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BOUNDARY HALF-STRIPS AND THE STRONG CHIP*

EMIL ERNST[†] AND MICHEL THÉRA[‡]

Abstract. When the subdifferential sum rule formula holds for the indicator functions ι_C and ι_D of two closed convex sets C and D of a locally convex space X , the pair (C, D) is said to have the strong conical hull intersection property (the strong CHIP). The specification of a well-known theorem due to Moreau to the case of the support functionals σ_C and σ_D subsumes the fact that the pair (C, D) has the strong CHIP whenever the inf-convolution of σ_C and σ_D is exact. In this article we prove, in the setting of Euclidean spaces, that if the pair (C, D) has the strong CHIP while the boundary of C does not contain any half-strip, then the inf-convolution of σ_C and σ_D is exact. Moreover, when the boundary of a closed and convex set C does contain a half-strip, it is possible to find a closed and convex set D such that the pair (C, D) has the strong CHIP while the inf-convolution of σ_C and σ_D is not exact. The validity of the converse of Moreau's theorem in Euclidean spaces is thus associated to the absence of half-strips within the boundary of concerned convex sets.

Key words. strong conical hull intersection property, convex programming with convex inequalities, Euclidean space, exact infimal convolution, qualification conditions

AMS subject classifications. 90C46, 90C51, 46N10, 49K40

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1. Introduction. This study concerns an application of a geometrical notion called the *strong conical hull intersection property* (strong CHIP) introduced by Deutsch, Li, and Swetits (see [6]). We say that the pair (C, D) of closed and convex subsets of some locally convex space X has the strong CHIP if the subdifferential of the sum and sum of the subdifferentials of their indicator functions coincide:

$$(1.1) \quad \partial(\iota_C + \iota_D) = \partial\iota_C + \partial\iota_D.$$

As customary, ι_A is the indicator function of a subset A of X and is defined by $\iota_A(x) = 0$ if $x \in A$, and $\iota_A(x) = +\infty$ otherwise. We also recall that the convex subdifferential is an operator from X into the topological dual X^* of X , which assigns to each extended-real-valued mapping Φ on X a set-valued operator between X and X^* defined by

$$\partial\Phi(x_0) = \{x^* \in X^* : \langle x - x_0, x^* \rangle + \Phi(x_0) \leq \Phi(x) \quad \forall x \in X\},$$

where $\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{R}$ is the duality pairing between X and X^* .

Recalling that the normal cone $N_C(x)$ at $x \in X$ to a closed convex set C of X is the set $\partial\iota_C(x)$, the strong CHIP for the pair (C, D) amounts to saying that

$$(1.2) \quad N_{C \cap D}(x) = N_C(x) + N_D(x) \quad \forall x \in C \cap D;$$

i.e., every normal direction to $C \cap D$ at some point x can be expressed as the sum of normal directions at x to C and D .

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This property is important in convex optimization because when we consider the problem of minimizing a convex functional Φ on the intersection of two sets C and D which have the strong CHIP, the optimality condition for \bar{x} to be a minimizer becomes

$$0 \in \partial\Phi(\bar{x}) + N_C(\bar{x}) + N_D(\bar{x}).$$

In the case of convex differentiable optimization, it becomes

$$-\nabla f(\bar{x}) \in N_C(\bar{x}) + N_D(\bar{x}).$$

Let us quote only the result proved when X is a Hilbert space by Deutsch (see [4]). It says that the strong CHIP is the weakest constraint qualification under which a minimizer \bar{x} of a convex function $\Phi : C_1 \cap C_2 \rightarrow R$ can be characterized using the subdifferential of Φ at x and the normal cones of C_1 and C_2 at x .

The existence of conditions ensuring that a pair of closed and convex sets has the strong CHIP is based on a classical result by Moreau ([9, Remarque 10.2]); the result initially published in [8] makes use of another key concept of convex analysis, namely the notion of *infimal convolution* (inf-convolution). Recall that, if Φ and Ψ are extended-real-valued lower semicontinuous convex functions over X (this class of functions from now on is denoted by $\Gamma_0(X)$), the inf-convolution of Φ and Ψ is the extended real-valued function $\Phi \square \Psi$ defined by

$$(1.3) \quad \Phi \square \Psi(x) = \inf_{y \in X} (\Phi(x - y) + \Psi(y)).$$

The infimal convolution between Φ and Ψ is said to be *exact* if $\Phi \square \Psi \in \Gamma_0(X)$ and the infimum is achieved in (1.3) whenever $\Phi \square \Psi(x) < +\infty$.

Let us also recall that given a convex closed set A in X , we note $\sigma_A : X^* \rightarrow R \cup \{+\infty\}$, the support function of A . It is defined by

$$\sigma_A(f) = \sup_{x \in A} \langle x, f \rangle.$$

Using this concept, Moreau's theorem states that the subdifferential sum formula (1.1) holds, provided that the inf-convolution of the support functionals of C and D is exact. Remark that an equivalent way of expressing the exactness of the inf-convolution of the support functionals σ_C and σ_D is to say that every linear functional $f \in X^*$ which is bounded above on $C \cap D$ may be expressed as the sum of two linear functionals f_1 and f_2 , bounded above on C and D respectively, such that

$$\sup_{x \in C \cap D} \langle x, f \rangle = \sup_{x \in C} \langle x, f_1 \rangle + \sup_{x \in D} \langle x, f_2 \rangle.$$

Let us observe that relation (1.2), i.e., the strong CHIP, is equivalent to the following property: *Every linear functional $f \in X^*$ which achieves its maximum on $C \cap D$ may be expressed as the sum of two linear functionals f_1 and f_2 , achieving their maximum on C and D , respectively, such that*

$$\max_{x \in (C \cap D)} \langle x, f \rangle = \max_{x \in C} \langle x, f_1 \rangle + \max_{x \in D} \langle x, f_2 \rangle.$$

The importance of Moreau's theorem comes from the fact that several very general *qualification conditions* are known to ensure the exactness of the inf-convolution of support functionals (the reader is referred for further information to the excellent

articles of Zălinescu [13], Gowda and Teboulle [10], and, respectively, Simons [11], in which he/she may find a clear picture of the topic, as well as self-contained proofs for most of the concerned results).

Accordingly, the result proved by Moreau gives the possibility of systematically specifying every qualification condition as a criterion for the strong CHIP (see for instance [5, Proposition 2.3].)

Let us remark that the exactness of the inf-convolution of the support functionals is stronger than the simple strong CHIP. Indeed, Moreau's condition concerns all the continuous linear functionals bounded above on the intersection $C \cap D$, while the strong CHIP is formulated only in terms of those elements from X^* which achieve their maximum on $C \cap D$. The question is thus raised of the validity of the converse to this theorem.

The converse to Moreau's theorem obviously holds for sets C and D such that every linear and continuous map bounded above on any one of the sets $C \cap D$, C , or D necessarily achieve their maximums on this set. On this ground, a first partial converse of the Moreau result has recently been proved by Bauschke, Borwein, and Li for Hilbert spaces (see [1, Proposition 6.4]); the result was extended to the setting of Banach spaces by Burachik and Jeyakumar [3, Proposition 4.2]. Their result states that, if C and D is a pair of closed and convex *cones* with the strong CHIP, then the inf-convolution of their support functionals is always exact.

