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THE OPEN STRING MCKAY CORRESPONDENCE FOR TYPE A SINGULARITIES

ANDREA BRINI, RENZO CAVALIERI, AND DUSTIN ROSS

ABSTRACT. We formulate a Crepant Resolution Correspondence for open Gromov–Witten invariants (OCRC) of toric Calabi–Yau orbifolds by viewing the open theories as sections of Givental’s symplectic vector space and the correspondence as a linear map of Givental spaces which identifies them. We deduce a Bryan–Graber-type statement for disk invariants and extend it to arbitrary genus zero topologies in the Hard Lefschetz case. The equivariant version of Iritani’s theory of integral structures leads us to conjecture a general form of the symplectomorphism entering the OCRC, which should arise from a geometric correspondence at the equivariant K -theory level. We give a complete proof of this in the case of minimal resolutions of threefold A_n -singularities. Our methods rely on a new description of the equivariant quantum D -modules underlying the Gromov–Witten theory of this class of targets.

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1. INTRODUCTION

1.1. Summary of Results. This paper proposes an approach to the Crepant Resolution Conjecture for open Gromov–Witten invariants, and supports it with a series of results and verifications about threefold A_n -singularities and their resolutions.

Let \mathcal{Z} be a smooth toric Calabi–Yau Deligne–Mumford stack with generically trivial stabilizers and let L be an Aganagic–Vafa brane (Sec. 2.2). Fix a Calabi–Yau torus action T on \mathcal{Z} and denote by $\Delta_{\mathcal{Z}}$ the free module over $H^{\bullet}(BT)$ spanned by the T -equivariant lifts of orbifold cohomology classes of Chen–Ruan degree at most two. We define (Sec. 3.1) a family of elements of Givental space,

$$\widehat{\mathcal{F}}_{L,\mathcal{Z}}^{\text{disk}} : H_T^{\bullet}(\mathcal{Z}) \rightarrow \mathcal{H}_{\mathcal{Z}} = H_T^{\bullet}(\mathcal{Z})((z^{-1})), \quad (1)$$

which we call the *winding neutral disk potential*. Upon appropriate specializations of the variable z , $\widehat{\mathcal{F}}_{L,\mathcal{Z}}^{\text{disk}}$ encodes disk invariants of (\mathcal{Z}, L) at any winding d .

Consider a *crepant resolution diagram* $\mathcal{X} \rightarrow X \leftarrow Y$, where X is the coarse moduli space of \mathcal{X} and Y is a crepant resolution of the singularities of X . A Lagrangian boundary condition L is chosen on \mathcal{X} and we denote by L' its transform in Y . Our version of the open crepant resolution conjecture is a comparison of the (restricted) winding neutral disk potentials.

Proposal (The OCRC). *There exists a $\mathbb{C}((z^{-1}))$ -linear map of Givental spaces $\mathbb{O} : \mathcal{H}_{\mathcal{X}} \rightarrow \mathcal{H}_Y$ and analytic functions $\mathfrak{h}_{\mathcal{X}} : \Delta_{\mathcal{X}} \rightarrow \mathbb{C}$, $\mathfrak{h}_Y : \Delta_Y \rightarrow \mathbb{C}$ such that*

$$\mathfrak{h}_Y^{1/z} \widehat{\mathcal{F}}_{L',Y}^{\text{disk}}|_{\Delta_Y} = \mathfrak{h}_{\mathcal{X}}^{1/z} \mathbb{O} \circ \widehat{\mathcal{F}}_{L,\mathcal{X}}^{\text{disk}}|_{\Delta_{\mathcal{X}}} \quad (2)$$

upon analytic continuation of quantum cohomology parameters.

Further, we conjecture (Conjecture 3.4) that both \mathbb{O} and \mathfrak{h}_{\bullet} are completely determined by the classical toric geometry of \mathcal{X} and Y . In particular, we give a prediction for the transformation \mathbb{O} depending on a choice of identification of the K -theory lattices of \mathcal{X} and Y .

When \mathcal{X} is a Hard Lefschetz Calabi–Yau orbifold, the OCRC extends to functions on all of $H_T^{\bullet}(\mathcal{Z})$. Together with WDVV, this gives a Bryan–Graber-type statement for potentials encoding invariants from genus 0 maps with an arbitrary number of boundary components:

Theorem 3.8. *Let $\mathcal{X} \rightarrow X \leftarrow Y$ be a Hard Lefschetz diagram for which the OCRC holds. Defining $\mathbb{O}^{\otimes n} = \mathbb{O}(z_1) \otimes \dots \otimes \mathbb{O}(z_n)$, we have:*

$$\widehat{\mathcal{F}}_{L',Y}^n = \mathbb{O}^{\otimes n} \circ \widehat{\mathcal{F}}_{L,\mathcal{X}}^n, \quad (3)$$

where $\widehat{\mathcal{F}}^n$ is the n -boundary components analog of $\widehat{\mathcal{F}}^{\text{disk}}$ defined in (86).

Consider now the family of threefold A_n singularities, where $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{C}$ and Y is its canonical minimal resolution.

Main Theorem. *The OCRC and Conjecture 3.4 hold for the A_n -singularities for any choice of Aganagic–Vafa brane on \mathcal{X} .*

The main theorem is an immediate consequence of Proposition 3.5 and Theorem 4.1. From it we deduce a series of comparisons of generating functions in the spirit of Bryan–Graber’s formulation of the CRC.

In (82) we define the *cohomological disk potential* $\mathcal{F}_L^{\text{disk}}$ - a cohomology valued generating function for disk invariants that “remembers” the twisting and the attaching fixed point of an orbi-disk map. We also consider the coarser *scalar disk potential* (see (51)), which keeps track of the winding of the orbimaps but forgets the twisting and attaching point. There are essentially two different choices for the Lagrangian boundary condition on \mathcal{X} ; the simpler case occurs when L intersects one of the effective legs of the orbifold. In this case we have the following result.

Theorem 4.5. *Identifying identically the winding parameters and setting $\mathbb{O}_{\mathbb{Z}}(\mathbf{1}_k) = P_{n+1}$ for every k , we have:*

$$\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}) = \mathbb{O}_{\mathbb{Z}} \circ \mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}). \quad (4)$$

It is immediate to observe that the scalar disk potentials coincide (Corollary 4.6).

The case when L intersects the ineffective leg of the orbifold is more subtle.

Theorem 4.3. *We exhibit a matrix $\mathbb{O}_{\mathbb{Z}}$ of roots of unity and a specialization of the winding parameters depending on the equivariant weights such that*

$$\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}) = \mathbb{O}_{\mathbb{Z}} \circ \mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}). \quad (5)$$

The comparison of scalar potentials in this case does not hold anymore. Because of the special form of the matrix $\mathbb{O}_{\mathbb{Z}}$ we deduce in Corollary 4.4 that the scalar disk potential for Y corresponds to the contribution to the potential for \mathcal{X} by the untwisted disk maps. As the A_n -singularities satisfy the Hard Lefschetz condition, it is an exercise in book-keeping to extend the statements of Theorems 4.3 and 4.5 to compare generating functions for arbitrary genus zero open invariants, even treating all boundary Lagrangian conditions at the same time.

In order to prove our main theorem, we must establish a fully equivariant version of the symplectomorphism of Givental spaces which verifies the closed CRC for the A_n geometries. Our analysis is centered on a new global description of the gravitational quantum cohomology of these targets which enjoys a number of remarkable features, and may have an independent interest *per se*.

Theorem 5.6. *By identifying the A -model moduli space with a genus zero double Hurwitz space, we construct a global quantum D -module $(\mathcal{F}_{\lambda,\phi}, T\mathcal{F}_{\lambda,\phi}, \nabla^{(g,z)}, H(\cdot, \cdot)_g)$ which is locally isomorphic to $\text{QDM}(\mathcal{X})$ and $\text{QDM}(Y)$ in appropriate neighborhoods of the orbifold and large complex structure points.*

1.2. Context, Motivation and Further Discussion. Open Gromov–Witten (GW) theory intends to study holomorphic maps from bordered Riemann surfaces, where the image of the boundary is constrained to lie in a Lagrangian submanifold of the target. While some general foundational work has been done [55, 63], at this point most of the results in the theory rely on additional structure. In [21, 22] Lagrangian Floer theory is employed to study the case when the boundary condition is a fiber of the moment map. In the toric context, a mathematical approach [14, 31, 47, 58] to construct operatively a virtual counting theory of open maps is via the use of localization. A variety of striking relations have been verified connecting open GW theory and several other types of invariants, including open B -model invariants and matrix models [3, 4, 8, 37, 50], quantum knot invariants [41, 52], and ordinary Gromov–Witten and Donaldson–Thomas theory via “gluing along the boundary” [2, 51, 53].

Since Ruan’s influential conjecture [59], an intensely studied problem in Gromov–Witten theory has been to determine the relation between GW invariants of target spaces related by a crepant birational transformation (CRC). The most general formulation of the CRC is framed in terms of Givental formalism ([28], [29, Conj 4.1]); the conjecture has been proved in a number of examples [24, 26, 28] and has by now gained folklore status, with a general proof in the toric setting announced for some time [25]. A natural question one can ask is whether similar relations exist in the context of open Gromov–Witten theory. Within the toric realm, physics arguments based on open mirror symmetry [8, 9, 16] have given strong indications that some version of the Bryan–Graber [18] statement of the crepant resolution conjecture should hold at the level of disk invariants. This was proven explicitly for the crepant resolution of the Calabi–Yau orbifold $[\mathbb{C}^3/\mathbb{Z}_2]$ in [20]. Around the same time, it was suggested [10, 11] that a general statement of a Crepant Resolution Conjecture for open invariants should have a natural formulation within Givental’s formalism, as in [26, 29]. Some implications of this philosophy were verified in [11] for the crepant resolution $\mathcal{O}_{\mathbb{P}^2}(-3)$ of the orbifold $[\mathbb{C}^3/\mathbb{Z}_3]$.

The OCRC we propose here is a natural extension to open Gromov–Witten theory of the Coates–Corti–Iritani–Tseng approach [28] to Ruan’s conjecture. The observation that the disk

function of [14, 58] can be interpreted as an endomorphism of Givental space makes the OCRC statement follow almost tautologically from the Coates–Corti–Iritani–Tseng/Ruan picture of the ordinary CRC via toric mirror symmetry [26]. The more striking aspect of our conjecture is then that the linear function \mathbb{O} comparing the winding neutral disk potentials is considerably simpler than the symplectomorphism $\mathbb{U}_\rho^{\mathcal{X}, Y}$ in the closed CRC and it is characterized in terms of *purely classical data*: essentially, the equivariant Chern characters of \mathcal{X} and Y . This is closely related to Iritani’s proposal [45] that the analytic continuation for the flat sections of the global quantum D -module is realized via the composition of K -theoretic central charges; our disk endomorphisms are very close to just being inverses to the Γ factors appearing in Iritani’s central charges and therefore “undo” most of the transcendental nature of $\mathbb{U}_\rho^{\mathcal{X}, Y}$.

Iritani’s proposal is inspired and consistent with the idea of global mirror symmetry, i.e. that there should be a global quantum D -module on the A -model moduli space which locally agrees with the Frobenius structure given by quantum cohomology. In order to verify Iritani’s proposal in the fully equivariant setting, we construct explicitly such a global structure. Motivated by the connection of the Gromov–Witten theory of A_n to certain integrable systems [12], we realize the Dubrovin local system as a system of one-dimensional hypergeometric periods. As a special feature of this case, structure constants of quantum cohomology are rational in exponentiated flat coordinates (or, equivalently, the inverse mirror map is a rational function of the B -model variables). Moreover, the n -dimensional oscillating integrals describing the periods of the system reduce to Euler–Pochhammer line integrals in the complex plane. As a consequence, the computation of the analytic continuation of flat sections is drastically simplified with respect to the standard toric mirror symmetry methods. Furthermore, in this context integral structures in equivariant cohomology emerge naturally from the interpretation of flat sections of the Dubrovin connection as twisted period maps. The Deligne–Mostow monodromy of hypergeometric periods translates then to an action of the colored braid group in equivariant K -theory. An enticing speculation is that, upon mirror symmetry, this may correspond to autoequivalences of $D_T^b(Y)$ and surject to the Seidel–Thomas braid group action [62] in the non-equivariant limit.

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2. BACKGROUND

This section gathers background for the formulation of the open string Crepant Resolution Conjecture of Section 3 and its proof in Section 5. We give a self-contained account of the quantum D -module/Givental space approach to the study of the closed string Crepant Resolution Conjecture in genus zero along the lines of Coates–Corti–Iritani–Tseng [26] and Iritani [45] (Section 2.1). Section 2.2 provides an overview of open Gromov–Witten theory for toric Calabi–Yau threefolds à la Katz–Liu as well as its extension to toric orbifolds. Section 2.3 collects relevant material on the classical and quantum geometry of A_n -resolutions.

The content of Section 2.1 is surveyed in Iritani’s excellent review article [46], to which the reader is referred for further details. For a more comprehensive introduction to the open Gromov–Witten theory for toric orbifolds, see e.g. [14, 58].

2.1. Quantum D -modules and the Crepant Resolution Conjecture. Let \mathcal{Z} be a smooth Deligne–Mumford stack with coarse moduli space Z and suppose that \mathcal{Z} carries an algebraic $T \simeq \mathbb{C}^*$ action with zero-dimensional fixed loci. Write $I\mathcal{Z}$ for the inertia stack of \mathcal{Z} , $\text{inv} : I\mathcal{Z} \rightarrow I\mathcal{Z}$ for its canonical involution and $i : I\mathcal{Z}^T \hookrightarrow I\mathcal{Z}$ for the inclusion of the T -fixed loci into $I\mathcal{Z}$. The equivariant Chen–Ruan cohomology ring $H(\mathcal{Z}) \triangleq H_{T,CR}^\bullet(\mathcal{Z})$ of \mathcal{Z} is a finite rank free module over the T -equivariant cohomology of a point $H_T(\text{pt}) \simeq \mathbb{C}[\nu]$, where $\nu = c_1(\mathcal{O}_{BT}(1))$; we define $N_{\mathcal{Z}} \triangleq \text{rank}_{\mathbb{C}[\nu]} H(\mathcal{Z})$ and will denote by $\Delta_{\mathcal{Z}}$ the free module over $\mathbb{C}[\nu]$ spanned by the T -equivariant lifts of Chen–Ruan cohomology classes having age-shifted degree at most two. We furthermore suppose that odd cohomology groups vanish in all degrees.

The T -action on \mathcal{Z} gives a non-degenerate inner product on $H(\mathcal{Z})$ via the equivariant orbifold Poincaré pairing

$$\eta(\theta_1, \theta_2)_{\mathcal{Z}} \triangleq \int_{I\mathcal{Z}^T} \frac{i^*(\theta_1 \cup \text{inv}^*\theta_2)}{e(N_{I\mathcal{Z}^T/I\mathcal{Z}})}, \quad (6)$$

and it induces a torus action on the moduli space $\overline{\mathcal{M}}_{g,n}(\mathcal{Z}, \beta)$ of degree β twisted stable maps [1, 23] from genus g orbicurves to \mathcal{Z} . For classes $\theta_1, \dots, \theta_n \in H(\mathcal{Z})$ and integers $r_1, \dots, r_n \in \mathbb{N}$, the Gromov–Witten invariants of \mathcal{Z}

$$\langle \sigma_{r_1}(\theta_1) \dots \sigma_{r_n}(\theta_n) \rangle_{g,n,\beta}^{\mathcal{Z}} \triangleq \int_{[\overline{\mathcal{M}}_{g,n}(\mathcal{Z}, \beta)]_T^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^* \theta_i \psi_i^{r_i}, \quad (7)$$

$$\langle \theta_1 \dots \theta_n \rangle_{g,n,\beta}^{\mathcal{Z}} \triangleq \langle \sigma_0(\theta_1) \dots \sigma_0(\theta_n) \rangle_{g,n,\beta}^{\mathcal{Z}}, \quad (8)$$

define a sequence of multi-linear functions on $H(\mathcal{Z})$ with values in the field of fractions $\mathbb{C}(\nu)$ of $H_T(\text{pt})$. The correlators (8) (respectively, (7) with $r_i > 0$) are the *primary* (respectively, *descendent*) Gromov–Witten invariants of \mathcal{Z} .

Fix a basis $\{\phi_i\}_{i=0}^{N_{\mathcal{Z}}-1}$ of $H(\mathcal{Z})$ such that $\phi_0 = \mathbf{1}_{\mathcal{Z}}$ and ϕ_j , $1 \leq j \leq b_2(\mathcal{Z})$ are untwisted Poincaré duals of T -equivariant divisors in \mathcal{Z} . Denote by $\{\phi^i\}_{i=0}^{N_{\mathcal{Z}}-1}$ the dual basis with respect to the pairing (6). Let $\tau = \sum \tau_i \phi_i$ denote a general point of $H(\mathcal{Z})$. The WDVV equation for primary Gromov–Witten invariants (8) defines a family of associative deformations \circ_{τ} of the T -equivariant Chen–Ruan cohomology ring of \mathcal{Z} via

$$\eta(\theta_1 \circ_{\tau} \theta_2, \theta_3)_{\mathcal{Z}} \triangleq \langle \langle \theta_1, \theta_2, \theta_3 \rangle \rangle_{0,3}^{\mathcal{Z}}(\tau) \quad (9)$$

where

$$\langle \langle \theta_1, \dots, \theta_k \rangle \rangle_{0,k}^{\mathcal{Z}}(\tau) \triangleq \sum_{\beta} \sum_{n \geq 0} \frac{\langle \theta_1, \dots, \theta_k, \overbrace{\tau, \tau, \dots, \tau}^{n \text{ times}} \rangle_{0,n+k,\beta}^{\mathcal{Z}}}{n!} \in \mathbb{C}((\nu)), \quad (10)$$

and the index β ranges over the cone of effective curve classes $\text{Eff}(\mathcal{Z}) \subset H_2(\mathcal{Z}, \mathbb{Q})$; we denote by $l_{\mathcal{Z}} \triangleq b_2(\mathcal{Z})$ its dimension.

By the Divisor Axiom [1] this can be rewritten as

$$\eta(\theta_1 \circ_{\tau} \theta_2, \theta_3)_{\mathcal{Z}} = \sum_{\beta \in \text{Eff}(\mathcal{Z}), n \geq 0} \frac{\langle \theta_1, \theta_2, \theta_3, \overbrace{\tau', \tau', \dots, \tau'}^{n \text{ times}} \rangle_{0,n+3,\beta}^{\mathcal{Z}}}{n!} e^{\tau_{0,2} \cdot \beta} \quad (11)$$

where we have decomposed $\tau = \sum_{i=0}^{N_{\mathcal{Z}}-1} \tau_i \phi_i = \tau_{0,2} + \tau'$ as

$$\tau_{0,2} = \sum_{i=1}^{l_{\mathcal{Z}}} \tau_i \phi_i, \quad (12)$$

$$\tau' = \tau_0 \mathbf{1}_{\mathcal{Z}} + \sum_{i=l_{\mathcal{Z}}+1}^{N_{\mathcal{Z}}-1} \tau_i \phi_i. \quad (13)$$

The quantum product (11) is a formal Taylor series in $(\tau', e^{\tau_{0,2}})$. Suppose that it is actually *convergent* in a contractible open set $U \ni (0, 0)$; this is the case for many toric orbifolds [27, 40] and, as we see explicitly, for all the examples of Section 2.3. Then the quantum product \circ_{τ} is an analytic deformation of the Chen–Ruan cup product \cup_{CR} , to which it reduces in the limit $\tau' \rightarrow 0$, $\Re(\tau_{0,2}) \rightarrow -\infty$. Thus, the holomorphic family of rings $H(\mathcal{Z}) \times U \rightarrow U$, together with the inner pairing (6) and the associative product (11), gives U the structure of a (non-conformal) Frobenius manifold $QH(\mathcal{Z}) \triangleq (U, \eta, \circ_{\tau})$ [34]; this is the *quantum cohomology ring* of \mathcal{Z} . We refer to the Chen–Ruan limit $\tau' \rightarrow 0$, $\Re(\tau_{0,2}) \rightarrow -\infty$ as the *large radius limit point* of \mathcal{Z} .

Assigning a Frobenius structure on U is tantamount to endowing the trivial cohomology bundle $TU \simeq H(\mathcal{Z}) \times U \rightarrow U$ with a flat pencil of affine connections [34, Lecture 6]. Denote by $\nabla^{(\eta)}$ the Levi–Civita connection associated to the Poincaré pairing on $H(\mathcal{Z})$; in Cartesian coordinates for $U \subset H(\mathcal{Z})$ this reduces to the ordinary de Rham differential $\nabla^{(\eta)} = d$. Consider then the one parameter family of covariant derivatives on TU

$$\nabla_X^{(\eta,z)} \triangleq \nabla_X^{(\eta)} + z^{-1} X \circ_{\tau}. \quad (14)$$

The fact that the quantum product is commutative, associative and integrable implies that $R_{\nabla^{(\eta,z)}} = T_{\nabla^{(\eta,z)}} = 0$ identically in z ; this is equivalent to the WDVV equations for the genus zero Gromov–Witten potential. The equation for the horizontal sections of $\nabla^{(\eta,z)}$,

$$\nabla^{(\eta,z)} \omega = 0, \quad (15)$$

is a rank- $N_{\mathcal{Z}}$ holonomic system of coupled linear PDEs. We denote by $\mathcal{S}_{\mathcal{Z}}$ the vector space of solutions of (15): a $\mathbb{C}((z))$ -basis of $\mathcal{S}_{\mathcal{Z}}$ is by definition given by the gradient of a flat frame $\tilde{\tau}(\tau, z)$ for the deformed connection $\nabla^{(\eta,z)}$. The Poincaré pairing induces a non-degenerate inner product $H(s_1, s_2)_{\mathcal{Z}}$ on $\mathcal{S}_{\mathcal{Z}}$ via

$$H(s_1, s_2)_{\mathcal{Z}} \triangleq \eta(s_1(\tau, -z), s_2(\tau, z))_{\mathcal{Z}}. \quad (16)$$

The triple $\text{QDM}(\mathcal{Z}) \triangleq (U, \nabla^{(\eta,z)}, H(\cdot, \cdot)_{\mathcal{Z}})$ defines a *quantum D-module* structure on U , and the system (15) is the *quantum differential equation* (in short, QDE) of \mathcal{Z} .

