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Braided Systems, Multi-Braided Tensor Products and Bialgebra Homologies

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Abstract

We introduce the notion of braided system and develop its representation and homology theories. The case of braided systems of associative algebras is presented in detail, giving an efficient tool for studying multi-braided tensor products of algebras and their actions. Braided systems encoding the structures of generalized crossed product and bialgebra are considered. For the latter, Hopf modules are identified as the corresponding multi-braided modules, Heisenberg double as the corresponding multi-braided tensor product of algebras, and Gerstenhaber-Schack and Panaite-Ştefan (co)homologies as particular cases of multi-braided (co)homologies. This “braided” interpretation offers a conceptual explication and a simplified proof of several algebraic phenomena concerning the structures above.

Keywords: braided systems; braided homology; multi-braided modules; Hopf algebras; bialgebra homology; Hopf modules; Heisenberg double; module algebras; (generalized) crossed products; multi-braided algebra tensor products; distributive laws; multi-quantum shuffle algebra.

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1 Introduction

In [15] we introduced general *representation and homology theories for a braided object* V in a monoidal category \mathcal{C} (for simplicity, the reader can think about braided \mathbb{k} -vector spaces). Here *braided* means endowed with a braiding, i.e. a morphism $\sigma : V \otimes V \rightarrow V \otimes V$ satisfying the *Yang-Baxter equation* (= *YBE*)

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \quad \sigma_1 := \sigma \otimes \text{Id}_V, \quad \sigma_2 := \text{Id}_V \otimes \sigma.$$

Further, for several basic algebraic structures (*associative algebras, Leibniz algebras, self-distributive structures*) on an object V we presented braidings on V encoding many algebraic properties of these structures. Trying to apply these ideas to more complicated structures (e.g. the *bialgebra* structure on H), one soon hits the limits of the concept of braiding. In particular, if one wants to recover the Gerstenhaber-Shack differential for H ([7]), defined on $\text{Hom}(H^{\otimes n}, H^{\otimes m}) \simeq H^{\otimes m} \otimes (H^*)^{\otimes n}$, one should be able to simultaneously handle objects of different nature (here H and H^*). This suggests the notion of braided system, central to this paper.

Concretely, a **rank r braided system** is a family of objects (V_1, \dots, V_r) in \mathcal{C} endowed with morphisms $\sigma_{i,j} : V_i \otimes V_j \rightarrow V_j \otimes V_i \forall i \leq j$ satisfying the (colored) Yang-Baxter equation on all the tensor products $V_i \otimes V_j \otimes V_k$ with $i \leq j \leq k$. This is a positive (the invertibility condition is relaxed), partial and local version of the notion of braiding. Note that our definition (for $r = 2$) recovers the *WXZ-systems* constructed by L. Hlavatý and L. Šnobl in [10] and motivated by the concept of quantum doubles. They classified such systems in dimension 2 and studied their symmetries.

In Sections 2 and 3 we develop in detail general *representation and homology theories for braided systems*, introducing the notions of *multi-braided modules* and *multi-braided differentials*, and generalizing the case of braided objects. Note that multi-braided modules play the role of *coefficients* in multi-braided differential complexes. The last sections are devoted to more or less general examples. In each of them, we look for a braided system encoding the given algebraic structure in the following sense:

multi-braiding	\leftrightarrow	algebraic structure
YBEs	\Leftrightarrow	defining relations
braided morphisms	\simeq	structural morphisms
invertibility	\Leftrightarrow	algebraic properties
BrSyst$_r(\mathcal{C})$	\leftrightarrow	Structure(\mathcal{C})
multi-braided modules	\simeq	usual modules
multi-braided differentials	\supseteq	usual differentials

Table 1: “Braided” interpretation for algebraic structures

The line **BrSyst $_r(\mathcal{C}) \leftrightarrow \text{Structure}(\mathcal{C})$** means that we want to recover the categories of the algebraic structures we are interested in (e.g. the category of bialgebras in \mathcal{C}) as subcategories of the category of rank r braided systems and braided morphisms in \mathcal{C} .

In the examples presented in this paper, all components V_i of braided systems are *unital associative algebras* (= *UAAs*) (V_i, μ_i, ν_i) , and the $\sigma_{i,i}$ braiding components are the “*associativity braidings*” $\sigma_{Ass} = \nu_i \otimes \mu_i$ (or their “right” versions $\sigma_{Ass}^r = \mu_i \otimes \nu_i$) from [15]. The braiding σ_{Ass} , presented in the category of vector spaces by a more eloquent formula $\sigma_{Ass}(a \otimes b) = 1 \otimes a \cdot b$, encodes the associativity structure. Such braided systems of UAAs are studied in detail in Section 4. They are shown to be in one-to-one correspondence with *multi-braided tensor products of algebras* $\overleftarrow{V} = V_r \underset{\xi}{\otimes} \dots \underset{\xi}{\otimes} V_1$. Concretely, morphisms $\xi_{i,j}$ for $i < j$ are shown to complete the associativity braidings $\sigma_{i,i}$ into a braided system structure if and only if a formula generalizing the familiar $r = 2$ case

$$\mu_{V_2 \otimes V_1} := (\mu_2 \otimes \mu_1) \circ (\text{Id}_2 \otimes \xi_{1,2} \otimes \text{Id}_1) : (V_2 \otimes V_1) \otimes (V_2 \otimes V_1) \rightarrow V_2 \otimes V_1$$

defines an associative multiplication on \overleftarrow{V} .

This braided tensor product construction (for $r = 2$) is at the heart of the *braided geometry*, introduced by S. Majid in a long series of papers in the 1990’s (cf. for example [17], [18], [19]).

Majid’s motivation was to develop an algebra analogue of the product of spaces in non-commutative geometry. A pleasant consequence of his work was the construction of new examples of non-commutative non-cocommutative Hopf algebras via the *bicrossproduct* construction (which is a particular case of braided tensor product).

The case of general r was independently treated by different authors from different viewpoints:

1. P.J. Martínez, J.L. Peña, F. Panaite and F. van Oystaeyen introduced in [11] the notion of *iterated twisted tensor products of algebras*. Their motivation lied in the braided geometry, like for Majid (namely, they wanted an algebra analogue of the product of r non-commutative spaces). They studied various *Hopf algebraic, geometric and physical examples*. They worked in the category of vector spaces.
2. In category theory, E. Cheng generalized J. Beck’s notion of distributive laws to that of *iterated distributive laws* ([1], [4]). Having category theory motivations in mind, she placed everything in the monoidal category of endofunctors of a given category, thus working with monads. Iterated distributive laws turned out to be very useful for describing *interchange laws in a strict n -category*.

Both approaches above study, in our language, the relation between

- the associativity of the multiplication on \widehat{V} , and
- the YBE for the $\xi_{i,j}$ ’s with $i < j$ and the naturality of the $\xi_{i,j}$ ’s with respects to multiplications μ_i and μ_j .

Our main contribution to this picture consists in a treatment of **all** the conditions involved in the definition of multi-braided tensor product (and not only the compatibilities between the $\xi_{i,j}$ ’s) in terms of braided systems, and hence in terms of YBEs. It is our associativity braiding that makes this possible. Such a unified treatment leads to a homology theory for braided systems of UAAs, thanks to our general multi-braided homology machinery.

structure ingredient	YBE on ...
associativity of μ_i	$V_i \otimes V_i \otimes V_i$
compatibility of $\xi_{i,j}$ and μ_i	$V_i \otimes V_i \otimes V_j$
compatibility of $\xi_{i,j}$ and μ_j	$V_i \otimes V_j \otimes V_j$
compatibilities between the $\xi_{i,j}$ ’s	$V_i \otimes V_j \otimes V_k$

Table 2: Instances of YBE encoding different ingredients of multi-braided tensor product structure

Representation-theoretical aspects of braided systems of UAAs present a particular interest. We identify *multi-braided modules* over such a system with *modules over the algebra* \widehat{V} . This should be considered in the context of the general principle of presenting “complicated” structures (e.g. Hopf modules) as “simpler” structures – algebra modules – over certain “complicated” algebras (e.g. Heisenberg doubles). The concrete examples we are interested in are the following:

“complicated” structure	corresponding “complicated” algebra
bimodule over an algebra A	enveloping algebra $A \otimes A^{op}$
Hopf module over a bialgebra H	Heisenberg double $\mathcal{H}(H) := H^* \widetilde{\otimes} H$
YD module over a bialgebra H	Drinfel’d double $\mathcal{D}(H) := H^* \widetilde{\otimes} H^{op}$
Hopf bimodule over a Hopf algebra H	algebras $\mathcal{X}(H)$, $\mathcal{Y}(H)$ and $\mathcal{Z}(H)$ ([5], [27])

Table 3: Algebras encoding Hopf and Yetter-Drinfel’d (bi)module structures

The symbol $\widetilde{\otimes}$ is used here to stress that braided tensor products are used. The first two examples are treated here, the last two are left for a follow-up paper [16].

In our braided system interpretation, these “complicated” algebras are precisely the multi-braided tensor products for carefully chosen braided systems of UAAs. Thus *guessing the right braided systems of UAAs* suffices for *constructing the right “complicated” algebra*, automatically proving its associativity. This considerably reduces the technical calculations, as we see here on the example of Heisenberg doubles and in [16] on other examples from the table. Note also that our general braided system theory gives convenient explicit *permutation rules* for components of a UAA braided system – and thus for components of a multi-braided tensor product. Further,

the *adjoint module* theory developed here in the context of braided systems allows to reinterpret different “complicated” structures on concrete braided differential complexes (for instance, the Hopf bimodule structure on the bar complex of a bialgebra with coefficients in a Hopf bimodule).

An important example – that of *generalized two-sided crossed products*, as defined by D. Bulacu, F. Panaite and F. Van Oystaeyen in [3] – is studied in Section 5. The non-trivial components of the braiding on the corresponding braided system are inspired from the classical *braiding on the category of Yetter-Drinfel’d modules*. This braiding is used in Section 6 in the construction of a rank 2 braided system encoding the finite-dimensional \mathbb{k} -*bialgebra* structure. The definition and some properties of this latter braided system are presented in the following table, including also detailed information about the associativity braiding σ_{Ass} for comparison:

structure	system	braiding	invertibility	br. modules	br. complexes
algebra A	(A)	σ_{Ass} or σ_{Ass}^r	no	algebra modules	bar, Hochschild
bialgebra H	$\overline{H}_{bi} :=$ (H, H^*)	$\sigma_{1,1} = \sigma_{Ass}^r,$ $\sigma_{2,2} = \sigma_{Ass},$ $\sigma_{1,2} = \sigma_{bi}$	$\exists \sigma_{1,2}^{-1}$ iff H is a Hopf algebra	Hopf modules	Gerstenhaber- Schack [7], Panaite-Ştefan [28]

Table 4: “Braided” ingredients of the algebra and bialgebra structures

Here $\sigma_{bi} : H \otimes H^* \rightarrow H^* \otimes H$ is defined, using Sweedler’s notation, by

$$\sigma_{bi}(h \otimes l) := \langle l_{(1)}, h_{(2)} \rangle l_{(2)} \otimes h_{(1)}.$$

Note the “braided” interpretation of the existence of the antipode: it turns out to be equivalent to the invertibility of the σ_{bi} braiding component.

Our braided system allows to recover the category of bialgebras in $\mathbf{vect}_{\mathbb{k}}$ as a subcategory of $\mathbf{BrSyst}_2(\mathbf{vect}_{\mathbb{k}})$ (endowed with some additional structure). The paper [24] by F.F. Nichita can also be interpreted in the same vein. The difference consists in the braiding he uses to encode the associativity, namely a generalization of the braiding $\widetilde{\sigma}_{Ass} = \nu \otimes \mu + \mu \otimes \nu - \text{Id}_{V \otimes 2}$ proposed by P. Nuss in [25], in the context of the descent theory for noncommutative rings. Although more complicated, this braiding has the advantage of being self-inverse. F.F. Nichita and collaborators studied its properties in [23], [2] and related papers. We stick however to our σ_{Ass} because of its *homological applications* (cf. the last column of the above table).

In [16] we suggest an alternative interpretation of \overline{H}_{bi} as the simplest case of a *Yetter-Drinfel’d system*, which is a special type of braided system of UAAs designed for dealing with YD modules.

We finish with a list of *other structures admitting a braided system interpretation*. All these points are work in progress.

1. The braided system we constructed for generalized two-sided crossed products gives in particular a “braided” treatment of H -(bi)(co)module algebras: it suffices to take the underlying bialgebra H or its dual H^* as the remaining components of the triple. Repeating our study of braided homology of bialgebras in this context, one recovers in particular D. Yau’s *deformation bicomplex of module algebras* ([34]). Our “braided” tools also allow to simplify A. Kaygun’s treatment of H -equivariant A -bimodule structures used for his construction of module algebra homology, called *Hopf-Hochschild homology* ([12]).
2. In a follow-up paper [16], we construct braided systems encoding the structures of *Hopf bi-modules* and *Yetter-Drinfel’d modules*, recovering and generalizing their usual (co)homologies ([26], [32], [28]). Several other features of these structures naturally appear in this “braided” interpretation, offering a new conceptual explanation for some of these. In particular, we naturally recover two definitions of *tensor products of Yetter-Drinfel’d modules*, proposed by L.A. Lambe and D.E. Radford in [13].
3. Resuming the braiding from [15] encoding the Jacobi identity and combining it with σ_{Ass} , one gets a rank 2 braided system encoding the non-commutative Poisson algebra structure. Its multi-braided homology will hopefully include B. Fresse’s *Poisson algebra homology* ([6]).
4. The braided system machinery can also be applied to the *quantum Koszul complexes* of D. Gurevich ([8]) and M. Wambst ([33]).

Notations and conventions

All the structures in this paper live in a strict monoidal category $(\mathcal{C}, \otimes, \mathbf{I})$; the reader can have in mind the category $\mathbf{Vect}_{\mathbb{k}}$ of \mathbb{k} -vector spaces or the category \mathbf{Mod}_R of R -modules for simplicity. Notation \mathbb{k} always stands for a field, and R for a commutative unital ring.

The tensor algebra of an R -module V is denoted by $T(V)$. We often call it the *tensor module* of V , emphasizing that it can be endowed with a multiplication different from the usual concatenation.

The word ‘‘monoidal’’ is omitted in the terms *braided / symmetric monoidal category* for brevity.

Given an object V in \mathcal{C} , we succinctly denote its tensor powers by $V^n := V^{\otimes n}$, $V^0 := \mathbf{I}$. Further, given a morphism $\varphi : V^l \rightarrow V^r$, the following notation is repeatedly used:

$$\varphi^i := \text{Id}_V^{\otimes(i-1)} \otimes \varphi \otimes \text{Id}_V^{\otimes(k-i+1)} : V^{k+l} \rightarrow V^{k+r}, \quad (1)$$

and similarly for morphisms on tensor products of different objects.

Working with a family of objects (V_1, V_2, \dots) , we put $\text{Id}_i := \text{Id}_{V_i}$.

All the category equivalences in the paper are ‘‘*structure equivalences*’’: they preserve the underlying objects and establish an equivalence of different structures on them.

The already classical *graphical calculus* is extensively used in this paper, with

- dots standing for objects in \mathcal{C} ,
- horizontal gluing corresponding to the tensor product,
- graph diagrams representing morphisms from the object which corresponds to the lower dots to that corresponding to the upper dots,
- vertical gluing standing for morphism composition, and vertical strands for identities.

All diagrams are to be read from bottom to top here.

Notations S_n, B_n, B_n^+ stand for the symmetric groups, the braid groups and the positive braid monoids respectively. Their standard generators are denoted by s_i and σ_i , $1 \leq i \leq n-1$.

2 Multi-braided vocabulary

2.1 Braided systems

The notion of braided system is a weak (in the three senses explained below) generalization of the common notion of braided object in a monoidal category.

Definition 2.1. → A *braided system* in \mathcal{C} is an ordered finite family $V_1, V_2, \dots, V_r \in \text{Ob}(\mathcal{C})$ endowed with a *braiding*, i.e. morphisms $\sigma_{i,j} : V_i \otimes V_j \rightarrow V_j \otimes V_i \quad \forall \quad 1 \leq \underline{i \leq j} \leq r$ satisfying the (*colored*) *Yang-Baxter equation* (=YBE)

$$(\sigma_{j,k} \otimes \text{Id}_i) \circ (\text{Id}_j \otimes \sigma_{i,k}) \circ (\sigma_{i,j} \otimes \text{Id}_k) = (\text{Id}_k \otimes \sigma_{i,j}) \circ (\sigma_{i,k} \otimes \text{Id}_j) \circ (\text{Id}_i \otimes \sigma_{j,k}) \quad (\text{YB})$$

on all the tensor products $V_i \otimes V_j \otimes V_k$ with $1 \leq \underline{i \leq j \leq k} \leq r$. Such a system is denoted by $((V_i)_{1 \leq i \leq r}; (\sigma_{i,j})_{1 \leq i < j \leq r})$ or briefly $(\overline{V}, \overline{\sigma})$.

