Equivalences between blocks of p-local Mackey algebras.

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Abstract

Let G be a finite group and $(K, \mathcal{O}, k)$ be a p-modular system. Let $R = \mathcal{O}$ or k. There is a bijection between the blocks of the group algebra and the blocks of the so-called p-local Mackey algebra $\mu_R^1(G)$. Let b be a block of RG with abelian defect group D. Let b' be its Brauer correspondant in $N_G(D)$. It is conjectured by Broué that the blocks RGb and $RN_G(D)b'$ are derived equivalent. Here we look at equivalences between the corresponding blocks of p-local Mackey algebras. We prove that an analogue of the Broué's conjecture is true for the p-local Mackey algebras in the following cases: for the principal blocks of p-nilpotent groups and for blocks with defect 1. We also point out the probable importance of *splendid* equivalences for the Mackey algebras.

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A.M.S. subject classification: 20C05, 18E30,16G10,16W99.

1 Introduction and preliminaries

1.1 Introduction

Let G be a finite group and $(K, \mathcal{O}, k)$ be a p-modular system. Let $R = \mathcal{O}$ or k. In their paper, Thévenaz and Webb proved that there is a bijection $b \mapsto b^\mu$ between the blocks of RG and the primitive central idempotents of $\mu_R^1(G)$, called the blocks of $\mu_R^1(G)$, where $\mu_R^1(G)$ is the so called p-local Mackey algebra.

This bijection preserves the defect groups (which are the same as the defect groups of the corresponding block of RG, see Section 5.3 of [5]). So using the Brauer correspondence, we have the following diagram: Let b be a block of RG with defect group D and b' be the Brauer correspondent of b in $RN_G(D)$.

$$
\begin{array}{ccc}
b \in Z(RG) & \longrightarrow & b^\mu \in Z(\mu_R^1(G)) \\
\downarrow & & \downarrow \\
b' \in Z(RN_G(D)) & \longrightarrow & b'^\mu \in Z(\mu_R^1(N_G(D))).
\end{array}
$$

If D is abelian, it is conjectured by Broué that the block algebras RGb and $RN_G(D)b'$ are derived equivalent. In [21], the Author proved that if two blocks of group algebras

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are *splendidly* derived equivalent, then the corresponding blocks of the so-called *cohomological* Mackey algebras are derived equivalent. The cohomological Mackey algebra is a quotient of the p-local Mackey algebra. So, it is a very natural question to ask if the same can happen for the corresponding p-local Mackey algebras which contain much more informations than their cohomological quotient (see Proposition 3.5 e.g.).

Question 1.1 (Bouc)**.** *Let G be a finite group and b be a block of $\mathcal{O}G$ with abelian defect group D. Let b' be its Brauer correspondent in $\mathcal{O}N_G(D)$. Is there a derived equivalence $D^b(\mu^1_{\mathcal{O}}(G)b^\mu) \cong D^b(\mu^1_{\mathcal{O}}(N_G(D))b'^\mu)$?*

Unfortunately, the tools which was developed for the cohomological Mackey algebras cannot be used here. Moreover Question 1.1 is much harder than its analogue for the cohomological Mackey algebras and is connected to deep questions of representation of finite groups such as the knowledge of the indecomposable p-permutation modules (see Section 6 and 7 of [4]). Here, we will not answer this question in general, but we consider it in the following two cases: first for the Mackey algebra of the principal block of p-nilpotent groups, and then for groups with Sylow p-subgroups of order p.
The main result of this paper is the following theorem:

Theorem 1.2. *The answer to Question 1.1 is affirmative in the following cases:*

- *For principal blocks of p-nilpotent groups. Here we do not assume the Sylow p-subgroups to be abelian. Moreover it is a Morita equivalence.*

- *When the p-local Mackey algebra is symmetric. That is for the blocks with defect 1.*

For a non principal block of a p-nilpotent group, or more generally for a nilpotent block it is not expect for the corresponding block of the Mackey algebra to be Morita equivalent to the Mackey algebra of its defect group.
Let b and c be two blocks of group algebras. Then it is probable that the corresponding blocks of p-local Mackey algebras will be equivalent only if there is an equivalence between b and c which respect the p-permutation modules (*splendid* equivalences e.g.). We do not have a proof of this fact, but there are some clues in this direction. Example 4.9 is exactly an example of blocks which are Morita equivalent but not 'splendidly' Morita equivalent. In this case, the corresponding blocks of p-local Mackey algebras are not Morita equivalent but derived equivalent. Other example can be found in Chapter 4 of [19]. Moreover the decomposition matrix of a block of p-local Mackey algebras involves p-permutation modules and Brauer quotient (see Proposition 3.5).
In the first part, we recall some basic definitions and results on Mackey functors. Since we will use several points of view on Mackey functors, we give a full proof of the well-known fact that all these points of view are equivalent. Then we recall the bijection between the blocks of the group algebra and the blocks of the p-local Mackey algebra. At the end of the second section, we explain how the decomposition matrix of the p-local Mackey algebra of a finite group G can be computed from the knowledge of the ordinary characters of some p-local subgroups of G and the indecomposable p-permutation

kG-modules.

In the last part, we investigate on some basic properties of equivalences between blocks of p-local Mackey algebras. We show that naive thoughts about these questions are seldom true. For example an equivalence between blocks of p-local Mackey algebras do not need to induce an equivalence for the cohomological quotient. If two blocks of group algebras are isomorphic (resp. Morita equivalent), then the corresponding blocks of p-local Mackey algebras do not need to be isomorphic (resp. Mortia equivalent).

We show that there is a Morita equivalence between the principal block of the p-local Mackey algebra of a p-nilpotent group G and the Mackey algebra of a Sylow p-subgroup of G. Moreover, we determine the structure of algebra of the block of the p-local Mackey algebra in this case. This can be view as an analogue of Puig's theorem (see [16] or [15]) for the Mackey algebras.

Then we will answer Question 1.1 for blocks with defect group of order p. The result uses the fact that the Mackey algebras are *symmetric* algebras in this situation. More precisely over the field k, they are *Brauer tree* algebras and over the valuation ring $\mathcal{O}$ they are *Green orders*.

It should be notice that all the equivalences between blocks of groups algebras which induce equivalences between the corresponding p-local Mackey algebras that we were able to construct are in fact *splendid*.

Notation: Let R be a commutative ring with unit. We denote by $R\text{-}Mod$ the category of (all) R-modules.

Let G be a finite group. Let p be a prime number. We denote by $(K, \mathcal{O}, k)$ a p-modular system, i-e $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal $\mathfrak{p}$, such that $\mathcal{O}/\mathfrak{p} = k$ is a field of characteristic p and $Frac(\mathcal{O}) = K$ is a field of characteristic zero. If $R = \mathcal{O}$ or k, then the direct summands of the permutation RG-modules are called *p-permutation* modules. We denote by $G\text{-}set$ the category of finite G-sets. If H is a subgroup of G then, we denote by $N_G(H)$ its normalizer in G. The quotient $N_G(H)/H$ will be, sometimes, denoted by $\overline{N}_G(H)$. We denote by Ω_G the disjoint union of all transitive G-sets.

Let V be an RG-module, for $R = \mathcal{O}$ or k. Let Q be a p-subgroup of G. Since we are dealing with Mackey functor we use the notation $V[Q]$ instead of the usual $V(Q)$ for the Brauer quotient of V (see Section 1 [7]).

If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are two functors, we denote by $F \dashv G$ the fact that F is a left adjoint of G.

N.B. We will denote by the same letter the block idempotents for the ring $\mathcal{O}$ and the field k.

