On the other hand, due to the module-comodule duality from Lemma 6.2, a right-right Hopf module structure over  $H$  can also be viewed as right module structures over the UAAs  $H$  and  $H^*$ , with the compatibility condition obtained by applying  $\text{Id}_M \otimes \text{ev}$  to the defining condition (25) of Hopf modules (tensored with  $\text{Id}_{H^*}$  on the right) and turning  $H$ -comodule structures into  $H^*$ -module structures. The condition obtained coincides with (30), implying  $\mathbf{Mod}_H^H \simeq \mathbf{Mod}_{\mathcal{B}(H)}$ .

In the Hopf algebra case, Point 2 gives the invertibility of  $\sigma_{bi}$ . The component permuting Proposition 4.10 proves then the desired equivalences.  $\square$

All the remarks following Theorem 2 remain relevant in the bialgebra case. One particular feature of the bialgebra setting is to be added to that list:

*Remark 6.7.* It is essential to work in the *groupoid*, and not just in the category of bialgebras, if one wants a bialgebra morphism  $H \rightarrow G$  to induce a morphism of dual bialgebras  $H^* \rightarrow G^*$  (and not just a morphism  $G^* \rightarrow H^*$ ), so that the functor (26) can be defined on morphisms.

Denote by  $\mathcal{H}'(H) := H \otimes_{\theta} H^*$  one of the braided tensor products of UAAs from the theorem. Then  $\mathcal{H}(H) := \mathcal{H}'(H^*)$  is the well-known *Heisenberg double* of the Hopf algebra  $H$  (cf. for example [21, 5]).

## Braided homology for bialgebras and Hopf modules

Our next goal is to write down explicit braided differentials for  $\mathcal{B}(H)$ . After detailed calculations with certain braided characters as coefficients, we discuss the general case of Hopf module coefficients.

First, for a bialgebra  $H$ , we study adjoint actions of the UAA  $H^*$  on the tensor powers of  $H$ :

**Lemma 6.8.** *The tensor powers of a bialgebra  $(H, \mu, \nu, \Delta, \varepsilon)$  in  $\mathbf{vect}_k$  can be endowed with an  $H^*$ -bimodule structure via the following formulas (cf. Fig. 15):*

$$\pi^{H^*} := \pi^{\varepsilon_{H^*}} = ev^1 \circ ev^2 \cdots \circ ev^n \circ (((\mu^*)^1)^{\circ(n-1)} \otimes (\omega_{2n}^{-1} \circ \Delta^{\otimes n})) \quad : \quad H^* \otimes H^n \rightarrow \otimes H^n,$$

$${}^{H^*}\pi := \varepsilon_{H^*}\pi = ev^{n+1} \circ ev^{n+2} \cdots \circ ev^{2n} \circ ((\omega_{2n}^{-1} \circ \Delta^{\otimes n}) \otimes ((\mu^*)^1)^{\circ(n-1)}) \quad : \quad H^n \otimes H^* \rightarrow \otimes H^n,$$

where notations (2) and (22) are used.

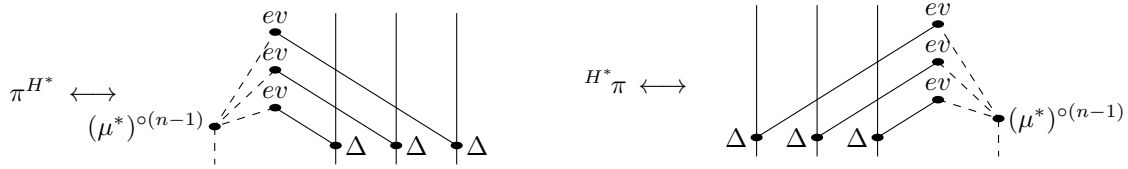


Figure 15:  $H^n$  as an  $H^*$ -bimodule

On the level of elements, the formulas can be written as

$$\pi^{H^*}(l \otimes h_1 \cdots h_n) = \langle l_{(1)}, h_{n(1)} \rangle \langle l_{(2)}, h_{n-1(1)} \rangle \cdots \langle l_{(n)}, h_{1(1)} \rangle h_{1(2)} \cdots h_{n(2)},$$

$${}^{H^*}\pi(h_1 \cdots h_n \otimes l) = \langle l_{(1)}, h_{n(2)} \rangle \langle l_{(2)}, h_{n-1(2)} \rangle \cdots \langle l_{(n)}, h_{1(2)} \rangle h_{1(1)} \cdots h_{n(1)}.$$