Remark 2.1. Notice that the assumption that the quantum product (11) is analytic in $(\tau', e^{\tau_{0,2}})$ around the large radius limit point translates into the statement that the QDE (15) has a Fuchsian singularity along $\cup_{i=1}^{l_{\mathcal{Z}}} \{q_i \triangleq e^{\tau_i} = 0\}$.

In the same way in which the genus zero primary theory of \mathcal{Z} defines a quantum D -module structure on $H(\mathcal{Z}) \times U$, the genus zero gravitational invariants (7) furnish a basis of horizontal sections of $\nabla^{(\eta,z)}$ [39]. For every $\theta \in H(\mathcal{Z})$, a flat section of the D -module is given by an $\text{End}(H(\mathcal{Z}))$ -valued function $S_{\mathcal{Z}}(\tau, z) : H(\mathcal{Z}) \rightarrow \mathcal{S}_{\mathcal{Z}}$ defined as

$$S_{\mathcal{Z}}(\tau, z)\theta \triangleq \theta - \sum_{k=0}^{N_{\mathcal{Z}}-1} \phi^k \left\langle \left\langle \phi_k, \frac{\theta}{z + \psi} \right\rangle \right\rangle_{0,2}^{\mathcal{Z}}(\tau) \quad (17)$$

where ψ is a cotangent line class and we expand the denominator as a geometric series $\frac{1}{z + \psi} = \frac{1}{z} \sum \left(-\frac{\psi}{z}\right)^k$. We call the pair $(\text{QDM}(\mathcal{Z}), S_{\mathcal{Z}})$ a *calibration* of the Frobenius structure $(H(\mathcal{Z}), \circ_{\tau}, \eta)$.

The flows of coordinate vectors for the flat frame of $TH(\mathcal{Z})$ induced by $S_{\mathcal{Z}}(\tau, z)$ give a basis of deformed flat coordinates of $\nabla^{(n, z)}$, which is defined uniquely up to an additive z -dependent constant. A canonical basis is obtained upon applying the String Axiom: define the J -function $J^{\mathcal{Z}}(\tau, z) : U \times \mathbb{C} \rightarrow H(\mathcal{Z})$ by

$$J^{\mathcal{Z}}(\tau, z) \triangleq z S_{\mathcal{Z}}(\tau, -z)^* \mathbf{1}_{\mathcal{Z}} \quad (18)$$

where $S_{\mathcal{Z}}(\tau, z)^*$ denotes the adjoint to $S_{\mathcal{Z}}(\tau, z)$ under $H(-, -)_{\mathcal{Z}}$. Explicitly,

$$J^{\mathcal{Z}}(\tau, z) = (z + \tau_0) \mathbf{1}_{\mathcal{Z}} + \tau_1 \phi_1 + \dots + \tau_{N_{\mathcal{Z}}} \phi_{N_{\mathcal{Z}}} + \sum_{k=0}^{N_{\mathcal{Z}}-1} \phi^k \left\langle \left\langle \frac{\phi_k}{z - \psi_{n+1}} \right\rangle \right\rangle_{0,1}^{\mathcal{Z}}(\tau). \quad (19)$$

Components of $J^{\mathcal{Z}}(\tau, z)$ in the ϕ -basis give flat coordinates of (14); this is a consequence of (18) combined with the String Equation. From (19), the undeformed flat coordinate system is obtained in the limit $z \rightarrow \infty$ as

$$\lim_{z \rightarrow \infty} (J^{\mathcal{Z}}(\tau, z) - z \mathbf{1}_{\mathcal{Z}}) = \tau. \quad (20)$$

By Remark 2.1, a loop around the origin in the variables $q_i = e^{\tau_i}$ gives a non-trivial monodromy action on the J -function. Setting $\tau' = 0$ in (19) and applying the Divisor Axiom then gives [30, Proposition 10.2.3]

$$\begin{aligned} J^{\mathcal{Z}, \text{small}}(\tau_{0,2}, z) &\triangleq J^{\mathcal{Z}}(\tau, z) \Big|_{\tau'=0} \\ &= z e^{\tau_1 \phi_1 / z} \dots e^{\tau_{l_{\mathcal{Z}}} \phi_{l_{\mathcal{Z}}} / z} \left(\mathbf{1}_{\mathcal{Z}} + \sum_{\beta, k} e^{\tau_1 \beta_1} \dots e^{\tau_{l_{\mathcal{Z}}} \beta_{l_{\mathcal{Z}}}} \phi^k \left\langle \frac{\phi_k}{z(z - \psi_1)} \right\rangle_{0,1,\beta}^{\mathcal{Z}} \right). \end{aligned} \quad (21)$$

In our situation where the T -action has only zero-dimensional fixed loci $\{P_i\}_{i=1}^{N_{\mathcal{Z}}}$, write

$$\phi_i \rightarrow \sum_{j=1}^{N_{\mathcal{Z}}} c_{ij}(\nu) P_j, \quad i = 1, \dots, l_{\mathcal{Z}}, \quad (22)$$

for the image of $\{\phi_i \in H^2(\mathcal{Z}, \mathbb{C})\}_{i=1}^{l_{\mathcal{Z}}}$ under the Atiyah–Bott isomorphism. The image of each ϕ_i is concentrated on the fixed point cohomology classes with trivial isotropy which are idempotents of the classical Chen–Ruan cup product on $H(\mathcal{Z})$. Therefore, the components of the J -function in the fixed points basis

$$J^{\mathcal{Z}, \text{small}}(\tau_{0,2}, z) =: \sum_{j=1}^{N_{\mathcal{Z}}} J_j^{\mathcal{Z}, \text{small}}(\tau_{0,2}, z) P_j \quad (23)$$

satisfy

$$J_j^{\mathcal{Z}, \text{small}}(\tau_{0,2}, z) = z e^{\sum_{i=1}^{l_{\mathcal{Z}}} \tau_i c_{ij} / z} (1 + \mathcal{O}(e^{\tau_{0,2}})) \quad (24)$$

where the $\mathcal{O}(e^{\tau_{0,2}})$ term on the right hand side is an analytic power series around $e^{\tau_{0,2}} = 0$ by (21) and the assumption of convergence of the quantum product. The localized basis $\{P_j\}_{j=1}^{N_{\mathcal{Z}}}$ therefore diagonalizes the monodromy around large radius: by (24), each $J_j^{\mathcal{Z}, \text{small}}(\tau_{0,2}, z)$ is an eigenvector of the monodromy around a loop in the q_i -plane encircling the large radius limit of \mathcal{Z} with eigenvalue $e^{2\pi i c_{ij} / z}$.

2.1.1. *Global mirror symmetry and the closed CRC.* Consider a toric Gorenstein orbifold \mathcal{X} , and let $X \leftarrow Y$ be a crepant resolution of its coarse moduli space. Ruan’s Crepant Resolution Conjecture can be phrased as the existence of a *global quantum D-module* underlying the quantum differential systems of \mathcal{X} and Y . This is a 4-tuple $(\mathcal{M}_A, F, \nabla, H(\cdot, \cdot)_F)$ with

- \mathcal{M}_A a complex quasi-projective variety
- $F \rightarrow \mathcal{M}_A$ a rank- $N_{\mathcal{Z}}$ holomorphic vector bundle on \mathcal{M}_A ;
- ∇ a flat $\mathcal{O}_{\mathcal{M}_A}$ -connection on F ;
- $H(\cdot, \cdot)_F \in \text{End}(F)$ a non-degenerate ∇ -flat inner product.

In the quantum D -module picture, the Crepant Resolution Conjecture states that there exist open subsets $V_{\mathcal{X}}, V_Y \subset \mathcal{M}_A$ and functions $\mathfrak{h}_{\mathcal{X}}, \mathfrak{h}_Y \in \mathcal{O}_{\mathcal{M}_A}$ such that the global D -module $(\mathcal{M}_A, F, \nabla, H(\cdot)_F)$ is locally isomorphic to $\text{QDM}(\mathcal{X})$ and $\text{QDM}(Y)$:

$$(\mathcal{M}_A, F, \nabla \circ \mathfrak{h}_{\mathcal{X}}^{1/z}, H(\cdot)_F)|_{V_{\mathcal{X}}} \simeq \text{QDM}(\mathcal{X}), \quad (25)$$

$$(\mathcal{M}_A, F, \nabla \circ \mathfrak{h}_Y^{1/z}, H(\cdot)_F)|_{V_Y} \simeq \text{QDM}(Y). \quad (26)$$

Notice that the Dubrovin connections on $TH(\mathcal{X})$ and $TH(Y)$ correspond to different trivialization of the global flat system ∇ when $\mathfrak{h}_{\mathcal{X}} \neq \mathfrak{h}_Y$. Any 1-chain ρ in \mathcal{M}_A gives an analytic continuation map of ∇ -flat sections $\mathbb{U}_{\mathcal{S}, \rho}^{\mathcal{X}, Y} : \Gamma(V_Y, \mathcal{O}(F)) \rightarrow \Gamma(V_{\mathcal{X}}, \mathcal{O}(F))$, which is an isometry of $H(\cdot)_F$ and identifies the quantum D -modules of \mathcal{X} and Y .

Remark 2.2. When $\mathfrak{h}_{\mathcal{X}} \neq \mathfrak{h}_Y$, the induced Frobenius structures on $H(\mathcal{X})$ and $H(Y)$ are inequivalent. A sufficient condition [28] for the two Frobenius structures to coincide is given by the Hard Lefschetz criterion for $\mathcal{X} \rightarrow X$:

$$\text{age}(\theta) - \text{age}(\text{inv}^*\theta) = 0 \quad (27)$$

for any class $\theta \in H(\mathcal{X})$.

Remark 2.3. Suppose that $c_1(\mathcal{Z}) \geq 0$ and that the coarse moduli space Z is a semi-projective toric variety given by a GIT quotient of $\mathbb{C}^{\dim \mathcal{Z} + l_{\mathcal{Z}}}$ by $(\mathbb{C}^*)^{l_{\mathcal{Z}}}$. In this setting, the global quantum D -module arises naturally in the form of the GKZ system associated to \mathcal{Z} [7, 25, 40]. The scaling factor $\mathfrak{h}_{\mathcal{Z}}^{1/z}$ then measures the discrepancy between the small J -function and the canonical basis-vector of solutions of the GKZ system (the I -function), restricted to zero twisted insertions:

$$\mathfrak{h}_{\mathcal{Z}}^{1/z}(\tau_{0,2})J^{\mathcal{Z}, \text{small}}(\tau_{0,2}, z) = I^{\mathcal{Z}}(\mathfrak{a}(\tau_{0,2}), z), \quad (28)$$

where $\mathfrak{a}(\tau_{0,2})$ is the inverse mirror map. As a consequence of (28), the scaling factor $\mathfrak{h}_{\mathcal{Z}}$ is determined by the toric data defining \mathcal{Z} [25, 28, 40]. Let $\Xi_i \in H^2(Z)$ be the T -equivariant Poincaré dual of the reduction to the quotient of the i^{th} coordinate hyperplane in $\mathbb{C}^{\dim \mathcal{Z} + l_{\mathcal{Z}}}$ and write $\zeta_i^{(j)} = \text{Coeff}_{\phi_j} \Xi_i \in \mathbb{C}[\nu]$ for the coefficient of the projection of Ξ_i along $\phi_j \in H(\mathcal{Z})$ for $j = 0, \dots, l_{\mathcal{Z}}$. Defining, for every β , $D_i(\beta) \triangleq \int_{\beta} \Xi_i$ and $J_{\beta}^{\pm} \triangleq \{j \in \{1, \dots, \dim \mathcal{Z} + l_{\mathcal{Z}}\} \mid \pm D_j(\beta) > 0\}$, we have

$$\pi_l = \log \mathfrak{a}_l + \sum_{\beta \in \text{Eff}(\mathcal{Z})} \mathfrak{a}^{\beta} \frac{\prod_{j_- \in J_{\beta}^-} (-1)^{D_{j_-}(\beta)} |D_{j_-}(\beta)|!}{\prod_{j_+ \in J_{\beta}^+} D_{j_+}(\beta)!} \sum_{k_- \in J_{\beta}^-} \frac{-\zeta_{k_-}^{(l)}}{D_{k_-}(\beta)}, \quad l = 1, \dots, l_{\mathcal{Z}}, \quad (29)$$

$$\mathfrak{h}_{\mathcal{Z}} = \exp \left[\sum_{\beta \in \text{Eff}(\mathcal{Z})} \mathfrak{a}^{\beta} \frac{\prod_{j_- \in J_{\beta}^-} (-1)^{D_{j_-}(\beta)} |D_{j_-}(\beta)|!}{\prod_{j_+ \in J_{\beta}^+} D_{j_+}(\beta)!} \sum_{k_- \in J_{\beta}^-} \frac{-\zeta_{k_-}^{(0)}}{D_{k_-}(\beta)} \right]. \quad (30)$$

2.1.2. *Givental's symplectic formalism.* The global quantum D -module picture is intimately connected to the CRC statement of [26, 29]. In view of our statement of the OCRC in Section 3, we find it useful to spell it out here. Givental's symplectic space $(\mathcal{H}_{\mathcal{Z}}, \Omega_{\mathcal{Z}})$ is the infinite dimensional vector space

$$\mathcal{H}_{\mathcal{Z}} \triangleq H(\mathcal{Z}) \otimes \mathcal{O}(\mathbb{C}^*) \quad (31)$$

along with the symplectic form

$$\Omega_{\mathcal{Z}}(f, g) \triangleq \text{Res}_{z=0} \eta(f(-z), g(z))_{\mathcal{Z}}. \quad (32)$$

A general point of $\mathcal{H}_{\mathcal{Z}}$ can be written as

$$\sum_{k \geq 0} \sum_{\alpha=0}^{N_{\mathcal{Z}}-1} q_{k,\alpha} \phi_{\alpha} z^k + \sum_{l \geq 0} \sum_{\beta=0}^{N_{\mathcal{Z}}-1} p_{l,\beta} \phi_{\beta} (-z)^{-k-1}. \quad (33)$$

Notice that $\{p_{l,\beta}, q_{k,\alpha}\}$ are Darboux coordinates for (32); call $\mathcal{H}_{\mathcal{Z}}^+$ the Lagrangian subpace spanned by $q_{k,\alpha}$. The generating function of genus zero descendent Gromov–Witten invariants of \mathcal{Z} ,

$$\mathcal{F}_0^{\mathcal{Z}} \triangleq \sum_{n=0}^{\infty} \sum_{\beta \in \text{Eff}(\mathcal{Z})} \sum_{\substack{a_1, \dots, a_n \\ r_1, \dots, r_n}} \frac{\prod_{i=1}^n \tau_{a_i, r_i}}{n!} \langle \sigma_{r_1}(\phi_{a_1}) \cdots \sigma_{r_n}(\phi_{a_n}) \rangle_{0, n, \beta}^{\mathcal{Z}}, \quad (34)$$

is the germ of an analytic function on $\mathcal{H}_{\mathcal{Z}}^+$ upon identifying $\tau_{0,1} = q_{0,1} + 1$, $\tau_{\alpha,n} = q_{\alpha,n}$; under the assumption of convergence of the quantum product, coefficients of monomials in $\tau_{\alpha,n}$ with $\deg_{\text{CR}} \phi_{\alpha} \neq 0$, $n > 0$ are analytic functions of $e^{\tau_{0,2}}$ in a neighbourhood of the origin. The graph of the differential of (34),

$$p_{l,\beta} = \frac{\partial \mathcal{F}_0^{\mathcal{Z}}}{\partial q^{l,\beta}}, \quad (35)$$

then yields a formal germ of a Lagrangian submanifold $\mathcal{L}_{\mathcal{Z}}$ (in fact, a ruled cone, as a consequence of the genus zero Gromov–Witten axioms), depending analytically on the small quantum cohomology variables $\tau_{0,2}$. By the equations defining the cone, the J -function $J^{\mathcal{Z}}(\tau, -z)$ yields a family of elements of $\mathcal{L}_{\mathcal{Z}}$ parameterized by $\tau \in H(\mathcal{Z})$, which is uniquely determined by its large z asymptotics $J(\tau, -z) = -z + \tau + \mathcal{O}(z^{-1})$. Conversely, the genus zero topological recursion relations imply that $\mathcal{L}_{\mathcal{Z}}$ can be reconstructed entirely from $J^{\mathcal{Z}}(\tau, z)$.

The Crepant Resolution Conjecture has a natural formulation in terms of morphisms of Givental spaces, as pointed out by Coates–Corti–Iritani–Tseang (CCIT) [26] and further explored by Coates–Ruan [29].

Conjecture 2.4 ([26], [29]). *There exists $\mathbb{C}((z^{-1}))$ -linear symplectic isomorphism of Givental spaces $\mathbb{U}_{\rho}^{\mathcal{X}, Y} : \mathcal{H}_{\mathcal{X}} \rightarrow \mathcal{H}_Y$, matching the Lagrangian cones of \mathcal{X} and Y upon a suitable analytic continuation of small quantum cohomology parameters:*

$$\mathbb{U}_{\rho}^{\mathcal{X}, Y}(\mathcal{L}_{\mathcal{X}}) = \mathcal{L}_Y. \quad (36)$$

This version of the CRC is equivalent to the quantum D -module approach via the fundamental solutions, which give a canonical z -linear identification

$$S_{\mathcal{Z}}(\tau, z) : \mathcal{H}_{\mathcal{Z}} \xrightarrow{\cong} S_{\mathcal{Z}}. \quad (37)$$

translating the analytic continuation map $\mathbb{U}_{S, \rho}^{\mathcal{X}, Y}$ to a linear isomorphism of Givental spaces which is symplectic, as $\mathbb{U}_{S, \rho}^{\mathcal{X}, Y}$ preserves the pairing (16).

Suppose now that $c_1(\mathcal{X}) = 0$, $\dim_{\mathbb{C}} \mathcal{X} = 3$ and assume further that the J -functions $J^{\mathcal{Z}}$, for \mathcal{Z} either \mathcal{X} or Y , and $\mathbb{U}_{\rho}^{\mathcal{X}, Y}$ admit well-defined non-equivariant limits,

$$J_{n\text{-eq}}^{\mathcal{Z}}(\tau, z) \triangleq \lim_{\nu \rightarrow 0} J^{\mathcal{Z}}(\tau, z), \quad \mathbb{U}_{\rho, 0}^{\mathcal{X}, Y} \triangleq \lim_{\nu \rightarrow 0} \mathbb{U}_{\rho}^{\mathcal{X}, Y}. \quad (38)$$

By homogeneity, $e^{-\tau_0/z} J_{n\text{-eq}}^{\mathcal{Z}}(\tau, z)$ is a Laurent polynomial of the form [30, §10.3.2]

$$J_{n\text{-eq}}^{\mathcal{Z}}(\tau, z) = e^{-\tau_0/z} \left(z + \sum_{i=1}^{N_{\mathcal{Z}}-1} \left(\tau_i + \frac{\mathfrak{f}_i^{\mathcal{Z}}(\tau)}{z} \right) \phi_i + \frac{\mathfrak{g}^{\mathcal{Z}}(\tau)}{z^2} \mathbf{1}_{\mathcal{Z}} \right), \quad (39)$$

where $\mathfrak{f}^{\mathcal{Z}}(\tau)$ and $\mathfrak{g}^{\mathcal{Z}}(\tau)$ are analytic functions around the large radius limit point of \mathcal{Z} . Restricting $J_{n\text{-eq}}^{\mathcal{Z}}(\tau, z)$ to $\Delta_{\mathcal{Z}}$ and picking up a branch ρ of analytic continuation of the quantum

parameters, the vector valued analytic function $\mathcal{I}_\rho^{\mathcal{X},Y}$ defined by

$$\begin{array}{ccc}
\Delta_{\mathcal{X}} & \xrightarrow{\mathcal{I}_\rho^{\mathcal{X},Y}} & \Delta_Y \\
J_{n\text{-eq}}^{\mathcal{X}}|_{\Delta_{\mathcal{X}}} \downarrow & & \downarrow J_{n\text{-eq}}^Y|_{\Delta_Y} \\
\mathcal{H}_{\mathcal{X}} & \xrightarrow{\mathfrak{h}_{\mathcal{X}}^{1/z} \mathbb{U}_{\rho,0}^{\mathcal{X},Y} \mathfrak{h}_Y^{-1/z}} & \mathcal{H}_Y
\end{array} \tag{40}$$

gives an analytic isomorphism¹ between neighbourhoods $V_{\mathcal{X}}$, V_Y of the projections of the large radius points of \mathcal{X} and Y to $\Delta_{\mathcal{X}}$ and Δ_Y . When \mathcal{X} satisfies the Hard–Lefschetz condition, the coefficients of $\mathbb{U}_{\rho}^{\mathcal{X},Y}$ contain only non-positive powers of z [29] and the non-equivariant limit coincides with the $z \rightarrow \infty$ limit; then the isomorphism $\mathcal{I}_\rho^{\mathcal{X},Y}$ extends to an affine linear change of variables $\widehat{\mathcal{I}}_\rho^{\mathcal{X},Y} : H(\mathcal{X}) \rightarrow H(Y)$ at the level of the full cohomology rings of \mathcal{X} and Y , which is an isomorphism of Frobenius manifolds.