2 Basic results on Mackey functors.

Let G be a finite group, and R be a commutative ring. There are several definitions of Mackey functors for G over a ring R, the first one was introduced by Green in [11]:

Definition 2.1. A Mackey functor for G over R consists of the following data:

- For every subgroup H of G, an R-module denoted by $M(H)$.

- For subgroups $H \subseteq K$ of G, a morphism of R-modules
  $t_H^K : M(H) \to M(K)$ called transfer, or induction, and a morphism of R-modules
  $r_H^K : M(K) \to M(H)$ called restriction.

- For every subgroup H of G, and each element x of G, a morphism of R-modules
  $c_{x,H} : M(H) \to M(^xH)$ called conjugacy map.

such that:

1. *Triviality axiom*: For each subgroup H of G, and each element $h \in H$, the morphisms r_H^H, t_H^H et $c_{h,H}$ are the identity morphism of $M(H)$.

2. *Transitivity axiom*: If $H \subseteq K \subseteq L$ are subgroups of G, then $t_K^L \circ t_H^K = t_H^L$ and $r_H^K \circ r_K^L = r_H^L$. Moreover if x and y are elements of G, then $c_{y,^xH} \circ c_{x,H} = c_{yx,H}$.

3. *Compatibility axioms:* If $H \subseteq K$ are subgroups of G, and if x is an element of G, then $c_{x,K} \circ t_H^K = t_{^xH}^{^xK} \circ c_{x,H}$ et $c_{x,H} \circ r_H^K = r_{^xH}^{^xK} \circ c_{x,K}$.

4. *Mackey axiom:* If $H \subseteq K \supseteq L$ are subgroups of G, then
$$r_H^K \circ t_L^K = \sum_{x \in [H\backslash K/L]} t_{H \cap {^xL}}^H \circ c_{x, H^x \cap L} \circ r_{H^x \cap L}^L.$$

   where $[H\backslash K/L]$ is a set of representatives of the double cosets $H\backslash K/L$.

In particular, for each subgroup H of G, the R-module $M(H)$ is an $N_G(H)/H$-module. A morphism f between two Mackey functors M and N is the data of a R-linear morphism $f(H) : M(H) \to N(H)$ for every subgroup H of G. These morphisms are compatible with transfer, restriction and conjugacy maps. We denote by $Mack_R(G)$ the category of Mackey functors for G over R.

Example 2.2. Let V be an RG-module. The fixed point functor FP_V is the Mackey functor for G over R defined as follows:
For $H \leqslant G$, then $FP_V(H) = V^H := \{ v \in V \; ; \; h \cdot v = v \; \forall \; h \in H \; \}$. If $H \leqslant K \leqslant G$, we have $V^K \subseteq V^H$, so we define the restriction map r_H^K as the inclusion map. The transfer map $t_H^K : V^H \to V^K$ is defined by $t_H^K(v) = \sum_{k \in [K/H]} k \cdot v$ where $[K/H]$ is a set of representatives of K/H. The conjugacy maps are induced by the action of G on V.

It is not hard to see that the construction $V \mapsto FP_V$ is a functor from RG-Mod to $Mack_R(G)$.
Conversely we have an obvious functor $ev_1 : Mack_R(G) \to RG$-Mod given by the evaluation at the subgroup $\{1\}$.

Proposition 2.3. *[22] The functor ev_1 is a left adjoint to the functor $V \mapsto FP_V$. I.e. there is a natural isomorphism:*

$$Hom_{Mack_R(G)}(M, FP_V) \cong Hom_{RG}(M(1), V)$$

for a Mackey functor M and an RG-module V.

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Remark 2.4. The full subcategory of $Mack_R(G)$ consisting of fixed point functors FP_V is equivalent to the category $RG\text{-}Mod$ (see proof of Theorem 18.1 [22]).

An other definition of Mackey functors was given by Dress in [10]:

Definition 2.5. A bivariant functor $M = (M^*, M_*)$ from $G\text{-}set$ to $R\text{-}Mod$ is a pair of functors from $G\text{-}set$ to $R\text{-}Mod$ such that M^* is a contravariant functor, and M_* is a covariant functor. If X is a finite G-set, then the image by the covariant and by the contravariant part coincide. We denote by $M(X)$ this image. A Mackey functor for G over R is a bivariant functor from $G\text{-}set$ to $R\text{-}Mod$ such that:

- Let X and Y be two finite G-sets, i_X and i_Y the canonical injection of X (resp. Y) in $X \sqcup Y$. Then $M^*(i_X) \oplus M^*(i_Y)$ and $(M_*(i_X), M_*(i_Y))$ are inverse isomorphisms, from $M(X) \oplus M(Y)$ to $M(X \sqcup Y)$.

- If

$$
\begin{array}{ccc}
X & \xrightarrow{\ a\ } & Y \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\
Z & \xrightarrow{\ d\ } & T
\end{array}
$$

  is a pull back diagram of G-sets, then the diagram

$$
\begin{array}{ccc}
M(X) & \xleftarrow{M^*(a)} & M(Y) \\
\downarrow{\scriptstyle M_*(b)} & & \downarrow{\scriptstyle M_*(c)} \\
M(Z) & \xleftarrow{M^*(d)} & M(T)
\end{array}
$$

  is commutative.

A morphism between two Mackey functors is a natural transformation of bivariant functors.

Example 2.6 (2.4.1 [3]). If X is a finite G-set, the category of G-sets over X is the category with objects (Y, ϕ) where Y is a finite G-set and ϕ is a morphism from Y to X. A morphism f from (Y, ϕ) to (Z, ψ) is a morphism of G-sets $f : Y \to Z$ such that $\psi \circ f = \phi$.

The Burnside functor at X, written $B(X)$, is the Grothendieck group of the category of G-sets over X, for relations given by disjoint union. This is a Mackey functor for G over R by extending scalars from $\mathbb{Z}$ to R. We will denote RB the Burnside functor after scalars extension.

If X is a G-set, the Burnside group $RB(X^2)$ has a ring structure. A G-set Z over $X \times X$ is the data of a G-set Z and a map $(b \times a)$ from Z to $X \times X$, denoted by $(X \xleftarrow{b} Y \xrightarrow{a} X)$. The product of (the isomorphism class of) $(X \xleftarrow{\alpha} Y \xrightarrow{\beta} X)$ and (the isomorphism class

of)($X \xleftarrow{\gamma} Z \xrightarrow{\delta} X$) is given by (the isomorphism class of) the pullback along β and γ.

$$P$$

$$Y \qquad\qquad Z$$

$$\alpha \quad \beta \qquad \gamma \quad \delta$$

$$X \qquad\qquad X \qquad\qquad X$$
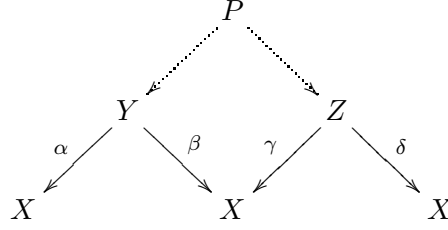
The identity of this ring is (the isomorphism class)

$$X$$

$$X \qquad\qquad X$$
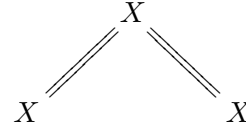
In the rest of the paper, we will use the same notation for a G-set over $X \times X$ and its isomorphism class in $RB(X \times X)$.

We need a last definition of Mackey functors which was given by Thévenaz and Webb in [22], and uses the Mackey algebra.

Definition 2.7. The Mackey algebra $\mu_R(G)$ for G over R is the unital associative algebra with generators t_H^K, r_H^K and $c_{g,H}$ for $H \leqslant K \leqslant G$ and $g \in G$, with the following relations:

- $\sum_{H \leqslant G} t_H^H = 1_{\mu_R(G)}.$

- $t_H^H = r_H^H = c_{h,H}$ for $H \leqslant G$ and $h \in H$.