*Proof.* In the proof of Theorem 4, we noticed that  $A = (H^*, (\mu^*)^{co}) \in {}_{H^{cop}}\mathbf{ModAlg}$  and  $C = (H^{cop}, \Delta^{cop}, \Delta^{cop}) \in {}^{H^{cop}}\mathbf{ModAlg}^{H^{cop}}$  can be fed into Proposition 5.2. Symmetry considerations allow to complete this data by  $B = (H^*, (\mu^*)^{co}) \in \mathbf{ModAlg}_{H^{cop}}$ . Together with the counit  $\varepsilon_{H^*} = (\nu_H)^*$  of  $H^*$ , which is an algebra character of  $H^*$ , and hence of  $A$  and  $B$ , this data can then be fed into Proposition 5.6. The actions from Proposition 5.6 are precisely the desired ones.  $\square$

Interchanging the roles of  $H$  and  $H^*$ , one gets  $H$ -bimodules  $((H^*)^m, \pi^H, {}^H\pi)$ . By abuse of notation, we define, for all  $m, n \in \mathbb{N}$  for which this makes sense, the following morphisms from  $H^n \otimes (H^*)^m$  to  $H^{(n-1)} \otimes (H^*)^m$  or to  $H^n \otimes (H^*)^{(m-1)}$ :

$$\begin{aligned} {}^{H^*}\pi &:= {}^{H^*}\pi \otimes \text{Id}_{H^*}^{\otimes(m-1)}, & \pi^{H^*} &:= (\pi^{H^*} \otimes \text{Id}_{H^*}^{\otimes(m-1)}) \circ \tau_{H^n \otimes (H^*)^{(m-1)}, H^*}, \\ \pi^H &:= \text{Id}_H^{\otimes(n-1)} \otimes \pi^H, & {}^H\pi &:= (\text{Id}_H^{\otimes(n-1)} \otimes {}^H\pi) \circ \tau_{H, H^{(n-1)} \otimes (H^*)^m}. \end{aligned}$$

**Lemma 6.9.** *The endomorphisms  ${}^{H^*}\pi$ ,  $\pi^{H^*}$ ,  $\pi^H$  and  ${}^H\pi$  of  $T(H) \otimes T(H^*)$  pairwise commute.*

*Proof.* Lemma 6.8 implies the commutativity of  ${}^{H^*}\pi$  and  $\pi^{H^*}$ . Replacing  $H$  with  $H^*$ , one gets the commutativity of  ${}^H\pi$  and  $\pi^H$ . Next, returning to the braided interpretation of the adjoint actions,  $\pi^H$  corresponds to pulling the rightmost  $H$ -strand to the right of all the  $H^*$ -strands (using  $\sigma_{bi}$ ) and applying  $\varepsilon_H$ , while  ${}^{H^*}\pi$  corresponds to pulling the leftmost  $H^*$ -strand to the left of all the  $H$ -strands and applying  $\varepsilon_{H^*}$ . Thus  $\pi^H$  and  ${}^{H^*}\pi$  commute. The case of  $\pi^{H^*}$  and  ${}^H\pi$  is analogous.

In order to prove the commutativity of the two remaining pairs, consider the linear isomorphisms

$$\Delta_n \otimes \text{Id}_{H^*}^{\otimes m} : H^n \otimes (H^*)^m \xrightarrow{\sim} (H^{op})^n \otimes ((H^{op})^*)^m, \quad \Delta_n := \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \\ & & & & 1 \end{pmatrix} \in S_n,$$

where  $S_n$  acts on  $H^n$  by component permutation. These isomorphisms transport the endomorphisms  ${}^{H^*}\pi$ ,  $\pi^{H^*}$ ,  $\pi^H$  and  ${}^H\pi$  of  $H^{\otimes n} \otimes (H^*)^{\otimes m}$  to, respectively,  $(H^{op})^*\pi$ ,  $\pi^{(H^{op})^*}$ ,  ${}^{H^{op}}\pi$  and  $\pi^{H^{op}}$ . Thus the commutativity of  $(H^{op})^*\pi$  and  $\pi^{(H^{op})^*}$  induces that of  ${}^{H^*}\pi$  and  ${}^H\pi$ , and similarly for  $\pi^{H^*}$  and  $\pi^H$ .  $\square$