2.1.3. Integral structures and the CRC. In [45], Iritani uses K -groups to define an integral structure in the quantum D-module associated to the Gromov–Witten theory of a smooth Deligne–Mumford stack \mathcal{Z} ; we recall the discussion in [45, 46], adapting it to the equivariant setting.

Write $K(\mathcal{Z})$ for the Grothendieck group of topological vector bundles $V \rightarrow \mathcal{Z}$ and consider the map $\Psi : K(\mathcal{Z}) \rightarrow H(\mathcal{Z}) \otimes \mathbb{C}((z^{-1}))$ given by

$$\Psi(V) \triangleq (2\pi)^{-\frac{\dim \mathcal{Z}}{2}} z^{-\mu} \widehat{\Gamma}_{\mathcal{Z}} \cup (2\pi i)^{\deg/2} \text{inv}^* \text{ch}(V), \tag{42}$$

where $\text{ch}(V)$ is the orbifold Chern character, \cup is the topological cup product on $I\mathcal{Z}$, and

$$\widehat{\Gamma}_{\mathcal{Z}} \triangleq \bigoplus_v \prod_f \prod_\delta \Gamma(1 - f + \delta), \tag{43}$$

$$\mu \triangleq \left(\frac{1}{2} \deg(\phi) - \frac{3}{2} \right) \phi, \tag{44}$$

where the sum in (43) is over all connected components of the inertia stack, the left product is over the eigenbundles in a decomposition of the tangent bundle $T\mathcal{Z}$ with respect to the stabilizer action (with f the weight of the action on the eigenspace), and the right product is over all of the Chern roots δ of the eigenbundle. Via the fundamental solution (17) this induces a map to the space of flat sections of $\text{QDM}(\mathcal{Z})$; its image is a lattice [45] in $\mathcal{S}_{\mathcal{Z}}$, which Iritani dubs the *K-theory integral structure* of $QH(\mathcal{Z}) = (H(\mathcal{Z}), \eta, \circ_\tau)$. This implies the existence of an integral local system underlying $\text{QDM}(\mathcal{Z})$ induced by the K -theory of \mathcal{Z} .

Iritani’s theory has important implications for the Crepant Resolution Conjecture. At the level of integral structures, the analytic continuation map $\mathbb{U}_{\mathcal{S},\rho}^{\mathcal{X},Y}$ of flat sections should be induced

¹Explicitly, matrix entries $(\mathbb{U}_{\rho,0}^{\mathcal{X},Y})_{ij}$ of $\mathbb{U}_{\rho,0}^{\mathcal{X},Y}$ are monomials in z ; call u_{ij} the coefficient of such monomial. Then (40) boils down to the statement that quantum cohomology parameters τ_i^\bullet in Δ_\bullet for $i = 1, \dots, l_Y$ are identified as

$$\tau_i^Y = (\mathcal{I}_\rho^{\mathcal{X},Y} \tau^\mathcal{X})_i \triangleq u_{i0} + \sum_{j=1}^{l_Y} u_{ij} \tau_j^\mathcal{X} + \sum_{k=l_Y+1}^{N_Y-1} u_{ik} \mathfrak{f}_k^\mathcal{X}(\tau^\mathcal{X}). \tag{41}$$

Since $\deg(\mathbb{U}_{\rho,0}^{\mathcal{X},Y})_{ij} > 0$ for $j > l_Y$, in the Hard Lefschetz case the condition that the coefficients of $\mathbb{U}_{\rho}^{\mathcal{X},Y}$ are Taylor series in $1/z$ implies that $u_{ik} = 0$ for $k > l_Y$.

by an isomorphism $\mathbb{U}_{K,\rho}^{\mathcal{X},Y} : K(Y) \rightarrow K(\mathcal{X})$ at the K -group level,

$$\begin{array}{ccc}
K(\mathcal{X}) & \xrightarrow{\mathbb{U}_{K,\rho}^{\mathcal{X},Y}} & K(Y) \\
\downarrow S_{\mathcal{X}}(x,z)\Psi_{\mathcal{X}} & & \downarrow S_Y(t,z)\Psi_Y \\
\mathcal{S}_{\mathcal{X}} & \xrightarrow{\mathfrak{h}_Y^{1/z}\mathbb{U}_{S,\rho}^{\mathcal{X},Y}\mathfrak{h}_{\mathcal{X}}^{-1/z}} & \mathcal{S}_Y
\end{array} \tag{45}$$

The Crepant Resolution Conjecture can then be phrased in terms of the existence of an identification of the integral structures underlying quantum cohomology. In [45], it is conjectured that $\mathbb{U}_{K,\rho}^{\mathcal{X},Y}$ should be induced by a natural geometric correspondence between K -groups (see also [7] for earlier work in this context). In terms of Givental’s symplectic formalism, we have

$$\mathbb{U}_{\rho}^{\mathcal{X},Y} = \Psi_Y \circ \mathbb{U}_{K,\rho}^{\mathcal{X},Y} \circ \Psi_{\mathcal{X}}^{-1}. \tag{46}$$

2.2. Open Gromov–Witten theory. For a three-dimensional toric Calabi–Yau variety, open Gromov-Witten invariants are defined “via localization” in [31, 47]. This theory has been first introduced for orbifold targets in [14] and developed in full generality in [58] (see also [37] for recent results in this context). Boundary conditions are given by choosing special type of Lagrangian submanifolds introduced by Aganagic–Vafa in [4]. These Lagrangians are defined locally in a formal neighborhood of each torus invariant line: in particular if p is a torus fixed point adjacent to the torus fixed line l , and the local coordinates at p are (z, u, v) , then L is defined to be the fixed points of the anti-holomorphic involution

$$(z, u, v) \rightarrow (1/\bar{z}, \bar{z}\bar{u}, \bar{z}\bar{v}) \tag{47}$$

defined away from $z = 0$. Boundary conditions can then be thought of as “formal” ways of decorating the web diagram of the toric target.

Loci of fixed maps are described in terms of closed curves mapping to the compact edges of the web diagram in the usual way and disks mapping rigidly to the torus invariant lines with Lagrangian conditions. Beside Hodge integrals coming from the contracting curves, the contribution of each fixed locus to the invariants has a factor for each disk, which is constructed as follows. The map from the disk to a neighborhood of its image is viewed as the quotient via an involution of a map of a rational curve to a canonical target. The obstruction theory in ordinary Gromov-Witten theory admits a natural \mathbb{Z}_2 action, and the equivariant Euler class of the involution invariant part of the obstruction theory is chosen as the localization contribution from the disk [14, Section 2.2], [58, Section 2.4]. This construction is efficiently encoded via the introduction of a “disk function”, which we now review in the context of cyclic isotropy (see [58, Section 3.3] for the general case of finite abelian isotropy groups).

Let \mathcal{Z} be a three-dimensional CY toric orbifold, p a fixed point such that a neighborhood is isomorphic to $[\mathbb{C}^3/\mathbb{Z}_{n+1}]$, with representation weights (m_1, m_2, m_3) and CY torus weights (w_1, w_2, w_3) . Define $n_e = (n+1)/\gcd(m_1, n+1)$ to be the size of the effective part of the action along the first coordinate axis. There exist a map from an orbi-disk mapping to the first coordinate axis with winding d and twisting k if the compatibility condition

$$\frac{d}{n_e} - \frac{km_1}{n+1} \in \mathbb{Z} \tag{48}$$

is satisfied. In this case the positively oriented disk function is

$$D_k^+(d; \vec{w}) = \left(\frac{n_e w_1}{d}\right)^{\text{age}(k)-1} \frac{n_e}{d(n+1) \left[\frac{d}{n_e}\right]!} \frac{\Gamma\left(\frac{dw_2}{n_e w_1} + \left\langle \frac{km_3}{n+1} \right\rangle + \frac{d}{n_e}\right)}{\Gamma\left(\frac{dw_2}{n_e w_1} - \left\langle \frac{km_2}{n+1} \right\rangle + 1\right)}. \tag{49}$$

The negatively oriented disk function is obtained by switching the indices 2 and 3. By renaming the coordinate axes this definition applies to the general boundary condition.

In [58] the disk function is used to construct the GW orbifold topological vertex, a building block for open and closed GW invariants of \mathcal{Z} . The disk potential is efficiently expressed in terms of the disk and of the J function of \mathcal{Z} . Fix a Lagrangian boundary condition L which we assume to be on the first coordinate axis in the local chart ($\cong [\mathbb{C}^3/\mathbb{Z}_{n+1}]$) around the point p . Denote by $\{\mathbf{1}_{\mathbf{p},\mathbf{k}}\}_{\mathbf{k}=1,\dots,n+1}$ the part of the localized basis for $H(\mathcal{Z})$ supported at p . Raising indices using the orbifold Poincaré pairing, and extending the disk function to be a cohomology valued function

$$\mathcal{D}^+(d; \vec{w}) = \sum_{\mathbf{k}=1}^{n+1} D_{\mathbf{k}}^+(d; \vec{w}) \mathbf{1}_{\mathbf{p}}^{\mathbf{k}}, \quad (50)$$

the (genus zero) *scalar disk potential* is obtained by contraction with the J function:

$$\begin{aligned} F_L^{\text{disk}}(\tau, y, \vec{w}) &\triangleq \sum_d \frac{y^d}{d!} \sum_n \frac{1}{n!} \langle \tau, \dots, \tau \rangle_{0,n}^{L,d} \\ &= \sum_d \frac{y^d}{d!} \left(\mathcal{D}^+(d; \vec{w}), J^{\mathcal{Z}} \left(\tau, \frac{gw_1}{d} \right) \right)_{\mathcal{Z}}, \end{aligned} \quad (51)$$

where we denoted by $\langle \tau, \dots, \tau \rangle_{0,n}^{L,d}$ the disk invariants with boundary condition L , winding d and n general cohomological insertions.

Remark 2.5. We may consider the disk potential relative to multiple Lagrangian boundary conditions. In that case, we define the disk function by adding the disk functions for each Lagrangian, and we introduce a winding variable for each boundary condition.

Remark 2.6. It is not conceptually difficult (but book-keeping intensive) to express the general genus zero open potential in terms of appropriate contractions of arbitrary copies of these disk functions with the full descendant Gromov-Witten potential of \mathcal{Z} .

2.3. A_n resolutions.

2.3.1. GIT Quotients. Here we review the relevant toric geometry concerning our targets. Let $\mathcal{X} \triangleq [\mathbb{C}^3/\mathbb{Z}_{n+1}]$ be the 3-fold A_n singularity and Y its resolution. The toric fan for \mathcal{X} has rays $(0, 0, 1)$, $(1, 0, 0)$, and $(1, n+1, 0)$, while the fan for Y is obtained by adding the rays $(1, 1, 0)$, $(1, 2, 0), \dots, (1, n, 0)$. The divisor class group is described by the short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{M^T} \mathbb{Z}^{n+3} \xrightarrow{N} \mathbb{Z}^3 \longrightarrow 0, \quad (52)$$

where

$$M = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & \ddots & & & \vdots & \\ 0 & \dots & 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 0 \\ 0 & 1 & 2 & \dots & n+1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad (53)$$

Both \mathcal{X} and Y are GIT quotients:

$$\mathcal{X} = \left[\frac{\mathbb{C}^{n+3} \setminus V(x_1 \cdot \dots \cdot x_n)}{(\mathbb{C}^*)^n} \right], \quad (54)$$

$$Y = \frac{\mathbb{C}^{n+3} \setminus V(I_1, \dots, I_n)}{(\mathbb{C}^*)^n}, \quad (55)$$

where

$$I_i = \prod_{j=0, j \neq i-1, i}^{n+1} x_j, \quad (56)$$

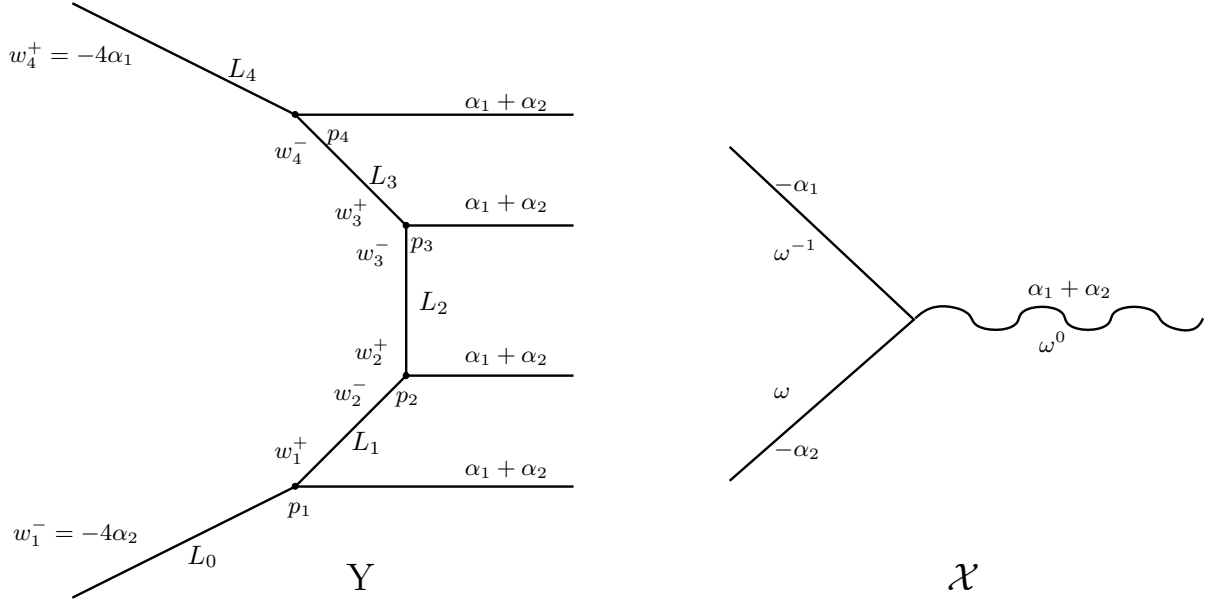


FIGURE 1. The toric web diagrams for Y and \mathcal{X} for $n = 3$. Fixed points and invariants lines are labelled, together with the relevant torus and representation weights.

and the torus action is specified by M . From the quotient (54), we can compute pseudo-coordinates on the orbifold

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_0 x_1^{\frac{n}{n+1}} x_2^{\frac{n-1}{n+1}} \cdots x_n^{\frac{1}{n+1}} \\ x_1^{\frac{1}{n+1}} x_2^{\frac{2}{n+1}} \cdots x_n^{\frac{n}{n+1}} x_{n+1} \\ x_{n+2} \end{bmatrix}. \quad (57)$$

These coordinates are only defined up to a choice of $(n+1)^{\text{st}}$ root of unity for each x_i . This accounts for a residual $\mathbb{Z}_{n+1} \subset (\mathbb{C}^*)^n$ acting with dual representations on the first two coordinates. We identify this residual \mathbb{Z}_{n+1} as the subgroup generated by $(\omega, \omega^2, \dots, \omega^n) \in (\mathbb{C}^*)^n$, where $\omega = e^{\frac{2\pi i}{n+1}}$. This realizes the quotient (54) as the 3-fold A_n singularity where $\mathbb{Z}_{n+1} = \langle \omega \rangle$ acts by $\omega \cdot (z_1, z_2, z_3) = (\omega z_1, \omega^{-1} z_2, z_3)$.

Remark 2.7. The weights of the \mathbb{Z}_{n+1} action on the corresponding fibers of $T\mathcal{X}$ are inverse to the weights on the local coordinates because a local trivialization of the tangent bundle is given by $\frac{\partial}{\partial z^\alpha}$ where z^α are the local coordinates.

The geometry of the space Y is captured by the toric web diagram in Figure 1. In particular, Y has $n+1$ torus fixed points (corresponding to the $n+1$ 3-dimensional cones in the fan) and a chain of n torus invariant lines connecting these points. We label the points p_1, \dots, p_{n+1} where p_i corresponds to the cone spanned by $(0, 0, 1)$, $(1, i-1, 0)$, and $(1, i, 0)$ and we label the torus invariant lines by L_1, \dots, L_n where L_i connects p_i to p_{i+1} . We also denote by L_0 and L_{n+1} the torus invariant (affine) lines corresponding to the 2-dimensional cones spanned by the rays $(1, 0, 0)$, $(0, 0, 1)$ and $(1, n, 0)$, $(0, 0, 1)$, respectively. From the quotient (55) we compute homogeneous coordinates on the line L_i

$$\begin{bmatrix} x_0^i x_1^{i-1} \cdots x_{i-1} \\ x_{n+1}^{n+1-i} x_{n-i}^{n-i} \cdots x_{i+1} \end{bmatrix} \quad (58)$$

where $p_i \leftrightarrow [0 : 1]$ and $p_{i+1} \leftrightarrow [1 : 0]$.

On the resolution, $H_2(Y)$ is generated by the torus invariant lines L_i . Define $\gamma_i \in H^2(Y)$ to be dual to L_i . The γ_i form a basis of $H^2(Y)$; denote the corresponding line bundles by $\mathcal{O}(\gamma_i)$. Note that $\mathcal{O}(\gamma_i)$ restricts to $\mathcal{O}(1)$ on L_i and \mathcal{O} on L_j if $j \neq i$ and this uniquely determines the line bundle $\mathcal{O}(\gamma_i)$. On the orbifold, line bundles correspond to \mathbb{Z}_{n+1} equivariant line bundles on \mathbb{C}^3 . We denote \mathcal{O}_k the line bundle where \mathbb{Z}_{n+1} acts on fibers with weight ω^k ; then, for example, $T_{\mathcal{X}} = \mathcal{O}_{-1} \oplus \mathcal{O}_1 \oplus \mathcal{O}_0$ where the subscripts are computed modulo $n+1$ (c.f. Remark 2.7).

2.3.2. Classical equivariant geometry. Given that we are working with noncompact targets, all of our quantum computations utilize Atiyah-Bott localization with respect to an additional $T = \mathbb{C}^*$ action on our spaces. Let T act on \mathbb{C}^{n+3} with weights $(\alpha_1, 0, \dots, 0, \alpha_2, -\alpha_1 - \alpha_2)$. Then the induced action on the orbifold and resolution can be read off from the local coordinates in (57) and (58). In particular, the three weights on the fibers of $T_{\mathcal{X}}$ are $-\alpha_1, -\alpha_2, \alpha_1 + \alpha_2$. The T -equivariant Chen-Ruan cohomology $H(\mathcal{X})$ is by definition the T -equivariant cohomology of the inertia stack $\mathcal{I}\mathcal{X}$. The latter has components $\mathcal{X}_1, \dots, \mathcal{X}_n, \mathcal{X}_{n+1}$, the last being the untwisted sector²:

$$\begin{aligned}\mathcal{X}_k &= [\mathbb{C}/\mathbb{Z}_{n+1}], \quad 1 \leq k \leq n, \\ \mathcal{X}_{n+1} &= [\mathbb{C}^3/\mathbb{Z}_{n+1}]\end{aligned}\tag{59}$$

Writing $\mathbf{1}_k$, $k = 1, \dots, n+1$ for the fundamental class of \mathcal{X}_k we obtain a $\mathbb{C}(\nu)$ basis of $H(\mathcal{X})$; the age-shifted grading assigns degree 0 to the fundamental class of the untwisted sector, and degree 1 to every twisted sector. The Atiyah-Bott localization isomorphism is trivial, i.e. the fundamental class on each twisted sector is identified with the unique T -fixed point on that sector. We abuse notation and use $\mathbf{1}_k$ to also denote the fixed point basis. The equivariant Chen-Ruan pairing in orbifold cohomology is

$$\eta(\mathbf{1}_i, \mathbf{1}_j)_{\mathcal{X}} = \frac{\delta_{i,n+1}\delta_{j,n+1} + \alpha_1\alpha_2\delta_{i+j,n+1}}{\alpha_1\alpha_2(\alpha_1 + \alpha_2)(n+1)}.\tag{60}$$

On the resolution Y , the three weights on the tangent bundle at p_i are

$$(w_i^-, w_i^+, \alpha_1 + \alpha_2) \triangleq ((i-1)\alpha_1 + (-n+i-2)\alpha_2, -i\alpha_1 + (n+1-i)\alpha_2, \alpha_1 + \alpha_2).\tag{61}$$

Moreover, $\mathcal{O}(\gamma_j)$ is canonically linearized via the homogeneous coordinates in (57). The weight of $\mathcal{O}(\gamma_i)$ at the fixed point p_i is

$$\begin{cases} (n+1-j)\alpha_2 & i \leq j, \\ j\alpha_1 & i > j. \end{cases}\tag{62}$$

Denote by $\{P_i\}_{i=1}^{n+1}$ the equivariant cohomology classes corresponding to the fixed points of Y . Choosing the canonical linearization given in (62), the Atiyah-Bott localization isomorphism on Y is given by

$$\gamma_j \longrightarrow \sum_{i \leq j} (n+1-j)\alpha_2 P_i + \sum_{i > j} j\alpha_1 P_i,\tag{63}$$

$$\gamma_{n+1} \longrightarrow \sum_{i=1}^{n+1} P_i.\tag{64}$$

where γ_{n+1} is the fundamental class on Y . Genus zero, degree zero GW invariants are given by equivariant triple intersections on Y ,

$$\langle \gamma_i, \gamma_j, \gamma_k \rangle_{0,3,0}^Y = \int_Y \gamma_i \cup \gamma_j \cup \gamma_k.\tag{65}$$

²While it is more common to index the untwisted sector by 0, we make this choice of notation for the sake of the computations of Section 5, where certain matrices are triangular with this ordering.