- The *rank* of a braided system is the number r of its components.
- A *braided morphism* $\overline{f} : (\overline{V}, \overline{\sigma}) \rightarrow (\overline{W}, \overline{\xi})$ between two braided systems in \mathcal{C} of the same rank r is a collection of morphisms $(f_i \in \text{Hom}_{\mathcal{C}}(V_i, W_i))_{1 \leq i \leq r}$ respecting the braiding, i.e.

$$(f_j \otimes f_i) \circ \sigma_{i,j} = \xi_{i,j} \circ (f_i \otimes f_j) \quad \forall \quad 1 \leq \underline{i \leq j} \leq r. \quad (2)$$

- The category of rank r braided systems and braided morphisms in \mathcal{C} is denoted by $\mathbf{BrSyst}_r(\mathcal{C})$, or $\mathbf{Br}(\mathcal{C})$ in the rank one case.
- Rank 1 braided systems are called *braided objects* in \mathcal{C} .
- For given $1 \leq s \leq t \leq r$, the *braided (s, t)-subsystem* of $(\overline{V}, \overline{\sigma})$, denoted by $(\overline{V}, \overline{\sigma})[s, t]$, is the subfamily V_s, \dots, V_t with the corresponding components $\sigma_{i,j}$ of $\overline{\sigma}$.

The notion of braiding on \overline{V} thus defined is

1. **positive**, i.e. we do not demand the invertibility of any of the $\sigma_{i,j}$'s; the term *pre-braiding* was used in our previous papers to emphasize this;
2. **partial**, i.e. defined only on certain couples of objects (this is underlined in the definition);

3. **local**, i.e., contrary to the usual notion of braiding in a monoidal category, no naturality conditions are imposed.

Graphically, the $\sigma_{i,j}$ component of the braiding is depicted as



Figure 1: A braiding component

Note that we write i and j instead of V_i and V_j in order not to overload the diagrams.

According to the definition, one allows a strand to overcross only the strands colored with a smaller or equal index $i \in \{1, 2, \dots, r\}$. The diagrammatical counterpart of Equation (YB) is the (colored) third Reidemeister move, which is at the heart of knot theory:



Figure 2: Yang-Baxter equation \longleftrightarrow Reidemeister move III

Each component of a braided system is clearly a braided objects in \mathcal{C} . Pursuing this remark, one gets

Proposition 2.2. *Given a braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$, one has, $\forall r \in \mathbb{N}$, a fully faithful functor*

$$\begin{aligned} (\mathbf{Br}(\mathcal{C}))^{\times r} &\hookrightarrow \mathbf{BrSyst}_r(\mathcal{C}), \\ (V_i, \sigma_i)_{1 \leq i \leq r} &\longmapsto (V_1, \dots, V_r; \sigma_{i,i} := \sigma_i, \sigma_{i,j} := c_{V_i, V_j} \ \forall i < j), \\ (f_i : V_i \rightarrow W_i)_{1 \leq i \leq r} &\longmapsto \overline{f} := (f_i)_{1 \leq i \leq r}. \end{aligned} \quad (3)$$

Proof. There are three kinds of tensor products on which one should check (YB) in order to verify that (3) defines a braided system:

1. $V_i \otimes V_i \otimes V_i$. Use YBE for σ_i here.
2. $V_i \otimes V_i \otimes V_j$ or $V_i \otimes V_j \otimes V_j$ for $i < j$. Use the naturality of c with respect to σ_i or σ_j .
3. $V_i \otimes V_j \otimes V_k$ for $i < j < k$. Use YBE for the braiding c .

As for morphisms, condition (2) is automatic for $i < j$ thanks to the naturality of c , and for $i = j$ it is equivalent to f_i being a braided morphism. Thus our functor is well defined, full and faithful on morphisms. \square

The following elementary observation will be useful in what follows:

Observation 2.3. *If our category \mathcal{C} is preadditive, then one has, $\forall r \in \mathbb{N}$, a category automorphism*

$$\begin{aligned} \mathbf{BrSyst}_r(\mathcal{C}) &\xrightarrow{\sim} \mathbf{BrSyst}_r(\mathcal{C}), \\ ((V_i)_{1 \leq i \leq r}; (\sigma_{i,j})_{1 \leq i < j \leq r}) &\longleftrightarrow ((V_i)_{1 \leq i \leq r}; (-\sigma_{i,j})_{1 \leq i < j \leq r}) =: (\overline{V}, -\overline{\sigma}), \\ (f_i : V_i \rightarrow W_i)_{1 \leq i \leq r} &\longleftrightarrow (f_i : V_i \rightarrow W_i)_{1 \leq i \leq r}. \end{aligned}$$

2.2 Multi-braided modules

Like for any algebraic structure, the *representation theory* of braided systems is important to study.

Definition 2.4. \rightarrow A **right multi-braided module** over $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$ is an object M equipped with morphisms $\overline{\rho} := (\rho_i : M \otimes V_i \rightarrow M)_{1 \leq i \leq r}$ satisfying

$$\rho_j \circ (\rho_i \otimes \text{Id}_j) = \rho_i \circ (\rho_j \otimes \text{Id}_i) \circ (\text{Id}_M \otimes \sigma_{i,j}) : M \otimes V_i \otimes V_j \rightarrow M \quad \forall 1 \leq i < j \leq r \quad (4)$$

or, graphically,

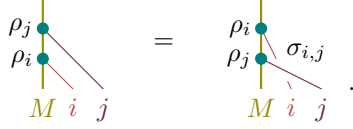


Figure 3: Multi-braided module

- Left multi-braided modules and left/right multi-braided comodules, as well as multi-braided (co)module morphisms, are defined in the usual way.
- The category of right multi-braided modules and their morphisms is denoted by $\mathbf{Mod}_{(\overline{V}, \overline{\sigma})}$. We use notation ${}_{(\overline{V}, \overline{\sigma})}\mathbf{Mod}$ in the left case and $\mathbf{Mod}^{(\overline{V}, \overline{\sigma})}$ and $(\overline{V}, \overline{\sigma})\mathbf{Mod}$ in the co-cases.
- We talk about **braided (co)modules** in the rank 1 case.

Observation 2.5. A multi-braided $(\overline{V}, \overline{\sigma})$ -module can be seen as a braided $(V_i, \sigma_{i,i})$ -module $\forall 1 \leq i \leq r$, these structures being compatible $\forall 1 \leq i < j \leq r$ in the sense of (4).

In the following sections and a subsequent paper we interpret algebra (bi)modules, Hopf (bi)modules, Yetter-Drinfel'd modules and other structures as multi-braided modules over appropriate braided systems.

We next study multi-braided module structures on the unit object \mathbf{I} of \mathcal{C} , which are particularly important in practice.

Observation 2.6. The notions of right and left multi-braided $(\overline{V}, \overline{\sigma})$ -modules coincide for \mathbf{I} . Condition (4) takes in this case a simpler form

$$(\rho_j \otimes \rho_i) \circ \sigma_{i,j} = \rho_i \otimes \rho_j : V_i \otimes V_j \rightarrow \mathbf{I}.$$

Definition 2.7. A right (= left) multi-braided $(\overline{V}, \overline{\sigma})$ -module structure on \mathbf{I} is called a **multi-braided character**, or just a **braided character** in the rank 1 case.

Example 2.8. If \mathcal{C} is preadditive, then a braided character ε_i on any V_i extended to other components by zero trivially becomes a multi-braided character on $(\overline{V}, \overline{\sigma})$.

2.3 Invertibility questions

The invertibility of some of the $\sigma_{i,j}$'s, often encountered in practice, can be helpful in extending braided structures. It allows to interchange the corresponding components of a braided system without changing the module category:

Proposition 2.9. Take $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$ with $\sigma_{p,p+1}$ invertible for some p . Then:

1. The family $(V_1, \dots, V_{p-1}, V_{p+1}, V_p, V_{p+2}, \dots, V_r)$, equipped with the old $\sigma_{i,j}$'s on the tensor products $V_i \otimes V_j$ with $(i, j) \neq (p+1, p)$ and with $\sigma_{p,p+1}^{-1}$ on $V_{p+1} \otimes V_p$, is a braided system, denoted by $s_p(\overline{V}, \overline{\sigma})$.
2. The categories of multi-braided modules for the original and the rearranged system are equivalent:

$$\mathbf{Mod}_{(\overline{V}, \overline{\sigma})} \simeq \mathbf{Mod}_{s_p(\overline{V}, \overline{\sigma})}.$$

As usual in this paper, the equivalence of module categories from the proposition preserves the underlying objects of \mathcal{C} .

Proof. 1. One has to check the following new instances of the YBE:

- (a) On $V_i \otimes V_{p+1} \otimes V_p$ with $i < p$, i.e., using notation (1),

$$\sigma_{i,p+1}^2 \circ \sigma_{i,p}^1 \circ (\sigma_{p,p+1}^2)^{-1} = (\sigma_{p,p+1}^1)^{-1} \circ \sigma_{i,p}^2 \circ \sigma_{i,p+1}^1.$$

Composing both sides with $\sigma_{p,p+1}^1$ on the left and with $\sigma_{p,p+1}^2$ on the right, one gets

$$\sigma_{p,p+1}^1 \circ \sigma_{i,p+1}^2 \circ \sigma_{i,p}^1 = \sigma_{i,p}^2 \circ \sigma_{i,p+1}^1 \circ \sigma_{p,p+1}^2,$$

which is precisely the YBE on $V_i \otimes V_p \otimes V_{p+1}$. The latter holds by the definition of braided system.

- (b) On $V_{p+1} \otimes V_p \otimes V_j$ with $j > p + 1$. This case is similar to the previous one.
(c) On $V_{p+1} \otimes V_{p+1} \otimes V_p$. Manipulations similar to case 1a iterated twice lead to the YBE on $V_p \otimes V_{p+1} \otimes V_{p+1}$.
(d) On $V_{p+1} \otimes V_p \otimes V_p$. This case is similar to the previous one.
2. Given an object M equipped with morphisms $\rho_i : M \otimes V_i \rightarrow M$, the list of compatibility conditions (4) one has to check for $(\bar{V}, \bar{\sigma})$ differs from the list for $s_p(\bar{V}, \bar{\sigma})$ only in the conditions for components $p, p + 1$:

$$\rho_{p+1} \circ (\rho_p \otimes \text{Id}_{p+1}) = \rho_p \circ (\rho_{p+1} \otimes \text{Id}_p) \circ (\text{Id}_M \otimes \sigma_{p,p+1})$$

versus

$$\rho_p \circ (\rho_{p+1} \otimes \text{Id}_p) = \rho_{p+1} \circ (\rho_p \otimes \text{Id}_{p+1}) \circ (\text{Id}_M \otimes \sigma_{p,p+1}^{-1}).$$

These two conditions are clearly equivalent. So the identity functor of \mathcal{C} and the transposition s_p of the components of $\bar{\rho}$ give the demanded category equivalence. \square

Remark 2.10. More generally, given a $\theta \in S_r$ and a $(\bar{V}, \bar{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$ with all the $\sigma_{i,j}$'s invertible for i and j permuted by θ , the family $(V_{\theta^{-1}(1)}, \dots, V_{\theta^{-1}(r)})$, equipped with the old $\sigma_{i,j}$'s on the tensor products $V_i \otimes V_j$ with $i < j$, $\theta(i) < \theta(j)$ and with $\sigma_{i,j}^{-1}$ on $V_j \otimes V_i$ with $i < j$, $\theta(i) > \theta(j)$, is a braided system. Its module category is equivalent to that of $(\bar{V}, \bar{\sigma})$.

One thus obtains a **partial S_r -action** on $\mathbf{BrSyst}_r(\mathcal{C})$. This explains

Notation 2.11. The braided system from the above remark is denoted by $\theta(\bar{V}, \bar{\sigma})$.

Corollary 2.12. *Let $(\bar{V}, \bar{\sigma})$ be a braided system in an additive monoidal \mathcal{C} , with $\sigma_{i,j}$ invertible for all $s \leq i < j \leq t$. Then one can glue the objects V_s, \dots, V_t together into $V_{s:t} := \bigoplus_{i=s}^t V_i$ and extend the braiding onto $(V_1, \dots, V_{s-1}, V_{s:t}, V_{t+1}, \dots, V_r)$, putting $\sigma|_{V_j \otimes V_i} := \sigma_{i,j}^{-1} \forall s \leq i < j \leq t$.*

Note that the invertibility of the $\sigma_{i,i}$'s is not required here even for $s \leq i \leq t$.

Proof. It suffices to consider the case $s = t - 1 =: p$, the general case then following by induction. The only instances of (YB) appearing here in addition to those coming from the braided systems $(\bar{V}, \bar{\sigma})$ and $s_p(\bar{V}, \bar{\sigma})$ are those on $V_p \otimes V_{p+1} \otimes V_p$ and $V_{p+1} \otimes V_p \otimes V_{p+1}$. They are proved by the same argument as in Point 1a (or 1b) of the proof of Proposition 2.9, taking $i = p$ (or $j = p + 1$). \square

Remark 2.13. The corollary recovers in particular the **gluing procedure** for Yang-Baxter operators (or, in our terms, for braided objects) described by S. Majid and M. Markl in [20].

3 A homology theory for braided systems

We now generalize the braided homology theory, developed in [15] for a braided object in \mathcal{C} equipped with a left and right braided modules, to the braided system setting.

In this section \mathcal{C} is additive monoidal. In particular, one can interpret the collection $\bar{\sigma}$ as a **partial braiding**, still denoted by $\bar{\sigma}$, on

$$V := V_1 \oplus V_2 \oplus \dots \oplus V_r,$$

and the family $\bar{\rho}$ defining a right multi-braided $(\bar{V}, \bar{\sigma})$ -module M as a morphism $\rho : M \otimes V \rightarrow M$.

3.1 Quantum shuffle structures: a “multi-version”

We start by showing that the collection $\bar{\sigma}$ suffices for defining a partial version of quantum (co)shuffle structures, defined in M. Rosso’s pioneer papers [29], [30].

Definition 3.1. The permutation sets

$$Sh_{p,q} := \left\{ \theta \in S_{p+q} \left| \begin{array}{l} \theta(1) < \theta(2) < \dots < \theta(p), \\ \theta(p+1) < \theta(p+2) < \dots < \theta(p+q) \end{array} \right. \right\}$$

or, more generally,

$$Sh_{p_1,p_2,\dots,p_k} := \left\{ \theta \in S_{p_1+p_2+\dots+p_k} \left| \begin{array}{l} \theta(1) < \theta(2) < \dots < \theta(p_1), \\ \theta(p_1+1) < \dots < \theta(p_1+p_2), \\ \dots, \\ \theta(p+1) < \dots < \theta(p+p_k) \end{array} \right. \right\}$$

where $p = p_1 + p_2 + \dots + p_{k-1}$, are called **shuffle sets**.

In other words, one permutes $p_1 + p_2 + \dots + p_k$ elements preserving the order within k consecutive blocks of size p_1, p_2, \dots, p_k , just like when shuffling cards, which explains the name.

Recall further the projection

$$\begin{aligned} B_n^+ &\twoheadrightarrow S_n, \\ \sigma_i &\longmapsto s_i, \end{aligned}$$

and its set-theoretical (i.e. not preserving the monoid structure) section, called *Matsumoto section*:

$$\begin{aligned} S_n &\hookrightarrow B_n^+, \\ \theta = s_{i_1} s_{i_2} \dots s_{i_k} &\longmapsto \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k}, \end{aligned}$$

where $s_{i_1} s_{i_2} \dots s_{i_k}$ is any of the shortest words representing $\theta \in S_n$.

Notation 3.2. We denote by T_θ the image of $\theta \in S_n$ under this inclusion.

Now let us return to the context of braided systems.

Definition 3.3. \twoheadrightarrow An **ordered tensor product** for $(\bar{V}, \bar{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$ is a tensor product of the form

$$V_1^{\otimes m_1} \otimes V_2^{\otimes m_2} \otimes \dots \otimes V_r^{\otimes m_r}, \quad m_i \geq 0.$$

\twoheadrightarrow A **reversely ordered tensor product** is one of the form

$$V_r^{\otimes m_r} \otimes V_{r-1}^{\otimes m_{r-1}} \otimes \dots \otimes V_1^{\otimes m_1}, \quad m_i \geq 0.$$

\twoheadrightarrow The **degree** of such a tensor product is simply the sum $\sum_{i=1}^r m_i$.