Further, recall the *bar* and (the dual of the) *cobar differentials* on  $T(H) \otimes T(H^*)$ :

$$d_{bar}(h_1 \dots h_n \otimes l_1 \dots l_m) = \sum_{i=1}^{n-1} (-1)^i h_1 \dots h_{i-1} (h_i \cdot h_{i+1}) h_{i+2} \dots h_n \otimes l_1 \dots l_m, \quad (31)$$

$$d_{cob}(h_1 \dots h_n \otimes l_1 \dots l_m) = \sum_{i=1}^{m-1} (-1)^i h_1 \dots h_n \otimes l_1 \dots l_{i-1} (l_i \cdot l_{i+1}) l_{i+2} \dots l_m. \quad (32)$$

**Proposition 6.10.** *For a finite-dimensional  $\mathbb{k}$ -linear bialgebra  $(H, \mu, \nu, \Delta, \varepsilon)$ , the bigraded vector space  $T(H) \otimes T(H^*) = \bigoplus_{n,m \in \mathbb{N}} H^n \otimes (H^*)^m$  can be endowed with four bicomplex structures, presented in Table 5. Being a bicomplex means here satisfying*

$$d_{n-1,m} \circ d_{n,m} = 0, \quad d'_{n,m-1} \circ d'_{n,m} = 0, \quad d_{n,m-1} \circ d'_{n,m} + d'_{n-1,m} \circ d_{n,m} = 0.$$

	$d_{n,m} : H^n \otimes (H^*)^m \rightarrow H^{(n-1)} \otimes (H^*)^m$	$d'_{n,m} : H^n \otimes (H^*)^m \rightarrow H^n \otimes (H^*)^{(m-1)}$
1.	$d_{bar}$	$(-1)^n d_{cob}$
2.	$d_{bar} + (-1)^n \pi^H$	$(-1)^n d_{cob} + (-1)^n ({}^H \pi)$
3.	$d_{bar} + {}^H \pi$	$(-1)^n d_{cob} + (-1)^{n+m} \pi^{H^*}$
4.	$d_{bar} + (-1)^n \pi^H + {}^H \pi$	$(-1)^n d_{cob} + (-1)^n ({}^H \pi) + (-1)^{n+m} \pi^{H^*}$

Table 5: Bicomplex structures on  $T(H) \otimes T(H^*)$

*Proof.* 1. Maps  $d_{bar}$  and  $d_{cob}$  are well known to be differentials (see also their interpretation as braided differentials in Theorem 2). They affect different parts of  $T(H) \otimes T(H^*)$  ( $T(H)$  and, respectively,  $T(H^*)$ ), and thus commute. The sign  $(-1)^n$  then assures the anticommutativity. 2. Return to the braided system  $\overline{H}_{bi}$ , which we no longer consider as bipointed. The counit  $\varepsilon_H$  of  $H$  is an algebra character, hence a braided character for  $(H, \sigma_{Ass}^r(H))$ . Extended to  $H^*$  by zero, it becomes a braided character for  $\overline{H}_{bi}$  (cf. Example 2.8). Similarly,  $\varepsilon_{H^*}$  extended to  $H$  by zero is also a braided character for  $\overline{H}_{bi}$ . Choosing them as coefficients, one gets the following braided bidifferential, which coincides with the desired one up to a sign:

$$\varepsilon^{H^*} d = (-1)^n d_{cob} + (-1)^n ({}^H \pi), \quad d^{\varepsilon^H} = -(d_{bar} + (-1)^n \pi^H).$$

3. Symmetrically, one gets a bidifferential  $((-1)^m d_{bar} + (-1)^m ({}^H \pi), d_{cob} + (-1)^m \pi^{H^*})$ , hence  $(d_{bar} + {}^H \pi, (-1)^n d_{cob} + (-1)^{n+m} \pi^{H^*})$ . 4. The last point follows from the three preceding ones using an elementary observation:

**Lemma 6.11.** *Take an Abelian group  $(S, +, 0, a \mapsto -a)$  endowed with an operation  $\cdot$ , distributive with respect to  $+$ . Then, for any  $a, b, c, d, e, f \in S$ ,*

$$(a+b) \cdot (d+e) = (a+c) \cdot (d+f) = a \cdot d = b \cdot f + c \cdot e = 0 \\ \implies (a+b+c) \cdot (d+e+f) = 0.$$

*Proof.*  $(a+b+c) \cdot (d+e+f) = (a+b) \cdot (d+e) + (a+c) \cdot (d+f) - a \cdot d + (b \cdot f + c \cdot e)$ .  $\square$