However, it is not necessary to impose to every linear and continuous map which is bounded above on any one of the sets $C \cap D$, C , or D , to achieve its maximum on this set, in order to ensure the validity of the converse to the above mentioned Moreau's theorem. It is the aim of this article to clearly define the best conditions under which the converse of Moreau's theorem holds. More precisely, we characterize all the closed and convex subsets C of an Euclidean space X such that the following converse of Moreau's theorem holds: If, for some closed and convex set D , the pair (C, D) has the strong CHIP, then the inf-convolution of σ_C and σ_D is exact.

Our main result states that the validity of the converse of Moreau's theorem is ensured if and only if the boundary of the closed and convex subset C of the Euclidean space X does not contain any half-strip (by *half-strip* we mean, as customary, the convex hull of two disjoint and parallel half-lines). Note that the class of closed and convex sets without boundary half-strips is rather large, as it contains—the list is not exhaustive—all the bounded sets, the strictly convex sets, or even the continuous sets in the sense of Gale and Klee—sets such that their support functional is continuous except at the origin (see [7]).

The outline of the paper is as follows. The case of closed and convex sets without boundary half-strips is considered in section 2. We prove (Theorem 2.3) that, if the pair (C, D) has the strong CHIP, and if the boundary of one of the sets, say C , does not contain any half-strip, then the inf-convolution of the support functions of C and D must be exact.

The last section is concerned with convex sets which do admit at least one boundary half-strip. If the boundary of a closed and convex set C contains some half-strip, then we give a construction of a closed and convex set D such that the pair (C, D) has the strong CHIP, while the inf-convolution of σ_C and σ_D fails to be exact.

2. Convex sets without boundary half-strips. Now let us first collect some conditions ensuring in every reflexive Banach space the validity of the converse of Moreau's theorem.

PROPOSITION 2.1. *Let C and D be a pair of closed and convex subsets of a reflexive Banach space X . If C and D have the strong CHIP, and at least one of the following conditions holds:*

- (i) $C \cap D$ is bounded;
- (ii) $C \cap D$ is a flat;
- (iii) $C \cap D$ is a half-line,

then the inf-convolution of the support functions of C and D is exact. In other words, the converse of Moreau's theorem is valid.

Proof of Proposition 2.1. We need the following standard convex analysis result.

LEMMA 2.2. *Let C and D be two closed and convex subsets of a locally convex space X , and consider an element y of X^* expressed as the sum $y = y_1 + y_2$ of two normal vectors $y_1 \in N_C(x)$ and $y_2 \in N_D(x)$, for some $x \in C \cap D$. Then the inf-convolution of the support functionals is exact at y , that is,*

$$(2.1) \quad \sigma_C \square \sigma_D(y) = \sigma_C(y_1) + \sigma_D(y_2) = \langle x, y \rangle.$$

Proof of Lemma 2.2. As $y_1 \in \partial \iota_C(x)$ and $y_2 \in \partial \iota_D(x)$, relation $y = y_1 + y_2$ implies that $y \in \partial \iota_{C \cap D}(x)$. Thus

$$\sigma_C(y_1) = \langle x, y_1 \rangle, \quad \sigma_D(y_2) = \langle x, y_2 \rangle, \quad \sigma_{C \cap D}(y) = \langle x, y \rangle,$$

and hence

$$(2.2) \quad \sigma_{C \cap D}(y) = \sigma_C(y_1) + \sigma_D(y_2).$$

Recall that $\sigma_{C \cap D} \leq \sigma_C \square \sigma_D$, which means that

$$(2.3) \quad \sigma_{C \cap D}(y) \leq \sigma_C \square \sigma_D(y).$$

Finally, use the definition of the inf-convolution and the fact that $y = y_1 + y_2$ to deduce that

$$(2.4) \quad \sigma_C \square \sigma_D(y) \leq \sigma_C(y_1) + \sigma_D(y_2).$$

Relation (2.1) follows from relations (2.2), (2.3), and (2.4). \square

Let us now return to the proof of Proposition 2.1 and consider that case (i) holds; i.e., we suppose that the pair (C, D) has the strong CHIP and $C \cap D$ is bounded.

As, in addition, X is a reflexive Banach space, it is easy to see that, for every $y \in X^*$, there is an $x \in C \cap D$ such that $y \in \partial \iota_{C \cap D}(x)$. The pair (C, D) has the strong CHIP, and thus $y = y_1 + y_2$, for some $y_1 \in \partial \iota_C(x)$ and $y_2 \in \partial \iota_D(x)$; we may therefore apply Lemma 2.2 and deduce that

$$(2.5) \quad \sigma_C \square \sigma_D(y) = \sigma_C(y_1) + \sigma_D(y_2) = \langle x, y \rangle.$$

On one hand, from relation (2.5) we observe that the $\Gamma_0(X^*)$ -functional $\sigma_C \square \sigma_D$ is real-valued on X^* and thus, as X^* is a reflexive Banach space, it follows that $\sigma_C \square \sigma_D$ is continuous. Taking into account that relation (2.5) implies, on the other hand, that the infimum is always attained in the expression of the inf-convolution, we conclude that the inf-convolution of the support functions σ_C and σ_D is exact.

Case (ii). Let L be the closed subspace of X parallel to the flat $C \cap D$ (that is, $C \cap D = x_0 + L$ for every $x_0 \in C \cap D$), and factorize X with respect to L .

The quotient space X/L , say \hat{X} , is again a reflexive Banach space. Since $x_0 + L \subset C$ for every $x_0 \in C$, and $x_0 + L \subset D$ for every $x_0 \in D$, it follows that \hat{C} and \hat{D} , the

quotients of the sets C and D are closed, and convex subsets of \hat{X} ; moreover, it is straightforward to prove that the pair (\hat{C}, \hat{D}) has the strong CHIP if and only if the same holds for the pair (C, D) , and that the inf-convolution of the support functions of \hat{C} and \hat{D} is exact if and only if the inf-convolution of the support functions of C and D is exact.

But the intersection between \hat{C} and \hat{D} reduces to a singleton, and case (ii) is proved by applying the conclusion of case (i) to the pair (\hat{C}, \hat{D}) .

Case (iii). Set $x_0 + R_+\bar{x}$ for the half-line $C \cap D$. Obviously, when $y \in X^*$ and $\langle \bar{x}, y \rangle \leq 0$ we have $y \in \partial \iota_{C \cap D}(x_0)$. Use the fact that the pair (C, D) has the CHIP to deduce that $y = y_1 + y_2$ for some $y_1 \in \partial \iota_C(x_0)$ and $y_2 \in \partial \iota_D(x_0)$, together with Lemma 2.2, to infer that

$$(2.6) \quad \sigma_C \square \sigma_D(y) = \sigma_C(y_1) + \sigma_D(y_2) = \langle x_0, y \rangle \quad \forall y \in X^*, \langle \bar{x}, y \rangle \leq 0.$$

In order to obtain a similar relation for the case $\langle \bar{x}, y \rangle > 0$, note that, for every $z \in X^*$ such that $\langle \bar{x}, z \rangle > 0$ it holds that $\sigma_C(z) = \sigma_D(z) = +\infty$. Moreover, the inequality $\langle \bar{x}, z + v \rangle > 0$ means that at least one of the inequalities $\langle \bar{x}, z \rangle > 0$ and $\langle \bar{x}, v \rangle > 0$ holds. Combining these two facts, we deduce that

$$\sigma_C(z) + \sigma_D(v) = +\infty \quad \forall z, v \in X^* \text{ such that } \langle \bar{x}, z + v \rangle > 0,$$

which means that

$$(2.7) \quad \sigma_C \square \sigma_D(y) = +\infty \quad \forall y \in X^*, \langle \bar{x}, y \rangle > 0.$$

Combining relations (2.6) and (2.7) yields

$$\sigma_C \square \sigma_D = \iota_{\{y \in X^* : \langle \bar{x}, y \rangle \leq 0\}} + \langle x_0, \cdot \rangle,$$

which means that the inf-convolution of σ_C and σ_D is the sum between the indicator function of a half-space and a linear and continuous functional, and clearly belongs to $\Gamma_0(X^*)$. Use once more relation (2.6) to see that the infimum in the expression of the inf-convolution is achieved, and conclude that the inf-convolution of σ_C and σ_D is exact. \square

Apparently, Proposition 2.1 lists three completely disparate conditions, each one being sufficient in its own way for the validity of the converse of Moreau's theorem. The geometric notion of a *half-strip*, that is, a convex hull of two parallel and disjoint half-lines, allows us to spot a common property of cases (i), (ii), and (iii) in Proposition 2.1.

