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Titre en anglais: Forward-backward systems of stochastic differential equations generated by Bernstein diffusions.

Titre en français: Systèmes progressifs-rétrogrades d'équations différentielles stochastiques engendrés par des diffusions de Bernstein.

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Abstract. In this Note we present new results which bring about hitherto unknown relations between certain Bernstein diffusions wandering in bounded convex domains of Euclidean space on the one hand, and processes which typically occur in forward-backward systems of stochastic differential equations on the other hand. A key point in establishing such relations is the fact that the Bernstein diffusions we consider are actually reversible Itô diffusions.

Résumé. Dans cette Note nous présentons des résultats nouveaux mettant en évidence certaines relations jusqu'ici inconnues entre certaines diffusions de Bernstein évoluant dans des domaines convexes bornés de l'espace euclidien d'une part, et des processus apparaissant typiquement dans des systèmes progressifs-rétrogrades d'équations différentielles stochastiques d'autre part. Un point clé permettant d'établir de telles relations est que les diffusions de Bernstein que nous considérons sont en fait des diffusions d'Itô réversibles.

Version française abrégée

La numérotation, la notation et la terminologie que nous utilisons ici ainsi que les hypothèses (L), (V) et (IF) se réfèrent directement aux formules, à la notation, à la terminologie et aux hypothèses (L), (V), et (IF) de la version principale en anglais. Nous nous intéressons à mettre en évidence certaines relations existant entre les diffusions de Bernstein associées à une classe d'équations aux dérivées partielles paraboliques linéaires déterministes et certain processus apparaissant dans des systèmes progressifs-rétrogrades d'équations différentielles stochastiques de type Itô.

Théorème. *Supposons que les hypothèses (L), (V) and (IF) soient satisfaites, et que le champ vectoriel l soit conservatif dans D . Alors il existe une mesure de probabilité μ sur $\mathcal{B}(\bar{D}) \times \mathcal{B}(\bar{D})$, un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ et un processus stochastique $(A_\tau, B_\tau, C_\tau)_{\tau \in [0, T]}$ sur $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ à valeurs dans $\bar{D} \times \mathbb{R}^d \times \mathbb{R}^{d^2}$ tels que les propriétés suivantes soient valables:*

(a) *Le processus $(A_\tau, B_\tau, C_\tau)_{\tau \in [0, T]}$ est la solution unique en loi du système progressif-rétrograde dont les équations différentielles stochastiques sont de la forme (3) et (4), où f et g sont donnés par (10) et (11), respectivement.*

(b) *La densité de la distribution du vecteur aléatoire initial $\eta : \Omega \mapsto D$ est $x \mapsto \varphi(x)v_\psi(x, 0)$ quel que soit $x \in D$, tandis que $\kappa : \Omega \mapsto \mathbb{R}^d$ est de la forme*

$$\kappa = \nabla_x \ln \psi(A_T).$$

(c) *Le processus $A_{\tau \in [0, T]}$ est une diffusion de Bernstein réversible dans \bar{D} .*

La démonstration de ce résultat repose tout d'abord sur le fait établi en [18] suivant lequel il est possible d'associer aux équations paraboliques (5) et (6) un processus de Bernstein $Z_{\tau \in [0, T]}$ possédant les propriétés d'une diffusion d'Itô réversible dans D . Dans un premier temps nous remarquons que cette diffusion est solution faible de l'équation d'Itô progressive (13) dans laquelle apparaît le champ vectoriel de dérive (12). Dans un deuxième temps nous vérifions que le processus $c^*(Z_\tau, \tau)_{\tau \in [0, T]}$ est solution faible de l'équation d'Itô rétrograde (14). Enfin nous montrons que le processus $(Z_\tau, c^*(Z_\tau, \tau), \nabla c^*(Z_\tau, \tau))_{\tau \in [0, T]}$, où $\nabla c^*(Z_\tau, \tau)$ est défini par (15), est solution d'un système progressif-rétrograde au sens de la définition 2.1. L'identification de $(A_\tau, B_\tau, C_\tau)_{\tau \in [0, T]}$ au processus précédent conduit alors au résultat désiré. À notre connaissance le résultat énoncé est nouveau et basé sur la définition 1.2 qui tient compte du fait que les processus de Wiener y apparaissant ne sont pas donnés *a priori* mais sont déterminés *a posteriori* par les diffusions de Bernstein elles-mêmes. Nous obtenons également un résultat dual du résultat énoncé ci-dessus en considérant le processus $(Z_\tau, c(Z_\tau, \tau), \nabla c(Z_\tau, \tau))_{\tau \in [0, T]}$ où apparaît le champ vectoriel de dérive (16). Dans ce cas c'est le processus $Z_{\tau \in [0, T]}$ qui est solution faible de l'équation rétrograde (17), alors que $c(Z_\tau, \tau)_{\tau \in [0, T]}$ satisfait à une équation d'Itô progressive. Nous donnons les démonstrations détaillées de tous ces résultats en [5].