Now take  $S = \text{End}_{\mathbb{k}}(T(H) \otimes T(H^*))$  with the usual addition and the operation  $a \cdot b := a \circ b$  (for proving that the two morphisms from the fourth line of our table are differentials), or the operation  $a \cdot b := a \circ b + b \circ a$  (for proving that the two morphisms anti-commute). Choose  $a = d_{bar}$ ,  $b = (-1)^n \pi^H$ ,  $c = {}^H \pi$ ,  $d = d_{bar}$  or  $d = (-1)^n d_{cob}$ , etc. The equalities of the type  $b \cdot f + c \cdot e = 0$  follow from the pairwise anti-commutativity of  $(-1)^n ({}^H \pi)$ ,  $(-1)^{n+m} \pi^{H^*}$ ,  $(-1)^n \pi^H$  and  ${}^H \pi$  (guaranteed by Lemma 6.9), and the remaining ones from Points 1-3.  $\square$

One recognizes in  $d_{bar} + (-1)^n \pi^H + {}^H \pi$  the Hochschild differential for  $H$  with (right) coefficients in the  $H$ -bimodule  $T(H^*)$  (cf. Lemma 6.8), and similarly for  $(-1)^n d_{cob} + (-1)^n ({}^H \pi) + (-1)^{n+m} \pi^{H^*}$ . Thus the last bicomplex from Table 5 defines the *Gerstenhaber-Schack bialgebra homology*; cf. [7] for the first mention, R. Taillefer's thesis [30] for detailed calculations and a comparison with other bialgebra homologies, and M. Mastnak and S. Witherspoon's paper [20] for explicit formulas and the passage from  $\text{Hom}_{\mathbb{k}}(H^m, H^n)$  to  $H^n \otimes (H^*)^m$ .

Now, instead of the braided characters  $\varepsilon_H$  and  $\varepsilon_{H^*}$  for  $\mathcal{B}(H)$ , take general braided modules  $(M, \rho, \delta) \in \mathbf{Mod}_H^H \simeq \mathbf{Mod}_{\mathcal{B}(H)}$  and  $(N, \lambda, \gamma) \in {}_H^H \mathbf{Mod} \simeq {}_{\mathcal{B}(H)} \mathbf{Mod}$ . On  $M \otimes H^n \otimes (H^*)^m \otimes N$ ,

define maps  $\pi^H$  and  $H^*\pi$  using adjoint actions as before:

$$\begin{aligned}\pi^H(a \otimes h_1 \dots h_n \otimes l_1 \dots l_m \otimes b) &= \\ &\langle l_{1(1)}, h_{n(m+1)} \rangle \langle l_{2(1)}, h_{n(m)} \rangle \dots \langle l_{m(1)}, h_{n(2)} \rangle \langle b_{-1}, h_{n(1)} \rangle a \otimes h_1 \dots h_{n-1} \otimes l_{1(2)} \dots l_{m(2)} \otimes b_0, \\ H^*\pi(a \otimes h_1 \dots h_n \otimes l_1 \dots l_m \otimes b) &= \\ &\langle l_{1(1)}, h_{n(2)} \rangle \langle l_{1(2)}, h_{n-1(2)} \rangle \dots \langle l_{1(n)}, h_{1(2)} \rangle \langle l_{1(n+1)}, a_1 \rangle a_0 \otimes h_{1(1)} \dots h_{n(1)} \otimes l_2 \dots l_m \otimes b.\end{aligned}$$

Further, let  $H\pi$  be the action  $\rho$  applied to the two leftmost factors, and let  $\pi^{H^*}$  be the action  $\lambda$  applied to the two rightmost factors. We still denote by  $d_{bar}$  and  $d_{cob}$  the differentials (31)-(32) tensored with  $\text{Id}_M$  on the left and with  $\text{Id}_N$  on the right. Repeating the argument of Proposition 6.10 for these maps, one shows that  $d_{bar} + (-1)^n \pi^H + H\pi$  and  $(-1)^n d_{cob} + (-1)^n (\pi^{H^*}) + (-1)^{n+m} \pi^{H^*}$  define a bicomplex on  $M \otimes T(H) \otimes T(H^*) \otimes N$ . If  $N$  is finite dimensional, then one can see  $M \otimes H^n \otimes (H^*)^m \otimes N$  as  $\text{Hom}(N^* \otimes H^m, M \otimes H^n)$ , with  $N^* \in \mathbf{Mod}_H^H$ . One recovers (a variation of) the *deformation (co)homology of Hopf modules*, defined by Panaite-Ştefan in [27].