With $i \leq j \leq k < n + 1$, (63)-(64) yield

$$\langle \gamma_{n+1}, \gamma_{n+1}, \gamma_{n+1} \rangle_{0,3,0}^Y = \frac{1}{(n+1)\alpha_1\alpha_2(\alpha_1 + \alpha_2)}, \quad (66)$$

$$\langle \gamma_{n+1}, \gamma_{n+1}, \gamma_i \rangle_{0,3,0}^Y = 0, \quad (67)$$

$$\langle \gamma_{n+1}, \gamma_i, \gamma_j \rangle_{0,3,0}^Y = \frac{i(n+1-j)}{-(n+1)(\alpha_1 + \alpha_2)}, \quad (68)$$

$$\langle \gamma_i, \gamma_j, \gamma_k \rangle_{0,3,0}^Y = -\frac{ij(n+1-k)\alpha_1 + i(n+1-j)(n+1-k)\alpha_2}{(n+1)(\alpha_1 + \alpha_2)}. \quad (69)$$

The T -equivariant pairing $\eta(\gamma_i, \gamma_j)_Y$ is given by (68) and diagonalizes in the fixed point basis:

$$\eta(P_i, P_j)_Y = \frac{\delta_{i,j}}{w_i^- w_i^+ (\alpha_1 + \alpha_2)}. \quad (70)$$

2.3.3. Quantum equivariant geometry. We compute the genus 0 GW invariants of Y via localization (extending the computations of [17] to a more general torus action):

$$\langle \gamma_{i_1}, \dots, \gamma_{i_l} \rangle_{0,l,\beta} = \begin{cases} -\frac{1}{d^3} & \text{if } \beta = d(L_j + \dots + L_k) \text{ with } j \leq \min\{i_\alpha\} \leq \max\{i_\alpha\} \leq k, \\ 0 & \text{else.} \end{cases} \quad (71)$$

Denote by $\Phi = \sum_{i=1}^{n+1} t_i \gamma_i$ a general cohomology class $\Phi \in H(Y)$. The equivariant three-point correlators used to define the quantum cohomology can be computed from (66)-(69), (71), and the divisor equation (with $i \leq j \leq k \leq n + 1$):

$$\langle \langle \gamma_i, \gamma_j, \gamma_k \rangle \rangle_{0,3}^Y(t) = \int_Y \gamma_i \cup \gamma_j \cup \gamma_k - \sum_{l \leq i \leq k \leq m} \frac{e^{t_l + \dots + t_m}}{1 - e^{t_l + \dots + t_m}}. \quad (72)$$

The equivariant quantum cohomology of \mathcal{X} can then be computed from the following result, which is proven in the appendix of [26].

Theorem 2.8 (Coates-Corti-Iritani-Tseng). *Let $\log \rho : [0, 1] \rightarrow H^2(Y)$ be a path in $H^2(Y)$ such that ρ is a straight line in the Kähler cone of Y connecting*

$$\rho_i(0) = 0, \quad (73)$$

$$\rho_i(1) = \omega^{-i}. \quad (74)$$

Then upon analytic continuation in the quantum parameters t_i along ρ , the quantum products for \mathcal{X} and Y coincide after the affine-linear change of variables

$$t_i = \left(\widehat{\mathcal{I}}_\rho^{\mathcal{X}, Y} x \right)_i = \begin{cases} -\frac{2\pi i}{n+1} + \sum_{k=1}^n \frac{\omega^{-ik}(\omega^{\frac{k}{2}} - \omega^{-\frac{k}{2}})}{n+1} x_k & 0 < i < n+1 \\ x_{n+1}, & i = n+1 \end{cases} \quad (75)$$

and the linear isomorphism $\mathbb{U}_{\rho,0}^{\mathcal{X}, Y} : H_T^{\text{orb}}(\mathcal{X}) \rightarrow H_T(Y)$

$$\mathbf{1}_k \rightarrow \sum_{i=1}^n \frac{\omega^{-ik}(\omega^{\frac{k}{2}} - \omega^{-\frac{k}{2}})}{n+1} \gamma_i,$$

$$\mathbf{1}_{n+1} \rightarrow \gamma_{n+1}.$$

Furthermore, $\mathbb{U}_{\rho,0}^{\mathcal{X}, Y}$ preserves the equivariant Poincaré pairings of \mathcal{X} and Y .

3. THE OPEN CREPANT RESOLUTION CONJECTURE

3.1. The disk function, revisited. We reinterpret the disk function as an endomorphism of Givental space. First we homogenize Iritani's Gamma class (43) and make it of degree zero:

$$\widehat{\Gamma}_{\mathcal{Z}}(z) \triangleq z^{-\frac{1}{2} \deg \widehat{\Gamma}_{\mathcal{Z}}} \widehat{\Gamma}_{\mathcal{Z}} \triangleq \sum \widehat{\Gamma}_{\mathcal{Z}}^k \mathbf{1}_{\mathbf{p}, \mathbf{k}}, \quad (76)$$

where the second equality defines $\widehat{\Gamma}_{\mathcal{Z}}^k$ as the $\mathbf{1}_{\mathbf{p}, \mathbf{k}}$ -coefficient of $\widehat{\Gamma}_{\mathcal{Z}}(z)$. With notation as in Section 2.2,

$$\widehat{\mathcal{D}}_{\mathcal{Z}}^+(z; \vec{w})(\mathbf{1}_{\mathbf{p}, \mathbf{k}}) \triangleq \frac{\pi}{w_1(n+1) \sin\left(\pi \left(\left\langle \frac{km_3}{n+1} \right\rangle - \frac{w_3}{z}\right)\right)} \frac{1}{\widehat{\Gamma}_{\mathcal{Z}}^k} \mathbf{1}_{\mathbf{p}, \mathbf{k}}. \quad (77)$$

The natural basis of inertia components gives a basis of eigenvectors for the linear transformation $\widehat{\mathcal{D}}_{\mathcal{Z}}^+ : \mathcal{H}_{\mathcal{Z}} \rightarrow \mathcal{H}_{\mathcal{Z}}$.

Lemma 3.1. *The k -th eigenvalue of $\widehat{\mathcal{D}}_{\mathcal{Z}}^+$ coincides with $D_k(d; \vec{w})$ when $z = n_e w_1/d$ and the winding/twisting compatibility condition is met:*

$$\delta_{1, \exp\left(2\pi i \left(\frac{d}{n_e} - \frac{km_1}{n+1}\right)\right)} \left(\widehat{\mathcal{D}}_{\mathcal{Z}}^+ \left(\frac{n_e w_1}{d}; \vec{w}\right) (\mathbf{1}_{\mathbf{p}, \mathbf{k}}, \mathbf{1}_{\mathbf{p}}^{\mathbf{k}})\right)_{\mathcal{Z}} = D_k(d; \vec{w}) \quad (78)$$

Proof. This formula follows from the explicit expression of $\widehat{\Gamma}_{\mathcal{Z}}$ in the localization/inertia basis, manipulated via the identity $\Gamma(\star)\Gamma(1-\star) = \frac{\pi}{\sin(\pi\star)}$. The Calabi-Yau condition $w_1 + w_2 + w_3 = 0$ is also used. The δ factor encodes the degree/twisting condition. \square

Let now $\mathcal{X} \rightarrow X \leftarrow Y$ be a diagram of toric Calabi–Yau threefolds for which the CCIT/Coates–Ruan version of the closed crepant resolution conjecture holds. Choose a Lagrangian boundary condition $L_{\mathcal{X}}$ in \mathcal{X} and denote by L_Y the transform of such condition in Y ; notice that in general this can consist of several Lagrangian boundary conditions. We have the following

Proposition 3.2. *There exists a $\mathbb{C}((z^{-1}))$ -linear transformation $\mathbb{O} : \mathcal{H}_{\mathcal{X}} \rightarrow \mathcal{H}_Y$ of Givental spaces such that*

$$\widehat{\mathcal{D}}_Y^+ \circ \mathbb{U}_{\rho}^{\mathcal{X}, Y} = \mathbb{O} \circ \widehat{\mathcal{D}}_{\mathcal{X}}^+. \quad (79)$$

This proposition is trivial, as \mathbb{O} can be constructed as $\widehat{\mathcal{D}}_Y^+ \circ \mathbb{U}_{\rho}^{\mathcal{X}, Y} \circ (\widehat{\mathcal{D}}_{\mathcal{X}}^+)^{-1}$. However we observe that interesting open crepant resolution statements follow from this simple fact, and that \mathbb{O} is a simpler object than $\mathbb{U}_{\rho}^{\mathcal{X}, Y}$, and for a good reason: our disk function almost completely “undoes” the transcendental part in Iritani's central charge. We make this precise in the following observation.

Lemma 3.3. *Referring to equations (43) and (77) for the relevant definitions, we have*

$$\begin{aligned} \Theta_{\mathcal{Z}}(\mathbf{1}_{\mathbf{p}, \mathbf{k}}) &\triangleq z^{-\mu} \widehat{\Gamma}_{\mathcal{Z}} \cup \widehat{\mathcal{D}}_{\mathcal{Z}}^+(\mathbf{1}_{\mathbf{p}, \mathbf{k}}) \\ &= \frac{z^{\frac{3}{2}\pi}}{w_1(n+1) \sin\left(\pi \left(\left\langle \frac{km_3}{n+1} \right\rangle - \frac{w_3}{z}\right)\right)} \mathbf{1}_{\mathbf{p}, \mathbf{k}} \end{aligned} \quad (80)$$

In the hypotheses and notation of Proposition 3.2, note that \mathcal{X} and Y must be related by variation of GIT, and therefore they are quotients of a common space $Z = \mathbb{C}^{l_Y+3}$. Consider a *grade restriction window*³ $\mathfrak{W} \subset K(Z)$: a set of equivariant bundles on Z that descend bijectively to bases for $K(\mathcal{X}) \otimes \mathbb{C}$ and $K(Y) \otimes \mathbb{C}$. Combining Lemma 3.3 with Iritani's proposal (46), we obtain the following prediction.

³We borrow the terminology from [61]. See also [5] and [42].

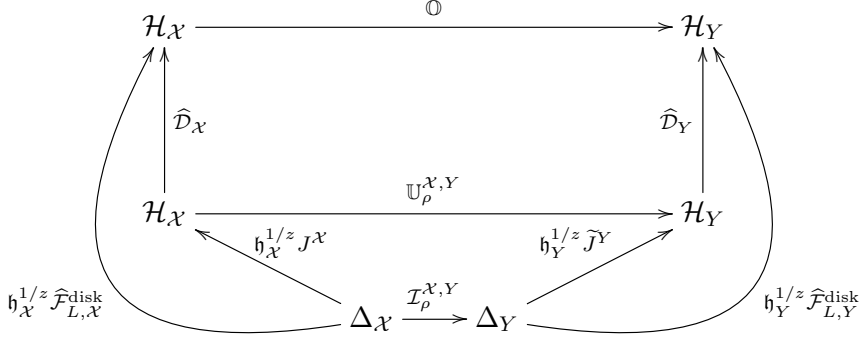


FIGURE 2. Open potential comparison diagram. In the Hard Lefschetz case this same diagram holds with the \mathfrak{h} factors omitted, and Δ_\bullet identified with the full cohomologies of either target.

Conjecture 3.4. Denote by $\widehat{CH}_\bullet = z^{-\frac{1}{2} \deg CH_\bullet}$ the (homogenized) matrix of Chern characters in the bases given by \mathfrak{W} . Let Θ_\bullet be as in equation (80). Then:

$$\mathbb{O} = \Theta_Y \circ \widehat{CH}_Y \circ \widehat{CH}_X^{-1} \circ \Theta_X^{-1}. \quad (81)$$

We verify Conjecture 3.4 for the resolution of A_n singularities in Section 4. We also note that while we are formulating the statement in the case of cyclic isotropy to keep notation lighter, it is not hard to write an analog prediction in a completely general toric setting.

Having modified our perspective on the disk functions, we also update our take on open disk invariants to remember the twisting of the map at the origin of the disk. In correlator notation, denote $\langle \tau, \dots, \tau \rangle_{0,n}^{L,d,k}$ the disk invariants with Lagrangian boundary condition L , winding d , twisting k and n cohomology insertions. We then define the *cohomological disk potential* as a cohomology valued function, which is expressed as a composition of the J function with the disk function (77):

$$\begin{aligned} \mathcal{F}_L^{\text{disk}}(\tau, y, \vec{w}) &\triangleq \sum_d \frac{y^d}{d!} \sum_n \frac{1}{n!} \langle \tau, \dots, \tau \rangle_{0,n}^{L,d,k} \mathbf{1}_{\mathbf{p},\mathbf{k}}, \\ &= \sum_d \delta_{1, \exp(2\pi i (\frac{d}{n_e} - \frac{km_1}{n+1}))} \frac{y^d}{d!} \widehat{\mathcal{D}}_Z^+ \circ J^Z \left(\tau, \frac{n_e w_1}{d}, \vec{w} \right). \end{aligned} \quad (82)$$

We define a section of Givental space that contains equivalent information to the disk potential:

$$\widehat{\mathcal{F}}_L^{\text{disk}}(t, z, \vec{w}) \triangleq \widehat{\mathcal{D}}_Z^+ \circ J^Z(\tau, z; \vec{w}). \quad (83)$$

We call $\widehat{\mathcal{F}}_L^{\text{disk}}(t, z, \vec{w})$ the *winding neutral disk potential*. For any pair of integers k and d satisfying (48), the twisting k and winding d part of the disk potential is obtained by substituting $z = \frac{n_e w_1}{d}$. A general “disk crepant resolution statement” that follows from the closed CRC is a comparison of winding-neutral potentials, as illustrated in Figure 2.

Proposition 3.5. Let $\mathcal{X} \rightarrow X \leftarrow Y$ be a diagram for which the CCIT/Coates–Ruan form of the closed crepant resolution conjecture holds and identify quantum parameters in $\Delta_{\mathcal{X}}$ and Δ_Y via $\mathcal{I}_\rho^{\mathcal{X},Y}$ as in (40). Then:

$$\mathfrak{h}_Y^{1/z} \widehat{\mathcal{F}}_{L,Y}^{\text{disk}}|_{\Delta_Y} = \mathfrak{h}_X^{1/z} \mathbb{O} \circ \widehat{\mathcal{F}}_{L,X}^{\text{disk}}|_{\Delta_X}. \quad (84)$$

Assume further that \mathcal{X} satisfies the Hard Lefschetz condition and identify cohomologies via the affine linear change of variables $\widehat{\mathcal{I}}_\rho^{\mathcal{X},Y}$. Then:

$$\widehat{\mathcal{F}}_{L,Y}^{\text{disk}} = \mathbb{O} \circ \widehat{\mathcal{F}}_{L,X}^{\text{disk}}. \quad (85)$$

Here we also understand that the winding-neutral disk potential of Y is analytically continued appropriately (we suppressed the tilde to avoid excessive proliferation of superscripts).

Remark 3.6. At the level of cohomological disk potentials, the normalization factors \mathfrak{h}_X and \mathfrak{h}_Y enter as a redefinition of the winding number variable y in (82) depending on small quantum cohomology parameters; this is the manifestation of the so-called *open mirror map* in the physics literature on open string mirror symmetry [3, 8, 11, 50].

Remark 3.7. The statement of the proposition in principle hinges on the very possibility to identify quantum parameters as in (40)-(41). In fact, the existence of the non-equivariant limits of $\mathbb{U}_\rho^{X,Y}$ and the J -functions in our case is guaranteed by the fact that we restrict to torus actions acting trivially on the canonical bundle of X and Y ; see e.g. [53].

3.2. The Hard Lefschetz OCRC. In the Hard Lefschetz case the comparison of disk potentials naturally extends to the full genus zero open potential. We first define the *n -holes winding neutral potential*, a function from $H(\mathcal{Z})$ to the n -th tensor power of Givental space $\mathcal{H}_{\mathcal{Z}}^{\otimes n} = H(\mathcal{Z})((z_1^{-1})) \otimes \dots \otimes H(\mathcal{Z})((z_n^{-1}))$:

$$\widehat{\mathcal{F}}_L^n(\tau, z_1, \dots, z_n, \vec{w}) \triangleq \widehat{\mathcal{D}}_{\mathcal{Z}}^{+\otimes n} \circ J_n^{\mathcal{Z}}(\tau, z_1, \dots, z_n; \vec{w}), \quad (86)$$

where $J_n^{\mathcal{Z}}$ encodes n -point descendent invariants:

$$J_n^{\mathcal{Z}}(\tau, \mathbf{z}; \vec{w}) \triangleq \left\langle \left\langle \frac{\phi_{\alpha_1}}{z_1 - \psi_1}, \dots, \frac{\phi_{\alpha_n}}{z_n - \psi_n} \right\rangle \right\rangle_{0,n} \phi^{\alpha_1} \otimes \dots \otimes \phi^{\alpha_n}. \quad (87)$$

In (87), we denoted $\mathbf{z} = (z_1, \dots, z_n)$ and a sum over repeated Greek indices is intended. Just as in the disk case, one can now define a winding neutral open potential by summing over all integers n and a cohomological open potential by introducing winding variables and summing over appropriate specializations of the z variables. For a pair of spaces X and Y in a Hard Lefschetz CRC diagram then the respective potentials can be compared as in Section 3.1 - this all follows from the comparison of the n -holes winding neutral potential, which we now spell out with care.

Theorem 3.8. *Let $X \rightarrow X \leftarrow Y$ be a Hard Lefschetz diagram for which the closed crepant resolution conjecture holds. With all notation as in Proposition 3.5, and $\mathbb{O}^{\otimes n} = \mathbb{O}(z_1) \otimes \dots \otimes \mathbb{O}(z_n)$, we have:*

$$\widehat{\mathcal{F}}_{L,Y}^n = \mathbb{O}^{\otimes n} \circ \widehat{\mathcal{F}}_{L,X}^n \quad (88)$$

Proof. The proof follows from the fact that the n -th tensor power of the symplectomorphism $\mathbb{U}_\rho^{X,Y}$ compares the J_n 's:

$$J_n^Y = \mathbb{U}_\rho^{X,Y \otimes n} \circ J_n^X. \quad (89)$$

For \mathcal{Z} either X or Y , define

$$\begin{aligned} L_{\mathcal{Z}}^n(\tau, \mathbf{z}) &\triangleq dJ_n^{\mathcal{Z}}(\tau, \mathbf{z}) = \left(\frac{\partial J_n^{\mathcal{Z}}(\tau, \mathbf{z})}{\partial \tau_\beta} \right) \otimes \phi^\beta \\ &= \left\langle \left\langle \frac{\phi_{\alpha_1}}{z_1 - \psi_1}, \dots, \frac{\phi_{\alpha_n}}{z_n - \psi_n}, \phi_\beta \right\rangle \right\rangle_{0,n+1} \phi^{\alpha_1} \otimes \dots \otimes \phi^{\alpha_n} \otimes \phi^\beta. \end{aligned} \quad (90)$$

Here d is the total differential, and the second equality is its expression in coordinates with the natural identification of $d\tau_\beta$ with ϕ^β . Since the total differential is a coordinate independent operator, we have the following.

Lemma 3.9. *After the change of variables and the linear isomorphism prescribed in the closed CRC (e.g. Theorem 2.8 for the A_n resolutions), we have an equivalence of operators:*

$$\sum_{1 \leq k \leq n+1} \left(\frac{\partial(-)}{\partial x_k} \right) \otimes \mathbf{1}^k = \sum_{1 \leq i \leq n+1} \left(\frac{\partial(-)}{\partial t_i} \right) \otimes \gamma^i. \quad (91)$$

Now we deduce (89) by induction. The result holds for $n = 1$ (this is the statement of the closed CRC), assume it holds for all $m < n$. Define $\widehat{\mathbb{U}}^m(z_1, \dots, z_{m+1}) \triangleq \mathbb{U}_\rho^{\mathcal{X}, Y \otimes m}(z_1, \dots, z_m) \otimes \mathbb{U}_0(z_{m+1})$. It follows from Lemma 3.9 that $\widehat{\mathbb{U}}^m((L_{\mathcal{X}}^m(x, \mathbf{z})) = L_Y^m(t, \mathbf{z})$ for all $m < n$.