1 Statement of the main results

The theory of Bernstein processes goes back to [2] which elaborates on the seminal contribution that was set forth in the very last section of [17]. It was subsequently thoroughly developed in [11], and has ever since played an important rôle in probability theory and in various areas of mathematical physics as testified for instance by the works [4]-[7], [10], [16], [18], [19] and the many references therein. There are several equivalent ways to define a Bernstein process, but we settle here for a variant which is tailored to our needs. Let $D \subset \mathbb{R}^d$ be a bounded open convex subset whose smooth boundary ∂D is $\mathcal{C}^{2+\alpha}$ with $\alpha \in (0, 1)$; let us write $\overline{D} := D \cup \partial D$ and let $T \in (0, +\infty)$ be arbitrary.

Definition 1.1. We say the \overline{D} -valued process $Z_{\tau \in [0, T]}$ defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Bernstein process if the following conditional expectations satisfy the relation

$$\mathbb{E}(h(Z_r) | \mathcal{F}_s^+ \vee \mathcal{F}_t^-) = \mathbb{E}(h(Z_r) | Z_s, Z_t) \quad (1)$$

for every Borel measurable function $h : \overline{D} \rightarrow \mathbb{R}$, and for all r, s, t satisfying $r \in (s, t) \subset [0, T]$. In (1), \mathcal{F}_s^+ denotes the σ -algebra generated by the Z_τ 's for all $\tau \in [0, s]$, while \mathcal{F}_t^- is that generated by the Z_τ 's for all $\tau \in [t, T]$.

According to [11] Bernstein processes are neither Markovian nor reversible in general, but there are special classes of them which are reversible Itô diffusions as we shall see below. Since this conveys the idea that such diffusions can evolve forward and backward in time while satisfying Itô equations for some suitably constructed Wiener processes, they constitute the bare minimum we need to establish relations with certain forward-backward systems of stochastic differential equations. Following [1], [3] and [14] which were written in the context of stochastic optimal control, the theory of such systems has been considerably developed in recent times and has led to various applications in mathematical finance and partial differential equations (see e.g. [8], [9], [12], [13], [15] and their references). In this Note we adopt the standard definition for them which can be found for instance in [8] or [13], up to the essential difference that the Wiener processes which occur in our context are not *a priori* given but rather determined *a posteriori* by the Bernstein diffusions themselves. This leads to the following notion, where

$$f, g : \overline{D} \times \mathbb{R}^d \times [0, T] \mapsto \mathbb{R}^d \quad (2)$$

are continuous functions, and where we write $X = (X_1, \dots, X_d)$ for any vector $X \in \mathbb{R}^{d^2}$ with each $X_i \in \mathbb{R}^d$ and $|\cdot|$ for the Euclidean norm in \mathbb{R}^d or \mathbb{R}^{d^2} :

Definition 1.2. We say the $\overline{D} \times \mathbb{R}^d \times \mathbb{R}^{d^2}$ -valued process $(A_\tau, B_\tau, C_\tau)_{\tau \in [0, T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a solution to a forward-backward system of stochastic differential equations with initial condition η and final condition κ if the following four conditions hold:

(a) The process $W_{\tau \in [0, T]}$ defined by

$$W_\tau := A_\tau - A_0 - \int_0^\tau d\sigma f(A_\sigma, B_\sigma, \sigma)$$

is a Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ relative to its natural increasing filtration $\mathcal{F}_{\tau \in [0, T]}$, and $(A_\tau, B_\tau, C_\tau)_{\tau \in [0, T]}$ is progressively measurable with respect to $\mathcal{F}_{\tau \in [0, T]}$.

(b) We have

$$A_t = \eta + \int_0^t d\tau f(A_\tau, B_\tau, \tau) + W_t \quad (3)$$

\mathbb{P} -a.s. for every $t \in [0, T]$, where $\eta : \Omega \mapsto D$ is \mathcal{F}_0 -measurable and satisfies $\mathbb{E}|\eta|^2 < +\infty$.

(c) We have

$$B_{t,i} = \kappa_i - \int_t^T d\tau g_i(A_\tau, B_\tau, \tau) - \int_t^T (C_{\tau,i}, dW_\tau)_{\mathbb{R}^d} \quad (4)$$

\mathbb{P} -a.s. for every $t \in [0, T]$ and every $i \in \{1, \dots, d\}$, where $\kappa : \Omega \mapsto \mathbb{R}^d$ is \mathcal{F}_T -measurable and satisfies $\mathbb{E}|\kappa|^2 < +\infty$. In (4), $(\cdot, \cdot)_{\mathbb{R}^d}$ stands for the Euclidean inner product in \mathbb{R}^d and the second integral is the forward Itô integral defined with respect to $\mathcal{F}_{\tau \in [0, T]}$, which we assume to be well-defined.