\twoheadrightarrow The direct sum of the (reversely) ordered tensor products of degree n is denoted by $T(\bar{V})_n^\rightarrow$ (respectively, $T(\bar{V})_n^\leftarrow$).

In \mathbf{Mod}_R , the $T(\bar{V})_n^\rightarrow$ ’s (respectively, $T(\bar{V})_n^\leftarrow$ ’s) sum up to

$$\begin{aligned} T(\bar{V})_n^\rightarrow &:= T(V_1) \otimes T(V_2) \otimes \dots \otimes T(V_r), \\ T(\bar{V})_n^\leftarrow &:= T(V_r) \otimes T(V_{r-1}) \otimes \dots \otimes T(V_1). \end{aligned}$$

The last ingredient we need is a partial B_n^+ -action on V^n for $(\bar{V}, \bar{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$. For a generator σ_i of B_n^+ and a summand $V_{k_1} \otimes \dots \otimes V_{k_n}$ of V^n , with $k_i \leq k_{i+1}$, it is defined by

$$\begin{aligned} \sigma_i &\longmapsto \text{Id}_{k_1} \otimes \dots \otimes \text{Id}_{k_{i-1}} \otimes \sigma_{k_i, k_{i+1}} \otimes \text{Id}_{k_{i+2}} \otimes \dots \otimes \text{Id}_{k_n} \\ &\in \text{Hom}_{\mathcal{C}}(V_{k_1} \otimes \dots \otimes V_{k_n}, V_{k_1} \otimes \dots \otimes V_{k_{i+1}} \otimes V_{k_i} \otimes \dots \otimes V_{k_n}). \end{aligned}$$

This action agrees with the common graphical depiction of the positive braid monoid elements.

Notation 3.4. The partial action described above is denoted by $B_n^+ \ni b \mapsto b^{\bar{\sigma}}$.

Armed with all these definitions and notations, one is ready for writing down the main definition of this section:

Definition 3.5. Take $(\bar{V}, \bar{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$.

→ The family of morphisms

$$\Delta_{\bar{\sigma}}^{p,q} := \sum_{\theta \in Sh_{p,q}} (T_\theta)^{\bar{\sigma}} : T(\bar{V})_p^{\leftarrow} \otimes T(\bar{V})_q^{\leftarrow} \rightarrow T(\bar{V})_{p+q}^{\leftarrow}, \quad (5)$$

where for given reversely ordered tensor products W in $T(\bar{V})_p^{\leftarrow}$ and U in $T(\bar{V})_q^{\leftarrow}$ the summation runs only over θ for which the action $(T_\theta)^{\bar{\sigma}}$ is defined on $W \otimes U$ and takes values in the $T(\bar{V})_{p+q}^{\leftarrow}$ part of V^{p+q} , is called the **multi-quantum shuffle multiplication**.

→ The family of morphisms

$$\Theta_{\bar{\sigma}}^{p,q} := \sum_{\theta \in Sh_{p,q}} (T_{\theta^{-1}})^{\bar{\sigma}} : T(\bar{V})_{p+q}^{\rightarrow} \rightarrow T(\bar{V})_p^{\rightarrow} \otimes T(\bar{V})_q^{\rightarrow} \quad (6)$$

is called the **multi-quantum coshuffle comultiplication**.

→ More generally, replacing $Sh_{p,q}$ by Sh_{p_1, \dots, p_k} , one gets morphisms $\Delta_{\bar{\sigma}}^{p_1, \dots, p_k}$ and $\Theta_{\bar{\sigma}}^{p_1, \dots, p_k}$.

Note that even when its source is an ordered tensor product, the target of $\Theta_{\bar{\sigma}}^{p,q}$ is not a single tensor product of ordered tensor products, but their direct sum in general. This explains why we need additive categories here.

Lemma 3.6. *The morphisms above are well-defined, and give an associative multiplication (respectively, a coassociative comultiplication).*

Proof. It is sufficient to observe that if (reversely) ordered tensor products are fed into the formulas from the definition, then the braiding $\bar{\sigma}$ is applied only to components $V_i \otimes V_j$ with $i \leq j$. The verification of the (co)associativity repeats the classical one for the rank 1 case (see [30] or [14]). \square

Remark 3.7. One should think about the definition (5) as a dual version of the more natural definition (6). This gives in particular a better understanding of the condition on θ in the first definition.

3.2 Multi-braided differentials

We now explain what we mean by a *homology theory* for a braided system $(\bar{V}, \bar{\sigma})$ in \mathcal{C} :

Definition 3.8. → A **degree -1 differential** for $(\bar{V}, \bar{\sigma})$ is a family of morphisms $\{d_n : T(\bar{V})_n^{\rightarrow} \rightarrow T(\bar{V})_{n-1}^{\rightarrow}\}_{n>0}$, satisfying $d_{n-1} \circ d_n = 0 \quad \forall n > 1$.

→ A **bidegree -1 bidifferential** for $(\bar{V}, \bar{\sigma})$ consists of two families of morphisms $\{d_n, d'_n : T(\bar{V})_n^{\rightarrow} \rightarrow T(\bar{V})_{n-1}^{\rightarrow}\}_{n>0}$, satisfying

$$d_{n-1} \circ d_n = d'_{n-1} \circ d'_n = d'_{n-1} \circ d_n + d_{n-1} \circ d'_n = 0 \quad \forall n > 1.$$

→ These notions allow versions “with coefficients”: one simply works on $M \otimes T(\bar{V})_n^{\rightarrow} \otimes N$ (with some objects M and N) instead of $T(\bar{V})_n^{\rightarrow}$.

→ A collection of objects $(X_n)_{n \geq 0}$ in \mathcal{C} endowed with morphisms $(d_n : X_n \rightarrow X_{n-1})_{n>0}$ and, eventually, $(d'_n : X_n \rightarrow X_{n-1})_{n>0}$ satisfying the conditions above is called a **differential (bi)complex**. Morphisms between differential complexes (X_n, d_n) and (Y_n, b_n) are collections of morphisms $(f_n : X_n \rightarrow Y_n)_{n \geq 0}$ such that

$$b_n \circ f_n = f_{n-1} \circ d_n \quad \forall n \geq 1,$$

and similarly for differential bicomplexes.

Everything is now ready for constructing a multi-version of the braided differentials from [15]:

Theorem 1. *Take a braided system $(\bar{V}, \bar{\sigma})$ in an additive monoidal category \mathcal{C} , equipped with a right and, respectively, left multi-braided $(\bar{V}, \bar{\sigma})$ -modules $(M, \bar{\rho})$ and $(N, \bar{\lambda})$. The families of morphisms*

$$\begin{aligned} (\rho d)_n &:= (\rho \otimes \text{Id}_{T(\bar{V})_{n-1}^{\rightarrow} \otimes N}) \circ (\text{Id}_M \otimes \Theta_{\bar{\sigma}}^{1, n-1} \otimes \text{Id}_N), \\ (d^\lambda)_n &:= (-1)^{n-1} (\text{Id}_{M \otimes T(\bar{V})_{n-1}^{\rightarrow}} \otimes \lambda) \circ (\text{Id}_M \otimes \Theta_{\bar{\sigma}}^{n-1, 1} \otimes \text{Id}_N) \end{aligned}$$

from $M \otimes T(\bar{V})_n^{\rightarrow} \otimes N$ to $M \otimes T(\bar{V})_{n-1}^{\rightarrow} \otimes N$ define a bidegree -1 tensor bidifferential with coefficients.

Proof. The verifications use

1 the coassociativity of $\underline{\Theta}_{-\bar{\sigma}}$, and

2 the definition of multi-braided modules, reformulated in a preadditive \mathcal{C} as

$$\rho \circ (\rho \otimes \text{Id}_V) \circ (\text{Id}_M \otimes \underline{\Theta}_{-\bar{\sigma}} 1,1) = 0, \quad \lambda \circ (\text{Id}_V \otimes \lambda) \circ (\underline{\Theta}_{-\bar{\sigma}} 1,1 \otimes \text{Id}_N) = 0.$$

Concretely, writing $\underline{\Theta}_{-\bar{\sigma}}$ instead of $\text{Id}_M \otimes \underline{\Theta}_{-\bar{\sigma}} \otimes \text{Id}_N$ or $\underline{\Theta}_{-\bar{\sigma}} \otimes \text{Id}_N$ for brevity, one calculates

$$\begin{aligned} (\rho d)_{n-1} \circ (\rho d)_n &= (\rho \otimes \text{Id} \dots) \circ \underline{\Theta}_{-\bar{\sigma}} 1, n-2 \circ (\rho \otimes \text{Id} \dots) \circ \underline{\Theta}_{-\bar{\sigma}} 1, n-1 \\ &= (\rho \otimes \text{Id} \dots) \circ (\rho \otimes \underline{\Theta}_{-\bar{\sigma}} 1, n-2) \circ \underline{\Theta}_{-\bar{\sigma}} 1, n-1 \\ &= (\rho \otimes \text{Id} \dots) \circ (\rho \otimes \text{Id} \dots) \circ (\text{Id}_{M \otimes V} \otimes \underline{\Theta}_{-\bar{\sigma}} 1, n-2) \circ \underline{\Theta}_{-\bar{\sigma}} 1, n-1 \\ &\stackrel{1}{=} (\rho \otimes \text{Id} \dots) \circ (\rho \otimes \text{Id} \dots) \circ (\text{Id}_M \otimes \underline{\Theta}_{-\bar{\sigma}} 1,1 \otimes \text{Id} \dots) \circ \underline{\Theta}_{-\bar{\sigma}} 2, n-2 \\ &= ((\rho \circ (\rho \otimes \text{Id}_V) \circ (\text{Id}_M \otimes \underline{\Theta}_{-\bar{\sigma}} 1,1)) \otimes \text{Id} \dots) \circ \underline{\Theta}_{-\bar{\sigma}} 2, n-2 \\ &\stackrel{2}{=} 0 \circ \underline{\Theta}_{-\bar{\sigma}} 2, n-2 = 0, \end{aligned}$$

and similarly for d^λ .

Further,

$$(d^\lambda)_{n-1} \circ (\rho d)_n = (-1)^{n-2} (\rho \otimes \text{Id} \dots \otimes \lambda) \circ \underline{\Theta}_{-\bar{\sigma}} 1, n-2, 1 = -(\rho d)_{n-1} \circ (d^\lambda)_n. \quad \square$$

Our proof of $(\rho d)_{n-1} \circ (\rho d)_n = 0$ can be informally interpreted as follows: the coassociative comultiplication by a square zero coelement is a differential.

Pictorially, $(\rho d)_n$ is a signed sum (due to the use of the negative braiding $-\bar{\sigma}$) of terms

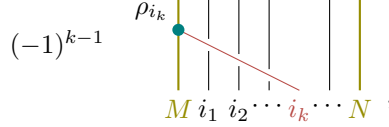


Figure 4: Multi-braided left differential

and similarly for $(d^\lambda)_n$. The sign can be interpreted here via the intersection number of the diagram.

Corollary 3.9. *Any \mathbb{Z} -linear combination of the families $(\rho d)_n$ and $(d^\lambda)_n$ from the theorem is a degree -1 tensor differential.*

Definition 3.10. The *(bi)differentials* from the above theorem and corollary are called **multi-braided**.

Remark 3.11. ✓ The constructions from the theorem are **functorial**, in the sense that a braided morphism $\bar{f} : (\bar{V}, \bar{\sigma}) \rightarrow (\bar{V}', \bar{\sigma}')$ and morphisms $\varphi : M \rightarrow M', \psi : N \rightarrow N'$ between multi-braided modules $(M, \bar{\rho}) \in \mathbf{Mod}_{(\bar{V}, \bar{\sigma})}$ and $(M', \bar{\rho}') \in \mathbf{Mod}_{(\bar{V}', \bar{\sigma}')}$ (respectively, $(N, \bar{\lambda}) \in \mathbf{Mod}_{(\bar{V}, \bar{\sigma})}$ and $(N', \bar{\lambda}') \in \mathbf{Mod}_{(\bar{V}', \bar{\sigma}')}$), compatible with \bar{f} (i.e., for instance, $\rho'_i \circ (\varphi \otimes f_i) = \varphi \circ \rho_i \forall i$), define a differential bicomplex morphism

$$\varphi \otimes \bar{f}^{\otimes n} \otimes \psi : M \otimes T(\bar{V})_n^{\rightarrow} \otimes N \rightarrow M' \otimes T(\bar{V}')_n^{\rightarrow} \otimes N'.$$

✓ Applying the categorical duality to this theorem, one gets a **cohomology theory** for $(\bar{V}, \bar{\sigma})$ with coefficients in multi-braided $(\bar{V}, \bar{\sigma})$ -comodules. Note that one should work with $T(\bar{V})_n^{\leftarrow}$ in the dual settings, since a braiding on the system (V_1, \dots, V_r) in \mathcal{C}^{op} is the same thing as a braiding on the reversed system (V_r, \dots, V_1) in \mathcal{C} .

- ✓ The braided bidifferentials can be shown to come from a **pre-bisimplicial structure** (more famous under the name of **precubical structure**), becoming weakly bisimplicial if the braided system is moreover endowed with a “good” comultiplication (i.e. multi-braided, $\bar{\sigma}$ -cocommutative and coassociative). See [15] for all the definitions and proofs in the braided object case; the generalization to higher rank does not present any particular difficulties.
- ✓ Braided differentials $({}^\rho d)_n$ (or $(d^\lambda)_n$) can be defined with coefficients on one side only, i.e. on $M \otimes T(\bar{V})_n^\rightarrow$ (or $T(\bar{V})_n^\rightarrow \otimes N$), by analogous formulas.

3.3 Adjoint multi-braided modules

The theory of adjoint braided modules from [15], including its homological consequences, has a multi-version as well.

Notation 3.12. The evident morphism from $(V_{i_1} \otimes \cdots \otimes V_{i_s}) \otimes (V_{j_1} \otimes \cdots \otimes V_{j_t})$ to $(V_{j_1} \otimes \cdots \otimes V_{j_t}) \otimes (V_{i_1} \otimes \cdots \otimes V_{i_s})$ induced by $\bar{\sigma}$ is denoted by $\bar{\sigma}$. (Here we suppose $i_n \leq j_m \forall n, m$, so that $\bar{\sigma}$ is applicable to $V_{i_n} \otimes V_{j_m}$.)

Proposition 3.13. Take $(\bar{V}, \bar{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$ and $(M, \bar{\rho}) \in \mathbf{Mod}_{(\bar{V}, \bar{\sigma})}$. Fix numbers $1 \leq s \leq t \leq r$, and denote by $(\bar{V}', \bar{\sigma})$ the braided (s, t) -subsystem $(\bar{V}, \bar{\sigma})[s, t]$.

1. For any $n \in \mathbb{N}$, $M \otimes T(\bar{V}')_n^\rightarrow$ becomes a multi-braided $(\bar{V}, \bar{\sigma})[t, r]$ -module via the morphisms

$$\begin{aligned} {}^\rho \pi_i &:= (\rho_i \otimes \text{Id}_{T(\bar{V}')_n^\rightarrow}) \circ (\text{Id}_M \otimes \bar{\sigma}_{T(\bar{V}')_n^\rightarrow, V_i}) : \\ &M \otimes T(\bar{V}')_n^\rightarrow \otimes V_i \rightarrow M \otimes T(\bar{V}')_n^\rightarrow, \quad t \leq i \leq r. \end{aligned}$$

2. The multi-braided differential ${}^\rho d$ on $M \otimes T(\bar{V}')_*^\rightarrow$ respects the multi-braided $(\bar{V}, \bar{\sigma})[t, r]$ -module structure described above.