THEOREM 2.3. *Let C and D two closed and convex subsets of the Euclidean space X , and assume that the boundary of the set C does not contain any half-strip. If the pair (C, D) has the strong CHIP, then the inf-convolution of σ_C and σ_D is exact (in other words, the converse of Moreau's theorem holds).*

Proof of Theorem 2.3. When the intersection $C \cap D$ meets the interior of C , we specify the well-known Moreau–Rockafellar internal point condition (see [9, Chap. 6, section 6.8]) to prove that the inf-convolution of the support functionals is exact.

If $C \cap D$ is a part of the boundary of C , use—as the boundary of C does not contain any half-strip—the obvious fact that the only closed and convex subsets of an Euclidean space which do not contain any half-strip are the bounded sets, the half-lines, and the lines, and completely prove Theorem 2.3 by making use of Proposition 2.1. \square

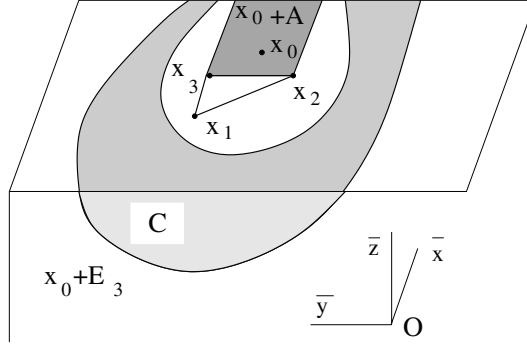


FIG. 3.1. Closed and convex set with a boundary half-strip.

3. Convex sets with boundary half-strips. The following result completes the analysis initiated in Theorem 2.3.

THEOREM 3.1. *Let C be a closed and convex subset of the Euclidean space X ; assume moreover that the boundary of C contains at least one half-strip. Then there is a closed and convex set D such that the pair (C, D) has the CHIP while the inf-convolution between σ_C and σ_D is not exact.*

In other words, the existence of at least one boundary half-strip prevents the converse of Moreau's theorem from holding.

3.1. Construction and properties of the set D . Our strategy in this section is to construct the set D . The following easy result will be useful. It says that when X is Euclidean, every closed and convex set with a boundary half-strip may be contained within some half-space such that its boundary half-strip lies within the hyperplane which delimits this half-space.

PROPOSITION 3.2. *Let C be a closed and convex subset of an Euclidean space X such that its boundary contains a half-strip. Then, there is an orthonormal basis of X , $B = \{b_1, b_2, \dots, b_n\}$, a positive parameter $a > 0$, and x_0 , an element of X , such that*

$$x_0 + A \subset C \subset x_0 + E_3,$$

where A is the half-strip defined as

$$A = \{x \in X : 0 \leq x \cdot b_1, -2a \leq x \cdot b_2 \leq 2a, 0 = x \cdot b_i \quad \forall i \geq 3\},$$

and the half-space E_3 is given by the relation

$$E_3 = \{x \in X : x \cdot b_3 \leq 0\}.$$

Proof of Proposition 3.2. Let x_1, x_2, \bar{x} in X be such that $\|\bar{x}\| = 1$ and the half-strip spanned by the half-lines $x_1 + R_+\bar{x}$ and $x_2 + R_+\bar{x}$ lies within the boundary of the set C . Assume (if necessary after changing x_1 into x_2) that $x_2 \cdot \bar{x} \geq x_1 \cdot \bar{x}$, and set $x_3 = x_1 + [(x_2 - x_1)\bar{x}]\bar{x}$.

Clearly, $x_3 \in x_1 + R_+\bar{x}$; as the half-lines $x_1 + R_+\bar{x}$ and $x_2 + R_+\bar{x}$ are disjoint, it follows that x_3 and x_2 cannot coincide. Set

$$a = \frac{\|x_3 - x_2\|}{4} \quad \text{and} \quad \bar{y} = \frac{x_3 - x_2}{4a} = \frac{x_3 - x_2}{\|x_3 - x_2\|}.$$

Note that

$$(3.1) \quad \bar{y} \cdot \bar{x} = \frac{1}{4a}(x_3 \cdot \bar{x} - x_2 \cdot \bar{x}) = \frac{1}{4a}(x_1 \cdot \bar{x} + (x_2 - x_1)\bar{x} \cdot \bar{x} - x_2 \cdot \bar{x}) = 0.$$

Finally, put

$$x_0 = \frac{x_3 + x_2}{2} + a\bar{x} = \frac{1}{2}(x_3 + a\bar{x}) + \frac{1}{2}(x_2 + a\bar{x});$$

as $x_3 + a\bar{x} \in x_1 + R_+\bar{x}$ and $(x_2 + a\bar{x}) \in (x_2 + R_+\bar{x})$, we deduce that x_0 belongs to the half-strip spanned by the half-lines $x_1 + R_+\bar{x}$ and $x_2 + R_+\bar{x}$, and thus belongs to the boundary of C .

Since x_0 is a boundary point of a closed and convex subset of an Euclidean space, it is well known that there is some linear mapping $\bar{z} \in X$, $\|\bar{z}\| = 1$, which achieves its maximum on C at x_0 ,

$$(3.2) \quad \bar{z} \cdot x_0 \geq \bar{z} \cdot x \quad \forall x \in C.$$

Apply relation (3.2) for $x = x_0 - a\bar{x} = \frac{x_3 + x_2}{2}$ to deduce that $\bar{z} \cdot \bar{x} \geq 0$, and then for $x = x_0 + a\bar{x} = \frac{1}{2}(x_3 + 2a\bar{x}) + \frac{1}{2}(x_2 + 2a\bar{x})$ to obtain $\bar{z} \cdot \bar{x} \leq 0$ and therefore conclude that

$$(3.3) \quad \bar{z} \cdot \bar{x} = 0.$$

Similarly, put $x_0 - a\bar{x} - 2a\bar{y} = x_2$ for x in relation (3.2), and using also relation (3.3), deduce that $\bar{z} \cdot \bar{y} \geq 0$. Finally, putting $x = x_0 - a\bar{x} + 2a\bar{y} = x_3$ in relation (3.2), and also taking into account relation (3.3), we infer $\bar{z} \cdot \bar{y} \leq 0$, that is,

$$(3.4) \quad \bar{z} \cdot \bar{y} = 0.$$

Relations (3.1), (3.3), and (3.4) prove that it is possible to complete the set $\{\bar{x}, \bar{y}, \bar{z}\}$ up to $B = \{b_1, b_2, \dots, b_n\}$, an orthonormal basis of X .