## 7 A braided interpretation of Hopf bimodules

In this section, the braided system of UAAs  $\mathcal{B}(H)$  for a bialgebra  $H$  is upgraded to a more complicated rank 4 braided system  $\mathcal{B}'(H)$ . Braided  $\mathcal{B}'(H)$ -modules are identified as *Hopf bimodules* over  $H$ , or else as modules over the algebras  $\mathcal{X}$ ,  $\mathcal{Y}$  and  $\mathcal{Z}$  of Cibils-Rosso-Panaite. These three algebras are included into a list of 24 braided tensor products of UAAs, shown pairwise isomorphic by component permuting techniques. Braided bidifferentials for  $\mathcal{B}'(H)$  are shown to generalize the *Hopf bimodule (co)homology* of Ospel-Taillefer.

### Hopf bimodules as braided modules

**Definition 7.1.** In a braided category  $\mathcal{C}$ , a *Hopf bimodule* over  $H \in \mathbf{Bialg}(\mathcal{C})$  is an object  $M$  endowed with a bimodule structure  $\rho : M \otimes H \rightarrow M$ ,  $\lambda : H \otimes M \rightarrow M$  and a bicomodule structure  $\delta : M \rightarrow M \otimes H$ ,  $\gamma : M \rightarrow H \otimes M$ , satisfying (25) and three other *Hopf compatibility conditions*:

$$\begin{aligned}\delta \circ \lambda &= (\lambda \otimes \mu) \circ (\text{Id}_H \otimes_{\mathcal{C}H, M} \text{Id}_H) \circ (\Delta \otimes \delta) & : H \otimes M &\rightarrow M \otimes H, \\ \gamma \circ \rho &= (\mu \otimes \rho) \circ (\text{Id}_H \otimes_{\mathcal{C}M, H} \text{Id}_H) \circ (\gamma \otimes \Delta) & : M \otimes H &\rightarrow H \otimes M, \\ \gamma \circ \lambda &= (\mu \otimes \lambda) \circ (\text{Id}_H \otimes_{\mathcal{C}H, H} \text{Id}_M) \circ (\Delta \otimes \gamma) & : H \otimes M &\rightarrow H \otimes M\end{aligned}$$

(cf. Fig. 16). The category of Hopf bimodules over  $H$  and their morphisms is denoted by  ${}^H_H\mathbf{Mod}_H^H$ .

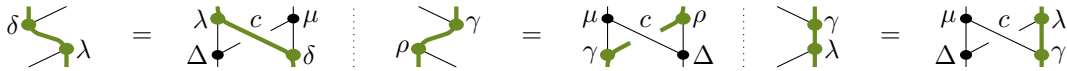


Figure 16: Hopf compatibility conditions

We now return to our category  $\mathcal{C} = \mathbf{vect}_k$ , omitted as usual for simplicity.

**Theorem 5.** 1. One has a fully faithful functor

$$\boxed{\mathcal{F}' : {}^*\mathbf{Bialg} \hookrightarrow {}^*\mathbf{BrSyst}_4^{\updownarrow}} \quad (33)$$

$$\begin{aligned}(H, \mu, \nu, \Delta, \varepsilon) &\longmapsto \mathcal{B}'(H) := (V_1 := H, V_2 := H^{op}, V_3 := H^*, V_4 := (H^{cop})^*); \\ \sigma_{i,i} &:= \sigma_{Ass}(V_i), \sigma_{1,2} := \tau_{H, H^{op}}, \sigma_{3,4} := \tau_{H^*, (H^{cop})^*}, \\ \sigma_{1,3} &:= \sigma_{bi}(H), \sigma_{2,3} := \sigma_{bi}(H^{op}), \sigma_{1,4} := \sigma_{bi}(H^{cop}), \sigma_{2,4} := \sigma_{bi}(H^{op, cop}); \\ \nu, \nu, \varepsilon^*, \varepsilon^*; \varepsilon, \varepsilon, \nu^*, \nu^*), \\ f &\longmapsto (f, f, (f^{-1})^*, (f^{-1})^*),\end{aligned}$$

where  $\tau$  is the transposition of the corresponding factors, and  $\sigma_{bi}(A)$  denotes the map  $\sigma_{bi}$  from Theorem 4 for the bialgebra  $A$  (cf. Fig. 17).