Proof. Let us prove the compatibility relation (4) for ${}^\rho \pi_i$ and ${}^\rho \pi_j$ with $t \leq i \leq j \leq r$. Working on $M \otimes T(\bar{V}')_n^\rightarrow \otimes V_i \otimes V_j$, one has

$$\begin{aligned} &{}^\rho \pi_i \circ ({}^\rho \pi_j \otimes \text{Id}_i) \circ (\text{Id}_{\dots} \otimes \sigma_{i,j}) \\ &= (\rho_i \otimes \text{Id}_{\dots}) \circ (\text{Id}_M \otimes \bar{\sigma}_{T(\bar{V}')_n^\rightarrow, V_i}) \circ (((\rho_j \otimes \text{Id}_{\dots}) \circ (\text{Id}_M \otimes \bar{\sigma}_{T(\bar{V}')_n^\rightarrow, V_j})) \otimes \text{Id}_i) \circ (\text{Id}_{\dots} \otimes \sigma_{i,j}) \\ &= (\rho_i \otimes \text{Id}_{\dots}) \circ (\rho_j \otimes \text{Id}_{\dots}) \circ (\text{Id}_M \otimes \bar{\sigma}_{T(\bar{V}')_n^\rightarrow, V_j \otimes V_i}) \circ (\text{Id}_{\dots} \otimes \sigma_{i,j}) \\ &\stackrel{\boxed{1}}{=} (\rho_i \otimes \text{Id}_{\dots}) \circ (\rho_j \otimes \text{Id}_{\dots}) \circ (\text{Id}_M \otimes \sigma_{i,j} \otimes \text{Id}_{\dots}) \circ (\text{Id}_M \otimes \bar{\sigma}_{T(\bar{V}')_n^\rightarrow, V_i \otimes V_j}) \\ &\stackrel{\boxed{2}}{=} (\rho_j \otimes \text{Id}_{\dots}) \circ (\rho_i \otimes \text{Id}_{\dots}) \circ (\text{Id}_M \otimes \bar{\sigma}_{T(\bar{V}')_n^\rightarrow, V_i \otimes V_j}) \\ &= {}^\rho \pi_j \circ ({}^\rho \pi_i \otimes \text{Id}_j), \end{aligned}$$

where $\boxed{1}$ is a repeated application of (YB), and $\boxed{2}$ follows from the relation (4) for ρ_i and ρ_j . The compatibility relation for ${}^\rho \pi_i$ and ${}^\rho d$, with $t \leq i \leq r$, is verified similarly. \square

Definition 3.14. The multi-braided modules from the above proposition are called **adjoint**.

Applied to a braided object (V, σ) endowed with a braided character, Proposition 3.13 endows all the tensor powers V^n with a braided (V, σ) -module structure, hence the term **adjoint**.

4 A protoexample: braided systems of associative algebras

In this section \mathcal{C} is again monoidal, not necessarily preadditive. We study braided systems whose components V_i are unital associative algebras (= **UAA**s) in \mathcal{C} . We start by recalling a braiding $\sigma_{i,i}$ on V_i encoding the associativity. It was introduced and studied in detail in [15]. Then we try to complete these $\sigma_{i,i}$'s into a braiding on the whole \bar{V} . Such braided systems are proved to be in one-to-one correspondence with *multi-braided tensor products of algebras*, and multi-braided modules

over such systems are shown to coincide with modules over the corresponding tensor product algebras. The “component permuting” Proposition 2.9 describes then the rules for permuting the factors of a tensor product algebra. Concrete examples illustrating the advantages of our braided system approach follow in subsequent sections.

4.1 A braiding encoding the associativity

A technical definition is first necessary.

Definition 4.1. \rightarrow Denote by $\mathbf{BrSyst}_r^\bullet(\mathcal{C})$ the category of

✓ $(\overline{V}, \overline{\sigma}) \in \mathbf{BrSyst}_r(\mathcal{C})$ endowed with distinguished morphisms $\overline{\nu} := (\nu_i : \mathbf{I} \rightarrow V_i)_{1 \leq i \leq r}$ called *units*, and

✓ morphisms from $\mathbf{BrSyst}_r(\mathcal{C})$ preserving all the units.

Objects $(\overline{V}, \overline{\sigma}, \overline{\nu})$ of $\mathbf{BrSyst}_r^\bullet(\mathcal{C})$ are called rank r *pointed braided systems*.

\rightarrow In the rank 1 case, notation $\mathbf{Br}^\bullet(\mathcal{C})$ is used.

\rightarrow A *right multi-braided module* over $(\overline{V}, \overline{\sigma}, \overline{\nu}) \in \mathbf{BrSyst}_r^\bullet(\mathcal{C})$ is a right multi-braided $(\overline{V}, \overline{\sigma})$ -module $(M, \overline{\rho})$ satisfying moreover $\rho_i \circ (\text{Id}_M \otimes \nu_i) = \text{Id}_M \ \forall 1 \leq i \leq r$ (morally, “the units act by identity”). The category of such modules and their morphisms is denoted by $\mathbf{Mod}_{(\overline{V}, \overline{\sigma}, \overline{\nu})}$. Similar definitions and notations are assumed for left modules.

Notation 4.2. The category of unital associative algebras and algebra morphisms in \mathcal{C} is denoted by $\mathbf{Alg}(\mathcal{C})$.

We next show that different aspects of associativity are extremely well captured by a braiding encoding it, called the *associativity braiding* in what follows:

Theorem 2 ([15]). 1 *One has a fully faithful functor*

$$\mathbf{Alg}(\mathcal{C}) \hookrightarrow \mathbf{Br}^\bullet(\mathcal{C})$$

$$(V, \mu, \nu) \mapsto (V, \sigma_{Ass}(V), \nu),$$

$$f \mapsto f,$$

where

$$\sigma_{Ass}(V) := \nu \otimes \mu : V \otimes V = \mathbf{I} \otimes V \otimes V \rightarrow V \otimes V. \quad (7)$$

2 *The YBE for the associativity braiding $\sigma_{Ass}(V)$ is equivalent to the associativity for μ , under the assumption that ν is a unit for μ .*

3 *The braiding $\sigma_{Ass}(V)$ is idempotent: $\sigma_{Ass} \circ \sigma_{Ass} = \sigma_{Ass}$.*

4 *For a UAA (V, μ, ν) in \mathcal{C} , one has an equivalence of right module categories*

$$\mathbf{Mod}_{(V, \mu, \nu)} \xrightarrow{\sim} \mathbf{Mod}_{(V, \sigma_{Ass}(V), \nu)}$$

$$(M, \rho) \longleftrightarrow (M, \rho),$$

where on the left one considers UAA modules, and on the right the pointed version of multi-braided modules.

5 *Given a module $(M, \rho) \in \mathbf{Mod}_{(V, \mu, \nu)} \simeq \mathbf{Mod}_{(V, \sigma_{Ass}(V), \nu)}$, the left braided differential ρd on $(M \otimes V^n)_{n \geq 0}$ coincides with the **bar differential** with coefficients in M .*

Remark 4.3. ✓ Composing the functor from 1 with the evident forgetful functor, one gets a perhaps more elegant functor $\mathbf{Alg}(\mathcal{C}) \rightarrow \mathbf{Br}(\mathcal{C})$. The additional **pointed** structure on the target category is necessary if one wants the fullness property.

✓ The equivalence in 2 holds only up to a mild condition concerning units; such “**normalization**” conditions often appear in our “braided” study of structures.

✓ Point 3 shows that the braiding $\sigma_{Ass}(V)$ is **highly non-invertible** in general. This explains why we choose the positive notion of braiding (i.e. without invertibility axiom).

✓ Point 4 applied to $M = \mathbf{I}$ ensures that an algebra character is always a braided character.

- ✓ Dualizing, one interpretes the category of **coalgebras** in \mathcal{C} as a subcategory of *co-pointed* (= endowed with a distinguished co-element) braided objects via the fully faithful functor

$$\begin{aligned} \mathbf{coAlg}(\mathcal{C}) &\hookrightarrow \mathbf{Br}_\bullet(\mathcal{C}), \\ (V, \Delta, \varepsilon) &\longmapsto (V, \sigma_{\mathbf{coAlg}}(V) = \varepsilon \otimes \Delta, \varepsilon), \\ f &\longmapsto f. \end{aligned}$$

The algebra-coalgebra duality can now be seen inside the category of bipointed braided objects $\mathbf{Br}_\bullet(\mathcal{C})$, since the latter is self-dual (the notion of braiding being so) and encompasses both $\mathbf{Alg}(\mathcal{C})$ and $\mathbf{coAlg}(\mathcal{C})$ (the missing structure of (co)unit can be taken zero):

$$\mathbf{coAlg}(\mathcal{C}) \hookrightarrow \mathbf{Br}_\bullet(\mathcal{C}) \hookleftarrow \mathbf{Alg}(\mathcal{C}).$$

- ✓ In the theorem, the associativity braiding can be replaced with its “*right version*” $\sigma_{Ass}^r(V) := \mu \otimes \nu$. In this case one should take left modules as coefficients in the last point. The diagrams of the two associativity braidings are vertically symmetric:



Figure 5: Associativity braidings: σ_{Ass} and its vertical mirror version σ_{Ass}^r

The rest of the paper is devoted to similar results for more complicated algebraic structures.

4.2 Multi-braided tensor products of algebras

We now show that the familiar tensor product of UAAs in a braided category can be generalized to the setting of a braided system with all the $\sigma_{i,i}$ components being the associativity braidings described above.

Some general technical definitions are first due. Recall notation φ^i from (1).

Definition 4.4. → Given a $V \in \text{Ob}(\mathcal{C})$, a morphism pair $(\eta : \mathbf{I} \rightarrow V, \varepsilon : V \rightarrow \mathbf{I})$ is called *normalized* if $\varepsilon \circ \eta = \text{Id}_{\mathbf{I}}$.

→ Given $V, W \in \text{Ob}(\mathcal{C})$, a morphism $\xi : V \otimes W \rightarrow W \otimes V$ is called *natural with respect to a morphism* $\varphi : V^n \rightarrow V^m$ (or $\psi : W^n \rightarrow W^m$) if

$$\xi^1 \circ \dots \circ \xi^m \circ (\varphi \otimes \text{Id}_W) = (\text{Id}_W \otimes \varphi) \circ \xi^1 \circ \dots \circ \xi^n,$$

or, respectively,

$$\xi^m \circ \dots \circ \xi^1 \circ (\text{Id}_V \otimes \psi) = (\psi \otimes \text{Id}_V) \circ \xi^n \circ \dots \circ \xi^1.$$

In the case $V = W$ both conditions are required.

Example 4.5. The naturality condition for $n = 1, m = 2$ and $V = W$ graphically means

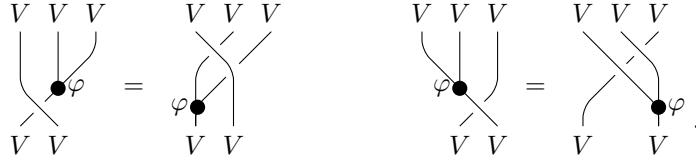


Figure 6: Naturality

This is precisely the definition of a braided comultiplication on (V, σ) . Note also that these graphical conditions appear in the study of diagrams of trivalent graphs.

Theorem 3. Take r UAAs $(V_i, \mu_i, \nu_i)_{1 \leq i \leq r}$ in a monoidal category \mathcal{C} , each unit ν_i being a part of a normalized pair (ν_i, ε_i) , and, for each couple of subscripts $1 \leq i < j \leq r$, take a morphism $\xi_{i,j}$ natural with respect to ν_i and ν_j . The following statements are then equivalent:

- 1 The morphisms $\xi_{i,i} := \sigma_{Ass}(V_i) \forall 1 \leq i \leq r$ complete the $\xi_{i,j}$'s and the ν_i 's into a pointed braided system structure on \overleftarrow{V} .
- 2 Each $\xi_{i,j}$ is natural with respect to μ_i and μ_j , and, for each triple $i < j < k$, the $\xi_{i,j}$'s satisfy the Yang-Baxter equation on $V_i \otimes V_j \otimes V_k$.
- 3 A UAA structure on

$$\overleftarrow{V} := V_r \otimes V_{r-1} \otimes \cdots \otimes V_1$$

can be defined by putting

$$\mu_{\overleftarrow{V}} := (\mu_r \otimes \mu_{r-1} \otimes \cdots \otimes \mu_1) \circ \xi^{2r-2} \circ (\xi^{2r-4} \circ \xi^{2r-3}) \circ \cdots \circ (\xi^2 \circ \cdots \circ \xi^{r-1} \circ \xi^r), \quad (8)$$

$$\nu_{\overleftarrow{V}} := \nu_r \otimes \nu_{r-1} \otimes \cdots \otimes \nu_1. \quad (9)$$

Here ξ^p means the morphism $\xi_{i,j}$ applied at the positions p and $p+1$ of the tensor product, i and j being the subscripts of the components of \overleftarrow{V} which are currently at these positions.

Note the inverse component order in the definition of \overleftarrow{V} , ensuring that (8) is well-defined.

Proof. We show that Points 1 and 3 are both equivalent to the (intermediate) Point 2.

Start with 1. YBE on each $V_i \otimes V_i \otimes V_i$ is automatic via Theorem 2. On $V_i \otimes V_i \otimes V_j, i < j$, YBE becomes

$$(\xi_{i,j} \otimes \text{Id}_i) \circ (\text{Id}_i \otimes \xi_{i,j}) \circ (\nu_i \otimes \mu_i \otimes \text{Id}_j) = (\text{Id}_j \otimes \nu_i \otimes \mu_i) \circ (\xi_{i,j} \otimes \text{Id}_i) \circ (\text{Id}_i \otimes \xi_{i,j}),$$

or, graphically,

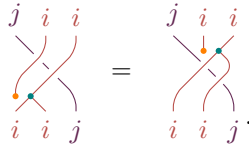


Figure 7: YBE for $V_i \otimes V_i \otimes V_j$

The naturality of $\xi_{i,j}$ with respect to the units permits to “pull” the short strand out of the crossing on the left diagram. The equation obtained is equivalent to $\xi_{i,j}$ being natural with respect to μ_i (compose this equation with $\text{Id}_j \otimes \mu_i$ to get one of the implications), or, graphically, to

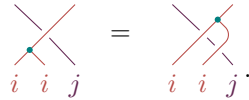


Figure 8: Naturality with respect to μ_i

Similarly, YBE on $V_i \otimes V_j \otimes V_j, i < j$, is equivalent to $\xi_{i,j}$ being natural with respect to μ_j . This terminates the proof of the equivalence $1 \Leftrightarrow 2$.

Let us now show that $3 \Leftrightarrow 2$. We use shortcut notations

$$\iota_j := \nu_r \otimes \cdots \otimes \nu_{j+1} \otimes \text{Id}_j \otimes \nu_{j-1} \otimes \cdots \otimes \nu_1 : V_j \rightarrow \overleftarrow{V} \quad \forall 1 \leq j \leq r. \quad (10)$$

Given a collection of $\xi_{i,j}$'s satisfying the conditions of Point 2, one verifies (for instance graphically) that the morphisms from Point 3 define a UAA structure on \overleftarrow{V} . This is a generalization of the verifications usually made while defining the tensor product of algebras in a braided category. To show that all the conditions from Point 2 are indeed necessary, consider the associativity condition for $\mu_{\overleftarrow{V}}$ composed with

- either $\iota_i \otimes \iota_j \otimes \iota_k : V_i \otimes V_j \otimes V_k \rightarrow \overleftarrow{V}^{\otimes 3}$ on the right and the ϵ_t 's at all the positions except for i, j, k on the left;
- or $\iota_i \otimes \iota_i \otimes \iota_j : V_i \otimes V_i \otimes V_j \rightarrow \overleftarrow{V}^{\otimes 3}$ on the right and the ϵ_t 's at all the positions except for i, j on the left;

→ or $\iota_i \otimes \iota_j \otimes \iota_j : V_i \otimes V_j \otimes V_j \rightarrow \overleftarrow{V}^{\otimes 3}$ on the right and the ϵ_i 's at all the positions except for i, j on the left.

Using the naturality of the ξ 's with respect to the units and the defining property of a normalized pair, in the first case one gets YBE for the ξ 's on $V_i \otimes V_j \otimes V_k$ with $i < j < k$, in the second and third cases – the naturality of $\xi_{i,j}$ with respect to μ_i and μ_j respectively, with $i < j$. \square

The theorem gives a “braided” $\boxed{1}$, an “algebraic” $\boxed{3}$ and a “mixed” $\boxed{2}$ interpretations of the same phenomenon. In practice, it is usually convenient to use $\boxed{1}$ or $\boxed{2}$ in order to check the associativity of $\mu_{\overleftarrow{V}}$, considerably simplifying the associativity verification for complicated structures.