The proof of Proposition 3.2 will be completed if we remark that the set $x_0 + A$ is nothing but the half-strip spanned by the half-lines $(x_3 + a\bar{x}) + R_+\bar{x}$ and $(x_2 + a\bar{x}) + R_+\bar{x}$, and thus lies within the boundary of C , while relation (3.2) implies that C is a part of the half-space $x_0 + E_3$. \square

The basis B , the parameter a , and the element x_0 thus defined allow us to proceed to the construction of the set D . Let us first define the set F ,

$$F = (P_1 + S) \cap (P_2 + T),$$

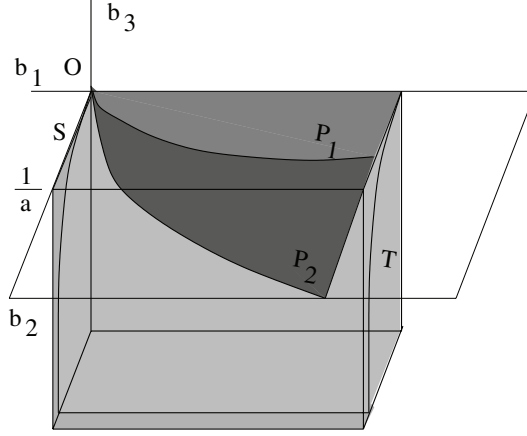
where $P_1 \subset P_2$ are the sets bordered by two plane parabolae:

$$P_1 = \left\{ x \in X : x \cdot b_1 \leq -\frac{a(x \cdot b_2)^2}{4}, x \cdot b_i = 0 \quad \forall i \geq 3 \right\},$$

$$P_2 = \left\{ x \in X : x \cdot b_1 \leq -\frac{a(x \cdot b_2)^2}{8}, x \cdot b_i = 0 \quad \forall i \geq 3 \right\},$$

S is an orthogonal box in X :

$$S = \{x \in X : x \cdot b_1 \leq 0, -1 \leq ax \cdot b_2 \leq 1, x \cdot b_3 \leq 0\},$$

FIG. 3.2. Sets needed in constructing the set D .

T is a closed and convex subset of S :

$$T = \left\{ x \in X : x \cdot b_1 \leq 0, -1 < ax \cdot b_2 < 1, x \cdot b_3 \leq -\frac{a^2(x \cdot b_2)^2}{1 - a^2(x \cdot b_2)^2} \right\},$$

and, as customary, Z^* means the polar set of some subset Z of X ,

$$Z^* = \{x \in X : x \cdot y \leq 1 \quad \forall y \in Z\}.$$

Set now

$$D = x_0 + F^* = x_0 + ((P_1 + S) \cap (P_2 + T))^*.$$

This definition grants to the set F (and thus D) several geometrical properties which are crucial for our purpose.

Let us first notice that F is contained in the half-space $E_1 = \{x \in X : x \cdot b_1 \leq 0\}$; accordingly, the half-line R_+b_1 lies within F^* , and thus the half-line $x_0 + R_+b_1$ is a part of both sets C and $D = x_0 + F^*$. It follows that

$$\sigma_C(x) = \sigma_D(x) = +\infty \quad \forall x \in X \text{ such that } x \cdot b_1 > 0.$$

Let $x \in X$ be such that $x \cdot b_1 = 0$; if y is such that $y \cdot b_1 \neq 0$, then either $y \cdot b_1 > 0$ or $(x - y) \cdot b_1 > 0$, so

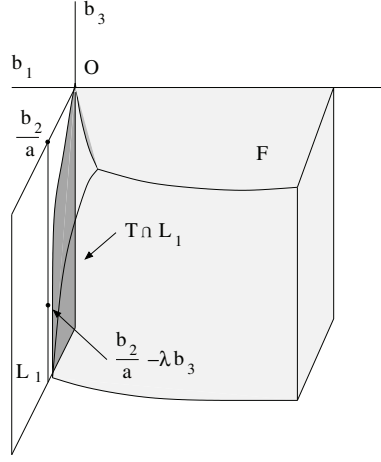
$$(3.5) \quad \sigma_C(y) + \sigma_D(x - y) = +\infty \quad \forall x \in X, x \cdot b_1 = 0, y \in X, y \cdot b_1 \neq 0;$$

hence for every element $x \in X$ such that $x \cdot b_1 = 0$ it results that

$$(3.6) \quad \sigma_C \square \sigma_D(x) = \inf_{y \in X, y \cdot b_1 = 0} \sigma_C(y) + \sigma_D(x - y).$$

The hyperplane $L_1 = \{x \in X : x \cdot b_1 = 0\}$ thus plays a very important role in computing the inf-convolution of the support functions σ_C and σ_D . The following lemma describes the intersection between the set F and L_1 ; for convenience, we state the result in terms of

$$\gamma_F(x) = \inf_{s > 0} \left\{ \frac{1}{s} : sx \in F \right\},$$


 FIG. 3.3. Intersection between F and L_1 .

the gauge function of the set F .

LEMMA 3.3. *The set F is a closed and convex subset of the Euclidean space X . Moreover, $F \cap L_1 = T \cap L_1$, and thus*

$$(3.7) \quad \gamma_F(x) \geq a|x \cdot b_2| \quad \forall x \in L_1,$$

$$(3.8) \quad \gamma_F(x) > 1 \quad \forall x \in L_1 \text{ such that } a|x \cdot b_2| = 1,$$

and

$$(3.9) \quad \lim_{\lambda \rightarrow +\infty} \gamma_F\left(\frac{b_2}{a} - \lambda b_3\right) = 1.$$

Proof of Lemma 3.3. Recall that the sum $Z_1 + Z_2$ of two closed and convex subsets Z_1 and Z_2 of an Euclidean space is always convex. This sum is moreover closed, provided that Z_1 and $-Z_2$ do not contain two parallel half-lines (see [12, Corollary 9.1.2]). This is obviously the case for the pairs of closed and convex sets P_1 and S , as well as P_2 and T , and thus the sets $P_1 + S$ and $P_2 + T$ are closed and convex, and the same clearly holds also for the set F , which is their intersection.

It has already been noticed that all the sets P_1 , P_2 , S , and T lay within E_1 ; accordingly, the sum of two elements x_1 and x_2 from either P_1 and S , or P_2 and T , is contained within the delimiting hyperplane L_1 if and only if both elements x_1 and x_2 belong to L_1 . In other words,

$$(P_1 + S) \cap L_1 = (P_1 \cap L_1) + (S \cap L_1),$$

$$(P_2 + T) \cap L_1 = (P_2 \cap L_1) + (T \cap L_1),$$

and note that $P_1 \cap L_1 = P_2 \cap L_1 = \{0\}$ to deduce that

$$\begin{aligned} F \cap L_1 &= ((P_1 + S) \cap L_1) \cap ((P_2 + T) \cap L_1) \\ &= (S \cap L_1) \cap (T \cap L_1) = (S \cap T) \cap L_1. \end{aligned}$$

Recall that $T \subset S$, and deduce that $F \cap L_1 = T \cap L_1$.