For \mathcal{Z} either \mathcal{X} or Y , the WDVV relations give:

$$\begin{aligned} & D(1, n+1|2, n+2) \left\langle \left\langle \frac{\phi_{\alpha_1}}{z_1 - \psi_1} \cdots \frac{\phi_{\alpha_n}}{z_n - \psi_n} \mathbf{1}_{\mathcal{Z}} \cdot \mathbf{1}_{\mathcal{Z}} \right\rangle \right\rangle_{0, n+2} \\ &= D(1, 2|n+1, n+2) \left\langle \left\langle \frac{\phi_{\alpha_1}}{z_1 - \psi_1} \cdots \frac{\phi_{\alpha_n}}{z_n - \psi_n} \mathbf{1}_{\mathcal{Z}} \cdot \mathbf{1}_{\mathcal{Z}} \right\rangle \right\rangle_{0, n+2}, \end{aligned} \quad (92)$$

where $D(i, j|k, l)$ is the divisor in the moduli space which separates the points i, j from k, l (pull-back of the class of a boundary point in $\overline{M}_{0,4}$). Expanding (92), we have

$$\begin{aligned} & \sum_{\substack{1 \in J \\ 2 \notin J}} \left(\left\langle \left\langle \mathbf{1}_{\mathcal{Z}} \cdot \prod_{i \in J} \frac{\phi_{\alpha_i}}{z_i - \psi_i} \cdot \phi_\beta \right\rangle \right\rangle_{0, |J|+2} \phi^\beta, \left\langle \left\langle \mathbf{1}_{\mathcal{Z}} \cdot \prod_{i \notin J} \frac{\phi_{\alpha_i}}{z_i - \psi_i} \cdot \phi_\epsilon \right\rangle \right\rangle_{0, |J^c|+2} \phi^\epsilon \right)_{\mathcal{Z}} \\ &= \sum_{\substack{1, 2 \in J \\ J^c \neq \emptyset}} \left(\left\langle \left\langle \prod_{i \in J} \frac{\phi_{\alpha_i}}{z_i - \psi_i} \cdot \phi_\beta \right\rangle \right\rangle_{0, |J|+1} \phi^\beta, \left\langle \left\langle \mathbf{1}_{\mathcal{Z}} \cdot \mathbf{1}_{\mathcal{Z}} \cdot \prod_{i \notin J} \frac{\phi_{\alpha_i}}{z_i - \psi_i} \cdot \phi_\epsilon \right\rangle \right\rangle_{0, |J^c|+3} \phi^\epsilon \right)_{\mathcal{Z}} \end{aligned} \quad (93)$$

where the sum index J ranges over subsets of $\{1, \dots, n\}$. Applying the string equation to eliminate the fundamental class insertions and summing over all α_i , we obtain the relation

$$\begin{aligned} & \sum_{\substack{1 \in J \\ 2 \notin J}} \left(\left(\sum_{i \in J} \frac{1}{z_i} \right) L_{\mathcal{Z}}^{|J|}(\tau, \mathbf{z}_J), \left(\sum_{i \notin J} \frac{1}{z_i} \right) L_{\mathcal{Z}}^{|J^c|}(\tau, \mathbf{z}_{J^c}) \right)_{\mathcal{Z}} \\ &= \sum_{\substack{1, 2 \in J \\ J^c \neq \emptyset}} \left(L_{\mathcal{Z}}^{|J|}(\tau, \mathbf{z}_J), \left(\sum_{i \notin J} \frac{1}{z_i} \right)^2 L_{\mathcal{Z}}^{|J^c|}(\tau, \mathbf{z}_{J^c}) \right)_{\mathcal{Z}} + \left(\sum_{1 \leq i \leq n} \frac{1}{z_i} \right) J_n^{\mathcal{Z}}(\tau, \mathbf{z}) \end{aligned} \quad (94)$$

where $(-, -)_{\mathcal{Z}}$ is extended by applying Poincaré pairing on the last coordinate and tensoring the remaining coordinates in the appropriate order. Equation (94) allows us to write J_n for either \mathcal{X} or Y in terms of L^m with $m < n$. Since $\widehat{\mathbb{U}}^m$ identifies $L_{\mathcal{X}}^m$ with L_Y^m and \mathbb{U}_0 preserves the Poincaré pairing, it follows that $\mathbb{U}_\rho^{\mathcal{X}, Y \otimes n}(\mathbf{z})(J_n^{\mathcal{X}}(x, \mathbf{z})) = J_n^Y(t, \mathbf{z})$. \square

4. OCRC FOR A_n RESOLUTIONS

For the pairs $(\mathcal{X}, Y) = ([\mathbb{C}^3/\mathbb{Z}_{n+1}], A_n)$, Propositions 3.5 and 3.8 imply a Bryan-Graber type CRC statement comparing the open GW potentials. Notice that since \mathcal{X} is a Hard Lefschetz orbifold we do not have to deal with the normalization factors \mathfrak{h}_\bullet . In Sections 4.2 and 4.3 we study the two essentially distinct types of Lagrangian boundary conditions.

4.1. Equivariant $\mathbb{U}_\rho^{\mathcal{X}, Y}$ and Integral Structures. We write the equivariant version of the symplectomorphism $\mathbb{U}_\rho^{\mathcal{X}, Y}$.

Theorem 4.1. *With notation as in Theorem 2.8, let $\widetilde{J}^Y(z)$ denote the analytic continuation of J^Y along the path ρ to the point $\rho(1)$ composed with the identification (75) of quantum parameters. Then the linear transformation*

$$\mathbb{U}_\rho^{\mathcal{X}, Y} \mathbf{1}_{\mathbf{k}} = \sum_i P_i \frac{1}{(n+1)} \frac{\widehat{\Gamma}_Y^i}{\widehat{\Gamma}_{\mathcal{X}}^k} \left(\sum_{j=0}^{i-1} \omega^{-jk} e^{2\pi i \frac{j\alpha_1}{z}} + \sum_{j=i}^n \omega^{-jk} e^{2\pi i \frac{(n+1-j)\alpha_2}{z}} \right) \quad (95)$$

is a linear isomorphism of Givental spaces such that

$$\tilde{J}^Y = \mathbb{U}_{K,\rho}^{\mathcal{X},Y} \circ J^{\mathcal{X}}. \quad (96)$$

The proof of this theorem in the equivariant setting is a consequence of our computations in Section 5.

Remark 4.2. In (42) we can replace the operator $z^{-\mu}$ by the ‘‘homogenization operator’’ $z^{-\frac{1}{2} \deg}$ since the additional $z^{\frac{3}{2}}$ part contributes two canceling scalar factors.

We observe now that this result is compatible with Iritani’s proposal (46). We first describe the canonical identification $\mathbb{U}_{K,\rho}^{\mathcal{X},Y}$. Denote by $\mathcal{O}(\lambda_k)$ the geometrically trivial line bundle on \mathbb{C}^{n+3} where the torus $(\mathbb{C}^*)^n$ acts via the k th factor with weight -1 and the torus T acts trivially. We define our grade restriction window $\mathfrak{W} \subset K(\mathbb{C}^{n+3})$ to be the subgroup generated by the $\mathcal{O}(\lambda_k)$. Using the description of the local coordinates in Section 2.3.1, we compute that the quotient (54) identifies $\mathcal{O}(\lambda_k)$ with \mathcal{O}_{-k} (with trivial T -action) and the quotient (55) identifies $\mathcal{O}(\lambda_k)$ with $\mathcal{O}(\gamma_k)$ (with canonical linearization (62)). Therefore, we define $\mathbb{U}_{K,\rho}^{\mathcal{X},Y}$ by identifying

$$\mathcal{O}_Y \leftrightarrow \mathcal{O}_{\mathcal{X}} \quad (97)$$

$$\mathcal{O}(\gamma_k) \leftrightarrow \mathcal{O}_{-k} \quad (98)$$

where the T -linearizations are trivial on the orbifold and canonical on the resolution.

On the orbifold, all of the bundles \mathcal{O}_j are linearized trivially, so the higher Chern classes vanish. The orbifold Chern characters are:

$$(2\pi i)^{\deg/2} I^* \text{ch}(\mathcal{O}_j) = \sum_{k=1}^{n+1} \omega^{-jk} \mathbf{1}_k. \quad (99)$$

The Γ class is

$$z^{-\frac{1}{2} \deg} \hat{\Gamma}_{\mathcal{X}} = \Gamma \left(1 + \frac{\alpha_1 + \alpha_2}{z} \right) \quad (100)$$

$$\cdot \left[\sum_{k=1}^n \Gamma \left(1 - \frac{k}{n+1} - \frac{\alpha_1}{z} \right) \Gamma \left(\frac{k}{n+1} - \frac{\alpha_2}{z} \right) \frac{\mathbf{1}_k}{z} + \Gamma \left(1 - \frac{\alpha_1}{z} \right) \Gamma \left(1 - \frac{\alpha_2}{z} \right) \mathbf{1}_{n+1} \right] \quad (101)$$

On the resolution, the Chern roots at each P_i are the weights of the action on the fiber above that point:

$$(2\pi i)^{\deg/2} \text{ch}(\mathcal{O}(\gamma_j)) = \sum_{i=1}^j e^{2\pi i(n+1-j)\alpha_2} P_i + \sum_{i=j+1}^{n+1} e^{2\pi i j \alpha_1} P_i \quad (102)$$

and

$$(2\pi i)^{\deg/2} \text{ch}(\mathcal{O}) = \sum_{i=1}^{n+1} P_i. \quad (103)$$

The Γ class is

$$z^{-\frac{1}{2} \deg} \hat{\Gamma}_Y = \Gamma \left(1 + \frac{\alpha_1 + \alpha_2}{z} \right) \left[\sum_{i=1}^{n+1} \Gamma \left(1 + \frac{w_i^+}{z} \right) \Gamma \left(1 + \frac{w_i^-}{z} \right) P_i \right] \quad (104)$$

With this information one can compute (46) and obtain the formula in Theorem 4.1.

We now derive explicit disk potential CRC statements for the two distinct types of Lagrangian boundary conditions.

4.2. **L intersects the ineffective axis.** We impose a Lagrangian boundary condition on the gerby leg of the orbifold (the third coordinate axis - $m_3 = 0$); correspondingly there are $n + 1$ boundary conditions L' on the resolution, intersecting the horizontal torus fixed lines in Figure 1.

Theorem 4.3. *Consider the cohomological disk potentials $\mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w})$ and $\mathcal{F}_{L',Y}^{\text{disk}}(t, y_{P_1}, \dots, y_{P_{n+1}}, \vec{w})$. Choosing the bases $\mathbf{1}_k$ and P_i (where k and i both range from 1 to $n + 1$), define a linear transformation $\mathbb{O}_{\mathbb{Z}} : H(\mathcal{X}) \rightarrow H(Y)$ by the matrix*

$$\mathbb{O}_{\mathbb{Z}}{}_{i,k} = \begin{cases} -\omega^{\left(\frac{1}{2}-i\right)k} & k \neq n+1 \\ -1 & k = n+1. \end{cases} \quad (105)$$

After the identification of variables from Theorem 2.8, and the specialization of winding parameters

$$y_{P_i} = e^{\pi i \left[\frac{w_i^- + (2i-1)\alpha_1}{\alpha_1 + \alpha_2} \right]} y \quad (106)$$

we have

$$\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}) = \mathbb{O}_{\mathbb{Z}} \circ \mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}). \quad (107)$$

Proof. From equation (77), we have

$$\widehat{\mathcal{D}}_{L,\mathcal{X}}^+(z; \vec{w})(\mathbf{1}_k) = \sum_{k=1}^{n+1} \frac{\pi \mathbf{1}_k}{(n+1)(\alpha_1 + \alpha_2) \sin\left(\pi \left(\left\langle \frac{k}{n+1} \right\rangle + \frac{\alpha_2}{z}\right)\right)} \widehat{\Gamma}_{\mathcal{X}}^k \quad (108)$$

and

$$\widehat{\mathcal{D}}_{L',Y}^+(z; \vec{w})(P_i) = \sum_{i=1}^{n+1} \frac{\pi P_i}{(\alpha_1 + \alpha_2) \sin\left(\pi \left(-\frac{w_i^-}{z}\right)\right)} \widehat{\Gamma}_Y^i \quad (109)$$

The transformation \mathbb{O} is now obtained as $\widehat{\mathcal{D}}_Y^+ \circ \mathbb{U}_{\rho}^{\mathcal{X},Y} \circ (\mathcal{D}_{\mathcal{X}}^+)^{-1}$:

$$\mathbb{O}(\mathbf{1}_k) = \sum_{i=1}^{n+1} \left[\frac{\sin\left(\pi \left(\left\langle \frac{k}{n+1} \right\rangle + \frac{\alpha_2}{z}\right)\right)}{\sin\left(\pi \left(-\frac{w_i^-}{z}\right)\right)} \left(\sum_{j=0}^{i-1} \omega^{-jk} e^{2\pi i \frac{j\alpha_1}{z}} + \sum_{j=i}^n \omega^{-jk} e^{2\pi i \frac{(n+1-j)\alpha_2}{z}} \right) \right] P_i. \quad (110)$$

We now specialize $z = \frac{\alpha_1 + \alpha_2}{d}$, for $d \in \mathbb{Z}$. The i, k coefficient for $k \neq n + 1$ is:

$$\begin{aligned} \mathbb{O}_{i,k} &= \frac{\sin\left(\pi \left(\frac{k}{n+1} + 1 - \frac{d\alpha_1}{\alpha_1 + \alpha_2}\right)\right)}{\sin\left(\pi \left(-d \frac{(n+1)\alpha_1}{\alpha_1 + \alpha_2} + d(n-i+2)\right)\right)} \left(\sum_{j=0}^{i-1} \omega^{-jk} e^{2\pi i j \frac{\alpha_1}{\alpha_1 + \alpha_2} d} \right. \\ &\quad \left. + \sum_{j=i}^n \omega^{-jk} e^{2\pi i (n+1-j) \left(1 - \frac{\alpha_1}{\alpha_1 + \alpha_2}\right) d} \right) \\ &= (-1)^{d(n-i+2)} \frac{\omega^{k/2} e^{-\pi i \frac{\alpha_1}{\alpha_1 + \alpha_2} d} - \omega^{-k/2} e^{\pi i \frac{\alpha_1}{\alpha_1 + \alpha_2} d}}{e^{\pi i (n+1) \frac{\alpha_1}{\alpha_1 + \alpha_2} d} - e^{-\pi i (n+1) \frac{\alpha_1}{\alpha_1 + \alpha_2} d}} \sum_{j=-(n+1)+i}^{i-1} \omega^{-jk} e^{2\pi i j \frac{\alpha_1}{\alpha_1 + \alpha_2} d} \\ &= (-1)^{d(n-i+2)+1} \omega^{\left(\frac{1}{2}-i\right)k} e^{\pi i (2i-n-2) \frac{\alpha_1}{\alpha_1 + \alpha_2} d} \\ &= (-1)^d e^{d\pi i \left[n-i+2 + (2i-n-2) \frac{\alpha_1}{\alpha_1 + \alpha_2} \right]} \omega^{\left(\frac{1}{2}-i\right)k}. \end{aligned} \quad (111)$$

For $k = n + 1$

$$\mathbb{O}_{i,n+1} = (-1)^d e^{d\pi i \left[n-i+2 + (2i-n-2) \frac{\alpha_1}{\alpha_1 + \alpha_2} \right]}. \quad (112)$$

To go from the second to the third line of this string of equations one notes that the product of the numerator of the fraction with the summation gives a telescoping sum; the residual terms have a factor canceling the denominator and leaving the expression in the third line. It

is now immediate to see that we can incorporate the part of the transformation that depends multiplicatively on d into a specialization of the winding variables, and that the remaining linear map is precisely $\mathbb{O}_{\mathbb{Z}}$. \square

From this formulation of the disk CRC one can deduce a statement about scalar disk potentials which essentially says that the scalar potential of the resolution compares with the untwisted disk potential on the orbifold.

Corollary 4.4. *With all notation as in Theorem 4.3:*

$$\left(\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}), \sum_{i=1}^{n+1} P^i \right)_Y = -\frac{1}{n+1} \left(\mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}), \mathbf{1}^{n+1} \right)_{\mathcal{X}} \quad (113)$$

Proof. This statement amounts to the fact that the coefficients of all but the last column of matrix $\mathbb{O}_{\mathbb{Z}}$ add to zero. \square

4.3. L intersects the effective axis. We impose our boundary condition L on the first coordinate axis, which is an effective quotient of \mathbb{C} with representation weight $m_1 = -1$ and torus weight $-\alpha_1$. We can obtain results for the boundary condition on the second axis by switching α_1 with α_2 , m_1 with m_2 and $+$ with $-$ in the orientation of the disks. In this case there is only one corresponding boundary condition L' on the resolution, which intersects the (diagonal) non compact leg incident to P_{n+1} in Figure 1.

Theorem 4.5. *Consider the cohomological disk potentials $\mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w})$ and $\mathcal{F}_{L',Y}^{\text{disk}}(t, y_{P_{n+1}}, \vec{w})$. Choosing the bases $\mathbf{1}_{\mathbf{k}}$ and P_i (where k and i both range from 1 to $n+1$), define $\mathbb{O}_{\mathbb{Z}}(\mathbf{1}_{\mathbf{k}}) = P_{n+1}$ for every k . After the identification of variables from Theorem 2.8, and the identification of winding parameters $y = y_{P_{n+1}}$ we have*

$$\mathcal{F}_{L',Y}^{\text{disk}}(t, y, \vec{w}) = \mathbb{O}_{\mathbb{Z}} \circ \mathcal{F}_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}). \quad (114)$$

We obtain as an immediate corollary a comparison among scalar potentials.

Corollary 4.6. *Setting $y = y_{P_{n+1}}$, we have*

$$F_{L',Y}^{\text{disk}}(t, y, \vec{w}) = F_{L,\mathcal{X}}^{\text{disk}}(t, y, \vec{w}). \quad (115)$$

Proof. The orbifold disk endomorphism is:

$$\widehat{\mathcal{D}}_{\mathcal{X}}^+(z; \vec{w})(\mathbf{1}_{\mathbf{k}}) = \frac{\pi}{-\alpha_1(n+1) \sin\left(\pi\left(-\frac{\alpha_1+\alpha_2}{z}\right)\right)} \widehat{\Gamma}_{\mathcal{X}}^k \mathbf{1}_{\mathbf{k}} \quad (116)$$

The resolution disk endomorphism is

$$\widehat{\mathcal{D}}_{\mathcal{X}}^+(z; \vec{w})(P_i) = \frac{\pi}{-(n+1)\alpha_1 \sin\left(\pi\left(-\frac{\alpha_1+\alpha_2}{z}\right)\right)} \widehat{\Gamma}_Y^{n+1} \delta_{i,n+1} P_{n+1} \quad (117)$$

We can now compute \mathbb{O} :

$$\mathbb{O}(\mathbf{1}_{\mathbf{k}}) = \frac{1}{n+1} \left(\sum_{j=0}^n \omega^{-jk} e^{2\pi i j \frac{j\alpha_1}{z}} \right) P_{n+i}. \quad (118)$$

Specializing $z = \frac{-(n+1)\alpha_1}{d}$ for any positive integer d , we obtain:

$$\mathbb{O}_{n+1,k} = \frac{1}{n+1} \sum_{j=0}^n \omega^{-jk} e^{2\pi i j \frac{-d}{n+1}} = \delta_{k,-d \pmod{n+1}}, \quad (119)$$

which implies the statement of the theorem. \square

5. ONE-DIMENSIONAL MIRROR SYMMETRY

In this section we exhibit a novel mirror symmetry description of the equivariant quantum cohomology of A_n singularities in terms of a weak Frobenius manifold structure on a genus zero double Hurwitz space. In physics terminology, this is a logarithmic Landau–Ginzburg model on the sphere, akin to the Hori–Iqbal–Vafa spectral curves of non-equivariant Gromov–Witten theory [43] and the one-dimensional mirror of equivariant local $\mathbb{C}\mathbb{P}^1$ [13]. This enables us to give a complete description of the global quantum D -module and to determine explicitly the form of the isomorphism between the calibrated Frobenius structures at the large volume points of \mathcal{X} and Y .

5.1. Weak Frobenius structures on double Hurwitz spaces.

Definition 5.1. Let $\mathbf{x} \in \mathbb{Z}^{n+3}$ be a vector of integers adding to 0. The double Hurwitz space $\mathcal{H}_\lambda \triangleq \mathcal{M}_0(\mathbb{P}^1; \mathbf{x})$ parameterizes isomorphism classes of covers λ of the projective line by a smooth genus 0 curve C , with marked ramification profile over 0 and ∞ specified by \mathbf{x} . This means that the principal divisor of λ is of the form

$$(\lambda) = \sum x_i P_i.$$

We denote by π and λ the universal family and universal map, and by Σ_i the sections marking the i -th point in (λ) :

$$\begin{array}{ccc} \mathbb{P}^1 & \hookrightarrow & \mathcal{U} \xrightarrow{\lambda} \mathbb{P}^1 \\ \uparrow P_i & & \uparrow \Sigma_i \\ \downarrow & & \downarrow \pi \\ [\lambda] & \xrightarrow{pt.} & \mathcal{H}_\lambda \end{array} \quad (120)$$

Remark 5.1. A genus zero double Hurwitz space is naturally isomorphic to $M_{0,n+3}$, and is therefore an open set in affine space \mathbb{A}^n . This is the only case that we utilize and it may seem overly sophisticated to use the language of moduli spaces to then work on such a simple object. We choose to do so to connect to the work of Dubrovin [33, 34] and Romano [56] (after Saito [60]; see also [48]), who studied existence and construction of Frobenius structures on arbitrary double Hurwitz spaces.

Let $\phi \in \Omega_C^1(\log(\lambda))$ be a meromorphic one form having simple poles at the support of (λ) with constant residues; we call (λ, ϕ) respectively the *superpotential* and the *quasi-momentum differential* of \mathcal{H}_λ . Borrowing the terminology from [56, 57], we say that an analytic Frobenius manifold structure $(\mathcal{F}, \circ, \eta)$ on a complex manifold \mathcal{F} is *weak* if

- (1) the \circ -multiplication gives a commutative and associative unital \mathcal{O} -algebra structure on the space of holomorphic vector fields on \mathcal{F} ;
- (2) the metric η provides a flat pairing which is Frobenius w.r.t. to \circ ;
- (3) the algebra structure admits a *potential*, meaning that the 3-tensor

$$R(X, Y, Z) \triangleq \eta(X, Y \circ Z) \quad (121)$$

satisfies the integrability condition

$$(\nabla^{(\eta)} R)_{[\alpha\beta]\gamma\delta} = 0. \quad (122)$$

In particular, this encompasses non-quasihomogeneous solutions of WDVV, and solutions without a flat identity element.

Proposition 5.2 ([56]). For vector fields $X, Y, Z \in \mathfrak{X}(\mathcal{H}_\lambda)$, define the non-degenerate symmetric pairing g and quantum product \star as

$$g(X, Y) \triangleq \sum_{P \in \text{supp}(\lambda)} \text{Res}_P \frac{X(\log \lambda) Y(\log \lambda)}{d_\pi \log \lambda} \phi^2, \quad (123)$$

$$g(X, Y \star Z) \triangleq \sum_{P \in \text{supp}(\lambda)} \text{Res}_P \frac{X(\log \lambda) Y(\log \lambda) Z(\log \lambda)}{d_\pi \log \lambda} \phi^2, \quad (124)$$

where d_π denotes the relative differential with respect to the universal family (i.e. the differential in the fiber direction). Then the triple $\mathcal{F}_{\lambda,\phi} = (\mathcal{H}_\lambda, \star, g)$ endows \mathcal{H}_λ with a weak Frobenius manifold structure.