Definition 4.6. A braided system of the type described in the above theorem is called a **braided system of UAAs**, and the UAA \overleftarrow{V} is called the **multi-braided tensor product** of the UAAs V_1, \dots, V_r , denoted (abusively) by

$$\overleftarrow{V} = V_r \underset{\xi}{\otimes} V_{r-1} \underset{\xi}{\otimes} \cdots \underset{\xi}{\otimes} V_1.$$

Remark 4.7. In the theorem, one can replace the existence of the ϵ_i 's, used only to prove $3 \Rightarrow 2$, by demanding $\boxed{3}$ to hold for all (even non-consecutive) subsystems of \overleftarrow{V} . In this case, while proving $3 \Rightarrow 2$, one can work with an appropriate subsystem instead of composing with the ϵ_i 's in order to get to the desired tensor product. In particular, the existence of the ϵ_i 's is not necessary for $r = 2$.

Remark 4.8. Some or all of the morphisms $\xi_{i,i} = \sigma_{Ass}(V_i)$ can be replaced with their right versions $\sigma_{Ass}^r(V_i)$. The previous theorem still holds, with analogous proof.

Example 4.9. According to Proposition 2.2, for a braided category \mathcal{C} , the choice $\xi_{i,j} := c_{V_i, V_j}$ in the theorem above gives a braided system. In addition, the c_{V_i, V_j} 's are natural with respect to everything hence in particular to the units. In this case, the UAA structure on \overleftarrow{V} given by the theorem is the usual tensor product of algebras in a braided category.

4.3 Multi-braided modules as modules over algebras

The structure equivalence from Theorem 3 has an important counterpart on the level of modules:

Proposition 4.10. *In the settings of Theorem 3, suppose one of the three equivalent conditions satisfied for the $\xi_{i,j}$'s. The following categories are then equivalent:*

$$\mathbf{Mod}_{V_r \underset{\xi}{\otimes} V_{r-1} \underset{\xi}{\otimes} \cdots \underset{\xi}{\otimes} V_1} \simeq \mathbf{Mod}_{((V_1, \dots, V_r), \overleftarrow{V})}$$

Proof. According to Observation 2.5 combined with Point $\boxed{4}$ of Theorem 2, a multi-braided module structure over the pointed braided system described in Theorem 3 consists of module structures (M, ρ_i) over each UAA (V_i, μ_i, ν_i) , compatible in the sense of (4). The correspondence with modules (M, ρ) over the UAA $V_r \underset{\xi}{\otimes} V_{r-1} \underset{\xi}{\otimes} \cdots \underset{\xi}{\otimes} V_1$ can now be given by

$$\rho_j := \rho \circ (\text{Id}_M \otimes \iota_j),$$

$$\rho := \rho_1 \circ (\rho_2 \otimes \text{Id}_1) \circ \cdots \circ (\rho_r \otimes \text{Id}_{r-1} \otimes \cdots \otimes \text{Id}_1),$$

where the ι_j 's are defined in (10). The identity functor of \mathcal{C} and this structure correspondence give the desired category equivalence. \square

Consider now the situation when one of the $\xi_{i,i+1}$'s is invertible, allowing one to apply Proposition 2.9. One gets

Proposition 4.11. *In the settings of Theorem 3, suppose one of the $\xi_{i,i+1}$'s invertible. Then*

1. *UAAs $V_1, \dots, V_{i-1}, V_{i+1}, V_i, V_{i+2}, \dots, V_r$ endowed with the ξ 's one had for the system \overline{V} , completed with $\xi_{i,i+1}^{-1}$ on $V_{i+1} \otimes V_i$, still form a braided system of UAAs.*
2. *Further, the morphism*

$$\text{Id}_r \otimes \dots \otimes \text{Id}_{i+2} \otimes \xi_{i,i+1}^{-1} \otimes \text{Id}_{i-1} \otimes \dots \otimes \text{Id}_1,$$

abusively denoted by $\xi_{i,i+1}^{-1}$, gives an algebra isomorphism between the multi-braided UAA tensor products \overleftarrow{V} and (using one more abusive notation)

$$s_i(\overleftarrow{V}) := V_r \otimes_{\xi} \dots \otimes_{\xi} V_{i+2} \otimes_{\xi} V_i \otimes_{\xi^{-1}} V_{i+1} \otimes_{\xi} V_{i-1} \otimes_{\xi} \dots \otimes_{\xi} V_1.$$

3. *The last isomorphism is compatible with the category equivalence*

$$\begin{array}{ccccc} \mathbf{Mod}_{\overleftarrow{V}} & \simeq & \mathbf{Mod}_{(\overline{V}, \overline{\xi}, \overline{\nu})} & \simeq & \mathbf{Mod}_{s_i(\overline{V}, \overline{\xi}, \overline{\nu})} & \simeq & \mathbf{Mod}_{s_i(\overleftarrow{V})} \\ (M, \rho_{\overleftarrow{V}}) & & \longleftarrow & & \longrightarrow & & (M, \rho_{s_i(\overleftarrow{V})}) \end{array}$$

in the sense that $\rho_{\overleftarrow{V}} = \rho_{s_i(\overleftarrow{V})} \circ (\text{Id}_M \otimes \xi_{i,i+1}^{-1})$. The leftmost and rightmost categories in this chain of equivalences are the usual categories of UAA modules.

Proof. 1. Proposition 2.9 allows to interchange the components V_i and V_{i+1} of the pointed braided system $(\overline{V}, \overline{\xi}, \overline{\nu})$ from Point 1 of Theorem 3. The new pointed braided system $s_i(\overline{V}, \overline{\xi}, \overline{\nu})$ then satisfies again the conditions of Point 1 from Theorem 3. Moreover, $\xi_{i,i+1}^{-1}$ is natural with respect to the units since so is $\xi_{i,i+1}$. One thus gets the desired braided system of UAAs.

2. Theorem 3 (Point 3) then gives the multi-braided UAA tensor product $s_i(\overleftarrow{V})$. Applying YBE several times, one shows that, in order to see that $\xi_{i,i+1}^{-1}$ is an algebra morphism, it is sufficient to work with V_i and V_{i+1} only. Namely, one has to prove

$$\xi_{i,i+1}^{-1} \circ (\nu_{i+1} \otimes \nu_i) = \nu_i \otimes \nu_{i+1},$$

which follows from the naturality of $\xi_{i,i+1}^{-1}$ with respect to the units, and

$$\begin{aligned} & (\mu_i \otimes \mu_{i+1}) \circ (\text{Id}_i \otimes \xi_{i,i+1}^{-1} \otimes \text{Id}_{i+1}) \circ (\xi_{i,i+1}^{-1} \otimes \xi_{i,i+1}^{-1}) = \\ & \xi_{i,i+1}^{-1} \circ (\mu_{i+1} \otimes \mu_i) \circ (\text{Id}_{i+1} \otimes \xi_{i,i+1} \otimes \text{Id}_i) : \\ & (V_{i+1} \otimes V_i)^{\otimes 2} \rightarrow V_i \otimes V_{i+1}, \end{aligned}$$

or, graphically,

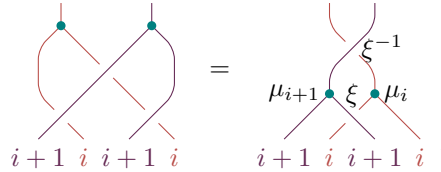


Figure 9: $\xi_{i,i+1}^{-1}$ is an algebra morphism

This relation follows from the naturality of $\xi_{i,i+1}$ (and hence $\xi_{i,i+1}^{-1}$) with respect to μ_i and μ_{i+1} (Point 2 of Theorem 3).

3. The equivalence of module categories is a consequence of (the proofs of) Propositions 2.9 and 4.10. \square

One thus obtains *partial S_r -actions* on braided systems of UAAs and multi-braided UAA tensor products of rank r . These actions are mutually compatible and induce equivalences of module categories.

4.4 A toy example: algebra bimodules

We upgrade here Theorem 2 to the rank 2 level. A **braided** category $(\mathcal{C}, \otimes, \mathbf{I}, c)$ is needed here.

First note that, for $(V, \mu, \nu) \in \mathbf{Alg}(\mathcal{C})$, the data $(\mu \circ c, \nu)$ define another UAA structure on V . Notation V^{op} is used for V endowed with this modified UAA structure. The associativity braiding becomes here $\sigma_{Ass}(V^{op}) = \nu \otimes (\mu \circ c)$. The *twisted multiplication* provides a useful transition between left and right module structures:

Lemma 4.12. *For $(V, \mu, \nu) \in \mathbf{Alg}(\mathcal{C})$, the following functors (extended on morphisms by identities) give an equivalence of module categories:*

$$\mathbf{Mod}_{V^{op}} \xleftarrow{\sim} {}_V\mathbf{Mod},$$

$$(M, \rho) \mapsto (M, \lambda(\rho) := \rho \circ c_{M,V}^{-1}), \quad (11)$$

$$(M, \rho(\lambda) := \lambda \circ c_{M,V}) \longleftarrow (M, \lambda). \quad (12)$$

Take now two UAAs (V, μ, ν) and (V', μ', ν') in \mathcal{C} . Returning to Example 4.9, one gets

Lemma 4.13. *The data $(V_1 = V, V_2 = V'; \sigma_{1,1} = \sigma_{Ass}(V), \sigma_{2,2} = \sigma_{Ass}(V'^{op}), \sigma_{1,2} = c_{V,V'})$ define a braided system of UAAs.*

The proofs of the above two lemmas are straightforward.

Notation 4.14. This braided system of UAAs is denoted by $\mathbf{Bimod}(V, V')$.

Applying Observation 2.5, the module category equivalence from Proposition 4.10 and permutation rules from Proposition 4.11 to $\mathbf{Bimod}(V, V')$, and keeping in mind Lemma 4.12, one obtains a nice interpretation for multi-braided modules over this system:

Proposition 4.15. *Take two UAAs (V, μ, ν) and (V', μ', ν') in a braided category \mathcal{C} . The following module categories are equivalent:*

$$\mathbf{Mod}_{V'^{op} \otimes V} \simeq \mathbf{Mod}_{\mathbf{Bimod}(V, V')} \simeq {}_{V'}\mathbf{Mod}_V \simeq \mathbf{Mod}_{s_2(\mathbf{Bimod}(V, V'))} \simeq \mathbf{Mod}_{V \otimes_{c^{-1}} V'^{op}},$$

where ${}_V\mathbf{Mod}_V$ is the usual category of (V', V) -bimodules.

One thus gets a multi-braided module treatment of **algebra bimodules**. The case $V' = V$ gives the familiar **enveloping algebra** of a UAA V , $V^e := V \otimes V^{op}$.

We finish by applying the ‘‘adjoint module’’ Proposition 3.13 to our bimodule context, choosing $s = 1, t = 2$. Recall notation φ^i from (1).

Proposition 4.16. *Take a bimodule $(M, \rho : M \otimes V \rightarrow M, \lambda : V' \otimes M \rightarrow M)$ over two UAAs V and V' in a braided category \mathcal{C} . The bar complex for V with coefficients in (M, ρ) on the left, i.e. $(M \otimes T(V), {}^\rho d)$, is a complex in ${}_V\mathbf{Mod}_V$. More precisely, the differentials $({}^\rho d)_n$ are bimodule morphisms, the bimodule structure on $M \otimes V^n$ being given by*

$$\rho_{bar} := \mu^{n+1} : M \otimes V^n \otimes V \rightarrow M \otimes V^n, \quad \lambda_{bar} := \lambda^1 : V' \otimes M \otimes V^n \rightarrow M \otimes V^n.$$

We call such structures **peripheral**, as they see only the extreme factors of a tensor product.

Proof. Plug the braided system $\mathbf{Bimod}(V, V')$ and $(M, \rho, \lambda) \in {}_{V'}\mathbf{Mod}_V \simeq \mathbf{Mod}_{\mathbf{Bimod}(V, V')}$ into Proposition 3.13. The braided $\mathbf{Bimod}(V, V')$ -multi-module structure on $M \otimes T(V)$ obtained this way is compatible, according to that proposition, with the braided differential ${}^\rho d$ and, due to Proposition 4.15, is equivalent to a (V', V) -bimodule structure on $M \otimes T(V)$. The latter structure can be explicitly written using Lemma 4.12 as follows:

$$\begin{aligned} {}^\rho \pi_1 &= \rho_1 \circ (\text{Id}_M \otimes \overline{\sigma}_{V^n, V}) \\ &= \text{Id}_M \otimes \text{Id}_{V^{n-1}} \otimes \mu : M \otimes V^n \otimes V \rightarrow M \otimes V^n, \\ \lambda({}^\rho(\lambda)\pi_2) &= \rho(\lambda)\pi_2 \circ c_{M \otimes V^n, V'}^{-1} \\ &= (\lambda \circ c_{M, V'})^1 \circ (\text{Id}_M \otimes \overline{\sigma}_{V^n, V'}) \circ c_{M \otimes V^n, V'}^{-1} \\ &= (\lambda \circ c_{M, V'})^1 \circ (\text{Id}_M \otimes c_{V^n, V'}) \circ c_{M \otimes V^n, V'}^{-1} \\ &= \lambda^1 : V' \otimes M \otimes V^n \rightarrow M \otimes V^n. \end{aligned} \quad \square$$

This bimodule structure on the bar complex is important for one of the constructions of the *Hochschild (co)homology*; namely, one considers the differential induced by the bar differential on the coinvariants.

5 Handling more structure: two-sided crossed products

We now present a rank 3 braided system of UAAs, where some of the $\xi_{i,j}$'s with $i < j$ are “structural”, i.e. coming from the algebraic structure on the V_i 's rather than from the underlying category. This example allows to reinterpret F. Panaite's braided treatment of *two-sided crossed products* $A\#H\#B$ from [27] and its extension to the case of *generalized two-sided crossed products* $A\blacktriangleright C\blacktriangleleft B$ (defined by D. Bulacu, F. Panaite and F. Van Oystaeyen in [3]) from [11]. In particular, we automatically obtain (via Proposition 4.11) six equivalent versions of the algebra $A\blacktriangleright C\blacktriangleleft B$. Pursuing further the “braided” ideas and using our results on adjoint multi-braided modules, we get, in the generalized crossed product setting, a (B, A) -bimodule structure on C^n , used in Section 6 for constructing bialgebra homologies.

5.1 Categorical bialgebras and module algebras

The categorical versions of some familiar algebraic notions are needed here:

Definition 5.1. \rightarrow A *bialgebra* structure in a braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$ is a UAA structure (μ, ν) and a counital coassociative coalgebra (= *coUAA*) structure (Δ, ε) on an object H , compatible in the following sense:

$$\Delta \circ \mu = (\mu \otimes \mu) \circ c_2 \circ (\Delta \otimes \Delta), \quad \Delta \circ \nu = \nu \otimes \nu, \quad (13)$$

$$\varepsilon \circ \mu = \varepsilon \otimes \varepsilon, \quad \varepsilon \circ \nu = \text{Id}_{\mathbf{I}}. \quad (14)$$

The category of bialgebras and bialgebra morphisms in \mathcal{C} is denoted by $\mathbf{Bialg}(\mathcal{C})$.

\rightarrow H is a *Hopf algebra* in \mathcal{C} if moreover it has an *antipode*, i.e. an endomorphism s of H satisfying

$$\mu \circ (s \otimes \text{Id}_H) \circ \Delta = \mu \circ (\text{Id}_H \otimes s) \circ \Delta = \nu \circ \varepsilon. \quad (s)$$

The category of Hopf algebras and their morphisms in \mathcal{C} is denoted by $\mathbf{HAlg}(\mathcal{C})$.

\rightarrow For a bialgebra H in \mathcal{C} , a *left H -module algebra* is a UAA (M, μ_M, ν_M) endowed with a left H -module structure $\lambda : H \otimes M \rightarrow M$, such that μ_M and ν_M are H -module morphisms:

$$\lambda \circ (\text{Id}_H \otimes \mu_M) = \mu_M \circ (\lambda \otimes \lambda) \circ c_2 \circ (\Delta \otimes \text{Id}_M^{\otimes 2}), \quad (15)$$

$$\lambda \circ (\text{Id}_H \otimes \nu_M) = \nu_M \circ \varepsilon. \quad (16)$$

Right H -module algebras, H -comodule algebras and H -bi(co)module algebras are defined similarly. The categories of module algebras and their morphisms are denoted by ${}_H\mathbf{ModAlg}$, \mathbf{ModAlg}_H , ${}^H\mathbf{ModAlg}$ etc.

Graphically, compatibility conditions (13), (15) and (16) mean

Figure 10: Bialgebra and module algebra relations

5.2 Two-sided crossed products as multi-braided tensor products

Everything is now ready for handling generalized two-sided crossed product.