This relation may be used in order to compute the value of $\gamma_F(x)$ for elements $x \in L_1$, since it obviously holds that

$$\gamma_F(x) = \gamma_{F \cap L_1}(x) = \gamma_{T \cap L_1}(x) \quad \forall x \in L_1.$$

Use the fact that

$$T \cap L_1 \subset S \cap L_1 \subset M = \{x \in X : x \cdot b_1 = 0, -1 \leq ax \cdot b_2 \leq 1\}$$

to deduce that

$$\gamma_{T \cap L_1}(x) \geq \gamma_M(x) = a|x \cdot b_2| \quad \forall x \in L_1,$$

that is, relation (3.7).

In order to prove relation (3.8), note that, for every $x \in T \cap L_1$ we have $a|x \cdot b_2| < 1$. Accordingly, relation $a|x \cdot b_2| = 1$ implies that $x \notin T \cap L_1$. Let us now use [12, Corollary 9.7.1], which says that $T \cap L_1 = \{y : \gamma_{T \cap L_1}(y) \leq 1\}$, and deduce that $\gamma_{T \cap L_1}(x) > 1$.

Finally, if $\lambda \geq 1$, standard computation shows

$$\frac{\sqrt{4\lambda^2 + 1} - 1}{2\lambda} \left(\frac{b_2}{a} - \lambda b_3 \right) \in T \cap L_1;$$

thus

$$(3.10) \quad \gamma_{T \cap L_1} \left(\frac{b_2}{a} - \lambda b_3 \right) \leq \frac{2\lambda}{\sqrt{4\lambda^2 + 1} - 1}.$$

Use relation (3.8) for $x = \left(\frac{b_2}{a} - \lambda b_3 \right)$ to see that

$$(3.11) \quad 1 < \gamma_{T \cap L_1} \left(\frac{b_2}{a} - \lambda b_3 \right);$$

relation (3.9) simply comes from relations (3.10) and (3.11). \square

An important step in proving that the pair of closed and convex sets (C, D) has the strong CHIP is to determine their intersection $C \cap D$.

LEMMA 3.4. *It holds that*

$$(3.12) \quad C \cap D = x_0 + (F + Rb_3)^*.$$

Proof of Lemma 3.4. Use the fact that $R_+(-b_3) \subset T \subset S$ to deduce that $R_+(-b_3) \subset F$, and thus that $F^* \subset (-E_3)$. Accordingly, $D \subset x_0 + (-E_3)$, and as $C \subset x_0 + E_3$, we obtain that

$$C \cap D \subset (x_0 + E_3) \cap (x_0 + (-E_3)) = x_0 + L_3,$$

where by L_3 we mean

$$L_3 = \{x \in X : x \cdot b_3 = 0\}.$$

In other words,

$$(3.13) \quad x_0 \cdot b_3 = x \cdot b_3 \quad \forall x \in C \cap D.$$

Consequently,

$$(3.14) \quad C \cap D = C \cap (D \cap (x_0 + L_3)) = C \cap (x_0 + (F^* \cap L_3)).$$

Recall (see [2, Chapter 4, section 1, Corollary of Proposition 3]) that, for every closed and convex sets A and B containing 0, it holds that $(A \cap B)^* = \overline{\text{co}}(A^* \cup B^*)$, where $\overline{\text{co}}(A)$ denotes the closed convex hull of the set A .

Thus, by using the bipolar theorem (see [2, Chapter 4, section 1, Proposition 3]) applied for the set F , and the obvious fact that $L_3^* = Rb_3$, we deduce that

$$(3.15) \quad (F^* \cap L_3)^* = \overline{\text{co}}(F^{**} \cup L_3^*) = \overline{\text{co}}(F \cup Rb_3).$$

It is well known that for every convex set A and flat W , $\overline{\text{co}}(A \cup W) = \overline{\text{co}}(A + W)$. Apply this relation to the convex set F and the one-dimensional flat Rb_3 to prove that $\overline{\text{co}}(F \cup Rb_3) = \overline{\text{co}}(F + Rb_3)$; by virtue of relation (3.15) it follows that

$$(F^* \cap L_3)^* = \overline{\text{co}}(F + Rb_3).$$

Accordingly, $F^* \cap L_3 = (F^* \cap L_3)^{**} = (\overline{\text{co}}(F + Rb_3))^*$; as the polar of any set coincides with the polar of its closure, we have

$$(3.16) \quad F^* \cap L_3 = (F + Rb_3)^*.$$

Let us prove that the set $(F + Rb_3)^*$ lies within C . Indeed, after an easy computation it results that

$$(3.17) \quad \begin{aligned} \overline{(T + Rb_3)} &= S + Rb_3 \\ &= N = \{x \in X : x \cdot b_1 \leq 0, -1 \leq ax \cdot b_2 \leq 1\}; \end{aligned}$$

thus $N \subset \overline{(F + Rb_3)}$, which means that $(F + Rb_3)^* \subset N^*$.

But as

$$N^* = \{x \in X : 0 \leq x \cdot b_1, -a \leq x \cdot b_2 \leq a, 0 = x \cdot b_i \quad \forall i \geq 3\},$$

we have (see Proposition 3.2) $N^* \subset A$, and thus

$$(3.18) \quad x_0 + (F + Rb_3)^* \subset (x_0 + N^*) \subset (x_0 + A) \subset C.$$

Relation (3.12) follows now from relations (3.14), (3.16), and (3.18). \square

It thus becomes necessary to determine the sum between the closed and convex set F and the line Rb_3 .

LEMMA 3.5. *It holds that*

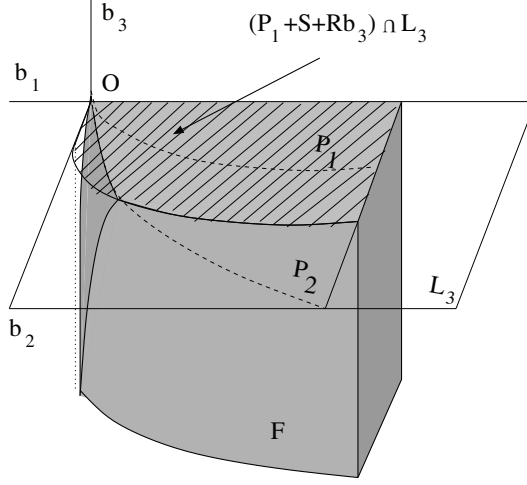
$$(3.19) \quad \{x \in (P_1 + S) : x \cdot b_1 < 0\} + Rb_3 \subset F + Rb_3 \subset (P_1 + S) + Rb_3;$$

accordingly,

$$(3.20) \quad C \cap D = x_0 + (P_1 + S + Rb_3)^*.$$

Moreover, the gauge functions γ_F and $\gamma_{P_1+S+Rb_3}$ fulfill the following property: For every $x \in X$ such that $x \cdot b_1 < 0$, there is $\theta(x) \geq 0$ such that

$$(3.21) \quad \gamma_{P_1+S+Rb_3}(x) = \gamma_F(x - \theta(x)b_3).$$

FIG. 3.4. The sum between F and the line Rb_3 .

Proof of Lemma 3.5. On the basis of formula (3.17), we claim that $S + Rb_3$ is a closed and convex set. Moreover, there are no parallel half-lines within P_1 and $-(S + Rb_3)$, so, using again [12, Corollary 9.1.2], we deduce that the set $P_1 + S + Rb_3$ is closed and convex.