Remark 5.3. Equations (123)-(124) are the Dijkgraaf–Verlinde–Verlinde formulae [32] for a topological Landau–Ginzburg model on a sphere with $\log \lambda(q)$ as its superpotential. The case in which $\lambda(q)$ itself is used as the superpotential gives rise to a *different* Frobenius manifold structure, which is the case originally studied in [34, Lecture 5]; the situation at hand is its Dubrovin-dual in the sense of [35], where g plays the role of the intersection form and \star the dual product.

5.1.1. *Twisted periods and the quantum differential equation.* The quantum D -module associated to $\mathcal{F}_{\lambda,\phi}$,

$$\nabla^{(g,z)}\omega = 0, \quad (125)$$

where

$$\nabla_X^{(g,z)}(Y, z) \triangleq \nabla_X^{(g)}Y + z^{-1}X \star Y \quad (126)$$

enjoys a neat description in terms of the Landau–Ginzburg data (λ, ϕ) : in particular, flat frames for (125) can be computed from the twisted Picard–Lefschetz theory of λ [13, 35, 38]. In contrast with the classical Picard–Lefschetz theory, this corresponds to considering cycles $\gamma \in H_1(\mathbb{C} \setminus H, \mathbf{L})$ in the *complement* of the zero-dimensional hypersurface $H = \lambda^{-1}(0)$ cut by λ , where the linear local system \mathbf{L} is defined by multiplication by $e^{2\pi i/z}$ when moving along a simple loop around any single point of H . Elements γ of the homology group with coefficients twisted by \mathbf{L} are the *twisted cycles* of λ .

Oscillating integrals around a basis of twisted cycles of the form

$$\Pi_{\lambda,\phi,\gamma}(z) \triangleq \int_\gamma \lambda^{1/z} \phi \quad (127)$$

are called *twisted periods*⁴ of $\mathcal{F}_{\lambda,\phi}$. Denote by $\text{Sol}_{\lambda,\phi}$ the solution space of (125),

$$\text{Sol}_{\lambda,\phi} = \{s \in \mathfrak{X}(\mathcal{F}_{\lambda,\phi}), \nabla^{(g,z)}s = 0\}. \quad (128)$$

We have the following

Proposition 5.4 (Dubrovin, [35]). *The solution space of the quantum differential equations of $\mathcal{F}_{\lambda,\phi}$ is generated by gradients of the twisted periods (127)*

$$\text{Sol}_{\lambda,\phi} = \text{span}_{\mathbb{C}((z))} \{ \nabla^{(g)} \Pi_{\lambda,\phi,\gamma} \}_{\gamma \in H_1(\mathbb{C} \setminus H, \mathbf{L})} \quad (129)$$

In particular, Proposition 5.4 implies that the quantum D -modules arising from weak Frobenius structures on genus zero double Hurwitz spaces are described by systems of period integrals of generalized hypergeometric type.

Remark 5.5. Since λ is a genus zero covering map, in an affine chart parametrized by $q \in \mathbb{C}$ its logarithm takes the form

$$\log \lambda = \sum_i a_i \log(q - q_i), \quad (130)$$

where $a_i \in \mathbb{Z}$. In fact, the existence of the weak Frobenius structure (123)-(124) extends [57] to the case where $d_\pi \log \lambda$ is a meromorphic function on C ; this in particular encompasses the case where $a_i \in \mathbb{C}$ in (130). As far as flat coordinates of the deformed connection $\nabla^{(g,z)}$ are concerned, Proposition 5.4 continues to hold, the only proviso being that the locally constant sheaf \mathbf{L} be replaced with the unique local system specified by the monodromy weights a_i/z in (127), (130).

⁴To be completely consistent with [35] we should more correctly call these the *twisted periods* of $\mathcal{F}_{e^\lambda,\phi}$. See Remark 5.3.

5.2. A one-dimensional Landau–Ginzburg mirror. It is known that the quantum D -modules associated to the equivariant Gromov–Witten theory of the A_n -singularity \mathcal{X} and its resolution Y admit a Landau–Ginzburg description in terms of n -dimensional oscillating integrals [6, 28, 39, 44]. We provide here an alternative description via one-dimensional twisted periods of a genus zero double Hurwitz space $\mathcal{F}_{\lambda, \phi}$.

Let \mathcal{M}_A be $M_{0, n+3}$. By choosing the last three sections to be the constant sections $0, 1, \infty$, we realize \mathcal{M}_A as an open subset of \mathbb{A}^n and trivialize the universal family. In homogeneous coordinates $[u_0 : \cdots : u_n]$ for \mathbb{P}^n ,

$$\mathcal{M}_A = \mathbb{P}^n \setminus \text{Proj} \frac{\mathbb{C}[u_0, \dots, u_n]}{\langle u_i(u_j - u_k) \rangle} \triangleq \mathbb{P}^n \setminus \text{discr} \mathcal{M}_A. \quad (131)$$

Let $\kappa_i = u_i/u_0$, $i = 1, \dots, n$ be a set of global coordinates on \mathcal{M}_A and q be an affine coordinate on the fibers of the universal family. We give $\mathbb{C} \times \mathcal{M}_A$ the structure of a one parameter family of double Hurwitz spaces by specifying the pair (λ, ϕ) ; we call κ_0 the coordinate in the first factor, and define

$$\lambda(\kappa_0, \dots, \kappa_n, q) = C_n(\kappa) \frac{q^{(n+1)\alpha_1}}{(1-q)^{\alpha_1+\alpha_2}} \prod_{k=1}^n (1-q\kappa_k)^{-\alpha_1-\alpha_2}, \quad (132)$$

$$\phi(q) = \frac{1}{\alpha_1 + \alpha_2} \frac{dq}{q}, \quad (133)$$

and

$$C_n(\kappa) \triangleq \prod_{j=0}^n \kappa_j^{\alpha_1}. \quad (134)$$

Then Eqs. (123)-(124) and (132)-(133) define a Frobenius structure $\mathcal{F}_{\lambda, \phi}$ on $\mathbb{C} \times \mathcal{M}_A$; the discriminant ideal in (131) coincides with the locus where the D -module (126) is singular, and the irreducible components $V(\kappa_i - \kappa_j)$, for $i, j > 0$, correspond to the loci where the \star -product (124) blows-up. We have the following

Theorem 5.6. (1) Let

$$\kappa_0 = e^{(t_{n+1} + \delta_Y)/\alpha_1}, \quad (135)$$

$$\kappa_j = \prod_{i=j}^n e^{t_i}, \quad 1 \leq j \leq n. \quad (136)$$

where δ_Y is an arbitrary constant. Then, in a neighbourhood V_Y of $\{e^{t_i} = 0\}$,

$$\mathcal{F}_{\lambda, \phi} \simeq QH_T(Y). \quad (137)$$

(2) Let

$$\kappa_0 = e^{(x_{n+1} + \delta_{\mathcal{X}})/\alpha_1}, \quad (138)$$

$$\kappa_j = \exp \left[-\frac{2i}{n+1} \left(\pi j + \sum_{k=1}^n e^{-\frac{i\pi k(j-1)}{n+1}} \sin \left(\frac{\pi j k}{n+1} \right) x_k \right) \right], \quad 1 \leq k \leq n. \quad (139)$$

where $\delta_{\mathcal{X}}$ is an arbitrary constant. Then, in a neighbourhood $V_{\mathcal{X}}$ of $\{x_i = 0\}$,

$$\mathcal{F}_{\lambda, \phi} \simeq QH_T(\mathcal{X}). \quad (140)$$

Proof. The proof is a straightforward computation from the Landau–Ginzburg formulae (123)-(124).

(1) Consider the three-point correlator $R(\kappa_i \partial_i, \kappa_j \partial_j, \kappa_k \partial_k)$, where $\partial_k \triangleq \frac{\partial}{\partial \kappa_k}$, and define

$$R_{i,j,k}^{(l)} \triangleq \text{Res}_{q=\kappa_l^{-1}} \frac{\kappa_i \frac{\partial \ln \lambda}{\partial \kappa_i} \kappa_j \frac{\partial \ln \lambda}{\partial \kappa_j} \kappa_k \frac{\partial \ln \lambda}{\partial \kappa_k} dq}{(\alpha_1 + \alpha_2)^2 q \frac{\partial \ln \lambda}{\partial q} q}. \quad (141)$$

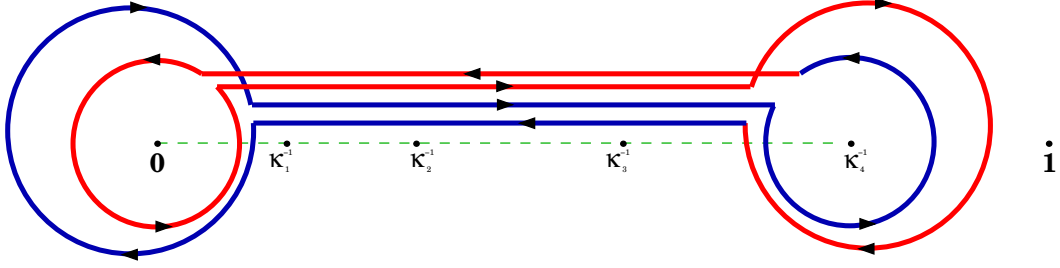


FIGURE 3. The double loop contour γ_4 for $n = 4$.

Inspection shows that $R_{ijk}^{(l)} = 0$ unless $l = i = j$, $l = i = k$ or $l = j = k$. Assume w.l.o.g. $l = j = i$, and suppose that $i, k > 0$. We compute

$$R_{i,i,k}^{(i)} = \frac{\kappa_i}{\kappa_k - \kappa_i} + \frac{\alpha_2}{\alpha_1 + \alpha_2}, \quad (142)$$

$$R_{i,i,i}^{(i)} = \frac{(n-1)\alpha_1 + \alpha_2}{\alpha_1 + \alpha_2} + \sum_{l \neq i}^{n+1} \frac{\kappa_l}{\kappa_i - \kappa_l}, \quad (143)$$

$$R_{0,i,i}^{(i)} = -\frac{1}{\alpha_1 + \alpha_2}. \quad (144)$$

Moreover, for all i, j and k we have

$$\begin{aligned} R_{i,j,k}^{(0)} &\triangleq \operatorname{Res}_{q=0} \frac{\kappa_i \frac{\partial \ln \lambda}{\partial \kappa_i} \kappa_j \frac{\partial \ln \lambda}{\partial \kappa_j} \kappa_k \frac{\partial \ln \lambda}{\partial \kappa_k}}{(\alpha_1 + \alpha_2)^2 q \frac{\partial \ln \lambda}{\partial q}} \frac{dq}{q}, \\ &= \frac{\alpha_1^{2-\delta_{i,n+1}-\delta_{j,n+1}-\delta_{k,n+1}}}{(n+1)(\alpha_1 + \alpha_2)^2} \end{aligned} \quad (145)$$

$$\begin{aligned} R_{i,j,k}^{(\infty)} &\triangleq \operatorname{Res}_{q=\infty} \frac{\kappa_i \frac{\partial \ln \lambda}{\partial \kappa_i} \kappa_j \frac{\partial \ln \lambda}{\partial \kappa_j} \kappa_k \frac{\partial \ln \lambda}{\partial \kappa_k}}{(\alpha_1 + \alpha_2)^2 q \frac{\partial \ln \lambda}{\partial q}} \frac{dq}{q}, \\ &= -\frac{(-\alpha_2)^{2-\delta_{i,n+1}-\delta_{j,n+1}-\delta_{k,n+1}}}{(n+1)(\alpha_1 + \alpha_2)^2}. \end{aligned} \quad (146)$$

It is immediate to see that (142)-(143) under the identification (136) imply that the quantum part of the three-point correlator $R(\partial_{t_1}, \partial_{t_2}, \partial_{t_3})$ coincides with that of $\langle\langle p_{i_1}, p_{i_2}, p_{i_3} \rangle\rangle_0^Y$ in (72). A tedious, but straightforward computation shows that (142)-(146) yield the expressions for the classical triple intersection numbers of Y .

(2) This is a consequence of the computation above and Theorem 2.8. □

Remark 5.7. The freedom of shift by $\delta_{\mathcal{X}}$ and δ_Y respectively along $H^0(\mathcal{X})$ and $H^0(Y)$ in (135), (138) is a consequence of the restriction of the String Axiom to the small phase space. We set $\delta_{\mathcal{X}} = \delta_Y = 0$ throughout this section, but it will turn out to be useful to reinstate the shifts in the computations of Section 5.4.

Remark 5.8. It should be possible to infer the form of the superpotential (132) from the equivariant GKZ system of \mathcal{X} and Y by arguments similar to the non-equivariant case (see e.g. [26, Appendix A]). The conceptual path we followed to conjecture the form (132) for a candidate dual Landau–Ginzburg model parallels the study of the equivariant local \mathbb{CP}^1 theory in [13]; there, the existence of a relation with a reduction of the 2-dimensional Toda hierarchy allows to derive a Landau–Ginzburg mirror model through the dispersionless Lax formalism for 2-Toda. More generally, (n, m) -graded reductions [57] of 2-Toda are believed to be relevant

for the equivariant Gromov–Witten theory of local $\mathbb{P}(n, m)$ [12]; the degenerate limit $m = 0$ corresponds to the threefold A_n singularity. In this case, the dispersionless 2-Toda Lax function reduces to (132).

5.3. The global quantum D -module. An immediate corollary of Theorem 5.6 and Proposition 5.4 is a concrete description of a global quantum D -module $(\mathcal{M}_A, F, \nabla, H(\cdot, \cdot)_g)$ interpolating between $\text{QDM}(\mathcal{X})$ and $\text{QDM}(Y)$. Let $F \triangleq T\mathcal{F}_{\lambda, \phi}$ be endowed with the family of connections $\nabla = \nabla^{(g, z)}$ as in (126) and for ∇ -flat sections s_1, s_2 let

$$H(s_1, s_2)_g = g(s_1(\kappa, -z), s_2(\kappa, z)) \quad (147)$$

Let now $V_{\mathcal{X}}$ and V_Y be neighbourhoods of $\{\kappa_i = \omega^{-i}\}$ and $\{\kappa_i = 0\}$ respectively. Then Theorem 5.6 can be rephrased as

$$(\mathcal{F}_{\lambda, \phi}, T\mathcal{F}_{\lambda, \phi}, \nabla^{(g, z)}, H(\cdot, \cdot)_g)|_{V_{\mathcal{X}}} \simeq \text{QDM}(\mathcal{X}), \quad (148)$$

$$(\mathcal{F}_{\lambda, \phi}, T\mathcal{F}_{\lambda, \phi}, \nabla^{(g, z)}, H(\cdot, \cdot)_g)|_{V_Y} \simeq \text{QDM}(Y), \quad (149)$$

that is, the twisted period system of $\mathcal{F}_{\lambda, \phi}$ is a global quantum D -module connecting the genus zero descendent theory of \mathcal{X} and Y ; the twisted periods (127) thus define a global flat frame for the quantum differential equations of \mathcal{X} and Y upon analytic continuation in the κ -variables,

$$\text{Sol}_{\lambda, \phi}|_{V_{\mathcal{X}}} = \mathcal{S}_{\mathcal{X}}, \quad \text{Sol}_{\lambda, \phi}|_{V_Y} = \mathcal{S}_Y. \quad (150)$$

A canonical basis of $\text{Sol}_{\lambda, \phi}$ can be constructed as follows. For the superpotential (132), the twisted homology $H_1(\mathbb{C} \setminus \lambda^{-1}(0), \mathbf{L})$ is generated [66] by Pochhammer double loop contours $\{\xi_i\}_{i=1}^{n+1}$ encircling the origin $q = 0$ and $q = \kappa_i^{-1}$, $i = 1, \dots, n+1$, as in Figure 3 (alternatively $\xi_i = [\rho_0, \rho_i]$, where the ρ 's are simple oriented loops around each of the punctures). Then the integrals

$$\begin{aligned} \Pi_i^{(n)}(\kappa, z) &\triangleq \frac{1}{(1 - e^{2\pi i a})(1 - e^{-2\pi i b})} \int_{\xi_i} \lambda^{1/z}(q) \frac{dq}{q} \\ &= \frac{C_n(\kappa)^{\frac{1}{z}}}{(1 - e^{2\pi i a})(1 - e^{-2\pi i b})} \int_{\xi_i} q^a (1 - q)^{-b} \prod_{k=1}^n (1 - q\kappa_k)^{-b} \frac{dq}{q} \\ &= \frac{C_n(\kappa)^{\frac{1}{z}} \kappa_i^{-a}}{(1 - e^{2\pi i a})(1 - e^{-2\pi i b})} \int_{\xi_{n+1}} q^a (1 - q)^{-b} (1 - q/\kappa_i)^{-b} \prod_{k \neq i}^n (1 - q\kappa_k/\kappa_i)^{-b} \frac{dq}{q} \end{aligned} \quad (151)$$

where we defined

$$a \triangleq \frac{(n+1)\alpha_1}{z}, \quad (152)$$

$$b \triangleq \frac{\alpha_1 + \alpha_2}{z}, \quad (153)$$

give a basis of twisted periods of $\mathcal{F}_{\lambda, \phi}$; when $\Re(a) > 0$, $\Re(b) < 1$ they reduce to line integrals along chains connecting $q = 0$ to $q = \kappa_i^{-1}$.

The integrals (151) can be given very explicit expressions in terms of known generalized hypergeometric functions [36]. Namely, we have

$$\begin{aligned} \Pi_i^{(n)}(\kappa, z) &= \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} C_n(\kappa)^{\frac{1}{z}} \kappa_i^{-a} \\ &\quad \times \Phi^{(n)}\left(a, b, 1+a-b; \frac{1}{\kappa_i}, \frac{\kappa_1}{\kappa_i}, \dots, \frac{\kappa_n}{\kappa_i}\right), \quad 1 \leq i \leq n, \end{aligned} \quad (154)$$

$$\Pi_{n+1}^{(n)}(\kappa, z) = \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} C_n(\kappa)^{\frac{1}{z}} \Phi^{(n)}(a, b, 1+a-b; \kappa_1, \dots, \kappa_n), \quad (155)$$

where we defined

$$\Phi^{(M)}(a, b, c, w_1, \dots, w_M) \triangleq F_D^{(M)}(a; b, \dots, b; c; w_1, \dots, w_M), \quad (156)$$

and $F_D^{(M)}(a; b_1, \dots, b_M; c; w_1, \dots, w_M)$ in (156) is the generalized hypergeometric Lauricella function of type D [49]:

$$F_D^{(M)}(a; b_1, \dots, b_M; c; w_1, \dots, w_M) \triangleq \sum_{i_1, \dots, i_M} \frac{(a)_{\sum_j i_j}}{(c)_{\sum_j i_j}} \prod_{j=1}^M \frac{(b_j)_{i_j} w_j^{i_j}}{i_j!}. \quad (157)$$

In (157), we used the Pochhammer symbol $(x)_m$ to denote the ratio $(x)_m = \Gamma(x+m)/\Gamma(x)$.

Remark 5.9. That flat sections of $\text{QDM}(\mathcal{X})$ and $\text{QDM}(Y)$ are solutions of a GKZ-type system, and therefore take the form of generalized hypergeometric functions in B -model variables, is a direct consequence of equivariant mirror symmetry for toric Deligne–Mumford stacks; see [26, Appendix A] for the case under study here, and [25] for the general case. Less expected, however, is the fact that flat sections of $\text{QDM}(\mathcal{X})$ and $\text{QDM}(Y)$ are hypergeometric functions in *exponentiated flat variables* for (6), that is, in A -model variables. This is a consequence of the particular form (72), (142)-(143) of the quantum product: this depends *rationally* on the variables in the Kähler cone for Y in such a way that the quantum differential equation (15) for Y (and therefore \mathcal{X} , via (75)) becomes a generalized hypergeometric system in exponentiated flat coordinates. From the vantage point of mirror symmetry, the rational dependence of the A -model three-point correlators on the quantum parameters can be regarded as an epiphenomenon of the Hard Lefschetz condition, which ensures that the inverse mirror map is a rational function of the B -model variables.

Remark 5.10. As a further surprising peculiarity of the case of A_n singularities, integral representations of the flat sections have a *simpler* description in A -model variables: the one-dimensional Euler integrals (151) replace here the n -fold Mellin-Barnes contour integrals that represent solutions of the corresponding GKZ system [26, 44]. This technical advantage is crucial for our calculations of Section 5.4. The reader may find a comparison of the Hurwitz mirror with the traditional approach of toric mirror symmetry in [15].

5.3.1. *Example: $n = 2$ and the Appell system.* In this case the quantum D -module has rank three. We factor out the dependence on $C_2(\kappa)$ in (154)-(155) for the flat coordinates of the deformed connection as

$$f(\kappa_1, \kappa_2, z) \triangleq (\kappa_0 \kappa_1 \kappa_2)^{-a/3} \tilde{t}(\kappa_0, \kappa_1, \kappa_2, z). \quad (158)$$

The flatness equations for $\nabla^{(g,z)}$ for $n = 2$ reduce to a hypergeometric Appell F_1 system [36] for f :

$$(\kappa_1 - \kappa_2) \partial_1 \partial_2 f - b(\partial_1 - \partial_2) f = 0, \quad (159)$$

$$\left[\kappa_1(1 - \kappa_1) \theta_1^2 + \kappa_2(1 - \kappa_1) \partial_{12} + (a + 1 - 2b) \partial_1 + \right. \\ \left. -(a + 1 + 2b) \kappa_1 \partial_1 - b \kappa_2 \partial_2 - ab \right] f = 0. \quad (160)$$

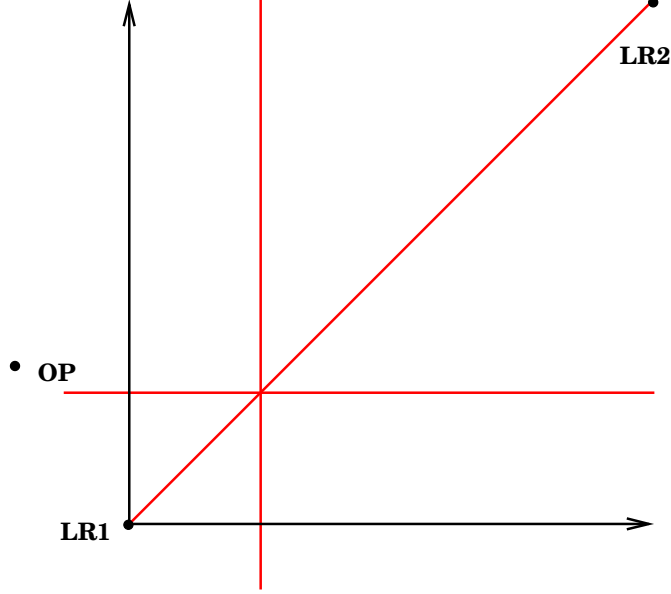


FIGURE 4. The Kähler moduli space of the A_2 singularity in A -model coordinates.