- $t_K^L t_H^K = t_H^L$, $r_H^K r_K^L = r_H^L$ for $H \subseteq K \subseteq L$.

- $c_{g',{}^g H} c_{g,H} = c_{g'g,H}$, for $H \leqslant G$ and $g, g' \in G$.

- $t_{{}^g H}^{{}^g K} c_{g,H} = c_{g,K} t_H^K$ and $r_{{}^g H}^{{}^g K} c_{g,K} = c_{g,H} r_H^K$, $H \leqslant K$, $g \in G$.

- $r_L^H t_K^H = \sum_{h \in [L \backslash H / K]} t_{L \cap {}^h K}^L c_{h, L^h \cap K} r_{L^h \cap K}^K$ for $L \leqslant H \geqslant K$.

- All the other products of generators are zero.

Definition 2.8. A Mackey functor for G over R is a left $\mu_R(G)$-module.

Proposition 2.9. *The Mackey algebra is a free R-module, of finite rank independent of R. The set of elements $t_K^H x r_{K^x}^L$, where H and L are subgroups of G, where $x \in [H \backslash G / L]$, and K is a subgroup of $H \cap {}^x L$ up to $(H \cap {}^x L)$-conjugacy, is an R-basis of $\mu_R(G)$.*

Proof. Section 3 of [22]. □

Let us recall that the Mackey algebra is isomorphic to a Burnside algebra:

Proposition 2.10 (Proposition 4.5.1 [3])**.** *The Mackey algebra $\mu_R(G)$ is isomorphic to $RB(\Omega_G^2)$, where $\Omega_G = \sqcup_{L \leqslant G} G/L$.*

Proof. Let $H \leqslant K$ be two subgroups of G, then we denote by π_H^K the natural surjection from G/H to G/K. If $g \in G$, then we denote by $\gamma_{H,g}$ the map from $G/\,{}^gH$ to G/H defined by $\gamma_{H,g}(xgHg^{-1}) = xgH$. The isomorphism β is defined on the generators of $\mu_R(G)$ by:

$$\beta(t_H^K) = \begin{array}{c} G/H \\ \pi_H^K \swarrow \qquad \parallel \\ \Omega_G \supset G/K \qquad\qquad G/H \subset \Omega_G \end{array}$$

$$\beta(r_H^K) = \begin{array}{c} G/H \\ \parallel \qquad \searrow \pi_H^K \\ \Omega_G \supset G/H \qquad\qquad G/K \subset \Omega_G \end{array}$$

$$\beta(c_{g,H}) = \begin{array}{c} G/\,{}^gH \\ \parallel \qquad \searrow \gamma_{H,g} \\ \Omega_G \supset G/\,{}^gH \qquad\qquad G/H \subset \Omega_G \end{array}$$

$\square$

We will make an intensive use of the connection between the different categories of Mackey functors, so let us recall the following well known result:

Proposition 2.11. *[22] The different definitions of Mackey functors for G over R are equivalent.*

Sketch of proof. The idea of the proof can be found in various places such as [22], or [3]. Here we propose a quite different, and more conceptual approach where we do not need to choose representatives. For this proof, we denote by $Mack_R^{(a)}(G)$ the category of Mackey functors for G over R in the sense of Green and by $Mack_R^{(b)}(G)$ the category of Mackey functors for Dress. The most cost-effective way seems to prove the following equivalences:

$$\begin{array}{ccc} Mack_R^{(a)}(G) & & Mack_R^{(b)}(G) \\ \cong \searrow & & \swarrow \cong \\ \mu_R(G)\text{-}Mod & \xrightarrow{\ \cong\ } & RB(\Omega_G^2)\text{-}Mod \end{array}$$

For the equivalence $Mack_R^a(G) \cong \mu_R(G)\text{-}Mod$, if $M \in Mack_R^{(a)}(G)$, then $\bigoplus_{H \leqslant G} M(H)$ is a $\mu_R(G)$-module. Conversely, if V is a $\mu_R(G)$-module, then one can define a Mackey functor M by: for $H \leqslant G$, then $M(H) := t_H^H V$. The Mackey functor structure of M is induced by the module structure of V.
The equivalence between $\mu_R(G)\text{-}Mod$ and $RB(\Omega_G^2)\text{-}Mod$ follows from the isomorphism of Proposition 2.10.

The equivalence $Mack_R^{(b)}(G) \cong RB(\Omega_G^2)\text{-}Mod$, is a bit more technical. Let M be a Mackey functor for G over R, using Dress definition. Then $M(\Omega_G)$ is a $RB(\Omega_G^2)$-module. The action of a G-set $Z = (\Omega_G \xleftarrow{b} Z \xrightarrow{a} \Omega_G)$ over $\Omega_G \times \Omega_G$ on $M(\Omega_G)$ is defined by: let $m \in M(\Omega_G)$,

$$Z \cdot m = M_*(b) \circ M^*(a)(m).$$

Let V be an $RB(\Omega_G^2)$-module, let X be a finite G-set. Then $RB_X = RB(\Omega_G \times X)$ is an $RB(\Omega_G^2)$-module. The action of $RB(\Omega_G^2)$ on RB_X is given by left multiplication. Let $f : X \to Y$ be a morphism between two finite G-sets. Let us denote by $\hat{f}$ the morphism $Id_{\Omega_G} \times f$. Let $\phi : Z \to X \times \Omega_G$ be a G-set over $X \times \Omega_G$ and let $\psi : W \to Y \times \Omega_G$ be a G-set over $Y \times \Omega_G$. Then there is a morphism of $RB(\Omega_G^2)$-modules: $RB(f)_* : RB_X \to RB_Y$ defined by:

$$RB_*(f)(\phi) = \phi \circ \hat{f}.$$

On the other direction, there is a morphism of $RB(\Omega_G^2)$-modules defined by:

$$RB^*(f)(\psi) = \pi,$$

where $\pi : P \to X \times \Omega_G$ is a G-set over $X \times \Omega_G$ such that the following diagram is a pullback diagram:

$$
\begin{array}{ccc}
P & \longrightarrow & W \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \psi} \\
X \times \Omega_G & \xrightarrow{\hat{f}} & Y \times \Omega_G
\end{array}
$$

then one can defined a Mackey functor Y_V for G over R by:

$$Y_V(X) = Hom_{RB(\Omega_G^2)}(RB_X, V).$$

The Mackey functor structure of Y_V is induced by the covariant and the contravariant part of RB.

The functor from $Mack_R^{(b)}(G)$ to $RB(\Omega_G^2)\text{-}Mod$ sending M to $M(\Omega_G)$ is fully faithful, indeed, if X is a finite G-set, then let $X = \bigsqcup_{H \leqslant G} n_H G/H$ be the decomposition of X as sum of transitive G-sets. If a transitive G-set G/H appears with a positive coefficient in X, then we denote by i_H^X the canonical injection from G/H to X. Moreover, we denote by $i_{G/H}$ the canonical injection from G/H to Ω_G. Let $\phi : M \to N$ be a morphism of Mackey functors, then since ϕ is a natural transformation of bivariant functors, and since $\sum_H n_H M_*(i_H^X) M^*(i_H^X) = Id_{M(X)}$, we have:

$$
\begin{aligned}
\phi_X &= \sum_{H \leqslant G} n_H N_*(i_H^X) \phi_{G/H} M^*(i_H^X) \\
&= \sum_{H \leqslant G} n_H N_*(i_H^X) N^*(i_{G/H}) \phi_{\Omega_G} M_*(i_{G/H}) M^*(i_H^X).
\end{aligned}
$$

So ϕ is characterized by its evaluation at Ω_G. Conversely with this formula, if we have $f : M(\Omega_G) \to N(\Omega_G)$ a morphism of $RB(\Omega_G^2)$-modules, then one can defined a morphism

of Mackey functors $\phi : M \to N$ such that $\phi_{\Omega_G} = f$. The fact that this construction gives a natural transformation of bivariant functors follows from the fact that f is a morphism of $RB(\Omega_G^2)$-modules.