Let us prove the second inclusion in (3.19). As $F \subset (P_1 + S)$, it clearly follows that

$$(3.22) \quad F + Rb_3 \subset P_1 + S + Rb_3.$$

To establish the first inclusion in relation (3.19), we prove and use the fact that, for every $x \in P_1 + S$ such that $x \cdot b_1 < 0$, there is $\lambda(x) \geq 0$ such that $(x - \lambda(x)b_3) \in F$.

When $-1 < ax \cdot b_2 < 1$, it is easy to see that the value

$$\lambda(x) = x \cdot b_3 + \frac{a^2(x \cdot b_2)^2}{1 - a^2(x \cdot b_2)^2}$$

does the job. Indeed, the element $y = (x \cdot b_1)b_1$ lies within both P_1 and P_2 , while

$$z = x - y - \lambda(x)b_3 = (x \cdot b_2)b_2 - \frac{a^2(x \cdot b_2)^2}{1 - a^2(x \cdot b_2)^2}b_3 + \sum_{i=4}^n (x \cdot b_i)b_i$$

is obviously contained in T , and thus in S . Accordingly,

$$x - \lambda(x)b_3 = y + (x - y - \lambda(x)b_3) \in (P_1 + S) \cap (P_2 + T) = F.$$

Let $x \in (P_1 + S)$ such that $x \cdot b_1 < 0$ and $a|x \cdot b_2| \geq 1$; to fix the ideas, admit that $ax \cdot b_2 \geq 1$. In order to define $\lambda(x)$ in this case, use the fact that x can be expressed as the sum $x = y + z$ of two elements y and z such that $y \in P_1$ and $z \in S$.

As $y \in P_1$, it follows that

$$(3.23) \quad y \cdot b_1 \leq -\frac{a(y \cdot b_2)^2}{4};$$

since for every $z \in S$ it holds that $z \cdot b_1 \leq 0$, we deduce that thus $x \cdot b_1 \leq y \cdot b_1$. We may accordingly infer from relation (3.23) that

$$(3.24) \quad x \cdot b_1 \leq -\frac{a(y \cdot b_2)^2}{4}.$$

Use once more the fact that $z \in S$, to conclude that $-1 \leq az \cdot b_2 \leq 1$. Recall that $ax \cdot b_2 \geq 1$, and deduce that

$$(3.25) \quad (y \cdot b_2)^2 = (x \cdot b_2 - z \cdot b_2)^2 \geq \left(x \cdot b_2 - \frac{1}{a}\right)^2;$$

from relation (3.24) and (3.25) it follows that

$$(3.26) \quad x \cdot b_1 \leq -\frac{a\left(x \cdot b_2 - \frac{1}{a}\right)^2}{4}.$$

Combine the fact that $x \cdot b_1 \neq 0$ with relation (3.26) and deduce that

$$x \cdot b_1 < -\frac{a\left(x \cdot b_2 - \frac{1}{a}\right)^2}{8};$$

accordingly, for some parameter α such that $0 < a\alpha < 1$, we have

$$(3.27) \quad x \cdot b_1 < -\frac{a\left(x \cdot b_2 - \alpha\right)^2}{8}.$$

We can now define $\lambda(x)$ as

$$\lambda(x) = \frac{a^2\alpha^2}{1 - a^2\alpha^2}.$$

Inequality (3.27) proves that the element $(x - \alpha b_2 - \sum_{i=4}^n (x \cdot b_i) b_i)$ belongs to the set P_2 ; as obviously

$$\alpha b_2 - \frac{a^2\alpha^2}{1 - a^2\alpha^2} b_3 + \sum_{i=4}^n (x \cdot b_i) b_i \in T,$$

we deduce that

$$(3.28) \quad \begin{aligned} & x - \lambda(x) b_3 \\ &= \left(x - \alpha b_2 - \sum_{i=4}^n (x \cdot b_i) b_i\right) + \left(\alpha b_2 - \frac{a^2\alpha^2}{1 - a^2\alpha^2} b_3 + \sum_{i=4}^n (x \cdot b_i) b_i\right) \\ &\in P_2 + T. \end{aligned}$$

Remark that the case $ax \cdot b_2 \leq -1$ is similar to the case $ax \cdot b_2 \geq 1$. Indeed, when $ax \cdot b_2 \leq -1$, one has

$$(y \cdot b_2)^2 \geq \left(x \cdot b_2 + \frac{1}{a}\right)^2$$

instead of relation (3.25). The parameter α now lies between $-\frac{1}{a}$ and 0, and fulfills

$$x \cdot b_1 < -\frac{a\left(x \cdot b_2 + \alpha\right)^2}{8},$$

and not relation (3.26).

As in the case $ax \cdot b_2 \geq 1$, x is the sum of two elements: one in P_2 , the other in T . However, when $ax \cdot b_2 \leq -1$, the element belonging to P_2 is $(x + \alpha b_2 - \sum_{i=4}^n (x \cdot b_i) b_i)$, and the one lying in T is $-\alpha b_2 - \frac{a^2 \alpha^2}{1 - a^2 \alpha^2} b_3 + \sum_{i=4}^n (x \cdot b_i) b_i$.

Noticing that $S + R_+(-b_3) = S$, we have $(P_1 + S + R_+ b_3) = (P_1 + S)$. Thus, as $x \in P_1 + S$, we deduce that

$$(3.29) \quad x - \lambda b_3 \in P_1 + S \quad \forall \lambda \geq 0;$$

from (3.28) and (3.29) it follows that

$$(x - \lambda(x) b_3) \in (P_1 + S) \cap (P_2 + T) = F,$$

and therefore for every $x \in P_1 + S$ such that $x \cdot b_1 < 0$, there is $\lambda(x) \geq 0$ such that $x - \lambda(x) b_3 \in F$.

Use this observation to prove that

$$\{x \in P_1 + S : x \cdot b_1 < 0\} + Rb_3 \subset F + Rb_3,$$

which, together with relation (3.22), yields relation (3.19).

Relation (3.19) implies that the set $P_1 + S + Rb_3$ is the closure of the set $F + Rb_3$. Recalling that the polar of any set coincides with the polar of its closure, we deduce that

$$(3.30) \quad (F + Rb_3)^* = (P_1 + S + Rb_3)^*,$$

and relation (3.20) follows from formulas (3.12) and (3.30).

It remains to prove relation (3.21). To begin with, notice that, from relation (3.19) it follows that $F \subset P_1 + S + Rb_3$, and thus

$$(3.31) \quad \gamma_{P_1 + S + Rb_3} \leq \gamma_F.$$

Let us first prove that $\gamma_{P_1 + S + Rb_3}(x)$ is real-valued for every $x \in X$ such that $x \cdot b_1 < 0$. In this respect, note that from relation (3.17) it results that $\{x \in X : x \cdot b_2 = x \cdot b_3 = 0\} \subset S + Rb_3$, and thus that

$$(3.32) \quad P_1 + \{x \in X : x \cdot b_1 = x \cdot b_2 = 0\} \subset (P_1 + S + Rb_3).$$

On the other hand, the set $P_1 + \{x \in X : x \cdot b_1 = x \cdot b_2 = 0\}$ contains all the elements $x \in X$ such that $x \cdot b_1 \leq -\frac{a(x \cdot b_2)^2}{4}$. As

$$\left(-\frac{4x \cdot b_1}{a(x \cdot b_2)^2} x \right) \cdot b_1 = -\frac{4(x \cdot b_1)^2}{a(x \cdot b_2)^2} = -\frac{a \left(\left(-\frac{4x \cdot b_1}{a(x \cdot b_2)^2} x \right) \cdot b_2 \right)^2}{4},$$

which means that

$$-\frac{4x \cdot b_1}{a(x \cdot b_2)^2} x \in P_1 + \{x \in X : x \cdot b_1 = x \cdot b_2 = 0\}.$$

Combine the previous relation with formula (3.32) to deduce that

$$\gamma_{P_1 + S + Rb_3}(x) \leq \frac{a(x \cdot b_2)^2}{|4x \cdot b_1|}.$$

Consequently, for every $x \in X$ such that $x \cdot b_1 < 0$ we have $\gamma_{P_1+S+Rb_3}(x) < +\infty$.