2. For a bialgebra  $H$ , one has category equivalences

${}^H_H\mathbf{Mod}_H^H$	$\xrightarrow{\sim}$	$\mathbf{Mod}_{\mathcal{B}'(H)}$	$\xrightarrow{\sim}$	$\mathbf{Mod}_{\mathcal{W}(H)}$
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$(M, \rho, \lambda, \delta, \gamma) \longmapsto (M; \rho, \mathcal{R}(\lambda), \delta^{co}, \mathcal{R}(\gamma^{co})) \longmapsto (M, \mathcal{R}(\gamma^{co}) \otimes \delta^{co} \otimes \mathcal{R}(\lambda) \otimes \rho)$   
 where  $\mathcal{R}$  is the correspondence from Lemma 4.12, and  $\mathcal{W}(H)$  is the braided tensor product of UAAs

$$\mathcal{W}(H) = (H^{cop})^* \underset{\xi}{\otimes} H^* \underset{\xi}{\otimes} H^{op} \underset{\xi}{\otimes} H.$$

3. If  $H$  is a Hopf algebra, then, for any permutation  $\theta \in S_4$ , one has category equivalences

$${}^H_H\mathbf{Mod}_H^H \xrightarrow{\sim} \mathbf{Mod}_{\theta\mathcal{B}'(H)} \xrightarrow{\sim} \mathbf{Mod}_{\theta \cdot \mathcal{W}(H)},$$

where the bipointed braided system  $\theta\mathcal{B}'(H)$  is obtained from  $\mathcal{B}'(H)$  by a component permutation from Remark 2.10, and the UAA  $\theta \cdot \mathcal{W}(H)$ , isomorphic to  $\mathcal{W}(H)$ , is obtained from  $\mathcal{W}(H)$  by a component permutation from Remark 4.11.

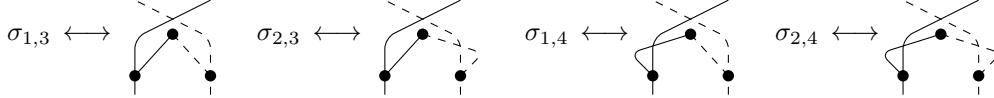


Figure 17: Some braiding components for  $\mathcal{B}'(H)$

*Proof.* Denote by  $\mathcal{F}'_{i,j}$  the composition of  $\mathcal{F}'$  with the forgetful functor  $For_{i,j} : {}^*\mathbf{BrSyst}_4^{\uparrow} \rightarrow {}^*\mathbf{BrSyst}_2^{\uparrow}$  which picks the  $i$ th and  $j$ th components,  $i < j$ . For  $i \leq 2 < j$  one recognizes in  $\mathcal{F}'_{i,j}$  the functor (26) from Theorem 4 and its slight modifications which send a bialgebra  $H$  to  $\mathcal{B}(H^{op})$ ,  $\mathcal{B}(H^{cop})$  or  $\mathcal{B}(H^{op,cop})$  (with some  $\sigma_{Ass}^r$ -type braiding components replaced with their  $\sigma_{Ass}$  versions, which has no importance here). Further,  $\mathcal{F}'_{1,2}(H)$  and  $\mathcal{F}'_{3,4}(H)$  coincide with the braided systems of UAAs  $\mathcal{BM}(H, H)$  and  $\mathcal{BM}(H^*, H^*)$  respectively. Hence all the  $\xi_{i,j}$  for  $i < j$  are natural with respect to the units and to the multiplications. They also satisfy the YBEs required by Point [2] of Theorem 3: indeed, on  $V_1 \otimes V_2 \otimes V_k$ ,  $k \in \{3, 4\}$ , it follows from the associativity of  $\mu$ , and on  $V_k \otimes V_3 \otimes V_4$ ,  $k \in \{1, 2\}$ , it follows from the coassociativity of  $\Delta$ . Theorem 3 then asserts that  $\mathcal{B}'(H)$  is a braided system of UAAs. It is clearly bipointed.

To show that  $\mathcal{F}'$  is well defined on morphisms, it suffices to check this statement for all the  $\mathcal{F}'_{i,j}$ ,  $i < j$ . For  $i \leq 2 < j$  it follows from Theorem 4. For  $\mathcal{F}'_{1,2}$  and  $\mathcal{F}'_{3,4}$ , notice that the  $\xi_{1,2}$  and  $\xi_{3,4}$  components of our braidings are simple transpositions, ensuring the defining property (3) of braided morphisms. Further, take a braided isomorphism  $(f, g, h, k) : \mathcal{B}'(H) \rightarrow \mathcal{B}'(K)$  for bialgebras  $H$  and  $K$ . Applying forgetful functors  $For_{i,j}$  with  $i \leq 2 < j$ , and using Theorem 4 again, one sees that  $f$  is a bialgebra isomorphism, and that  $f = g = (h^*)^{-1} = (k^*)^{-1}$ . Hence  $\mathcal{F}'$  is full and faithful.