Moreover the functor $M \mapsto M(\Omega_G)$ is dense. Indeed let V be an $RB(\Omega_G^2)$-module, then

$$
\begin{aligned}
Y_V(\Omega_G) &= Hom_{RB(\Omega_G^2)}(RB\Omega_G, V) \\
&= Hom_{RB(\Omega_G^2)}(RB(\Omega_G^2), V) \\
&\cong V.
\end{aligned}
$$

Now, this isomorphism is natural in V. So, the categories $Mack_R^{(b)}(G)$ and $RB(\Omega_G^2)\text{-}Mod$ are equivalents, and since $Y_V(\Omega_G) \cong V$, by unicity of the adjoint, the functor $V \mapsto Y_V$ is a quasi-inverse equivalence to the functor $M \mapsto M(\Omega_G)$. $\square$

In the rest of the paper, if no confusion is possible, we denote by $Mack_R(G)$ the category of Mackey functors for G over R for one of these three definitions.

3   Blocks of Mackey algebras

Let G be a finite group and $(K, \mathcal{O}, k)$ be a p-modular system for G which is "large enough" for all the $N_G(H)/H$ for $H \leqslant G$. In [22], Thévenaz and Webb proved that there is a bijection between the blocks of the group algebra $\mathcal{O}G$ and the blocks of $Mack_{\mathcal{O}}(G, 1)$, where $Mack_{\mathcal{O}}(G, 1)$ is the full subcategory of $Mack_{\mathcal{O}}(G)$ consisting of Mackey functors which are projective relatively to the p-subgroups of G.

The category $Mack_{\mathcal{O}}(G, 1)$ is equivalent to the category of $\mu_{\mathcal{O}}^1(G)$-modules where $\mu_{\mathcal{O}}^1(G)$ is the subalgebra of $\mu_{\mathcal{O}}(G)$ generated by the r_Q^H, t_Q^H, $c_{Q,x}$ where $Q \leqslant H \leqslant G$, $x \in G$ and Q is a p-group. This subalgebra is called the p-local Mackey algebra of G over $\mathcal{O}$. The same definitions hold for K or k.

Theorem 3.1. *The set of central primitive idempotents of the p-local Mackey algebra $\mu_{\mathcal{O}}^1(G)$ is in bijection with the set of the blocks of $\mathcal{O}G$.*

Remark 3.2. The bijection is moreover explicit. If b is a block of $\mathcal{O}G$ then Bouc gave an explicit formula for the corresponding central idempotent of $\mu_{\mathcal{O}}^1(G)$ denoted by b^μ (Theorem 4.5.2 [5]).

Using the equivalence of categories $Mack_{\mathcal{O}}(G, 1) \cong \mu_{\mathcal{O}}^1(G)\text{-}Mod$, we have a decomposition of $Mack_{\mathcal{O}}(G, 1)$ into a product of categories, which were called the blocks of $Mack_{\mathcal{O}}(G, 1)$ by Thévenaz and Webb in [22] Section 17. The formula of the idempotents of $\mu_R^1(G)$ (Remark 3.2) is rather technical but it is immediate to check that the action of a block idempotent b^μ on the evaluation at $\{1\}$ of a Mackey functor M (using Green's notation for evaluation) is given as follows: let $m \in M(1)$, writing the idempotent $b = \sum_{x \in G} b(x)x$ where $b(x) \in \mathcal{O}$, we have

$$
b^\mu \cdot m = \sum_{x \in G} b(x)c_{x,1}(m). \tag{1}
$$

Proposition 3.3. *Let $R = k$ or $\mathcal{O}$. The set of isomorphism classes of projective indecomposable Mackey functors in a block b^μ of $Mack_R(G, 1)$ is in bijection with the set of isomorphism classes of indecomposable p-permutation modules contained in the block RGb.*

Proof. We use Green's notation for the evaluation of Mackey functors. By Corollary 12.8 of [22], we know that the projective indecomposable Mackey functors of $Mack_k(G, 1)$ are in bijection with the indecomposable p-permutation kG-modules: if P is an indecomposable projective Mackey functor, then $P(1)$ is an indecomposable p-permutation module. Let Q be an other indecomposable projective Mackey functor. Then $P \cong Q$ if and only if $P(1) \cong Q(1)$. The same holds for the projective Mackey functors of $Mack_\mathcal{O}(G, 1)$. A projective indecomposable Mackey functor P is in the block b^μ if and only if $b^\mu \cdot P \neq 0$, but $b^\mu \cdot P$ is projective so:

$$b^\mu \cdot P \neq 0 \Leftrightarrow (b^\mu \cdot P)(1) \neq 0$$
$$\Leftrightarrow b \cdot (P(1)) \neq 0$$
$$\Leftrightarrow P(1) \text{ is in the block } b \text{ of } RG.$$

$\square$

We will use the following notation: $Mack_R(b)$ (resp. $\mu_R^1(b)$) for the category of Mackey functors which belong to the block b^μ (resp. the algebra $b^\mu \mu_R^1(G)$) for $R = \mathcal{O}$ or k.

3.1 Brauer construction for Mackey functors and decomposition matrices.

If H is a subgroup of G, then there is an induction functor from $Mack_R(H)$ to $Mack_R(G)$ denoted by Ind_H^G. There is also a restriction functor from $Mack_R(G)$ to $Mack_R(H)$ denoted by Res_H^G.

If N is a normal subgroup of G, there is an inflation functor $Inf_{G/N}^G$ from $Mack_R(G/N)$ to $Mack_R(G)$. For the definition of these three functors using Green's point of view see Section 4 and Section 5 of [23]. These three functors, using the Dress point of view, arrive from a more general setting, using the fact that a G-H-biset produces a functor, by 'pre-composition', from $Mack_R(H)$ to $Mack_R(G)$ (see Chapter 8 [3] for more details). Let D be a finite G-set. The Dress construction (see [10] or 1.2 [3]) at D is an endofunctor of the Mackey functors category given by the pre-composition by the cartesian product with D: if M is a Mackey functor for G in the sense of Dress, and if X is a finite G-set, then the Dress construction of M, denoted by M_D is:

$$M_D(X) = M(X \times D).$$

Let R be a commutative ring. Let Q be a p-subgroup of G. The Brauer construction for Mackey functors is a functor $Mack_R(G) \rightarrow Mack_R(\overline{N}_G(Q))$ denoted by $M \mapsto M^Q$.

If $M \in Mack_R(G)$, then for N/Q a subgroup of $\overline{N}_G(Q)$,

$$M^Q(N/Q) = M(N)/\sum_{Q \not\leq R < N} t_R^N(M(R)).$$

For more details about this functor and for the Mackey functor structure of M^Q see Section 5 [23], where M^Q is denoted by M^+. This functor generalizes the Brauer construction for modules since the evaluation at the subgroup $\{1\}$ of $\overline{N}_G(Q)$ is

$$M^Q(Q/Q) = \overline{M}(Q) := M(Q)/\sum_{R<Q} t_R^Q(M(R)).$$

Moreover, when $R = k$ is a field and V is a kG-module, then $\overline{FP_V}(Q) \cong V[Q]$, where $V[Q]$ is the Brauer construction for modules ([7] Section 1).