Let us first consider the case when $\gamma_{P_1+S+Rb_3}(x) = 0$, that is, when x belongs to a ray completely contained in $P_1 + S + Rb_3$. Taking into account the definitions of the sets P_1 and S , we have

$$[\gamma_{P_1+S+Rb_3}(x) = 0] \Leftrightarrow [x \cdot b_1 \leq 0 \text{ and } x \cdot b_2 = 0];$$

similarly,

$$[\gamma_F(x) = 0] \Leftrightarrow [x \cdot b_1 \leq 0, x \cdot b_2 = 0 \text{ and } x \cdot b_3 \leq 0].$$

In this case, $\theta(x) = \frac{x \cdot b_3 + |x \cdot b_3|}{2}$ obviously does the job.

Let us now turn to the case $\gamma_{P_1+S+Rb_3}(x) > 0$ and write that

$$\frac{x}{\gamma_{P_1+S+Rb_3}(x)} \in P_1 + S + Rb_3.$$

We deduce that there is a $\tilde{\lambda}(x) \in R$ such that

$$\frac{x}{\gamma_{P_1+S+Rb_3}(x)} - \tilde{\lambda}(x)b_3 \in P_1 + S.$$

Accordingly,

$$\left(\frac{x}{\gamma_{P_1+S+Rb_3}(x)} - \tilde{\lambda}(x)b_3 \right) - \lambda \left(\frac{x}{\gamma_{P_1+S+Rb_3}(x)} - \tilde{\lambda}(x)b_3 \right) b_3 \in F,$$

which means that

$$(3.33) \quad \gamma_F \left(x - \gamma_{P_1+S+Rb_3}(x) \left(\tilde{\lambda}(x) + \lambda \left(\frac{x}{\gamma_{P_1+S+Rb_3}(x)} - \tilde{\lambda}(x)b_3 \right) \right) b_3 \right) \\ \leq \gamma_{P_1+S+Rb_3}(x).$$

Relations (3.31), (3.33) and the obvious fact that

$$\gamma_{P_1+S+Rb_3}(x) = \gamma_{P_1+S+Rb_3}(x + \nu b_3) \quad \forall \nu \in R$$

prove relation (3.21) with

$$\theta(x) = \gamma_{P_1+S+Rb_3}(x) \left(\tilde{\lambda}(x) + \lambda \left(\frac{x}{\gamma_{P_1+S+Rb_3}(x)} - \tilde{\lambda}(x)b_3 \right) \right),$$

completing in this way the proof of Lemma 3.5. \square

3.2. The main result. We claim that *the pair of closed and convex sets C and D has the strong CHIP.*

PROPOSITION 3.6. *The pair of closed and convex subsets C and D of the Euclidean space X has the strong CHIP.*

Proof of Proposition 3.6. Let $x_1 \in C \cap D$ and $y \in \partial \iota_{C \cap D}(x_1)$, $y \neq 0$. Our aim is to express y as the sum of two elements y_1 and y_2 from $\partial \iota_C(x_1)$ and $\partial \iota_D(x_1)$.

Let us first remark that, since $(x_0 + A) \subset C \subset (x_0 + E_3)$ (see Proposition 3.2), it follows that

$$(3.34) \quad R_+ b_3 \subset \partial \iota_C(x)$$

for every $x \in x_0 + A$, in particular for every $x \in C \cap D$ (see Lemma 3.4 and relation (3.18)). Similarly, we deduce that

$$(3.35) \quad R_+(-b_3) \subset \partial \nu_D(x)$$

for every $x \in C \cap D$. The flat $\{x \in X : x \cdot b_i = 0 \ \forall 1 \leq i \leq 3\}$ obviously lies within T , and thus in S , hence in F . Thus D is contained within the flat $x_0 + \{x \in X : x \cdot b_i = 0 \ \forall i \geq 4\}$, and we deduce that

$$(3.36) \quad \{x \in X : x \cdot b_1 = x \cdot b_2 = x \cdot b_3 = 0\} \subset \partial \nu_D(x) \quad \forall x \in D.$$

From relations (3.34), (3.35), and (3.36) it follows that

$$(3.37) \quad \{x \in X : x \cdot b_1 = x \cdot b_2 = 0\} \subset (\partial \nu_C(x) + \partial \nu_D(x)) \quad \forall x \in (C \cap D).$$

We address first the case when $y \cdot b_1 = 0$. A standard computation shows that

$$(3.38) \quad (P_1 + S + Rb_3)^* = \left\{ x \in X : x \cdot b_1 \geq \frac{(x \cdot b_2)^2}{a - |x \cdot b_2|}, -a < x \cdot b_2 < a, x \cdot b_i = 0 \quad \forall i \geq 3 \right\}.$$

From the previous relation it follows that the set $(P_1 + S + Rb_3)^*$ is contained within the plane spanned by b_1 and b_2 . For every $y \in X$ such that $y \cdot b_1 = 0$ it follows that

$$(3.39) \quad x \cdot y = (x \cdot b_2)(y \cdot b_2) \quad \forall x \in (P_1 + S + Rb_3)^*.$$

The elements x_1 and x_0 are both in $C \cap D$; in view of relation (3.20) it appears that $(x_1 - x_0) \in (P_1 + S + Rb_3)^*$.

From relation (3.38) it follows that $|(x_1 - x_0) \cdot b_2| < a$. Set

$$\alpha = \frac{a + |(x_1 - x_0) \cdot b_2|}{2}, \quad z_1 = \frac{\alpha^2}{a - \alpha} b_1 - \alpha b_2, \quad z_2 = \frac{\alpha^2}{a - \alpha} b_1 + \alpha b_2;$$

thus $z_1, z_2 \in (P_1 + S + Rb_3)^*$ and $z_1 \cdot b_2 < (x_1 - x_0) \cdot b_2 < z_2 \cdot b_2$.

Recall that, as $y \in \partial \nu_{C \cap D}(x_1)$, the linear functional $X \ni x \rightarrow (x \cdot y) \in R$ achieves its maximum on $C \cap D$ at x_1 . Thus, on one hand,

$$(x_0 + z_1) \cdot y \leq x_1 \cdot y,$$

that is, in view of relation (3.39),

$$(z_1 \cdot b_2)(y \cdot b_2) \leq ((x_1 - x_0) \cdot b_2)(y \cdot b_2);$$

combine this relation with the fact that $z_1 \cdot b_2 < (x_1 - x_0) \cdot b_2$, and deduce that $y \cdot b_2 \geq 0$. On the other hand,

$$(x_0 + z_2) \cdot y \leq x_1 \cdot y,$$

that is, once more by virtue of relation (3.39),

$$(z_2 \cdot b_2)(y \cdot b_2) \leq ((x_1 - x_0) \cdot b_2)(y \cdot b_2);$$

in addition, as $(x_1 - x_0) \cdot b_2 < z_2 \cdot b_2$, we get $y \cdot b_2 \leq 0$.