Proposition 5.2. *Take a bialgebra H , a left H -module algebra (A, λ) , a right H -module algebra (B, ρ) and an H -bicomodule algebra $(C, \delta_l : C \rightarrow H \otimes C, \delta_r : C \rightarrow C \otimes H)$ in a symmetric category \mathcal{C} . Then*

1. The UAAs (B, C, A) together with morphisms

$$\begin{aligned}\xi_{1,2} &= (\text{Id}_C \otimes \rho) \circ (c_{B,C} \otimes \text{Id}_H) \circ (\text{Id}_B \otimes \delta_r), \\ \xi_{2,3} &= (\lambda \otimes \text{Id}_C) \circ (\text{Id}_H \otimes c_{C,A}) \circ (\delta_l \otimes \text{Id}_A), \\ \xi_{1,3} &= c_{B,A}\end{aligned}$$

form a braided system of UAAs.

2. Formulas (8)-(9) for the $\xi_{i,j}$'s above define a UAA structure on $A \otimes C \otimes B$.
3. One has a module category equivalence

$$\mathbf{Mod}_{(B,C,A)} \simeq \mathbf{Mod}_{A \otimes_{\xi} C \otimes_{\xi} B}. \quad (17)$$

Graphically, the $\xi_{1,2}$ and $\xi_{2,3}$ components of the braiding look as follows:

$$\xi_{1,2} = \begin{array}{c} \text{---} \rho \text{---} \\ \diagup \quad \diagdown \\ B \quad C \\ \text{---} \delta_r \text{---} \end{array} \quad \xi_{2,3} = \begin{array}{c} \text{---} \lambda \text{---} \\ \diagdown \quad \diagup \\ C \quad A \\ \text{---} \delta_l \text{---} \end{array}.$$

Figure 11: A braided system for a two-sided crossed product

Convention 5.3. The underlying braiding of a symmetric category is depicted here and afterwards by a solid crossing, in order to distinguish it from “structural” braidings proper to given objects.

Proof. The key point is to verify that the ξ 's satisfy the conditions of Point [2] of Theorem 3:

- ✓ YBE on $B \otimes C \otimes A$ follows from the left and the right H -coaction compatibility for C ;
- ✓ the naturality of the ξ 's with respect to μ_C is a consequence of the defining properties of H -bicomodule algebras for C ;
- ✓ the naturality of the ξ 's with respect to μ_A and μ_B can be deduced from the defining properties of H -module algebras for A and B .

As an example, we show in detail that $\xi_{1,2}$ is natural with respect to μ_B :

$$\begin{aligned}\xi_{1,2} \circ (\mu_B \otimes \text{Id}_C) & \\ \stackrel{1}{=} (\text{Id}_C \otimes \rho) \circ (c_{B,C} \otimes \text{Id}_H) \circ (\mu_B \otimes \delta_r) & \\ \stackrel{2}{=} (\text{Id}_C \otimes \rho) \circ (\text{Id}_C \otimes \mu_B \otimes \text{Id}_H) \circ (c_{B \otimes B, C} \otimes \text{Id}_H) \circ (\text{Id}_B^{\otimes 2} \otimes \delta_r) & \\ \stackrel{3}{=} (\text{Id}_C \otimes \mu_B) \circ (\text{Id}_C \otimes \rho \otimes \rho) \circ (\text{Id}_{C \otimes B} \otimes c_{B,H} \otimes \text{Id}_H) \circ (c_{B \otimes B, C} \otimes \Delta_H) \circ (\text{Id}_B^{\otimes 2} \otimes \delta_r) & \\ \stackrel{4}{=} (\text{Id}_C \otimes \mu_B) \circ (\text{Id}_C \otimes \rho \otimes \text{Id}_B) \circ (c_{B,C} \otimes \text{Id}_{H \otimes B}) \circ (\text{Id}_B \otimes \delta_r \otimes \rho) \circ (\text{Id}_B \otimes c_{B,C} \otimes \text{Id}_H) \circ (\text{Id}_B^{\otimes 2} \otimes \delta_r) & \\ \stackrel{5}{=} (\text{Id}_C \otimes \mu_B) \circ (\xi_{1,2} \otimes \text{Id}_B) \circ (\text{Id}_B \otimes \xi_{1,2}), & \end{aligned}$$

where we use

1. and 5. : the definition of $\xi_{1,2}$,
2. the naturality of c ,
3. the defining property of right H -module algebra for B ,
4. the defining property of right H -comodule for C and the naturality of c .

The reader is advised to draw diagrams in order to better follow these verifications.

Further, the naturality with respect to units follows from the defining properties of H -(co)module algebras as well. Point [1] from Theorem 3 then confirms that the ξ 's together with the σ_{Ass} 's form a braiding, while Point [3] proves the associativity of the multiplication (8).

Finally, Proposition 4.10 gives the required module category equivalence. \square

The tensor product algebra from the proposition is known as the *generalized two-sided crossed product* (cf. [3])

$$A \blacktriangleright C \blacktriangleleft B := A \otimes_{\xi} C \otimes_{\xi} B.$$

The choice $C = H$ (with both comodule structures given by Δ_H) gives the *two-sided crossed product* of F. Hausser and F. Nill (cf. [9]), usually denoted by

$$A \# H \# B := A \otimes_{\xi} H \otimes_{\xi} B.$$

Remark 5.4. Forgetting the B (or A) part of the structure and taking as C a left (respectively right) H -comodule, one obtains rank 2 braided systems, giving in particular a “braided” treatment of left and right ***crossed (or smash) products*** $A\#H := A \otimes_{\xi} H$ and $H\#B := H \otimes_{\xi} B$, as well as of their generalized versions.

We have thus obtained an alternative conceptual proof of the associativity of $A\blacktriangleleft C\blacktriangleright B$ and of the category equivalence (17), otherwise very technical.

Remark 5.5. If H is a Hopf algebra with an invertible antipode s , then all the ξ 's are invertible:

$$\begin{aligned}\xi_{1,2}^{-1} &= ((\rho \circ c_{H,B}) \otimes \text{Id}_C) \circ (s^{-1} \otimes c_{C,B}) \circ ((c_{C,H} \circ \delta_r) \otimes \text{Id}_B), \\ \xi_{2,3}^{-1} &= (\text{Id}_C \otimes (\lambda \circ c_{A,H})) \circ (c_{A,C} \otimes s^{-1}) \circ (\text{Id}_A \otimes (c_{H,C} \circ \delta_l)), \\ \xi_{1,3}^{-1} &= c_{A,B}.\end{aligned}$$

Proposition 4.11 then allows to **permute components of** $A \otimes_{\xi} C \otimes_{\xi} B$, giving six pairwise isomorphic UAAs, these isomorphisms being compatible with the equivalences of their module categories. In particular, one recovers the algebra isomorphism $A\#H\#B \simeq (A \otimes B) \bowtie H$ from [9].

Remark 5.6. Supposing the category \mathcal{C} moreover additive, one can start with an H -bimodule (C', δ_l, δ_r) and introduce an artificial trivial UAA structure by adding a **formal unit** $C := C' \oplus \mathbf{I}$, taking the zero multiplication on C' and making $\nu_C := \text{Id}_{\mathbf{I}} : \mathbf{I} \rightarrow C$ a unit. The bicomodule structure on C' extended to C by putting $\delta_l|_{\mathbf{I}} := \nu_H \otimes \nu_C$, $\delta_r|_{\mathbf{I}} := \nu_C \otimes \nu_H$ endows C with an H -bicomodule algebra structure. This formal trick will be useful in what follows.

5.3 Adjoint actions

We finish this example by applying the theory of adjoint multi-modules (Proposition 3.13) to the braided system of UAAs from Proposition 5.2, choosing trivial coefficients ($M = \mathbf{I}$).

A preliminary general lemma is first necessary:

Lemma 5.7. *Take a braided system $((V_1, \dots, V_r), \bar{\sigma})$ in a symmetric additive category \mathcal{C} , such that $\sigma_{1,r}$ is the underlying symmetric braiding c_{V_1, V_r} of \mathcal{C} . Take further two multi-braided characters $\bar{\epsilon}$ and $\bar{\zeta}$ for this braided system. Then the right braided V_r -module structure ${}^{\epsilon}\pi_r$ and the left braided V_1 -module structure π_1^{ζ} on $T(\bar{V})_n^{\rightarrow}$, $n \in \mathbb{N}$, commute:*

$${}^{\epsilon}\pi_r \circ (\pi_1^{\zeta} \otimes \text{Id}_r) = \pi_1^{\zeta} \circ (\text{Id}_1 \otimes {}^{\epsilon}\pi_r) : V_1 \otimes T(\bar{V})_n^{\rightarrow} \otimes V_r \rightarrow T(\bar{V})_n^{\rightarrow}.$$

Proof. The underlying braiding c is natural with respect to everything, in particular ϵ_r and ζ_1 . Both sides of the desired identity then equal

$$(\epsilon_r \otimes \text{Id}_{T(\bar{V})_n^{\rightarrow}} \otimes \zeta_1) \circ (\bar{\sigma}_{T(\bar{V})_n^{\rightarrow}, V_r} \otimes \text{Id}_1) \circ \bar{\sigma}_{V_1, T(\bar{V})_n^{\rightarrow} \otimes V_r}. \quad \square$$

Now return to the two-sided crossed products. Recall notation φ^i from (1).

Proposition 5.8. *In the settings of Proposition 5.2, choose algebra characters ϵ_A and ϵ_B for A and B . The tensor powers of C become then bimodules, $C^n \in {}_B\mathbf{Mod}_A \forall n \in \mathbb{N}$, via the formulas*

$$\begin{aligned}\epsilon^A \pi &= (\epsilon_A)^1 \circ \lambda^1 \circ (\text{Id}_H \otimes c_{C^n, A}) \circ (\mu^1)^{\circ(n-1)} \circ ((\omega_{2n}^{-1} \circ \delta_l^{\otimes n}) \otimes \text{Id}_A) : C^n \otimes A \rightarrow C^n, \\ \pi^{\epsilon_B} &= (\epsilon_B)^{n+1} \circ \rho^{n+1} \circ (c_{B, C^n} \otimes \text{Id}_H) \circ (\mu^{n+2})^{\circ(n-1)} \circ (\text{Id}_B \otimes (\omega_{2n}^{-1} \circ \delta_r^{\otimes n})) : B \otimes C^n \rightarrow C^n, \\ \text{where } \omega_{2n} &:= \begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n \\ 1 & 3 & \dots & 2n-1 & 2 & 4 & \dots & 2n \end{pmatrix} \in S_{2n},\end{aligned}\tag{18}$$

and S_{2n} acts on tensor products of objects from \mathcal{C} via the symmetric braiding c .

These actions are graphically depicted as



Figure 12: ${}_B\mathbf{Mod}_A$ structure on C^n

Proof. Note that for the Point 1 of Proposition 3.13 to hold,

- ✓ the additivity of \mathcal{C} is not necessary, and
- ✓ a multi-braided module $(M, \bar{\rho}) \in \mathbf{Mod}_{(\bar{V}, \bar{\sigma})[t, r]}$ (instead of $(M, \bar{\rho}) \in \mathbf{Mod}_{(\bar{V}, \bar{\sigma})}$) suffices.

That proposition and its right version applied to the braided system of UAAs (B, C, A) from Proposition 5.2 and to the algebra characters (hence braided characters) ϵ_A and ϵ_B give then a right braided A -module structure and a left braided B -module structure on C^n . Further, since the $\xi_{1,2}$ and $\xi_{2,3}$ components of the braiding on (B, C, A) are natural with respect to the units, the units of A and B act on C^n by identity. Theorem 2 then ensures that our braided A - and B -module structures on C^n are actually UAA module structures. One then verifies that these module structures coincide with the desired ones.

It remains to show that the actions of A and B commute, which is precisely the assertion of Lemma 5.7 in our setting. \square

6 Bialgebras

In this section we present a rank 2 braided system of UAAs \bar{H}_{bi} encoding the bialgebra structure, in the same sense that the braiding σ_{Ass} from Theorem 2 encodes the associativity (cf. table 1). This braided system is a particular case of the one presented in Section 5 for crossed products.

Note a new feature appearing in the bialgebra case compared to the algebra case. While the invertibility condition for σ_{Ass} is trivially false, the invertibility of the $\sigma_{1,2}$ component of \bar{H}_{bi} corresponds to an important algebraic property of the structure: it is in fact equivalent to the existence of the *antipode*.

The braided homology theory for \bar{H}_{bi} is very rich. In particular it is shown to include the familiar *Gerstenhaber-Schack bialgebra homology*. Multi-braided modules over \bar{H}_{bi} turn out to be precisely the familiar *Hopf modules*, so these are the natural candidates for coefficients in the bialgebra homology.

Except for some general observations, we specialize here to the category $\underline{\mathcal{C}} = \mathbf{vect}_{\mathbb{k}}$ of finite-dimensional \mathbb{k} -vector spaces. Note however that one could stay in the general setting of a braided category \mathcal{C} and choose a bialgebra in \mathcal{C} admitting a dual.

Several conventions are used when working in $\mathbf{vect}_{\mathbb{k}}$. Notation $T(V) := \bigoplus_{n \geq 0} V^n$ stands for the tensor space of $V \in \mathbf{vect}_{\mathbb{k}}$, with $V^0 := \mathbb{k}$. A simplified notation is used for tensors in V^n :

$$v_1 v_2 \dots v_n := v_1 \otimes v_2 \otimes \dots \otimes v_n \in V^n,$$

leaving the tensor product sign for

$$v_1 v_2 \dots v_n \otimes w_1 w_2 \dots w_m \in V^n \otimes W^m.$$

Sweedler's notation is used for comultiplications and comodule structures.

6.1 Duality: conventions and observations

We start with some conventions concerning dualities.

The dual space of $V \in \mathbf{vect}_{\mathbb{k}}$ is denoted by V^* . Letters h_i stay in this section for elements of V , l_j – for elements of V^* . The pairing \langle, \rangle is the *evaluation map* $ev : V^* \otimes V \rightarrow \mathbb{k}$, $l \otimes h \mapsto l(h)$. Multiplications on different spaces are simply denoted by \cdot when it does not lead to confusion.

Take \mathbb{k} -vector spaces V, W and a pairing $B : V \otimes W \rightarrow \mathbb{k}$ (e.g. the evaluation map). There are two common ways of extending it to $B : V^n \otimes W^m \rightarrow \mathbb{k}$:

$B(v_1 v_2 \dots v_n \otimes w_1 w_2 \dots w_n) :=$	
$B(v_1 \otimes w_n) \cdots B(v_n \otimes w_1)$	$B(v_1 \otimes w_1) \cdots B(v_n \otimes w_n)$
“rainbow”	“arched”

Table 5: Rainbow and arched dualities

The “arched” version is more common in literature, but it is the “rainbow” version we use in this work (like, for instance, D. Gurevich in [8]), avoiding unnecessary flips (in the diagram it is reflected by the absence of crossings). Similar conventions are used in the dual situation, i.e. for Casimir elements. The usual duality $V \mapsto V^*$ in $\mathbf{vect}_{\mathbb{k}}$ induces, with these conventions, a duality on morphisms which associates to an $f : V_1 \otimes \dots \otimes V_n \rightarrow W_1 \otimes \dots \otimes W_m$ its dual morphism $f^* : W_m^* \otimes \dots \otimes W_1^* \rightarrow V_n^* \otimes \dots \otimes V_1^*$ (note the inverse order of factors). In particular, the dual of a finite-dimensional \mathbb{k} -linear coalgebra V is always implicitly endowed here with the *induced algebra structure* via the evaluation map ev , extended to $V \otimes V$ and $V^* \otimes V^*$ using the “rainbow” pattern:

$$\langle l_1 l_2, h \rangle = \langle l_1, h_{(2)} \rangle \langle l_2, h_{(1)} \rangle \quad \forall h \in V, l_1, l_2 \in V^*,$$

or, graphically,

$$\begin{array}{c} \text{ev} \\ \bullet \\ \Delta^* \text{---} \text{---} \text{---} \\ \text{V}^* \text{V}^* \text{V} \end{array} = \begin{array}{c} \text{ev} \\ \bullet \\ \Delta \text{---} \text{---} \text{---} \\ \text{V}^* \text{V}^* \text{V} \end{array}.$$

Figure 13: Dual structures via the “rainbow” duality

Multiplication and (co)units are dualized in the same way.

Analyzing the graphical interpretation, one sees that, on the level of structures, the “rainbow” duality corresponds to the **central symmetry**, while the “arched” duality – to the **horizontal mirror symmetry**. Note that because of our non-conventional choice, we sometimes get formulas slightly different from the ones found in literature.