Lemma 3.4. *1. The functor $M \mapsto M^Q$ is left adjoint to the functor*
$$Ind_{N_G(Q)}^G Inf_{\overline{N}_G(Q)}^{N_G(Q)} : Mack_R(\overline{N}_G(Q)) \to Mack_R(G).$$

2. The functor $M \mapsto M^Q$ sends projective functors to projective functors.

3. Let H be a subgroup of G. Then $(Ind_H^G(M))^Q = 0$ if Q is not conjugate to a subgroup of H.
   Let $(K, \mathcal{O}, k)$ be a p-modular system, and $R = \mathcal{O}$ or k.

4. If $M \in Mack_R(G, 1)$, then $M^Q \in Mack_R(\overline{N}_G(Q), 1)$.

5. Let P be a projective Mackey functor of $Mack_k(G, 1)$ and L be a projective functor of $Mack_{\mathcal{O}}(G, 1)$ which lifts P. Then L^Q is a projective Mackey functor of $Mack_{\mathcal{O}}(G, 1)$ which lifts P^Q.

6. Let P be a projective Mackey functor of $Mack_k(G, 1)$. Then $P^Q(Q/Q) \cong P(1)[Q]$.

7. Let P be an indecomposable projective Mackey functor of $Mack_R(G, 1)$. Then the vertices of P are the maximal p-subgroups Q of G such that $P^Q \neq 0$.

Sketch of proof. 1. Theorem 5.1 of [23] with a different notation.

2. Since $M \mapsto M^Q$ is left adjoint to an exact functor, it sends projective objects to projective objects.

3. By successive adjunction: for $L \in Mack_R(\overline{N}_G(Q))$ and $M \in Mack_R(H)$, and using the Mackey formula for Mackey functors, we have:

$$Hom_{Mack_R(\overline{N}_G(Q))}((Ind_H^G M)^Q, L) \cong Hom_{Mack_R(H)}(M, Res_H^G Ind_{N_G(Q)}^G Inf_{\overline{N}_G(Q)}^{N_G(Q)} L)$$

$$\cong \bigoplus_{g \in [H \backslash G / N_G(Q)]} Hom_{Mack_R(H)}\left(M, Ind_{H \cap {}^g N_G(Q)}^H Iso(g) Res_{N_G(Q) \cap H^g}^{N_G(Q)} Inf_{\overline{N}_G(Q)}^{N_G(Q)} L\right).$$

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Here we denote by $Iso(g)$ the functor from the category $Mack_R(N_G(Q) \cap H^g)$ to the category $Mack_R(H \cap {}^gN_G(Q))$ induced by the conjugacy by g. The result now follows from the fact that $Res^{N_G(Q)}_{N_G(Q) \cap H^g} Inf^{N_G(Q)}_{\overline{N}_G(Q)} L = 0$ if Q is not a subgroup of H.

4. A Mackey functor M is in $Mack_k(G, 1)$ if and only if there exist a p-group P and a Mackey functor N for P such that $M \mid Ind^G_P(N)$. Moreover, it is not difficult to check that $(Ind^G_P N)^Q$ is a direct sum of Mackey functors induced from subgroups of conjugates of P ($\star$), so $M^Q \in Mack_k(\overline{N}_G(Q), 1)$.

   ($\star$) In order to prove this, one may continue the proof of the point 3 (or, in a more general setting, look at Lemme 3 of [2]).

5. Let M be a Mackey functor for $\overline{N}_G(Q)$. Then using successive adjunctions, we have:

$$Hom_{Mack_k(\overline{N}_G(Q))}(L^Q/\mathfrak{p}(L^Q), M) \cong Hom_{Mack_k(\overline{N}_G(Q))}(k \otimes_{\mathcal{O}} L^Q, M)$$
$$\cong Hom_{Mack_{\mathcal{O}}(\overline{N}_G(Q))}(L^Q, Hom_k(k, M))$$
$$\cong Hom_{Mack_{\mathcal{O}}(G)}(L, Ind^G_{N_G(Q)} Inf^{N_G(Q)}_{\overline{N}_G(Q)} Hom_k(k, M)).$$

However, $Ind^G_{N_G(Q)} Inf^{N_G(Q)}_{\overline{N}_G(Q)} Hom_k(k, M) \cong Hom_k(k, Ind^G_{N_G(Q)} Inf^{N_G(Q)}_{\overline{N}_G(Q)}(M))$, so

$$Hom_{Mack_k(\overline{N}_G(Q))}(L^Q/\mathfrak{p}(L^Q), M) \cong Hom_{Mack_{\mathcal{O}}(G)}(L, Hom_k(k, Ind^G_{N_G(Q)} Inf^{N_G(Q)}_{\overline{N}_G(Q)}(M)))$$
$$\cong Hom_{Mack_k(G)}(L/\mathfrak{p}L, Ind^G_{N_G(Q)} Inf^{N_G(Q)}_{\overline{N}_G(Q)}(M))$$
$$\cong Hom_{Mack_k(\overline{N}_G(Q))}((L/\mathfrak{p}L)^Q, M).$$

6. Lemme 5.10 of [4].

7. This is the first assertion of Theorem 3.2 of [7], using the fact that the vertices of a projective Mackey functor are the vertices of its evaluation at 1 (Corollary 12.8 [22]), and the fact that a p-local projective Mackey functor is non zero if and only if its evaluation at 1 is non zero.

$\square$

Let F be a field and let G be a finite group. By Theorem 8.3 of [23], the set of isomorphism classes of simple Mackey functors for the group G over the field F is in bijection with the set of pairs (H, V) where H runs through a set of representatives of conjugacy classes of subgroups of G and V through a set of isomorphism classes of $F\overline{N}_G(H)$-simple modules. We denote by $S_{H,V}$ the simple Mackey functor corresponding to the pair (H, V). Let us recall that:

$$S_{H,V} = Ind^G_{N_G(H)} Inf^{N_G(H)}_{\overline{N}_G(H)} S^{\overline{N}_G(H)}_{1,V},$$

where
$$S_{1,V}^G(K) = Im(t_1^K : V \to V^K).$$

Here t_1^K is the relative trace map, that is $t_1^K(v) = \sum_{k \in K} k \cdot v$, for $v \in V$.

If $(K, \mathcal{O}, k)$ is a p-modular system, then a simple Mackey functor $S_{H,V}$ over k is in $Mack_k(G, 1)$ if and only if H is a p-group (see the discussion after Proposition 9.6 of [22]). We denote by $P_{H,V}$ the projective cover of $S_{H,V}$.

Proposition 3.5 (Decomposition matrix of $\mu_{\mathcal{O}}^1(G)$). *Let G be a finite group, and $(K, \mathcal{O}, k)$ be a p-modular system which is "large enough" for the groups $\overline{N}_G(L)$ where L runs through the p-subgroups of G.*

The decomposition matrix of $\mu_{\mathcal{O}}^1(G)$ has rows indexed by the isomorphism classes of indecomposable p-permutation kG-modules, the columns are indexed by the ordinary irreducible characters of all the $\overline{N}_G(L)$ where L runs through the p-subgroups of G up to conjugacy.

Let χ be an ordinary character of the group $\overline{N}_G(L)$ and let W be an indecomposable p-permutation kG-module. Then the decomposition number $d_{\chi,W}$ is equal to

$$d_{\chi,W} = dim_K Hom_{K\overline{N}_L(Q)}(K \otimes_{\mathcal{O}} \widehat{W[L]}, K_\chi),$$

where $\widehat{W[L]}$ is the (unique) p-permutation $\mathcal{O}G$-module which lifts $W[L]$ and K_χ is the simple KG-module afforded by the character χ.