For $n = 2$, the twisted periods (154)-(155) reduce to Appell F_1 functions [36]

$$\begin{aligned} \Pi_1^{(2)}(\kappa_0, \kappa_1, \kappa_2, z) &= \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} C_2(\kappa)^{\frac{1}{z}} \kappa_1^{-a} \Phi^{(2)}\left(a, b, b, 1+a-b; \frac{1}{\kappa_1}, \frac{\kappa_2}{\kappa_1}\right) \\ &= \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} (\kappa_0\kappa_2)^{a/3} \kappa_1^{-a/3} F_1\left(a, b, b, 1+a-b, \frac{1}{\kappa_1}, \frac{\kappa_2}{\kappa_1}\right) \end{aligned} \quad (161)$$

$$\Pi_2^{(2)}(\kappa_0, \kappa_1, \kappa_2, z) = \Pi_1^{(2)}(\kappa_0, \kappa_2, \kappa_1, z) \quad (162)$$

$$\Pi_2^{(3)}(\kappa_0, \kappa_2, \kappa_1, z) = \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} (\kappa_0\kappa_1\kappa_2)^{a/3} F_1(a, b, b, 1+a-b, \kappa_1, \kappa_2) \quad (163)$$

where

$$F_1(a, b_1, b_2, c, x, y) \triangleq \sum_{i_1, i_2 \geq 0} \frac{(a)_{i_1+i_2}}{(c)_{i_1+i_2}} \frac{(b_1)_{i_1} x^{i_1}}{i_1!} \frac{(b_2)_{i_2} y^{i_2}}{i_2!}. \quad (164)$$

It is straightforward to check that (161)-(163) yield a complete set of solutions of (159)-(160).

In this case, irreducible components of the discriminant locus are given by the lines $\kappa_1 = \kappa_2$ and $\kappa_i = 0, 1, \infty$, $i = 1, 2$. Its moduli space is depicted in Figure 4. The large radius point of \mathcal{X} $(\kappa_1, \kappa_2) = (e^{4\pi i/3}, e^{2\pi i/3})$, denoted OP in Figure 4, is a regular point of the quantum D -module (159)-(160), and the Fuchsian singularities $(\kappa_1, \kappa_2) = (0, 0)$ and (∞, ∞) correspond to two copies of the large radius point (henceforth, LR) of Y , referred to as LR1 and LR2 in Figure 4. The Frobenius structure induced around the latter two points are canonically isomorphic to $QH_T(Y)$, and they are related to one another by the involution $\kappa_i \rightarrow -\kappa_i$. In contrast with the $n = 1$ case [15, 20], where the Appell system reduces to the Gauss ${}_2F_1$ -system, it is impossible here [36] to provide a local solution around LR of the Appell system (159)-(160) in terms of Appell F_1 -functions only; see Appendix A for a discussion of this point. Representing eigenvectors of the monodromy around LR in general in terms of the twisted period basis will be the subject of the first part of the proof of Theorem 4.1 in the next section.

5.4. Proof of Theorem 4.1. Let ρ be a straight line in \mathcal{M}_A connecting the large radius point $\{\kappa_j = 0\}$ of Y to the one of \mathcal{X} , given by $\{\kappa_j = \omega^{-j}\}$, with zero winding number around all irreducible components of the discriminant locus of \mathcal{M}_A . We compute the analytic continuation map $\mathbb{U}_\rho^{\mathcal{X}, Y} : \mathcal{H}_\mathcal{X} \rightarrow \mathcal{H}_Y$ that identifies the corresponding flat frames and Lagrangian cones upon

analytic continuation along ρ .

Define the period map Ω :

$$\begin{aligned} \Omega &: H_1(\mathbb{C} \setminus (\lambda), \mathbf{L}) \rightarrow \mathcal{O}_{\mathcal{F}_{\lambda, \phi}}, \\ &\quad \xi \rightarrow \int_{\xi} \lambda^{1/z} \phi, \end{aligned} \quad (165)$$

and denote by $\Pi^{(n)}$ as in (151) the image of the basis ξ of twisted cycles of Section 5.3 under the period map. The horizontality (17)-(18) of the J -functions of \mathcal{X} and Y , the String Equation for \mathcal{X} and Y , and Proposition (5.4) together state that $J^{\mathcal{X}}$, J^Y and $\Pi^{(n)}$ are three different $\mathbb{C}(e^{i\pi a}, e^{i\pi b}, z)$ -bases of deformed flat coordinates of $\nabla^{(g,z)}$ under the identifications (135)-(136), (138)-(139). This entails, for every ρ , the existence of two $\mathbb{C}(e^{i\pi a}, e^{i\pi b}, z)$ -linear maps A, B

$$\begin{aligned} \nabla^{(\eta_Y)} A \Omega &: H_1(\mathbb{C} \setminus (\lambda), \mathbf{L}) \rightarrow \mathcal{S}_Y, \\ \nabla^{(\eta_X)} B^{-1} \Omega &: H_1(\mathbb{C} \setminus (\lambda), \mathbf{L}) \rightarrow \mathcal{S}_X, \end{aligned} \quad (166)$$

such that

$$A \Pi^{(n)} = J_Y, \quad (167)$$

$$B J_{\mathcal{X}} = \Pi^{(n)}. \quad (168)$$

In particular,

$$\mathbb{U}_{\rho}^{\mathcal{X}, Y} = AB. \quad (169)$$

A sends the twisted period basis $\Pi^{(n)}$ to a basis of eigenvectors of the monodromy around the large radius point of Y normalized as in (24). We compute A by investigating the leading asymptotics of the twisted periods (154)-(155) around the large radius point of Y ; as in the example of Section 5.3.1, we denote the latter by LR.

In \mathbb{C}^m with coordinates (w_1, \dots, w_m) , let χ_i , for every $i = 1, \dots, m$, be a path connecting the point at infinity W_i^{∞} ,

$$W_i^{\infty} \triangleq (\overbrace{0, \dots, 0}^{i \text{ times}}, \overbrace{\infty, \dots, \infty}^{m-i \text{ times}}), \quad (170)$$

with zero winding number along $w_i = w_j$ ($i \neq j$) and $w_i = 0, 1$. We want to compute the analytic continuation along χ_i of the Lauricella function $F_D^{(m)}(a, b_1, \dots, b_n, c, w_1, \dots, w_i, w_{i+1}^{-1}, \dots, w_m^{-1})$ from an open ball centered on W_i^{∞} to the origin $W_0^{\infty} = (0, \dots, 0)$ in the sector where $w_i \ll 1$, $w_i/w_j \ll 1$ for $i < j$. One strategy to do this is by performing the continuation in each individual variable w_j , $j > i$ appearing in (157) through an iterated use of Goursat's identity (195). The final result is (201); we refer the reader to Appendix A for the details of the derivation.

In our case, Eq. (201) (see also Remark A.1) implies, around $w_i = \infty$, that

$$\begin{aligned} \Phi^{(m)}(a, b, c; w_1, \dots, w_m) &\sim \sum_{j=0}^{m-1} \Gamma \left[\begin{matrix} c, & a - jb, & (j+1)b - a \\ a, & b, & c - a \end{matrix} \right] \\ &\quad \prod_{i=1}^j (-w_{m-i+1})^{-b} (-w_{m-j})^{-a+jb} \\ &+ \prod_{j=1}^m (-w_j)^{-b} \Gamma \left[\begin{matrix} c, & a - mb \\ a, & c - mb \end{matrix} \right]. \end{aligned} \quad (171)$$

when $w_i \sim 0$, $w_i/w_j \sim 0$ for $j > i$. In particular, at the level of twisted periods this entails

$$\begin{aligned}
\Pi_{n-k}^{(n)} &\sim C_n(\kappa)^{\frac{1}{z}} \kappa_{n-k}^{-a} \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} \Phi^{(k+1)} \left(a, b, 1+a-b, \frac{\kappa_{n-k+1}}{\kappa_{n-k}}, \dots, \frac{\kappa_n}{\kappa_{n-k}}, \frac{1}{\kappa_{n-k}} \right) \\
&\sim C_n(\kappa)^{\frac{1}{z}} \kappa_{n-k}^{-a} \left\{ \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)} (-\kappa_{n-k})^a \right. \\
&\quad + \sum_{j=1}^k \frac{\Gamma(a-jb)\Gamma((j+1)b-a)}{\Gamma(b)} \left(-\frac{\kappa_{n+1-j}}{\kappa_{n-k}} \right)^{-a+jb} (-\kappa_{n-k})^b \prod_{i=1}^{j-1} \left(-\frac{\kappa_{n+1-i}}{\kappa_{n-k}} \right)^{-b} \\
&\quad + \left. \kappa_{n-k}^{(k+1)b} \frac{\Gamma(1-b)\Gamma(a-(k+1)b)}{\Gamma(1+a-(k+2)b)} \prod_{j=n-k+1}^n (-\kappa_j)^{-b} \right\} \\
&\sim C_n(\kappa)^{\frac{1}{z}} \left\{ \sum_{j=0}^k \frac{\Gamma(a-jb)\Gamma((j+1)b-a)}{\Gamma(b)} (-1)^a (\kappa_{n+1-j})^{-a+jb} \prod_{i=1}^{j-1} (\kappa_{n+1-i})^{-b} \right. \\
&\quad + \left. (-1)^{(k+1)b} \frac{\Gamma(1-b)\Gamma(a-(k+1)b)}{\Gamma(1+a-(k+2)b)} \kappa_{n-k}^{(k+1)b-a} \prod_{j=n-k+1}^n (\kappa_j)^{-b} \right\}. \tag{172}
\end{aligned}$$

in a neighbourhood of $\kappa = 0$ given by $|\kappa_i| \ll 1$, $\kappa_i/\kappa_j \ll 1$ for $j > i$; notice that in cohomology coordinates (136) for Y , this becomes an actual open ball $|q| \ll 1$ around the point of classical limit $q_i = e^{t_i} = 0$. Now, from the discussion of Section 2.3.1 and Eqns. (21), (136), around the limit point of classical cohomology the J -function of Y behaves as

$$J_{p_i}^Y = z C_n(\kappa)^{\frac{1}{z}} \kappa_i^{(n-i+1)b-a} \prod_{j=i+1}^n (\kappa_j)^{-b} (1 + \mathcal{O}(e^t)). \tag{173}$$

Then we can read off from (172)-(173) the decomposition of each twisted period $\Pi_i^{(n)}$ in terms of eigenvectors of the monodromy around LR, and in particular, in terms of the localized components of the J -function. Explicitly,

$$\Pi^{(n)} = A^{-1} J^Y, \tag{174}$$

where

$$A_{ji}^{-1} = \begin{cases} (-1)^{(n-i+1)b} \frac{\Gamma(1-b)\Gamma(a-(n-i+1)b)}{z\Gamma(1+a-(n-i+2)b)} & \text{for } i = j, \\ (-1)^a \frac{\Gamma(a-(n-i+1)b)\Gamma((n-i+2)b-a)}{z\Gamma(b)} & \text{for } j < i, \\ 0 & j > i. \end{cases} \tag{175}$$

Its inverse reads

$$A_{ij} = \begin{cases} e^{\pi i(n-i+1)b} \frac{z\Gamma(1+a-(n-i+2)b)\Gamma(1-a+(n-i+1)b)\sin(a+(n-i+1)b)}{\Gamma(1-b)\pi} & i = j, \\ e^{-i\pi(a-b(2n-2j+3))} \frac{z\sin(\pi b)\Gamma(1-a+b(n+1-i))\Gamma(1+a-b(n-i+2))}{\pi\Gamma(1-b)} & j > i, \\ 0 & j < i. \end{cases} \tag{176}$$

Consider now the situation at the orbifold point (as before, denoted OP) given by $\{\kappa_j = \omega^{-j}\}$. Since by (19),

$$J^{\mathcal{X}}(0, z) = z \mathbf{1}_0, \tag{177}$$

$$\frac{\partial J^{\mathcal{X}}}{\partial x_k}(0, z) = \mathbf{1}_{\frac{k}{n+1}}, \tag{178}$$

to compute the operator B in (166) it suffices to evaluate the expansion of the Lauricella functions (154)-(155) at OP to linear order in x_k , $k = 0, \dots, n$, where it is implicit that principal

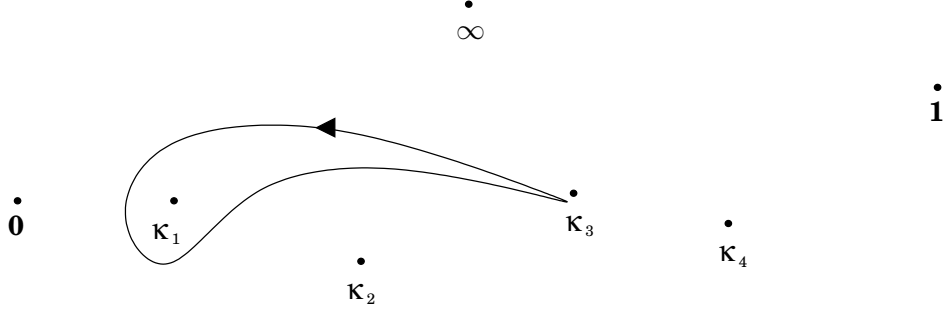


FIGURE 5. The path σ_{13} in $\pi_1(\mathcal{M}_A)$ for $n = 4$.

branches are chosen in (151) for all k ; this corresponds to the assumption that the 1-chain ρ has trivial winding number around the boundary divisors of \mathcal{M}_A . From (139), (151), one has

$$\begin{aligned} \Pi_j^{(n)}(\kappa, z) \Big|_{x=0} &= \omega^{(j-n/2)a} \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} \Phi^{(n)}(a, b, 1+a-b; \omega, \dots, \omega^n) \\ &= \frac{\omega^{(j-n/2)a}}{n+1} \frac{\Gamma(a/(n+1))\Gamma(1-b)}{\Gamma(1-b+a/(n+1))}, \quad j = 1, \dots, n+1. \end{aligned} \quad (179)$$

Similarly, a short computation shows that

$$\kappa_k \frac{\partial \Pi_j^{(n)}}{\partial \kappa_k}(0, z) = -\frac{\omega^{(j-n/2)a}}{n+1} \sum_{l=1}^n \omega^{(j-k)l} \frac{\Gamma\left(\frac{a+l}{n+1}\right)\Gamma(1-b)}{\Gamma\left(\frac{a+l}{n+1}-b\right)}, \quad (180)$$

$$\frac{\partial \Pi_j^{(n)}}{\partial x_k}(\kappa, z) \Big|_{x=0} = -\frac{\omega^{(j-n/2)a-jk+k/2}}{n+1} \frac{\Gamma\left(\frac{a-k}{n+1}+1\right)\Gamma(1-b)}{\Gamma\left(\frac{a-k}{n+1}+1-b\right)}. \quad (181)$$

In matrix form we have:

$$\Pi^{(n)} = BJ^{\mathcal{X}} = D_1 V D_2 J^{\mathcal{X}} \quad (182)$$

where

$$(D_1)_{jk} = \omega^{(j-n/2)a} \delta_{jk} \quad (183)$$

$$(D_2)_{jk} = \delta_{jk} \begin{cases} -\omega^{k/2} \frac{\Gamma\left(\frac{a-k}{n+1}+1\right)\Gamma(1-b)}{\Gamma\left(\frac{a-k}{n+1}+1-b\right)} & \text{for } 1 \leq k \leq n \\ \frac{\Gamma(a/(n+1))\Gamma(1-b)}{z\Gamma(1-b+a/(n+1))} & \text{for } k = n+1 \end{cases} \quad (184)$$

$$V_{jk} = \frac{\omega^{-jk}}{n+1} \quad (185)$$

Piecing (176) and (182)-(185) together yields⁵ (95), up to a scalar factor of ω^a . By Remark 5.7, this corresponds to our freedom of a String Equation shift along either of $H^0(\mathcal{X})$ and $H^0(Y)$. Setting $\delta_{\mathcal{X}} - \delta_Y = 2\pi i \alpha_1$ in (135), (138) concludes the proof. \square

5.5. Monodromy and equivariant integral structures. The expression (95) for the symplectomorphism $\mathbb{U}_{\rho}^{\mathcal{X}, Y}$ was obtained for the analytic continuation path ρ of Theorem 2.8. Fixing a reference point $m_0 = (\widetilde{\kappa}_1, \dots, \widetilde{\kappa}_n) \in \mathcal{M}_A$, for a general path $\rho \circ \sigma$ with $[\sigma] \in \pi_1(\mathcal{M}_A, m_0)$ we get a composition

$$\mathbb{U}_{\rho \circ \sigma}^{\mathcal{X}, Y} = \mathbb{U}_{\rho}^{\mathcal{X}, Y} M_{\sigma} \quad (186)$$

⁵This amounts to a rather tedious exercise in telescoping sums and additions of roots of unity. The computation can be made available upon request.

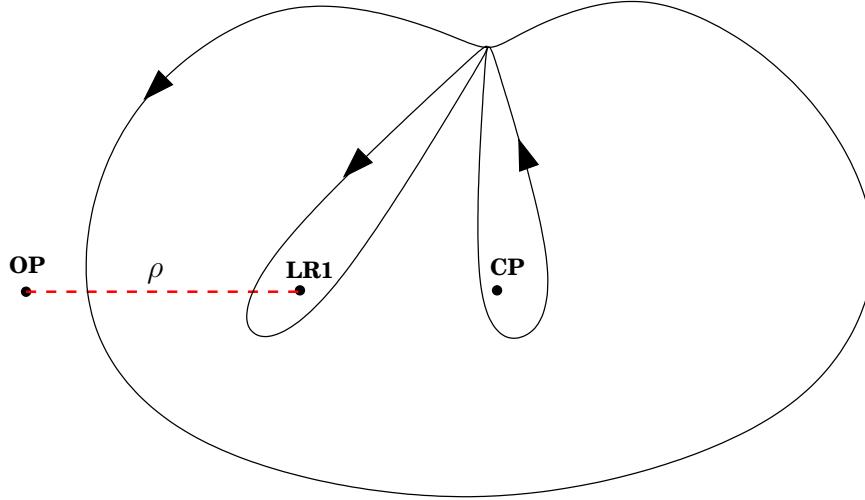


FIGURE 6. The Kähler moduli space of the A_1 singularity in A -model coordinates. LR1 and LR2 indicate the large radius points $\kappa = 0, \infty$ respectively, CP is the conifold point, and OP is the orbifold point. Circuits around LR1-2 and CP generate the monodromy group of the global quantum D -module. The dashed segment depicts the analytic continuation path ρ of Theorem 2.8 and 4.1.

where $M_\sigma : \pi_1(\mathcal{M}_A, m_0) \rightarrow \text{End}(\text{Sol}_{\lambda, \phi})$ is the monodromy representation of the fundamental group of \mathcal{M}_A in the space of solutions of the Lauricella system $F_D^{(n)}$.

By definition (131), \mathcal{M}_A is the configuration space of n distinct points in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Therefore, its fundamental group coincides with the genus zero pure mapping class group [19, 64]

$$\pi_1(\mathcal{M}_A) \simeq \text{PB}_{n+2}/Z(\text{PB}_{n+2}); \quad (187)$$

where PB_{n+2} denotes the pure braid group in $n+2$ strands and $Z(\text{PB}_{n+2}) \simeq \mathbb{Z}_2$ is its center. Writing $\tilde{\kappa}_i = 0, 1, \infty$ for $i = n+1, n+2$ and $n+3$ respectively, generators P_{ij} , $i = 1, \dots, n+3$, $j = 1, \dots, n$ of PB_{n+2} are in bijection with paths $\sigma_{ij} : [0, 1] \rightarrow \mathcal{M}_A$ given by lifts to \mathcal{M}_A of closed contours in the j^{th} affine coordinate plane that start at $\kappa_j = \tilde{\kappa}_j$, turn counterclockwise around $\tilde{\kappa}_i$ (and around no other point) and then return to their original position, as in Figure 5.

The image of the period map (165), by Proposition 5.4, is a lattice in $\text{Sol}_{\lambda, \phi}$:

$$\text{Sol}_{\lambda, \phi} = \nabla^{(g)} \Omega(H_1(\mathbb{C} \setminus (\lambda), \mathbf{L})) \otimes_{\mathbb{Z}(e^{i\pi a}, e^{i\pi b})} \mathbb{C}(e^{i\pi a}, e^{i\pi b}, z), \quad (188)$$

and by (176), (182)-(185) the induced morphism $H_1(\mathbb{C} \setminus (\lambda), \mathbf{L}) \simeq K(Y)$ is a lattice isomorphism. The monodromy action on $\text{Sol}_{\lambda, \phi}$, at the level of equivariant K -groups, is given by lattice automorphisms $\pi_1(\mathcal{M}_A) \rightarrow \text{Aut}_{\mathbb{Z}(e^{i\pi a}, e^{i\pi b})} K(Y)$; this can be verified explicitly from the form of the monodromy matrices in the twisted period basis [54]. It would be fascinating to trace the origin of this pure braid group action on the quantum D -module as coming from an action of the braid group at the level of the equivariant derived category of coherent sheaves on \mathcal{X} and Y , as in [62].