The same structure on V^* can be obtained via the dual *coevaluation map* $coev$ or via “twisted versions” $ev \circ \tau : V \otimes V^* \rightarrow \mathbb{k}$ and $\tau \circ coev : \mathbb{k} \rightarrow V \otimes V^*$, still with the “rainbow” extension on tensor products. Here τ is simply the transposition of factors V and V^* ; in the general settings of a symmetric category it should be replaced with the braiding c . It is common to simplify notations, writing just ev and $coev$ for the latter maps, which we do systematically.

Lemma 4.12 provides an important transition tool between *left* V -modules and *right* V^{op} -modules for an algebra V in $\mathbf{vect}_{\mathbb{k}}$. We now state an analogous transition lemma for modules and comodules. A classical general observation concerning “twisted” bialgebra structures is first due:

Observation 6.1. *Take a bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$ in a braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$. Then*

1. $H^{op} := (H, \mu^{op} := \mu \circ c^{-1}, \nu, \Delta, \varepsilon)$ and $H^{cop} := (H, \mu, \nu, \Delta^{cop} := c^{-1} \circ \Delta, \varepsilon)$ are bialgebras in $(\mathcal{C}, \otimes, \mathbf{I}, c^{-1})$, while $H^{op, cop} := (H, \mu \circ c^{-1}, \nu, c \circ \Delta, \varepsilon)$ and $H^{cop, op} := (H, \mu \circ c, \nu, c^{-1} \circ \Delta, \varepsilon)$ are bialgebras in $(\mathcal{C}, \otimes, \mathbf{I}, c)$.
2. If H is a Hopf algebra with the antipode s , then so are $H^{op, cop}$ and $H^{cop, op}$, with the same antipode s . If s is invertible, then s^{-1} becomes the antipode for H^{op} and H^{cop} .
3. Moreover, one has the following bialgebra or Hopf algebra isomorphisms:

$$(H^{op})^* \simeq (H^*)^{cop}, \quad (H^{cop})^* \simeq (H^*)^{op}, \quad (H^{op, cop})^* \simeq H^{cop, op}.$$

Lemma 6.2. For a coalgebra V in \mathbf{vect}_k , the following functors (extended on morphisms by identities) give an equivalence of module categories:

$$\begin{aligned} \mathbf{Mod}^V &\xleftarrow{\sim} \mathbf{Mod}_{V^*}, \\ (M, \delta) &\longmapsto (M, \delta^{co} := (\mathrm{Id}_M \otimes ev) \circ (\delta \otimes \mathrm{Id}_{V^*})), \\ (M, \rho^{co} := (\rho \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_M \otimes coev)) &\longleftarrow (M, \rho). \end{aligned} \quad (19)$$

$$(20)$$

If the ‘‘arched’’ version of structure dualities is used, one should take $\mathbf{Mod}_{(V^*)^{op}}$ on the right. The proof of the lemma is routine. Here is a graphical version of the (19) part of the equivalence:

Figure 14: Action-coaction duality

Convention 6.3. Here and afterwards thin lines stand for H , dashed lines for its dual H^* , and thick colored lines for different types of modules over them.

Similar equivalences hold for the categories of module (co)algebras:

Lemma 6.4. For a bialgebra H in \mathbf{vect}_k , the functors from Lemmas 4.12 and 6.2 (combined with identities on the algebra structures) induce the following category equivalences:

$$\begin{aligned} \mathbf{ModAlg}^H &\xleftarrow{\sim} \mathbf{ModAlg}_{(H^*)^{cop}}, \\ {}_H\mathbf{ModAlg} &\xleftarrow{\sim} \mathbf{ModAlg}_{H^{op}}. \end{aligned}$$

6.2 A braiding encoding the bialgebra structure

We now show how to include the groupoid ${}^*\mathbf{Bialg}(\mathbf{vect}_k)$ of bialgebras and bialgebra isomorphisms in \mathbf{vect}_k into the groupoid of *bipointed* rank 2 braided systems in \mathbf{vect}_k , just like it was done in Theorem 2 for UAAs and *pointed* rank 1 systems.

Definition 6.5. \rightarrow Given a monoidal category \mathcal{C} , denote by $\mathbf{BrSyst}_r(\mathcal{C})$ the category of $\checkmark (\bar{V}, \bar{\sigma}, \bar{\nu}) \in \mathbf{BrSyst}_r(\mathcal{C})$ endowed with distinguished morphisms $\bar{\varepsilon} := (\varepsilon_i : V_i \rightarrow \mathbf{I})_{1 \leq i \leq r}$ called *counits*, such that (ν_i, ε_i) is a normalized pair for each i , and \checkmark morphisms from $\mathbf{BrSyst}_r(\mathcal{C})$ preserving moreover all the counits. Objects $(\bar{V}, \bar{\sigma}, \bar{\nu}, \bar{\varepsilon})$ of $\mathbf{BrSyst}_r(\mathcal{C})$ are called rank r *bipointed braided systems*. \rightarrow The groupoid of rank r bipointed braided systems and their isomorphisms is denoted by ${}^*\mathbf{BrSyst}_r(\mathcal{C})$. \rightarrow In the rank 1 case, notations $\mathbf{Br}(\mathcal{C})$ and ${}^*\mathbf{Br}(\mathcal{C})$ are used.

We also have to recall the notion of Hopf modules in order to identify the multi-braided modules over the system presented here. A general categorical definition is given here for completeness:

Definition 6.6. In a braided category $(\mathcal{C}, \otimes, \mathbf{I}, c)$, a right module structure $\rho : M \otimes H \rightarrow M$ and a right comodule structure $\delta : M \rightarrow M \otimes H$ on $M \in \mathrm{Ob}(\mathcal{C})$ are said to form a (*right-right*) **Hopf module** structure over $H \in \mathbf{Bialg}(\mathcal{C})$ if they satisfy the *Hopf compatibility condition*

$$\delta \circ \rho = (\rho \otimes \mu) \circ (\mathrm{Id}_M \otimes c_{H,H} \otimes \mathrm{Id}_H) \circ (\delta \otimes \Delta) : M \otimes H \rightarrow M \otimes H. \quad (21)$$

The category of right-right Hopf modules over H and their morphisms is denoted by \mathbf{Mod}_H^H .

Condition (21) is graphically depicted as

Figure 15: Right-right Hopf compatibility condition

H itself gives an important example of an H -Hopf module, with $\rho := \mu_H, \delta := \Delta_H$.

We now return to our category $\mathcal{C} = \mathbf{vect}_k$, omitting it in most notations for simplicity.

Theorem 4. [1] *One has a fully faithful functor*

$$\mathbf{*Bialg} \hookrightarrow \mathbf{*BrSyst}_2 \bullet$$

$$\begin{aligned} (H, \mu, \nu, \Delta, \varepsilon) &\longmapsto \overline{H}_{bi} := (V_1 := H, V_2 := H^*; \\ &\sigma_{1,1} := \sigma_{Ass}^r(H), \sigma_{2,2} := \sigma_{Ass}(H^*), \sigma_{1,2} = \sigma_{bi}; \\ &\nu, \varepsilon^*; \varepsilon, \nu^*), \\ f &\longmapsto (f, (f^{-1})^*), \end{aligned}$$

where $\sigma_{bi}(h \otimes l) := \langle l_{(1)}, h_{(2)} \rangle l_{(2)} \otimes h_{(1)}$.

- [2] Take an $H \in \mathbf{vect}_k$ endowed with a UAA and a coUAA structures (μ, ν) and (Δ, ε) . Suppose the pair (ν, ε) normalized. Then the YBE on $H \otimes H \otimes H^*$ (or, symmetrically, on $H \otimes H^* \otimes H^*$) and the naturality of σ_{bi} with respect to units are equivalent to the bialgebra compatibility conditions (13)-(14) for H .
- [3] For a bialgebra H , the braiding component σ_{bi} is invertible if and only if H has an antipode:

$$\exists \sigma_{bi}^{-1} \iff H \in \mathbf{HAlg}$$

- [4] For a bialgebra H , one has an equivalence of module categories

$$\begin{aligned} \mathbf{Mod}_H^H &\xrightarrow{\sim} \mathbf{Mod}_{\overline{H}_{bi}} \xrightarrow{\sim} \mathbf{Mod}_{H^* \otimes H}^{\sigma_{bi}} \\ (M, \rho, \delta) &\longmapsto (M, \rho, \delta^{co}) \longmapsto (M, \delta^{co} \otimes \rho). \end{aligned}$$

If H is a Hopf algebra with the antipode s , then this chain of category equivalences can be continued on the left (putting: $\theta := \sigma_{bi}^{-1}$):

$$\mathbf{Mod}_{H \otimes H^*}^{\theta} \simeq \mathbf{Mod}_{s_1(\overline{H}_{bi})} \simeq \mathbf{Mod}_H^H.$$

Before proving the theorem, we present a graphical version of the braiding on \overline{H}_{bi} :

Figure 16: A braiding encoding the bialgebra structure

Remark 6.7. From the graphical interpretation of \overline{H}_{bi} it is clear that, applied to the dual bialgebra H^* instead of H , the construction from the theorem gives a vertical mirror version of \overline{H}_{bi} .

Proof. Consider $(H^*, \mu^*) \in {}^{H^*} \mathbf{ModAlg}$. (A left version of) Lemma 6.4 allows to transform this comodule algebra into a module algebra $(H^*, (\mu^*)^{co}) \in {}_{H^{cop}} \mathbf{ModAlg}$. Together with $(H^{cop}, \Delta^{cop}, \Delta^{cop}) \in {}^{H^{cop}} \mathbf{ModAlg}^{H^{cop}}$ these two structures can be fed into Proposition 5.2 as the A and the C parts of the structure (Remark 5.4 allows one to forget the B part). One verifies that the $\xi_{2,3}$ component of the braided system from that proposition is precisely the σ_{bi} described here. Further, H^{cop} and H share the same UAA structure. We thus recover \overline{H}_{bi} as a particular case of the braided system of UAAs constructed in Proposition 5.2. It is clearly bipointed. Moreover, one checks that the braiding on \overline{H}_{bi} suffices to recover all ingredients of the bialgebra structure on H .

To prove Point [1], it remains to understand isomorphisms of bipointed braided systems $(f, g) : \overline{H}_{bi} \rightarrow \overline{K}_{bi}$ for bialgebras H, K in \mathbf{vect}_k . By definition,

1. $f : H \rightarrow K$ and $g : H^* \rightarrow K^*$ are bijections;
2. f and g respect units and counits;
3. f respects the braiding $\sigma_{1,1} = \sigma_{Ass}^r$;

4. g respects the braiding $\sigma_{2,2} = \sigma_{Ass}$;
5. f and g respect σ_{bi} .

Due to Theorem 2 (Point 1), the first four points are equivalent to f and g being UAA isomorphisms compatible with counits (i.e. $\varepsilon_K \circ f = \varepsilon_H$ and $\nu_K^* \circ f = \nu_H^*$). The last point means

$$\sigma_{bi}(K) \circ (f \otimes g) = (g \otimes f) \circ \sigma_{bi}(H),$$

which can be explicitly written as

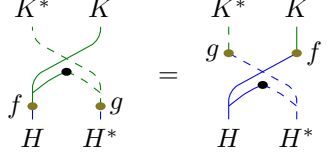


Figure 17: Braided morphisms $\overline{H}_{bi} \rightarrow \overline{K}_{bi}$: compatibility with $\sigma_{1,2}$

Applying $\nu_K^* \otimes \varepsilon_K$ to both sides, using the fact that f and g respect the counits, and playing with evaluation-coevaluation dualities, one deduces that $g^* \circ f = \text{Id}_H$, hence $g = (f^*)^{-1} = (f^{-1})^*$. Since g is a UAA isomorphism, so is g^{-1} , hence $f = (g^{-1})^*$ is a coUAA morphism, which completes its properties to show that it is a bialgebra isomorphism. Thus all bipointed braided system isomorphisms are of the form $(f, (f^{-1})^*)$ for a unique bialgebra isomorphism f , and for all bialgebra isomorphisms the couple $(f, (f^{-1})^*)$ is indeed a bipointed braided system isomorphism. The functor from the theorem is thus well defined, full and faithful, which finishes the proof of Point 1.

In Point 2, the compatibility between Δ and ν follow from the naturality of σ_{bi} with respect to ν by applying $\nu^* \otimes \text{Id}_H$ to the latter condition and using evaluation-coevaluation dualities. Symmetrically, the compatibility between μ and ε follow from the naturality of σ_{bi} with respect to ε^* . The converse (compatibility \implies naturality) is easy. Further, according to (the proof of) Theorem 3, the YBE on $H \otimes H \otimes H^*$ is equivalent to σ_{bi} being natural with respect to μ :

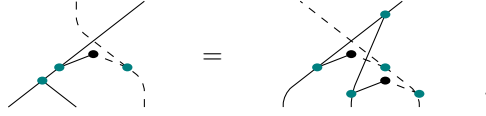


Figure 18: Naturality of σ_{bi} with respect to μ

Applying $\nu^* \otimes \text{Id}_H$ to both sides and playing with dualities, one recovers the bialgebra compatibility condition for μ and Δ . Conversely, the bialgebra compatibility condition suffices for showing the above naturality. By symmetry, one gets a proof for $H \otimes H^* \otimes H^*$.

The “if” part of Point 3 can be proved by checking an explicit formula for σ_{bi}^{-1}

$$\sigma_{bi}^{-1}(l \otimes h) = \langle l_{(1)}, s(h_{(2)}) \rangle h_{(1)} \otimes l_{(2)} \quad (22)$$

(or by using Remark 5.5 and Point 2 of Observation 6.1). The “only if” part is more delicate. Suppose the existence of σ_{bi}^{-1} and put

$$\tilde{s} := (((\varepsilon \otimes \nu^*) \circ \sigma_{bi}^{-1}) \otimes \text{Id}_H) \circ (\text{Id}_{H^*} \otimes c_{H,H}) \circ (\text{coev} \otimes \text{Id}_H) : H \rightarrow H.$$

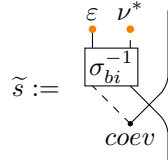


Figure 19: A candidate for the antipode

Let us prove that \tilde{s} is the antipode. The part

$$\mu \circ (\tilde{s} \otimes \text{Id}_H) \circ \Delta = \nu \circ \varepsilon \quad (23)$$

of the defining relation (s) is a direct consequence of $\sigma_{bi}^{-1} \circ \sigma_{bi} = \text{Id}_{H \otimes H^*}$ and the evaluation-coevaluation duality. One would expect to deduce the second part of (s) from $\sigma_{bi} \circ \sigma_{bi}^{-1} = \text{Id}_{H^* \otimes H}$, but surprisingly this does not seem to work. Some algebraic tricks come into play instead. Mimicking (22), set

$$\tilde{\sigma} := (\text{Id}_H \otimes (ev \circ (\tilde{s} \otimes \text{Id}_{H^*})) \otimes \text{Id}_{H^*}) \circ (\Delta \otimes \mu^*) \circ c_{H^*, H} : H^* \otimes H \rightarrow H \otimes H^*.$$

Relation (23) implies $\tilde{\sigma} \circ \sigma_{bi} = \text{Id}_{H \otimes H^*}$. But σ_{bi}^{-1} is the inverse of σ_{bi} , so

$$\tilde{\sigma} = \tilde{\sigma} \circ (\sigma_{bi} \circ \sigma_{bi}^{-1}) = (\tilde{\sigma} \circ \sigma_{bi}) \circ \sigma_{bi}^{-1} = \sigma_{bi}^{-1}.$$

This gives $\sigma_{bi} \circ \tilde{\sigma} = \text{Id}_{H^* \otimes H}$. Applying $\nu^* \otimes \varepsilon$ to both sides and playing with dualities, one recovers the second part of (s) for \tilde{s} .

We now move to Point 4. Equivalence $\mathbf{Mod}_{\overline{H}_{bi}} \xrightarrow{\sim} \mathbf{Mod}_{H^* \otimes_{\sigma_{bi}} H}$ follows from Point 3 of

Proposition 5.2. Further, according to Observation 2.5 combined with Point 4 of Theorem 2, a right multi-braided \overline{H}_{bi} -module means a right algebra H -module and a right algebra H^* -module structures ρ_H and ρ_{H^*} on M , compatible in the sense of (4):

$$\rho_{H^*} \circ (\rho_H \otimes \text{Id}_{H^*}) = \rho_H \circ (\rho_{H^*} \otimes \text{Id}_H) \circ (\text{Id}_M \otimes (\tau \circ (\text{Id}_H \otimes ev \otimes \text{Id}_{H^*}) \circ (\Delta \otimes \mu^*))).$$

On the other hand, the module-comodule duality Lemma 6.2 allows to interpret a right-right Hopf module over H as a right algebra H -module and a right algebra H^* -module structures, with the compatibility condition obtained by

- applying $\text{Id}_M \otimes ev$ to the definition (21) of Hopf modules, tensored with Id_{H^*} on the right,
- and using (19) to transform H -comodule structures into H^* -module structures.