Proof. Since $(K, \mathcal{O}, k)$ is a splitting system for $\mu_{\mathcal{O}}^1(G)$, and since $\mu_K^1(G)$ is a semi-simple algebra, the Cartan matrix is symmetric (Proposition 1.9.6 [1]). The rows of the decomposition matrix are indexed by the projective $\mu_k^1(G)$-modules, and the columns are indexed by the simples $\mu_K^1(G)$-modules. Instead of working with the p-local Mackey algebras, we use the Mackey functors point of view. Since the indecomposable projective Mackey functors of $Mack_k(G, 1)$ are in bijection with the indecomposable p-permutation kG-modules, we can index the rows of this matrix by the (isomorphism classes of) indecomposable p-permutation modules. The bijection send a p-local projective Mackey functor P to its evaluation $P(1)$ (with Green's notation for the evaluation). Moreover, the (isomorphism classes of) simple Mackey functors of $Mack_K(G, 1)$ are in bijection with the pairs (L, V), where L runs through the conjugacy classes of p-subgroups of G and V runs through the isomorphism classes of simple $KN_G(L)/L$-modules. So we can indexed the columns of the matrix by the set of ordinary characters of $N_G(L)/L$ when L runs through the conjugacy classes of p-subgroups of G.

Let L be a p-subgroup of G and χ be an ordinary character of $K\overline{N}_G(L)$. The corresponding simple Mackey functor for G over K, denoted by S_{L,K_χ} is isomorphic to $Ind_{N_G(L)}^G Inf_{\overline{N}_G(L)}^{N_G(L)} FP_{K_\chi}$ (see Lemma 9.2 [23]). Here, K_χ is the simple $K\overline{N}_G(L)$-module afforded by the character χ.

Let H be a p-subgroup of G and V be a simple $k\overline{N}_G(H)$-module. We denote by $P_{H,V}$ the corresponding projective indecomposable Mackey functor for G over k, and by $\widehat{P_{H,V}}$

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the indecomposable Mackey functor for G over $\mathcal{O}$ which lifts $P_{H,V}$.

We denote by M a Mackey functor for G over $\mathcal{O}$ such that M is $\mathcal{O}$-free and

$$K \otimes_{\mathcal{O}} M \cong S_{L,K_\chi}.$$

Then, the decomposition number indexed by S_{L,K_χ} and $P_{H,V}$ is:

$$
\begin{aligned}
d_{S_{L,K_\chi},P_{H,V}} &= dim_k Hom_{Mack_k(G,1)}(P_{H,V}, k \otimes_{\mathcal{O}} M) \\
&= rank_{\mathcal{O}} Hom_{Mack_{\mathcal{O}}(G,1)}(\widehat{P_{H,V}}, M) \\
&= dim_K Hom_{Mack_K(G,1)}(K \otimes_{\mathcal{O}} \widehat{P_{H,V}}, S_{L,K_\chi}) \\
&= dim_K Hom_{Mack_K(G,1)}(K \otimes_{\mathcal{O}} \widehat{P_{H,V}}, Ind^G_{N_G(L)} Inf^{N_G(L)}_{\overline{N}_G(L)} FP_{K_\chi}) \\
&= dim_K Hom_{Mack_K(N_G(L)/L,1)}\big((K \otimes_{\mathcal{O}} \widehat{P_{H,V}})^L, FP_{K_\chi}\big) \\
&= dim_K Hom_{KN_G(L)/L}\big((K \otimes_{\mathcal{O}} \widehat{P_{H,V}})^L(L/L), K_\chi\big).
\end{aligned}
$$

The two last equalities come from the fact that the two following pairs are pairs of adjoint functors: $ev_1 \dashv FP_-$ and $(-)^L \dashv ind^G_{N_G(L)} Inf^{N_G(L)}_{\overline{N}_G(L)}$.

Moreover, we have $(K \otimes_{\mathcal{O}} \widehat{P_{H,V}})^L(L/L) \cong K \otimes_{\mathcal{O}} ((\widehat{P_{H,V}})^L(L/L))$. By Lemma 3.4, the $\mathcal{O}\overline{N}_G(L)$-module $(\widehat{P_{H,V}})^L(L/L)$ is the unique (up to isomorphism) p-permutation module which lifts $(P_{H,V})^L(L/L) \cong P_{H,V}(1)[L]$, so:

$$d_{S_{L,K_\chi},P_{H,V}} = dim_K Hom_{K\overline{N}_L(Q)}(K \otimes_{\mathcal{O}} \widehat{W[L]}, K_\chi),$$

where W is the indecomposable p-permutation kG-module $P_{H,V}(1)$. $\square$

Remark 3.6. By Section 4.4 of [6], the sub-matrix indexed by the ordinary characters of G, and the (isomorphism classes of) indecomposable p-permutation kG-modules is the decomposition matrix of the cohomological Mackey algebra $co\mu_{\mathcal{O}}(G)$. The sub-matrix indexed by the ordinary characters of G and the isomorphism classes of indecomposable projective kG-modules is the decomposition matrix of $\mathcal{O}G$.

4 Equivalences between blocks of p-local Mackey algebras.

In this section we give some examples of equivalences between blocks of p-local Mackey algebras. We look at the case of p-nilpotent groups. We prove that the p-local algebra of the principal block of such a group is Morita equivalent to the Mackey algebra of its Sylow p-subgroup. But as in [17] the case of the non-principal block seems unexpectedly much more difficult. We give an example which proves that in general there is *no* Morita equivalence between the p-local Mackey algebra of the block and the Mackey algebra of the defect. Then we look at the case of a finite group with Sylow p-subgroups of order p. In this case the p-local Mackey algebras are symmetric algebras, they are more precisely Brauer tree algebras, so we can use the tools which were developed for Broué's conjecture for blocks with cyclic Sylow p-subgroups.

4.1 Basic results.

Since the p-local Mackey algebra and the cohomological Mackey algebra share a lot of properties, for example, they have the same number of simple modules in each block and the projective cohomological Mackey functors are the biggest cohomological quotients of the p-local projective Mackey functors, one may ask if an equivalence between blocks of p-local Mackey algebras induces in some sense, an equivalence between the corresponding blocks of the cohomological Mackey algebras. Such a result would allow us to use [21]. The following example shows that the situation is unfortunately not that simple.

Example 4.1. Let R be a commutative ring and $G = C_2$ be the (cyclic) group of order 2. Then a basis of $\mu_R(C_2)$ is given by: $t_{C_2}^{C_2}$, $t_1^{C_2} r_1^{C_2}$, $t_1^{C_2}$, $r_1^{C_2}$, t_1^1 and $t_1^1 x$ where $x \in C_2$ and $t_1^1 x$ means $t_1^1 c_{1,x}$. Then, there is an automorphism ϕ of $\mu_k(C_2)$ where ϕ is defined on the basis elements by: $\phi(t_{C_2}^{C_2}) = t_1^1$, $\phi(t_1^{C_2} r_1^{C_2}) = t_1^1 + t_1^1 x$, $\phi(t_1^{C_2}) = r_1^{C_2}$, $\phi(r_1^{C_2}) = t_1^{C_2}$, $\phi(t_1^1) = t_{C_2}^{C_2}$ and $\phi(t_1^1 x) = t_1^{C_2} r_1^{C_2} - t_{C_2}^{C_2}$. This gives an unitary automorphism of $\mu_R(C_2)$. By general results of Morita theory, the bimodule $_1\mu_R(C_2)_\phi$ induces a Morita self-equivalence of $\mu_R(C_2)$. Using the usual equivalence of categories between $\mu_R(C_2)$-Mod and $Mack_R(C_2)$ (in the sense of Green), it is not hard to check that this Morita self-equivalence induces an self-equivalence F of $Mack_R(C_2)$ which has the following property. Let M be a Mackey functor for C_2. Then $F(M)(1) = M(C_2)$ and $F(M)(C_2) = M(1)$. The maps $t_1^{C_2}$ and $r_1^{C_2}$ are exchanged. It is now easy to check that this functor F does not preserve the cohomological structure.