5.5.1. *Example: $n = 1$.* In this case the action on $K(Y)$ is given by the classical monodromy of the Gauss system for $c = a - b + 1$. With reference to Figure 6, we have in the standard basis

$\{\mathcal{O}_Y, \mathcal{O}_Y(1)\}$ for $K(Y)$,

$$M_{\text{LR1}} = \begin{pmatrix} e^{-ia\pi} (e^{2ia\pi} + e^{2ib\pi}) & e^{2ib\pi} \\ -1 & 0 \end{pmatrix}, \quad (189)$$

$$M_{\text{CP}} = \begin{pmatrix} 1 & -2ie^{-i(a-b)\pi} \sin(b\pi) \\ -e^{-i(a+2b)\pi} (-1 + e^{2ib\pi}) & e^{-2i(a+b)\pi} (-1 + e^{2ib\pi})^2 + 1 \end{pmatrix}, \quad (190)$$

$$M_{\text{LR2}} = \begin{pmatrix} 2 \cos(a\pi) & 1 - e^{-2ia\pi} (-1 + e^{2ib\pi}) \\ -1 & 2ie^{-i(a-b)\pi} \sin(b\pi) \end{pmatrix}, \quad (191)$$

for the large radius and the conifold monodromy of $\text{QDM}(Y)$. It is straightforward to check that they induce symplectic automorphisms of the Givental vector space $\mathcal{H}(Y)$.

Remark 5.11. In the non-equivariant limit the conifold monodromy becomes trivial. As a result, the monodromy group reduces to the integers, being generated by the Galois action around the large radius limit point of Y . This is consistent with the fact that $B_2 \simeq \mathbb{Z}$ in [62].

APPENDIX A. ANALYTIC CONTINUATION OF LAURICELLA $F_D^{(N)}$

Consider the Lauricella function $F_D^{(M+N)}(a; b_1, \dots, b_{M+N}; c; z_1, \dots, z_M, w_1, \dots, w_N)$ around $P = (0, 0, \dots, \infty, \dots, \infty)$. We are interested in the leading terms of the asymptotics of this function in the region Ω_{M+N} defined as

$$\Omega_{M+N} \triangleq B(P, \epsilon) \bigcap_{i < j} H_{ij} \quad (192)$$

given by the intersection of the ball $B(P, \epsilon)$ with the interior of the real hyperquadrics

$$H_{ij} \triangleq \{(z, w) \in \mathbb{C}^{M+N} \mid |w_i/w_j| < \epsilon\}. \quad (193)$$

As our interest is confined to the *leading* asymptotics only, we can assume w.l.o.g. that $M = 0$.

Following [36, Chapter 6], start from the power series expression (157) and perform the sum w.r.t. w_N

$$F_D^{(N)}(a; b_1, \dots, b_N; c; w_1, \dots, w_N) = \sum_{i_1, \dots, i_{N-1}} \frac{(a)_{\sum_{j=1}^{N-1} i_j}}{(c)_{\sum_{j=1}^{N-1} i_j}} \prod_{j=1}^{N-1} \frac{(b_j)_{i_j} w_j^{i_j}}{i_j!} {}_2F_1 \left(a + \sum_{j=1}^{N-1} i_j, b_N, c + \sum_{j=1}^{N-1} i_j, w_N \right). \quad (194)$$

The main idea then is to apply the connection formula for the inner Gauss function

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{(-z)^{-a} \Gamma(c) \Gamma(b-a) {}_2F_1(a, a-c+1; a-b+1; \frac{1}{z})}{\Gamma(b) \Gamma(c-a)} \\ &+ \frac{(-z)^{-b} \Gamma(c) \Gamma(a-b) {}_2F_1(b, b-c+1; -a+b+1; \frac{1}{z})}{\Gamma(a) \Gamma(c-b)} \end{aligned} \quad (195)$$

to analytically continue it to $|z| = |w_N| > 1$; in doing so, we fix a path of analytic continuation by choosing the principal branch for both the power functions $(-z)^{-a}$ and $(-z)^{-b}$ in (195) and continue ${}_2F_1(a, b; c; z)$ to $|z| > 1$ along a path that has winding number zero around the Fuchsian singularity at $z = 1$. As a power series in w_N the analytic continuation of (194) around

$w_N = \infty$ then reads

$$\begin{aligned}
F_D^{(N)}(a; b_1, \dots, b_N; c; w_1, \dots, w_N) &= (-w_N)^{-a} \Gamma \left[\begin{matrix} c, & b_N - a \\ b_N, & c - a \end{matrix} \right] \\
F_D^{(N)} \left(a; b_1, \dots, b_{N-1}, 1 - c + a; 1 - b_N + a, \frac{w_1}{w_N}, \dots, \frac{1}{w_N} \right) &+ (-w_N)^{-b_N} \Gamma \left[\begin{matrix} c, & a - b_N \\ a, & c - b_N \end{matrix} \right] \\
C_N^{(N-1)} \left(b_1, \dots, b_N, 1 - c + b_N; a - b_N, -w_1, -w_2, \dots, \frac{1}{w_N} \right), & \quad (196)
\end{aligned}$$

where we defined [36, Chapter 3]

$$C_N^{(k)}(b_1, \dots, b_N, a; a', x_1, \dots, x_N) \triangleq \sum_{i_1, \dots, i_N} (a)_{\alpha_N^{(k)}(\mathbf{i})} (a')_{-\alpha_N^{(k)}(\mathbf{i})} \prod_{j=1}^N \frac{(b_j)_{i_j} w_j^{i_j}}{i_j!} \quad (197)$$

and

$$\alpha_N^{(k)}(\mathbf{i}) \triangleq \sum_{j=k+1}^N i_j - \sum_{j=1}^k i_j, \quad (198)$$

$$\Gamma \left[\begin{matrix} a_1, & \dots, & a_m \\ b_1, & \dots, & b_n \end{matrix} \right] \triangleq \frac{\prod_{i=1}^m \Gamma(a_i)}{\prod_{i=1}^n \Gamma(b_i)}. \quad (199)$$

Now, notice that the $F_D^{(N-1)}$ function in the r.h.s. of (196) is analytic in Ω_N ; there is nothing more that should be done there. The analytic continuation of the $C_N^{(N-1)}$ function is instead much more involved (see [36] for a complete treatment of the $N = 3$ case); but as all we are interested in is the leading term of the expansion around P in Ω_N we isolate the $\mathcal{O}(1)$ term in its $1/w_N$ expansion to find

$$\begin{aligned}
&C_N^{(N-1)} \left(b_1, \dots, b_N, 1 - c + b_N; a - b_N, -w_1, -w_2, \dots, \frac{1}{w_N} \right) = \\
&= F_D^{(N-1)}(a - b_N, b_1, \dots, b_{N-1}, c - b_N; w_1, \dots, w_{N-1}) + \mathcal{O} \left(\frac{1}{w_N} \right) \quad (200)
\end{aligned}$$

We are done: by (200), the form of the leading terms in the expansion of $F_D^{(N)}$ inside Ω_N can be found recursively by iterating N times the procedure we have followed in (194)-(200); as at each step (195)-(200) generate one additional term, we end up with a sum of $N + 1$ monomials each having power-like monodromy around P . Explicitly:

$$\begin{aligned}
F_D^{(N)}(a; b_1, \dots, b_N; c; w_1, \dots, w_N) &\sim \sum_{j=0}^{N-1} \Gamma \left[\begin{matrix} c, & a - \sum_{i=N-j+1}^N b_i, & \sum_{i=n-j}^N b_i - a \\ a, & b_{N-j}, & c - a \end{matrix} \right] \\
&\quad \prod_{i=1}^j (-w_{N-i+1})^{-b_{N-i+1}} (-w_{N-j})^{-a + \sum_{i=N-j+1}^N b_i} \\
&\quad + \prod_{i=1}^N (-w_i)^{-b_i} \Gamma \left[\begin{matrix} c, & a - \sum_{i=1}^N b_j \\ a, & c - \sum_{i=1}^N b_j \end{matrix} \right]. \quad (201)
\end{aligned}$$

Remark A.1. The analytic continuation to some other sectors of the ball $B(P, \epsilon)$ is straightforward. In particular we can replace the condition $w_i/w_j \sim 0$ for $j > i$ by its reciprocal $w_j/w_i \sim 0$; this amounts to relabeling $b_i \rightarrow b_{N-i+1}$ in (201).

Remark A.2. When $a = -d$ for $d \in \mathbb{Z}^+$, the function $F_D^{(N)}$ reduces to a polynomial in w_1, \dots, w_N . In this case the arguments above reduce to a formula of Toscano [65] for Lauricella

polynomials:

$$F_D^{(N)}(-d; b_1, \dots, b_N; c; w_1, \dots, w_N) \\ = (-w_N)^d \frac{(b)_d}{(c)_d} F_D^{(N)}\left(-d; b_1, b_2, \dots, b_{N-1}; 1-d-c, 1-d-b_N, \frac{w_1}{w_N}, \dots, \frac{1}{w_N}\right). \quad (202)$$

REFERENCES

- [1] DAN ABRAMOVICH, TOM GRABER, AND ANGELO VISTOLI, Gromov-Witten theory of Deligne-Mumford stacks, *Amer. J. Math.*, 130 (2008), 1337–1398.
- [2] MINA AGANAGIC, ALBRECHT KLEMM, MARCOS MARIÑO, AND CUMRUN VAFA, The topological vertex, *Commun. Math. Phys.*, 254 (2005), 425–478. Preprint available at [hep-th/0305132](#).
- [3] MINA AGANAGIC, ALBRECHT KLEMM, AND CUMRUN VAFA, Disk instantons, mirror symmetry and the duality web, *Z. Naturforsch.*, A57 (2002), 1–28. Preprint available at [hep-th/0105045](#).
- [4] MINA AGANAGIC AND CUMRUN VAFA, Mirror symmetry, D-branes and counting holomorphic discs. Preprint available at [hep-th/0012041](#).
- [5] MATTHEW BALLARD, DAVID FAVERO, AND LUDMIL KATZARKOV, Variation of geometric invariant theory quotients and derived categories. Preprint available at [1203.6643](#).
- [6] VICTOR V. BATYREV AND DUCO VAN STRATEN, Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties, *Comm. Math. Phys.*, 168 (1995), 493–533.
- [7] LEV A. BORISOV AND R. PAUL HORJA, Mellin-Barnes integrals as Fourier-Mukai transforms, *Adv. Math.*, 207 (2006), 876–927.
- [8] VINCENT BOUCHARD, ALBRECHT KLEMM, MARCOS MARIÑO, AND SARA PASQUETTI, Remodeling the B-model, *Commun. Math. Phys.*, 287 (2009), 117–178. Preprint available at [arXiv:0709.1453](#).
- [9] VINCENT BOUCHARD, ALBRECHT KLEMM, MARCOS MARINO, AND SARA PASQUETTI, Topological open strings on orbifolds, *Commun. Math. Phys.*, 296 (2010), 589–623. Preprint available at [0807.0597](#).
- [10] ANDREA BRINI, A crepant resolution conjecture for open strings, talk at the BIRS Workshop on “New recursion formulae and integrability for Calabi-Yau manifolds”, October 26, 2011.
- [11] ———, Open topological strings and integrable hierarchies: Remodeling the A-model, *Commun. Math. Phys.*, 312 (2012), 735–780. Preprint available at [1102.0281](#).
- [12] ANDREA BRINI, GUIDO CARLET, STEFANO ROMANO, AND PAOLO ROSSI, Toric Gromov-Witten theory and integrable hierarchies, in preparation.
- [13] ANDREA BRINI, GUIDO CARLET, AND PAOLO ROSSI, Integrable hierarchies and the mirror model of local $\mathbb{C}P^1$, *Physica D*, 241 (2012), 2156–2167. Preprint available at [1105.4508](#).
- [14] ANDREA BRINI AND RENZO CAVALIERI, Open orbifold Gromov-Witten invariants of $[\mathbb{C}^3/\mathbb{Z}_n]$: localization and mirror symmetry, *Selecta Math. (N.S.)*, 17 (2011), 879–933. Preprint available at [1007.0934](#).
- [15] ANDREA BRINI, RENZO CAVALIERI, AND DUSTIN ROSS, in preparation.
- [16] ANDREA BRINI AND ALESSANDRO TANZINI, Exact results for topological strings on resolved $Y(p,q)$ singularities, *Commun. Math. Phys.*, 289 (2009), 205–252. Preprint available at [0804.2598](#).
- [17] JIM BRYAN AND AMIN GHOLAMPOUR, Root systems and the quantum cohomology of ADE resolutions, *Algebra Number Theory*, 2 (2008), 369–390.
- [18] JIM BRYAN AND TOM GRABER, The crepant resolution conjecture, *Proc. Sympos. Pure Math.*, 80 (2009), 23–42.
- [19] WERNER BURAU, Über Zopf-invarianten., *Abh. Math. Semin. Hamb. Univ.*, 9 (1932), 117–124.
- [20] RENZO CAVALIERI AND DUSTIN ROSS, Open Gromov-Witten theory and the crepant resolution conjecture, *Mich. Math. Journal*, to appear (2013). Preprint available at [1102.0717](#).
- [21] CHAN, KWOKWAI AND CHO, CHEOL-HYUN AND LAU, SIU-CHEONG AND TSENG, HSIAN-HUA, Lagrangian Floer superpotentials and crepant resolutions for toric orbifolds. Preprint available at [1208.5282](#).
- [22] CHAN, KWOKWAI AND LAU, SIU-CHEONG AND LEUNG, NAICHUNG CONAN AND TSENG, HSIAN-HUA, Open Gromov-Witten invariants and mirror maps for semi-Fano toric manifolds. Preprint available at [1112.0388](#).
- [23] WEIMIN CHEN AND YONGBIN RUAN, Orbifold Gromov-Witten theory, *Contemp. Math.*, 310 (2002), 25–85.
- [24] TOM COATES, On the crepant resolution conjecture in the local case, *Comm. Math. Phys.*, 287 (2009), 1071–1108.
- [25] TOM COATES, ALESSIO CORTI, HIROSHI IRITANI, AND HSIAN-HUA TSENG, The small quantum cohomology of toric Deligne-Mumford stacks, in preparation.
- [26] ———, Computing genus-zero twisted Gromov-Witten invariants, *Duke Math. J.*, 147 (2009), 377–438.
- [27] TOM COATES AND HIROSHI IRITANI, On the Convergence of Gromov-Witten Potentials and Givental’s Formula. Preprint available at [1203.4193](#).
- [28] TOM COATES, HIROSHI IRITANI, AND HSIAN-HUA TSENG, Wall-crossings in toric Gromov-Witten theory. I. Crepant examples, *Geom. Topol.*, 13 (2009), 2675–2744.

- [29] TOM COATES AND YONGBIN RUAN, Quantum cohomology and crepant resolutions: A conjecture, *Ann. Inst. Fourier*, to appear (2013). Preprint available at [0710.5901](#).
- [30] DAVID A. COX AND SHELDON KATZ, *Mirror symmetry and algebraic geometry*. Mathematical Surveys and Monographs, 68. American Mathematical Society, 1999.
- [31] DUILIU-EMANUEL DIACONESCU AND BOGDAN FLOREA, Localization and gluing of topological amplitudes, *Commun.Math.Phys.*, 257 (2005), 119–149. Preprint available at [hep-th/0309143](#).
- [32] ROBERT DIJKGRAAF, HERMAN L. VERLINDE, AND ERIK P. VERLINDE, Topological strings in $d < 1$, *Nucl.Phys.*, B352 (1991), 59–86.
- [33] BORIS DUBROVIN, Hamiltonian formalism of Whitham type hierarchies and topological Landau-Ginsburg models, *Commun.Math.Phys.*, 145 (1992), 195–207.
- [34] ———, Geometry of 2D topological field theories, in “*Integrable systems and quantum groups*” (*Montecatini Terme, 1993*), *Lecture Notes in Math.*, 1620 (1994), 120–348. Preprint available at [hep-th/9407018](#).
- [35] ———, On almost duality for Frobenius manifolds, in “*Geometry, topology, and mathematical physics*”, *Amer. Math. Soc. Transl. Ser. 2*, 212 (2004), 75–132.
- [36] HAROLD EXTON, *Multiple hypergeometric functions and applications*. Ellis Horwood Ltd., 1976.
- [37] BOHAN FANG, CHIU-CHU MELISSA LIU, AND HSIAN-HUA TSENG, Open-closed Gromov-Witten invariants of 3-dimensional Calabi-Yau smooth toric DM stacks. Preprint available at [1212.6073](#).
- [38] ALEXANDER GIVENTAL, Twisted Picard-Lefschetz formulas, *Funktsional. Anal. i Prilozhen.*, 22 (1988), 12–22, 96.
- [39] ———, Equivariant Gromov-Witten invariants, *Internat. Math. Res. Notices*, 13 (1996), 613–663.
- [40] ———, A mirror theorem for toric complete intersections, 160, 141–175.
- [41] RAJESH GOPAKUMAR AND CUMRUN VAFA, On the gauge theory/geometry correspondence, *Adv. Theor. Math. Phys.*, 3 (1999), 1415–1443. Preprint available at [hep-th/9811131](#).
- [42] M. HERBST, K. HORI, AND D. PAGE, B-type D-branes in toric Calabi-Yau varieties, *Homological mirror symmetry* 757, pp. 27–44. Springer, 2009.
- [43] KENTARO HORI, AMER IQBAL, AND CUMRUN VAFA, D-branes and mirror symmetry. Preprint available at [hep-th/0005247](#).
- [44] RICHARD PAUL HORJA, *Hypergeometric functions and mirror symmetry in toric varieties*. ProQuest LLC, Ann Arbor, MI, 1999.
- [45] HIROSHI IRITANI, An integral structure in quantum cohomology and mirror symmetry for toric orbifolds, *Adv. Math.*, 222 (2009), 1016–1079.
- [46] ———, Ruan’s conjecture and integral structures in quantum cohomology, *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)* 59, pp. 111–166. Math. Soc. Japan, 2010.
- [47] SHELDON H. KATZ AND CHIU-CHU MELISSA LIU, Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc, *Geom. Topol. Monogr.*, 8 (2002), 1–47. Preprint available at [math/0103074](#).
- [48] IGOR M. KRICHEVER, The tau function of the universal Whitham hierarchy, matrix models and topological field theories, *Commun. Pure Appl. Math.*, 47 (1994), 437. Preprint available at [hep-th/9205110](#).
- [49] G. LAURICELLA, Sulle funzioni ipergeometriche a piu variabili, *Rendiconti del Circolo Matematico di Palermo*, 7 (1893), 111–158.
- [50] W. LERCHE AND P. MAYR, On $N = 1$ mirror symmetry for open type II strings. Preprint available at [hep-th/0111113](#).
- [51] JUN LI, CHIU-CHU MELISSA LIU, KEFENG LIU, AND JIAN ZHOU, A Mathematical Theory of the Topological Vertex, *Geom. Topol.*, 13 (2009), 527–621. Preprint available at [math/0408426](#).
- [52] MARCOS MARIÑO AND CUMRUN VAFA, Framed knots at large N , *Contemp. Math.*, 310 (2002), 185–204. Preprint available at [hep-th/0108064](#).
- [53] D. MAULIK, A. OBLONKOV, A. OKOUNKOV, AND R. PANDHARIPANDE, Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds, *Inventiones Mathematicae* (2008), 1–45. Preprint available at [arXiv:0809.3976](#).
- [54] KATSUHISA MIMACHI AND TAKESHI SASAKI, Irreducibility and reducibility of Lauricella’s system of differential equations E_D and the Jordan-Pochhammer differential equation E_{JP} , *Kyushu J. Math.*, 66 (2012), 61–87.
- [55] R. PANDHARIPANDE, J. SOLOMON, AND J. WALCHER, Disk enumeration on the quintic 3-fold, *J. Amer. Math. Soc.*, 21 (2008), 1169–1209.
- [56] STEFANO ROMANO, Frobenius structures on double Hurwitz spaces. Preprint available at [1210.2312](#).
- [57] ———, *Special Frobenius structures on Hurwitz spaces and applications*. PhD Thesis in Mathematical Physics. SISSA, Trieste, 2012.
- [58] DUSTIN ROSS, Localization and gluing of orbifold amplitudes: The Gromov-Witten orbifold vertex, *Trans. of the AMS*, in press (2013). Preprint available at [math/1109.5995v3](#).
- [59] YONGBIN RUAN, The cohomology ring of crepant resolutions of orbifolds, *Contemp. Math.* 403, pp. 117–126. Amer. Math. Soc., 2006.

- [60] KYOJI SAITO, Period mapping associated to a primitive form, *Publ. Res. Inst. Math. Sci.*, 19 (1983), 1231–1264.
- [61] ED SEGAL, Equivalence between GIT quotients of Landau-Ginzburg B-models, *Comm. Math. Phys.*, 304 (2011), 411–432.
- [62] PAUL SEIDEL AND RICHARD THOMAS, Braid group actions on derived categories of coherent sheaves, *Duke Math. J.*, 108 (2001), 37–108.
- [63] JAKE SOLOMON, Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions. Preprint available at [math/0606429](https://arxiv.org/abs/math/0606429).
- [64] TOSHIAKI TERADA, Quelques propriétés géométriques du domaine de F_1 et le groupe de tresses colorées, *Publ. Res. Inst. Math. Sci.*, 17 (1981), 95–111.
- [65] LETTERIO TOSCANO, Sui polinomi ipergeometrici a più variabili del tipo F_D di Lauricella, *Matematiche (Catania)*, 27 (1973), 219–250.
- [66] E. T. WHITTAKER AND G. N. WATSON, *A course of modern analysis*. Cambridge Mathematical Library. Cambridge University Press, 1996.

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