Let us turn to modules. Take  $(M, \rho, \lambda, \delta, \gamma) \in {}^H_H\mathbf{Mod}_H^H$ . Transform left structures  $\lambda$  and  $\gamma$  into right ones  $\mathcal{R}(\lambda)$  and  $\mathcal{R}(\gamma)$ , and then comodule structures  $\delta$  and  $\mathcal{R}(\gamma)$  into module structures  $\delta^{co}$  and  $\mathcal{R}(\gamma)^{co} = \mathcal{R}(\gamma^{co})$ . One thus interprets the Hopf bimodule  $M$  over  $H$  as a module over UAAs  $H = V_1$ ,  $H^{op} = V_2$ ,  $H^* = V_3$  and  $(H^{cop})^* = V_4$ . Further, the four Hopf compatibility conditions coincide with the braided module compatibility conditions on  $V_i \otimes V_j$  with  $i \leq 2 < j$ , and left-right action (or coaction) compatibility conditions cover the case  $i = 1$ ,  $j = 2$  (respectively,  $i = 3$ ,  $j = 4$ ). Observation 2.5 then gives the desired category equivalence  ${}^H_H\mathbf{Mod}_H^H \simeq \mathbf{Mod}_{\mathcal{B}'(H)}$ .

The remaining assertions follow from the correspondence between braided modules and modules over braided tensor products (Proposition 4.9), the invertibility of  $\sigma_{bi}$  in the Hopf algebra case, the properties of twisted Hopf algebras (Observation 6.1; note that in the finite-dimensional case, an antipode is always invertible), and the component permuting Propositions 2.9 and 4.10.  $\square$

The category  ${}^H_H\mathbf{Mod}_H^H$  for a Hopf algebra  $H$  was already known to be equivalent to the categories of right modules over three UAAs: the twisted product

$$\mathcal{X}(H) = (H \otimes H^{op}) \underset{\otimes}{\otimes} (H^* \otimes (H^*)^{op})$$

of Cibils-Rosso ([5]), and the two-sided crossed product and the diagonal crossed product

$$\mathcal{Y}(H) = H^* \# (H^{op} \otimes H) \# (H^*)^{op},$$

$$\mathcal{Z}(H) = (H^* \otimes (H^*)^{op}) \bowtie (H^{op} \otimes H)$$

of F. Panaite ([26]). (Here we follow the notations of [26] adapted to our conventions: first, [26] uses the arched duality, so their dual bialgebra  $H^*$  corresponds to our  $(H^*)^{op,cop}$ ; second, they

interpret Hopf bimodules over  $H^*$  as left modules over  $\mathcal{X}(H)$  etc., which is equivalent to our interpretation of Hopf bimodules over  $H$  as right modules.) These three algebras are of the form  $\theta \cdot \mathcal{W}(H)$ , with as  $\theta$  permutations  $(1, 4)(2, 3)$ ,  $(1, 2, 3, 4)$  and  $(3, 4)$  respectively. Now, Point 3 of our theorem includes them into a family of  $\#S_4 = 24$  UAAs and gives an explicit isomorphism between any two of them, inducing an equivalence of their categories of modules (Remark 4.11). We thus generalize most results of [5, 26], conceptually explain them and minimize technical computations.

## Towards Hopf bimodule homology

In Proposition 4.15 we used braided adjoint actions to see that the bar complex with coefficients in a bimodule is a complex of bimodules. We now show an analogous property for Hopf bimodules:

**Proposition 7.2.** *Take a Hopf bimodule  $(M, \rho : M \otimes H \rightarrow M, \lambda : H \otimes M \rightarrow M, \delta : M \rightarrow M \otimes H, \gamma : M \rightarrow H \otimes M)$  over a bialgebra  $(H, \mu, \nu, \Delta, \varepsilon)$  in  $\mathbf{vect}_{\mathbb{k}}$ . The bar complex  $(M \otimes T(H), d_{\text{bar}})$  for  $H$  with coefficients in  $M$  is a complex in  ${}^H_H\mathbf{Mod}_H^H$ . In other words, the differentials  $(d_{\text{bar}})_n$  are Hopf bimodule morphisms, where a Hopf bimodule structure on  $M \otimes H^n$  is given by*

$$\begin{aligned} \rho_{\text{bar}} &:= \mu^{n+1} &: M \otimes H^n \otimes H &\rightarrow M \otimes H^n, & \left. \begin{array}{l} \rho_{\text{bar}} \\ \lambda_{\text{bar}} \end{array} \right\} & \text{peripheral} \\ \lambda_{\text{bar}} &:= \lambda^1 &: H \otimes M \otimes H^n &\rightarrow M \otimes H^n, & \left. \begin{array}{l} \rho_{\text{bar}} \\ \lambda_{\text{bar}} \end{array} \right\} & \text{actions} \\ \delta_{\text{bar}} &:= (\mu^{n+2})^{\circ n} \circ \omega_{2(n+1)}^{-1} \circ (\delta \otimes \Delta^{\otimes n}) &: M \otimes H^{\otimes n} &\rightarrow M \otimes H^{\otimes n} \otimes H, & \left. \begin{array}{l} \delta_{\text{bar}} \\ \gamma_{\text{bar}} \end{array} \right\} & \text{diagonal} \\ \gamma_{\text{bar}} &:= (\mu^1)^{\circ n} \circ \omega_{2(n+1)}^{-1} \circ (\gamma \otimes \Delta^{\otimes n}) &: M \otimes H^{\otimes n} &\rightarrow H \otimes M \otimes H^{\otimes n} & \left. \begin{array}{l} \delta_{\text{bar}} \\ \gamma_{\text{bar}} \end{array} \right\} & \text{coactions} \end{aligned}$$

(cf. Fig. 18), where  $\omega_{2(n+1)} \in S_{2(n+1)}$  is defined by (22), and  $S_{2(n+1)}$  acts on  $M \otimes H^{\otimes(2n+1)}$  by factor permutations.

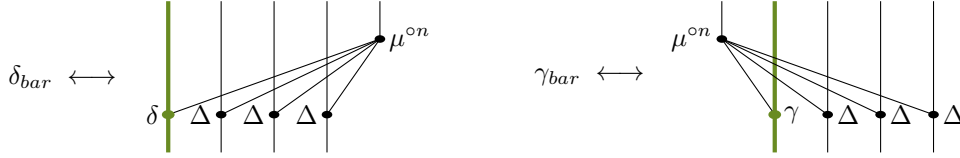


Figure 18: Diagonal bicomodule structure on the bar complex

*Proof.* Theorem 5 allows to see  $M$  as a braided module over  $\mathcal{B}'(H)$ . Proposition 3.11 for  $t = 1$  then gives a braided  $\mathcal{B}'(H)$ -module structure on  $M \otimes T(H)$ , compatible with the braided differential  $d$ . According to Theorem 2, the latter is precisely the bar differential. Using Theorem 5 once again, one transforms the braided  $\mathcal{B}'(H)$ -module structure on  $M \otimes T(H)$  into a Hopf bimodule structure over  $H$ , and shows that it coincides with the desired one.  $\square$

This Hopf bimodule structure on the bar complex, as well as its dual one on the cobar complex, are essential elements in the definition of the *Hopf bimodule (co)homology*, introduced by C. Ospel in the one-module case ([25]) and by R. Taillefer ([30, 31]) in the two-module case.

Now, take Hopf bimodules  $M \in {}^H_H\mathbf{Mod}_H^H \simeq \mathbf{Mod}_{\mathcal{B}'(H)}$  and  $N \in {}^{H^*}_{H^*}\mathbf{Mod}_{H^*}^{H^*} \simeq {}_{\mathcal{B}'(H)}\mathbf{Mod}$ . Mimicking the constructions for Hopf modules from the previous section, one gets a tetra-complex structure on the tetra-graded vector space  $M \otimes T(H) \otimes T(H^{op}) \otimes T(H^*) \otimes T((H^{cop})^*) \otimes N$ . If  $N$  is finite dimensional, then this space can be regarded as  $\text{Hom}(T(H) \otimes N^* \otimes T(H), T(H) \otimes M \otimes T(H))$ , with  $N^* \in {}^H_H\mathbf{Mod}_H^H$  (here in order to get rid of twisted (co)multiplications, we moved  $T(H^{op})$  to the left of  $M$ , reversing the order of its factors, and similarly for  $T((H^{cop})^*)$ ). This generalizes another (co)homological approach to Hopf bimodules from [31].

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