If $R = k$ is a field of characteristic 2, then the functor F exchanges the simples $S_{1,k}$ and $S_{C_2,k}$, so it exchanges the indecomposable projective Mackey functors $P_{1,k}$ and $P_{C_2,k}$. Moreover $P_{1,k}$ is in the full subcategory of $Mack_R(G)$ equivalent to kC_2-Mod and $P_{C_2,k}$ is not. So we have that F does not restrict to a self-equivalence of he full subcategory of cohomological Mackey functors or of the full subcategory equivalent to kC_2-Mod.

$$
\begin{array}{ccccc}
kC_2\text{-}Mod & \lhook\joinrel\longrightarrow & Comack_k(C_2) & \lhook\joinrel\longrightarrow & Mack_k(C_2) \\
\Big\downarrow{\scriptstyle\times} & & \Big\downarrow{\scriptstyle\times} & & \Big\downarrow{\scriptstyle F} \\
kC_2\text{-}Mod & \lhook\joinrel\longrightarrow & Comack_k(C_2) & \lhook\joinrel\longrightarrow & Mack_k(C_2)
\end{array}
$$

This example proves the following Proposition:

Proposition 4.2. *Let G and H be two finite groups, and k be a field of characteristic $p > 0$. An equivalence between $Mack_k(G,1)$ and $Mack_k(H,1)$ does not have to send cohomological Mackey functors to cohomological Mackey functors and does not have to respect the vertices of the projective indecomposable Mackey functors. In particular it does not have to induce an equivalence between the full-subcategories equivalent to kG-Mod and kH-Mod.*

Remark 4.3. This Proposition does not state that there exist finite groups G and H and blocks b and c of RG and RH such that $Mack_R(b)$ and $Mack_R(c)$ are equivalent and the category RGb-Mod and RHc-Mod are not equivalent. We do not have any results in this direction. This Proposition state that we have to be careful when we choose an

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equivalence between categories of Mackey functors if we want some results for the full subcategories of cohomological Mackey functors or of modules over the group algebra.

4.2 Principal block of p-nilpotent groups.

Lemma 4.4. *Let M and M' be two projective Mackey functors of $Mack_k(G, 1)$, and $f : M \to M'$ be a morphism. The morphism f is an isomorphism if and only if the morphism $f(1) : M(1) \to M'(1)$ is an isomorphism of kG-modules.*

Proof. Lemme 6.2 in [4]. □

Theorem 4.5. *Let $G = N \rtimes P$ be a p-nilpotent group, where P is a Sylow p-subgroup of G. Let b_0 be the principal block of kG. Then $Mack_k(b_0)$ is equivalent to $Mack_k(P)$.*

Remark 4.6. If b is a nilpotent block, for some $m \in \mathbb{N}$, we have an isomorphism of algebras $kGb \cong Mat(m, kP)$, where P is a defect group of the block b. This is not the case for the Mackey algebras. Example if $G = S_3$ and k is a field of characteristic 2. Let b be the principal block of kS_3, one can check that $dim_k(\mu_k(C_2)) = 6$ and $dim_k(\mu_k^1(b)) = 56$ (for a proof see Lemme 4.4.2 [19]).

Proof. Let us remark that we do not assume the Sylow p-subgroups to be abelian.
Recall that the principal block idempotent of kG is $b_0 = \frac{1}{|N|} \sum_{n \in N} n$, so we have: $kGb_0 \cong kP$. In consequence, there is a Morita equivalence $kGb_0\text{-}Mod \cong kP\text{-}Mod$. This equivalence is given by the following pair of adjoint functors $Res_P^G \dashv b_0 Ind_P^G$.
We use the fact that, in our situation, there are three pairs of adjoint functors $res_P^G \dashv Ind_P^G$. As we saw, there is one pair for the restriction and induction between the categories of modules over the group algebra. There is an other one between the categories of sets with a group action. Here, we denote by ϵ and η are the unit and co-unit of the usual adjunction $Ind_P^G \dashv Res_G^P$ between the categories of finite G-sets and finite P-sets (see Section 4 [23] for an explicit definition of these natural transformations).

Finally, at the level of Mackey functors, we also have a restriction and an induction functors. That is:
$$Res_P^G : Mack_k(b_0) \to Mack_k(P),$$
and
$$Ind_P^G : Mack_k(P) \to Mack_k(G).$$
Applying the block idempotent b_0^μ, we have a functor
$$b_0^\mu Ind_P^G : Mack_k(P) \to Mack_k(b_0).$$

1. The functor Res_P^G is both left and right adjoint to the functor $b_0^\mu Ind_P^G$, since it is the case for Ind_P^G and Res_P^G. Recall that the unit and the co-unit of the above adjunction are given by:

- The unit of the adjunction $Res_P^G \dashv b_0^\mu Ind_P^G$ is the natural transformation δ defined by: if $M \in Mack_k(G)$, then

$$\delta_M = b_0^\mu M^*(\epsilon) : M \to b_0^\mu Ind_P^G Res_P^G M.$$

  The co-unit is the natural transformation γ defined by: if $M \in Mack_k(P)$, then

$$\gamma_M = b_0^\mu M^*(\eta) : Res_P^G b_0^\mu Ind_P^G M \to M.$$

- For the adjunction $b_0^\mu Ind_P^G \dashv Res_P^G$, the unit is the natural transformation δ' defined by: if $M \in Mack_k(P)$, then

$$\delta_M' = M_*(\eta) : M \to Res_P^G b_0^\mu Ind_P^G M.$$

  The co-unit is the natural transformation γ' defined by: if $M \in Mack_k(G)$, then

$$\gamma_M' = b_0^\mu M_*(\epsilon) : b_0^\mu Ind_P^G Res_P^G M \to M.$$

2. Let M be a projective Mackey functor in $Mack_k(b_0)$, and let N be a projective Mackey functor in $Mack_k(P)$. Here we use Green's notation for the evaluation. We have a natural isomorphism of kG-modules

$$(Ind_P^G N)(1) \cong Ind_P^G(N(1)),$$

where the first induction functor is the induction of Mackey functors and the second is the induction of kP-modules. As well, we have an isomorphism of kP-modules

$$Res_P^G(N)(1) \cong Res_P^G(N(1)),$$

where the first restriction is a restriction of Mackey functors and the second is the restriction of kG-modules. The proof is straightforward but can be found in Proposition 5.2 [22]. So the evaluation at 1 of the units and co-units δ_M, γ_N and δ_N', γ_M' are the units and co-units of the adjunction $res_P^G \dashv Ind_P^G$ between the categories of modules over group algebras. Since the last adjunction is an equivalence of categories, we have that the following composition is equal to $Id_{M(1)}$:

$$M(1) \xrightarrow{\ \delta_M(1)\ } b_0^\mu Ind_P^G Res_P^G(M)(1) \xrightarrow{\ \gamma_M'(1)\ } M(1).$$

Moreover, the map $\gamma_N(1) \circ \delta_N'(1)$ is equal to $Id_{N(1)}$. By Lemma 4.4 the maps δ_M and γ_M' are two inverse isomorphisms and the maps γ_N and δ_N' are two inverse isomorphisms of Mackey functors.