We may thus conclude that, when $y \cdot b_1 = 0$, it results that $y \cdot b_2 = 0$, and formula (3.37) proves that y is the sum of two elements from $\partial\iota_C(x_1)$ and $\partial\iota_D(x_1)$.

Consider now the case when $y \cdot b_1 \neq 0$, which, taking into account the fact that the half-line $(x_0 + R_+ b_1)$ is contained (as already remarked) within $C \cap D$, amounts to saying that $y \cdot b_1 < 0$.

It is well known ([12, Theorem 14.5]) that, for every closed and convex set Z containing 0 it holds $\sigma_Z = \gamma_{Z^*}$. Use this relation for the set $(P_1 + S + Rb_3)^*$ to obtain

$$\sigma_{(P_1+S+Rb_3)^*} = \gamma_{P_1+S+Rb_3};$$

as (see 3.20) $(P_1 + S + Rb_3)^* = (C \cap D) - x_0$, it follows that

$$(3.40) \quad \sigma_{(C \cap D) - x_0} = \gamma_{P_1+S+Rb_3}.$$

Similarly,

$$(3.41) \quad \sigma_{D-x_0} = \gamma_F.$$

From relations (3.21), (3.40), and (3.41) it follows that there is some $\theta(y) \geq 0$ such that

$$\sigma_{(C \cap D) - x_0}(y) = \sigma_{D-x_0}(y - \theta(y)b_3);$$

thus

$$(3.42) \quad \sigma_{C \cap D}(y) = \sigma_D(y - \theta(y)b_3) + \theta(y)x_0 \cdot b_3.$$

As $y \in \partial\iota_{C \cap D}(x_1)$, it results that $\sigma_{C \cap D}(y) = x_1 \cdot y$; use relation (3.42) to see that

$$(3.43) \quad \sigma_D(y - \theta(y)b_3) + \theta(y)x_0 \cdot b_3 = x_1 \cdot y.$$

Relation (3.13) reads $x_0 \cdot b_3 = x_1 \cdot b_3$. Equality (3.43) may thus be stated as

$$\sigma_D(y - \theta(y)b_3) + \theta(y)x_1 \cdot b_3 = x_1 \cdot y,$$

that is,

$$\sigma_D(y - \theta(y)b_3) = x_1 \cdot (y - \theta(y)b_3).$$

This means that $(y - \theta(y)b_3) \in \partial\iota_D(x_1)$.

Recall (see relation (3.34)) that $\lambda b_3 \in \partial\iota_C(x_1)$ for every $\lambda \geq 0$, and express y as $y = \theta(y)b_3 + (y - \theta(y)b_3)$, that is, the sum of an element from $\partial\iota_C(x_1)$ and the sum of another element from $\partial\iota_D(x_1)$. \square

We finally claim that *the inf-convolution of the support functionals σ_C and σ_D is not exact at $\frac{b_2}{a}$* , a fact which completes the proof of Theorem 3.1.

PROPOSITION 3.7. *It holds that*

$$(3.44) \quad \sigma_C \square \sigma_D \left(\frac{b_2}{a} \right) = \frac{x_0 \cdot b_2}{a} + 1,$$

while

$$(3.45) \quad \sigma_C(y) + \sigma_D(z) > \frac{x_0 \cdot b_2}{a} + 1 \quad \forall y + z = \frac{b_2}{a}.$$

Proof of Proposition 3.7. Use the fact that $(x_0 + 2ab_2)$ and $(x_0 - 2ab_2)$ both belong to $x_0 + A$, and thus to C , to deduce that

$$(3.46) \quad \sigma_C(x) \geq \max((x_0 + 2ab_2) \cdot x, (x_0 - 2ab_2) \cdot x) = x_0 \cdot x + 2a|x \cdot b_2|.$$

From relation (3.7) it follows that

$$\gamma_F(x) \geq a|x \cdot b_2| \quad \forall x \in L_1.$$

Relation (3.41) reads $\sigma_{D-x_0} = \gamma_F$; hence, it results that

$$(3.47) \quad \sigma_D(x) = x_0 \cdot x + \gamma_F(x).$$

It follows that

$$(3.48) \quad \sigma_D(x) \geq x_0 \cdot x + a|x \cdot b_2| \quad \forall x \in L_1.$$

From relations (3.46) and (3.48) it results that

$$(3.49) \quad \begin{aligned} \sigma_C(x) + \sigma_D\left(\frac{b_2}{a} - x\right) &\geq x_0 \cdot x + 2a|x \cdot b_2| + x_0 \cdot \left(\frac{b_2}{a} - x\right) + a \left| \left(\frac{b_2}{a} - x\right) \cdot b_2 \right| \\ &\geq \frac{x_0 \cdot b_2}{a} + 1 + a|x \cdot b_2| \quad \forall x \in X, x \cdot b_1 = 0. \end{aligned}$$

By taking into account relations (3.6) and (3.49) we prove that

$$(3.50) \quad \sigma_C \square \sigma_D\left(\frac{b_2}{a}\right) \geq \frac{x_0 \cdot b_2}{a} + 1.$$

As $x_0 \in C \subset (x_0 + E_3)$, it follows that

$$\sigma_C(\lambda b_3) = \lambda x_0 \cdot b_3 \quad \forall \lambda \geq 0.$$

Use relation (3.47) to deduce that

$$\begin{aligned} \sigma_C(\lambda b_3) + \sigma_D\left(\frac{b_2}{a} - \lambda b_3\right) &= \lambda x_0 \cdot b_3 + \frac{x_0 \cdot b_2}{a} - \lambda x_0 \cdot b_3 + \gamma_F\left(\frac{b_2}{a} - \lambda b_3\right). \end{aligned}$$

From the previous equality, together with relation (3.9), it yields that

$$\lim_{\lambda \rightarrow \infty} \left(\sigma_C(\lambda b_3) + \sigma_D\left(\frac{b_2}{a} - \lambda b_3\right) \right) = \frac{x_0 \cdot b_2}{a} + 1,$$

which, combined with inequality (3.50), proves relation (3.44).

Finally, let $x \in L_1$ be such that $x \cdot b_2 = 0$. Then (see relation (3.8))

$$\gamma_F\left(\frac{b_2}{a} - x\right) > 1,$$

and, as obviously $\sigma_C(x) \geq x_0 \cdot x$, it results that

$$\begin{aligned}
 (3.51) \quad \sigma_C(x) + \sigma_D\left(\frac{b_2}{a} - x\right) & \\
 & \geq x_0 \cdot x + \frac{x_0 \cdot b_2}{a} - x_0 \cdot x + \gamma_F\left(\frac{b_2}{a} - x\right) \\
 & > \frac{x_0 \cdot b_2}{a} + 1 \quad \forall x \in X, x \cdot b_1 = x \cdot b_2 = 0.
 \end{aligned}$$

Use relation (3.49) to deduce that, for every $x \in X$ such that $x \cdot b_1 = 0$ and $x \cdot b_2 \neq 0$, it holds that

$$(3.52) \quad \sigma_C(x) + \sigma_D\left(\frac{b_2}{a} - x\right) > \frac{x_0 \cdot b_2}{a} + 1.$$

Relation (3.45) follows from relations (3.5), (3.52), and (3.51). \square

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