(d) We have

$$\mathbb{E} \int_0^T d\tau (|A_\tau|^2 + |B_\tau|^2 + |C_\tau|^2) < +\infty.$$

In order to show that there exist special classes of Bernstein processes which generate forward-backward systems in the above sense, we now consider parabolic initial-boundary value problems of the form

$$\begin{aligned} \partial_t u(x, t) &= \frac{1}{2} \Delta_x u(x, t) - (l(x, t), \nabla_x u(x, t))_{\mathbb{R}^d} - V(x, t)u(x, t), \\ (x, t) &\in D \times (0, T], \\ u(x, 0) &= \varphi(x), \quad x \in D, \\ \frac{\partial u(x, t)}{\partial n(x)} &= 0, \quad (x, t) \in \partial D \times (0, T] \end{aligned} \quad (5)$$

along with the corresponding adjoint final-boundary value problems

$$\begin{aligned} -\partial_t v(x, t) &= \frac{1}{2} \Delta_x v(x, t) + \operatorname{div}_x (v(x, t)l(x, t)) - V(x, t)v(x, t), \\ (x, t) &\in D \times [0, T), \\ v(x, T) &= \psi(x), \quad x \in D, \\ \frac{\partial v(x, t)}{\partial n(x)} &= 0, \quad (x, t) \in \partial D \times [0, T). \end{aligned} \quad (6)$$

In the preceding relations $n(x)$ denotes the unit outer normal vector at $x \in \partial D$, l is an \mathbb{R}^d -valued vector-field and V, φ, ψ are real-valued functions which satisfy the following hypotheses, respectively:

(L) For $l : \overline{D} \times [0, T] \mapsto \mathbb{R}^d$ we have $l_i \in \mathcal{C}^{2+\alpha, \frac{\alpha}{2}}(\overline{D} \times [0, T])$ for every $i \in \{1, \dots, d\}$.

(V) The function $V : \overline{D} \times [0, T] \mapsto \mathbb{R}$ is such that $V \in \mathcal{C}^{1, \frac{\alpha}{2}}(\overline{D} \times [0, T])$.

(IF) We have $\varphi, \psi \in \mathcal{C}^{2+\alpha}(\overline{D})$ with $\varphi > 0$ and $\psi > 0$ satisfying Neumann's boundary condition, that is,

$$\frac{\partial \varphi(x)}{\partial n(x)} = \frac{\partial \psi(x)}{\partial n(x)} = 0, \quad x \in \partial D.$$

Under such conditions there exists a Bernstein process $Z_{\tau \in [0, T]}$ wandering in \overline{D} which turns out to be a reversible Itô diffusion according to the theory developed in [18], to which we refer the reader for details. What this means is that $Z_{\tau \in [0, T]}$ may be considered simultaneously either as a forward or as a backward Markov diffusion on some probability space $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$, where \mathbb{P}_μ is uniquely determined by the positive function

$$\mu(E \times F) := \int_{E \times F} dx dy \varphi(x) g(y, T; x, 0) \psi(y) \quad (7)$$

defined for all $E, F \in \mathcal{B}(\overline{D})$ with $\mathcal{B}(\overline{D})$ the Borel σ -algebra on \overline{D} . In the preceding expression g stands for the parabolic Green function associated with (5), which is well-defined and positive for all $x, y \in \overline{D}$. In addition, (7) must satisfy

$$\int_{D \times D} dx dy \varphi(x) g(y, T; x, 0) \psi(y) = 1.$$

Furthermore the drift of $Z_{\tau \in [0, T]}$ is

$$b^*(x, t) = l(x, t) + \nabla_x \ln v_\psi(x, t)$$

in the forward case, and

$$b(x, t) = l(x, t) - \nabla_x \ln u_\varphi(x, t)$$

in the backward case, where u_φ is the unique positive classical solution to (5) and v_ψ the unique classical positive solution to (6). Finally $Z_{\tau \in [0, T]}$ satisfies two Itô equations in the weak sense, namely, the forward equation

$$Z_t = Z_0 + \int_0^t d\tau b^*(Z_\tau, \tau) + W_t^* \quad (8)$$

and the backward equation

$$Z_t = Z_T - \int_t^T d\tau b(Z_\tau, \tau) - W_t \quad (9)$$

\mathbb{P}_μ -a.s. for every $t \in [0, T]$, for two suitably constructed d -dimensional Wiener processes $W_{\tau \in [0, T]}^*$ and $W_{\tau \in [0, T]}$.