The two compatibility conditions coincide, implying $\mathbf{Mod}_H^H \simeq \mathbf{Mod}_{\overline{H}_{bi}}^H$.

In the Hopf algebra case Point 3 gives the invertibility of σ_{bi} . The component permuting Proposition 4.11 proves then the desired equivalences. \square

All the remarks following Theorem 2 remain relevant in the bialgebra case. One particular feature of the bialgebra setting is to be added to that list:

Remark 6.8. It is essential to work in the **groupoid**, and not just in the category of bialgebras, if one wants a bialgebra morphism $H \rightarrow G$ to induce a morphism of dual bialgebras $H^* \rightarrow G^*$, so that the functor from 1 can be defined on morphisms.

Denote by

$$\mathcal{H}'(H) := H \otimes_{\theta} H^*$$

one of the multi-braided tensor products of UAAs from the theorem. Then $\mathcal{H}(H) := \mathcal{H}'(H^*)$ is the well-known **Heisenberg double** of the Hopf algebra H (cf. for example [22] or [5]).

6.3 Braided homology for bialgebras

Our next goal is to write down explicit braided differentials for \overline{H}_{bi} . We do it for **partial characters** ε_H (the counit of H extended to H^* by zero) and ε_{H^*} (the counit of H^* , i.e. $(\nu_H)^*$, extended to H by zero); they are algebra characters and hence braided characters on the corresponding component \overline{H}_{bi} , and Example 2.8 justifies the extension by zero.

Some preliminary observations are necessary for our calculations. The first ones concern **twisted and dual variations** of Theorem 4. Applying that theorem to each of the bialgebras H^{op} , H^{cop} and $H^{op, cop}$ (which in our symmetric category $\mathcal{C} = \mathbf{vect}_{\mathbb{k}}$ coincides with $H^{cop, op}$) and their duals, one obtains several new braided system structures:

Corollary 6.9. For a finite-dimensional \mathbb{k} -bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$, one can construct the following braided systems on (H, H^*) :

$$\begin{aligned} \overline{H}^{op} : \quad & \sigma_{1,1} = (\mu \circ \tau) \otimes \nu, & \sigma_{2,2} &= \varepsilon^* \otimes \Delta^*, \\ & \sigma_{1,2} = \sigma_{bi}^{op} := \tau \circ (\text{Id}_H \otimes ev \otimes \text{Id}_{H^*}) \circ (\Delta \otimes (\tau \circ \mu^*)); \\ \overline{H}^{cop} : \quad & \sigma_{1,1} = \mu \otimes \nu, & \sigma_{2,2} &= \varepsilon^* \otimes (\Delta^* \circ \tau), \\ & \sigma_{1,2} = \sigma_{bi}^{cop} := \tau \circ (\text{Id}_H \otimes ev \otimes \text{Id}_{H^*}) \circ ((\tau \circ \Delta) \otimes \mu^*); \\ \overline{H}^{op,cop} : \quad & \sigma_{1,1} = (\mu \circ \tau) \otimes \nu, & \sigma_{2,2} &= \varepsilon^* \otimes (\Delta^* \circ \tau), \\ & \sigma_{1,2} = \sigma_{bi}^{op,cop} := \tau \circ (\text{Id}_H \otimes ev \otimes \text{Id}_{H^*}) \circ ((\tau \circ \Delta) \otimes (\tau \circ \mu^*)); \end{aligned}$$

the following braided system on (H^*, H) :

$$\begin{aligned} \overline{H}^r : \quad & \sigma_{1,1} = \Delta^* \otimes \varepsilon^*, & \sigma_{2,2} &= \nu \otimes \mu, \\ & \sigma_{1,2} = \sigma_{bi}^r := \tau \circ (\text{Id}_{H^*} \otimes ev \otimes \text{Id}_H) \circ (\mu^* \otimes \Delta); \end{aligned}$$

and the three “twisted” versions of the last structure.

Notations \overline{H}^r and σ_{bi}^r come from their interpretations as vertical mirror versions of \overline{H} and σ_{bi} . Here are graphical representations of the “twisted” bialgebra braidings:

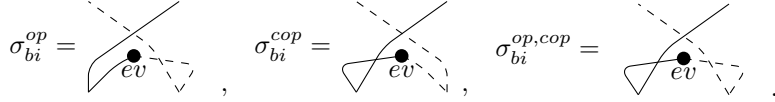


Figure 20: “Twisted” bialgebra braidings

We also need **adjoint actions** of H^* on the tensor powers of H , obtained via Proposition 5.8:

Lemma 6.10. The tensor powers of a finite-dimensional \mathbb{k} -bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$ can be endowed with an H^* -bimodule structure via formulas (cf. notations (1) and (18)):

$$\begin{aligned} \pi^{H^*} &:= \pi^{\varepsilon_{H^*}} = ev^1 \circ ev^2 \cdots ev^n \circ (((\mu^*)^1)^{\circ(n-1)} \otimes (\omega_{2n}^{-1} \circ \Delta^{\otimes n})): \\ & \quad H^* \otimes H^n \rightarrow \otimes H^n, \\ {}^{H^*} \pi &:= \varepsilon_{H^*} \pi = ev^{n+1} \circ ev^{n+2} \cdots ev^{2n} \circ ((\omega_{2n}^{-1} \circ \Delta^{\otimes n}) \otimes (((\mu^*)^1)^{\circ(n-1)})): \\ & \quad H^n \otimes H^* \rightarrow \otimes H^n. \end{aligned}$$

The H^* -actions are graphically depicted as

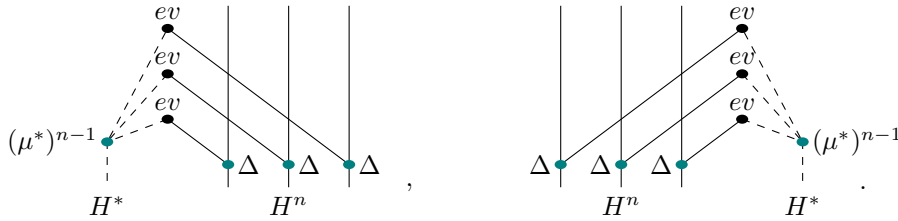


Figure 21: H^n as an H^* -bimodule

On the level of elements, the formulas can be written as

$$\begin{aligned} \pi^{H^*}(l \otimes h_1 \cdots h_n) &= \langle l_{(1)}, h_{n(1)} \rangle \langle l_{(2)}, h_{n-1(1)} \rangle \cdots \langle l_{(n)}, h_{1(1)} \rangle h_{1(2)} \cdots h_{n(2)}, \\ {}^{H^*} \pi(h_1 \cdots h_n \otimes l) &= \langle l_{(1)}, h_{n(2)} \rangle \langle l_{(2)}, h_{n-1(2)} \rangle \cdots \langle l_{(n)}, h_{1(2)} \rangle h_{1(1)} \cdots h_{n(1)}. \end{aligned}$$

Proof. In the proof of Theorem 4 we have noticed that $C = (H^{cop}, \Delta^{cop}, \Delta^{cop}) \in {}^{H^{cop}} \mathbf{ModAlg}^{H^{cop}}$ and $A = (H^*, (\mu^*)^{co}) \in {}_{H^{cop}} \mathbf{ModAlg}$ can be fed into Proposition 5.2, hence into Proposition 5.8. Symmetry considerations show that this data can be completed by $B = (H^*, (\mu^*)^{co}) \in \mathbf{ModAlg}_{H^{cop}}$. To conclude, notice that for this triple (A, B, C) and braided characters $\varepsilon_A = \varepsilon_B := \varepsilon_{H^*}$, the actions from Proposition 5.8 are precisely the desired ones. \square

Interchanging the roles of H and H^* , one gets an H -bimodule $((H^*)^{\otimes m}, \pi^H, H\pi)$. By abuse of notation, we define, for all $m, n \in \mathbb{N}$ for which this makes sense, the following morphisms from $H^n \otimes (H^*)^m$ to $H^{(n-1)} \otimes (H^*)^m$ or to $H^n \otimes (H^*)^{(m-1)}$:

$$\begin{aligned} {}^{H^*}\pi &:= {}^{H^*}\pi \otimes \text{Id}_{H^*}^{\otimes(m-1)}, & \pi^{H^*} &:= (\pi^{H^*} \otimes \text{Id}_{H^*}^{\otimes(m-1)}) \circ \tau_{H^n \otimes (H^*)^{(m-1)}, H^*}, \\ \pi^H &:= \text{Id}_H^{\otimes(n-1)} \otimes \pi^H, & H\pi &:= (\text{Id}_H^{\otimes(n-1)} \otimes H\pi) \circ \tau_{H, H^{(n-1)} \otimes (H^*)^m}. \end{aligned}$$

Lemma 6.11. *The endomorphisms ${}^{H^*}\pi, \pi^{H^*}, \pi^H$ and $H\pi$ of $T(H) \otimes T(H^*)$ pairwise commute.*

Proof. Lemma 6.10 implies the commutativity of ${}^{H^*}\pi$ and π^{H^*} . The commutativity of $H\pi$ and π^H follows by duality. Next, returning to the braided interpretation of the adjoint actions, π^H corresponds to pulling the rightmost H strand to the right of all the H^* strands (using σ_{bi}) and applying ε_H , while ${}^{H^*}\pi$ corresponds to pulling the leftmost H^* strand to the left of all the H strands and applying ε_{H^*} . Thus π^H and ${}^{H^*}\pi$ commute, and so do π^{H^*} and $H\pi$ by duality.

In order to prove the commutativity of the two remaining pairs, consider the linear isomorphism

$$\Delta_n \otimes \text{Id}_{H^*}^{\otimes m} : H^n \otimes (H^*)^m \xrightarrow{\sim} (H^{op})^n \otimes ((H^{op})^*)^m, \quad \Delta_n := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 2 & \\ & & & \ddots \\ & & & & n \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \in S_n,$$

where S_n acts on H^n via the flip τ . This isomorphism, extended to $T(H) \otimes T(H^*)$ by linearity, is denoted by Δ_* by abuse of notation (unfortunately, the common notation for Garside elements coincides with that for comultiplication). One checks that Δ_* transports the endomorphisms ${}^{H^*}\pi, \pi^{H^*}, \pi^H$ and $H\pi$ of $H^{\otimes n} \otimes (H^*)^{\otimes m}$ to, respectively, $(H^{op})^*\pi, \pi^{(H^{op})^*}, {}^{H^{op}}\pi$ and $\pi^{H^{op}}$. Thus the commutativity of $(H^{op})^*\pi$ and $\pi^{H^{op}}$ induces that of ${}^{H^*}\pi$ and $H\pi$, and similarly for π^{H^*} and π^H . \square

Further, recall the *bar* and *cobar differentials*

$$\begin{aligned} d_{bar}(h_1 \dots h_n \otimes l_1 \dots l_m) &= \sum_{i=1}^{n-1} (-1)^i h_1 \dots h_{i-1} (h_i \cdot h_{i+1}) h_{i+2} \dots h_n \otimes l_1 \dots l_m, \\ d_{cob}(h_1 \dots h_n \otimes l_1 \dots l_m) &= \sum_{i=1}^{m-1} (-1)^i h_1 \dots h_n \otimes l_1 \dots l_{i-1} (l_i \cdot l_{i+1}) l_{i+2} \dots l_m \end{aligned}$$

on $T(H) \otimes T(H^*)$. Note that we use the evaluation-coevaluation duality in order to transform the degree 1 cobar differential on $\text{End}_{\mathbb{k}}(T(H))$ into a degree -1 differential on $T(H) \otimes T(H^*)$.

Putting everything together, one gets

Proposition 6.12. *For a finite-dimensional \mathbb{k} -bialgebra $(H, \mu, \nu, \Delta, \varepsilon)$, one has the following bidifferential structures on $T(H) \otimes T(H^*)$:*

1.	d_{bar}	$(-1)^n d_{cob}$
2.	$d_{bar} + (-1)^n \pi^H$	$(-1)^n d_{cob} + (-1)^n ({}^{H^*}\pi)$
3.	$d_{bar} + {}^H\pi$	$(-1)^n d_{cob} + (-1)^{n+m} \pi^{H^*}$
4.	$d_{bar} + (-1)^n \pi^H + {}^H\pi$	$(-1)^n d_{cob} + (-1)^n ({}^{H^*}\pi) + (-1)^{n+m} \pi^{H^*}$

Table 6: Bidifferential structures on $T(H) \otimes T(H^*)$

The signs $(-1)^n$ etc. here are those for the component $H^n \otimes (H^*)^m$ of $T(H) \otimes T(H^*)$.

Proof. We prove the assertion for each pair of morphisms separately, keeping the order from the statement.

1. It is well known that d_{bar} and d_{cob} are differentials (this also follows from their interpretation in terms of braided differentials, cf. Point [5](#) of Theorem 2). They affect different components of $T(H) \otimes T(H^*)$ ($T(H)$ and, respectively, $T(H^*)$), and thus commute. The sign $(-1)^n$ then assures the anticommutativity.
2. Return to the braided system \overline{H}_{bi} . One calculate the braided differentials:

$$\varepsilon^{H^*} d = (-1)^n d_{cob} + (-1)^n ({}^{H^*}\pi), \quad d^{\varepsilon H} = -(d_{bar} + (-1)^n \pi^H),$$

obtaining the desired bidifferential.

3. Dually, one gets a bidifferential $((-1)^m d_{bar} + (-1)^m ({}^H\pi), d_{cob} + (-1)^m \pi^{H^*})$. Observe that multiplying the first differential by $(-1)^m$ and the second one by $(-1)^n$, one still gets a bidifferential, coinciding with the desired one.
4. The last point follows from the three preceding ones thanks to the following elementary observation:

Lemma 6.13. *Take an abelian group $(S, +, 0, a \mapsto -a)$ endowed with an operation \cdot distributive with respect to $+$. Then, for any $a, b, c, d, e, f \in S$ such that*

$$(a + b) \cdot (d + e) = (a + c) \cdot (d + f) = a \cdot d = b \cdot f + c \cdot e = 0,$$

one has

$$(a + b + c) \cdot (d + e + f) = 0.$$

Proof.

$$(a + b + c) \cdot (d + e + f) = (a + b) \cdot (d + e) + (a + c) \cdot (d + f) - a \cdot d + (b \cdot f + c \cdot e).$$

□

Now take $S = \text{End}_R(T(H) \otimes T(H^*))$ with the usual addition and the operation $a \cdot b := a \circ b$ (for proving that the two morphisms from the fourth line of our table are differentials), or the operation $a \cdot b := a \circ b + b \circ a$ (for proving that the two morphisms anti-commute). The equalities of the type $b \cdot f + c \cdot e = 0$ follow from the pairwise anti-commutativity of $(-1)^n ({}^{H^*}\pi)$, $(-1)^{n+m} \pi^{H^*}$, $(-1)^n \pi^H$ and ${}^H\pi$ (which is a consequence of Lemma 6.11), and the remaining ones from the preceding points of the proposition. □

One recognizes in $d_{bar} + (-1)^n \pi^H + {}^H\pi$ the Hochschild differential of H with the (right) coefficients in the H -bimodule $T(H^*)$ (cf. the dual version of Lemma 6.10), and similarly for $(-1)^n d_{cob} + (-1)^n ({}^{H^*}\pi) + (-1)^{n+m} \pi^{H^*}$. Thus the last bidifferential from the proposition defines the **bialgebra homology of M. Gerstenhaber and S.D. Schack**; cf. [7] where it was first introduced, R. Taillefer's thesis [31] for detailed calculations and a comparison with other bialgebra homologies, and M. Mastnak and S. Witherspoon's paper [21] for explicit formulas and the passage from $\text{Hom}_{\mathbb{k}}(H^m, H^n)$ to $H^n \otimes (H^*)^m$.

Remark 6.14. We have identified multi-braided modules over \overline{H}_{bi} as Hopf modules. Thus the multi-braided homology of \overline{H}_{bi} with coefficients gives in fact a homology theory for a pair (right-right Hopf module, left-left Hopf module). Repeating the manipulations from the previous proposition in this setting, one gets the **deformation (co)homology of Hopf modules** defined by F.F. Panaite and D. Ştefan in [28].

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