It is precisely all these properties along with some more refinements that lead to the desired relations with forward-backward systems. For instance, let us choose the functions in (2) as

$$f(x, y, t) = l(x, t) + y \quad (10)$$

and

$$g_i(x, y, t) = \partial_{x_i}(V(x, t) - \operatorname{div}_x l(x, t)) - (y, \nabla_x l_i(x, t))_{\mathbb{R}^d} \quad (11)$$

for every $i \in \{1, \dots, d\}$. One of our typical results is then the following:

Theorem. *Let us assume that Hypotheses (L), (V) and (IF) hold, and that the vector-field l is conservative in D . Then, there exist a probability measure μ on $\mathcal{B}(\overline{D}) \times \mathcal{B}(\overline{D})$, a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$ and a $\overline{D} \times \mathbb{R}^d \times \mathbb{R}^{d^2}$ -valued process $(A_\tau, B_\tau, C_\tau)_{\tau \in [0, T]}$ on $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$, such that the following properties hold:*

(a) *The process $(A_\tau, B_\tau, C_\tau)_{\tau \in [0, T]}$ is the unique solution to the forward-backward system whose stochastic differential equations are of the form (3) and (4) with f and g given by (10) and (11), respectively. The uniqueness is meant as uniqueness in law.*

(b) *The initial random vector $\eta : \Omega \mapsto D$ has the distribution density $x \mapsto \varphi(x)v_\psi(x, 0)$ for all $x \in D$, while $\kappa : \Omega \mapsto \mathbb{R}^d$ is of the form*

$$\kappa = \nabla_x \ln \psi(A_T).$$

(c) *The process $A_{\tau \in [0, T]}$ is a reversible Bernstein diffusion in \overline{D} .*

2 A brief sketch of the proof

We first consider the translated forward drift

$$c^*(x, t) := b^*(x, t) - l(x, t) = \nabla_x \ln v_\psi(x, t) \quad (12)$$

for all $(x, t) \in \overline{D} \times [0, T]$, and then rewrite (8) as

$$Z_t = Z_0 + \int_0^t d\tau (c^*(Z_\tau, \tau) + l(Z_\tau, \tau)) + W_t^*. \quad (13)$$

We can also verify that the process $c^*(Z_\tau, \tau)_{\tau \in [0, T]}$ satisfies the backward Itô equation

$$\begin{aligned} & c_i^*(Z_t, t) \\ &= c_i^*(Z_T, T) - \int_t^T d\tau (\partial_{x_i}(V(Z_\tau, \tau) - \operatorname{div}_x l(Z_\tau, \tau)) - (c^*(Z_\tau, \tau), \nabla_x l_i(Z_\tau, \tau))_{\mathbb{R}^d}) \\ & - \int_t^T (\nabla_x c_i^*(Z_\tau, \tau), d^+ W_\tau^*)_{\mathbb{R}^d} \end{aligned} \quad (14)$$

\mathbb{P}_μ -a.s. for every $t \in [0, T]$ and every $i \in \{1, \dots, d\}$, where the second integral in (14) is the forward Itô integral defined with respect to $W_{\tau \in [0, T]}^*$ and its natural increasing filtration $\mathcal{F}_{\tau \in [0, T]}^+$. It is then sufficient to choose $(A_\tau, B_\tau, C_\tau)_{\tau \in [0, T]} = (Z_\tau, c^*(Z_\tau, \tau), \nabla c^*(Z_\tau, \tau))_{\tau \in [0, T]}$, where we have written

$$\nabla c^*(Z_\tau, \tau)_{\tau \in [0, T]} := (\nabla_x c_1^*(Z_\tau, \tau), \dots, \nabla_x c_d^*(Z_\tau, \tau))_{\tau \in [0, T]}. \quad (15)$$

This choice shows indeed that we can identify (3) and (4) with (13) and (14), respectively, if we take (10) and (11) into account. The remaining statements follow from simple considerations.

We can obtain so to speak a dual result when we consider the process $(A_\tau, B_\tau, C_\tau)_{\tau \in [0, T]} = (Z_\tau, c(Z_\tau, \tau), \nabla c(Z_\tau, \tau))_{\tau \in [0, T]}$, where

$$c(x, t) := b(x, t) - l(x, t) = -\nabla_x \ln u_\varphi(x, t) \quad (16)$$

is the translated backward drift associated with $Z_{\tau \in [0, T]}$, which we consider this time as a backward Itô diffusion satisfying (9) or, equivalently,

$$Z_t = Z_T - \int_t^T d\tau (c(Z_\tau, \tau) + l(Z_\tau, \tau)) - W_t. \quad (17)$$

Indeed, in this case it is the process $c(Z_\tau, \tau)_{\tau \in [0, T]}$ that satisfies a forward Itô equation. We refer the reader to the forthcoming typescript [5] for details and complete proofs.

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