3. If $M \in Mack_k(b_0)$, let $P_\bullet$ be a projective resolution of M in $Mack_k(b_0)$, then we have the following commutative diagram,

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \delta_{P_1}} & & \downarrow{\scriptstyle \delta_{P_0}} & & \downarrow{\scriptstyle \delta_M} & & \\
\cdots & \longrightarrow & b_0 Ind_P^G Res_P^G(P_1) & \longrightarrow & b_0 Ind_P^G Res_P^G(P_0) & \longrightarrow & b_0 Ind_P^G Res_P^G(M) & \longrightarrow & 0
\end{array}
$$

Since the maps δ_{P_i} for $i \geqslant 0$ are isomorphisms, then δ_M is an isomorphism. By the same method, if $N \in Mack_k(P)$, then γ_N is an isomorphism. $\qquad\square$

Remark 4.7. One can see that we did not use the fact that the group G is p-nilpotent in the proof. Instead we use the fact that the restriction functor is an equivalence between the categories kGb_0-Mod and kP-Mod. More generally, let G be a finite group and let H be a subgroup of G. Let b be a block of kG and c be a block of kH. If the functor $c \cdot Res_H^G : kGb$-$Mod \to kHc$-Mod is an equivalence of categories, then $c^\mu Res_H^G : Mack_k(b) \to Mack_k(c)$ is an equivalence of categories.

Corollary 4.8. *There is an isomorphism of algebras $\mu_k^1(b_0) \cong kB(X^2)$ for the P-set $X \cong Iso_{G/N}^P Def_{G/N}^G \Omega_G$, and $kB(X^2)$ is the evaluation of the Burnside functor at X^2 (see example 2.6).*

Proof. By Theorem 4.5, we have an equivalence of categories $\mu_k^1(b_0)$-$Mod \cong \mu_k(P)$-Mod, so by Morita Theorem, there is an isomorphism of algebras $\mu_k^1(b_0) \cong End_{\mu_k(P)}(T)$, where T is the bimodule $Res_P^G(\mu_k^1(b_0))$. We will denote by B_0 the Mackey functor, in the sense of Dress, which corresponds to the $\mu_k^1(b_0)$-module $\mu_k^1(b_0)$ under the usual equivalence of categories. Since the finitely generated projective Mackey functors of $Mack_k(P)$ are exactly the Dress constructions kB_X of the Burnside functor where X is a finite P-set, we have

$$Res_P^G(B_0) \cong kB_X,$$

for some P-set X. In particular, using Green's notation for the evaluation, we have: $kX \cong Res_P^G(B_0)(1)$. But in the equivalence of categories between $\mu_k(G)$-Mod and $Mack_k(G)$, the evaluation at 1 correspond to the multiplication by the idempotent t_1^1. So we have: $Res_P^G(B_0(1)) = b_0 t_1^1 \mu_k^1(G)$.
A basis of $\mu_k^1(G)$ is given by $t_B^A x R_{B^x}^C$, where A and C are subgroups of G, the elements $x \in [A \backslash G / C]$ and B is a p-subgroup of $A \cap {}^x C$ up to $A \cap {}^x C$-conjugacy . So a basis of $t_1^1 \mu_k^1(G)$ is the set of $t_1^1 x r_1^H$ for $x \in G$ and $H \leqslant G$. Set $\gamma_{H,x} = t_1^1 b_0 x R_1^H$. We have that $\gamma_{H,nx} = \gamma_{H,x}$ and $\gamma_{H,xh} = \gamma_{H,x}$ for $x \in G$, $n \in N$ and $h \in H$. The set $\{\gamma_{H,x} \ ; \ H \leqslant G, \quad x \in G/NH\}$ is a $\mu_k(P)$-basis of $t_1^1 \mu_k^1(b_0)$. The action of $y \in P$ on a element $\gamma_{H,x}$ is given by $y.\gamma_{H,x} = \gamma_{H,yx}$. So,

$$kX \cong \bigoplus_{H \leqslant G} Res_P^G(kG/NH),$$

but for a p-group P, two permutation modules are isomorphic if and only if the corresponding P-sets are isomorphic by [9]. Hence

$$
\begin{aligned}
X &\cong \sqcup_{H \leqslant G} Res_P^G(G/NH) \\
&\cong \sqcup_{H \leqslant G} Res_P^G Ind_{G/N}^G Def_{G/N}^G(G/H) \\
&\cong Res_P^G(Inf_{G/N}^G Def_{G/N}^G(\Omega_G)) \\
&\cong Iso_{G/N}^P Def_{G/N}^G(\Omega_G).
\end{aligned}
$$

So $Res_P^G(B_0) \cong kB_X$, where $X = Iso_{G/N}^P Def_{G/N}^G(\Omega_G)$, and finally, we have:

$$\mu_k^1(b_0) \cong End_{Mack_k(P)}(kB_X) \cong kB(X^2),$$

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by adjunction property of kB_X see (Proposition 3.1.3 of [3] in a more general setting). $\square$

This can be viewed as an analogue of the isomorphism $kGb \cong Mat(n, kP)$ for nilpotent blocks.

For non principal blocks of p-nilpotent groups, or more generally for nilpotent blocks, the situation is much more complicate. The next example shows that, in general, if b is a nilpotent block with defect group D, then the Mackey algebra $\mu_k^1(b)$ is *not* Morita equivalent to the Mackey algebra $\mu_k(D)$.

Example 4.9. Let $G = SL_2(\mathbb{F}_3) \cong Q_8 \rtimes C_3$, and k be a field of characteristic 3. The group G is 3-nilpotent. Let b be the block idempotent such that the block kGb contains the simple kG-module W of dimension 2. Then $kGb\text{-}Mod \cong kC_3\text{-}Mod$. One can ask if the same happens to the corresponding blocks of the Mackey algebras. But the Cartan matrix of $\mu_k(C_3)$ is $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and the Cartan matrix of $\mu_k^1(b)$ is $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. So there is no Morita equivalence between these two algebras. One can compute these matrices by using Proposition 3.5 or see below.

By Theorem 4.11 they are in fact derived equivalent. In this example it is not difficult to make everything explicit: the projective indecomposable Mackey functors in the block b^μ are in bijection with the indecomposable p-permutation modules in the corresponding blocks. These indecomposable p-permutation modules are: $Ind_{Q_8}^G(W)$ and $Ind_{C_6}^G Inf_{C_2}^{C_6} k_\epsilon$ where $C_6 \cong Z(Q_8) \times C_3$, and k_ϵ is the non trivial simple kC_2-module. So the projective indecomposable Mackey functors are $P = Ind_{Q_8}^G FP_W$ and $Q = Ind_{C_6}^G kB_\epsilon$, where kB_ϵ is a direct summand of the Dress construction of the Burnside functor for C_6: kB_{C_6/C_3}.

More precisely, for each C_6-sets, if Y is any finite C_6-set, then $kB_{C_6/C_3}(Y) = kB(Y \times C_6/C_3)$ is a right kC_2-module, since

$$C_6/C_3 \cong End_{C_6-set}(C_6/C_3) \mapsto End_{C_6-set}(Y \times C_6/C_3).$$

So $kB_\epsilon(Y) = kB(Y \times C_6/C_3) \otimes_{k(C_6/C_3)} k_\epsilon$. Now this it is not difficult to compute the Cartan matrix of the block.

The derived equivalence can be given by a two terms complex:

$$0 \longrightarrow P^2 \xrightarrow{(\pi,0)} Q \longrightarrow 0$$

where π is the morphism of maximal rank between P and Q.

It should be notice that in this example the Morita equivalence between kGb and kC_3 is *not* given by a p-permutation bimodule. If we want to find an equivalence between these two algebras which respects the subcategories of p-permutation modules, we need to replace the bimodule which gives the Morita equivalence by a splendid tilting complex (see [17] Section 7.4).

4.3 Groups with Sylow p-subgroup of order p.

Lemma 4.10. *Let b be a block of RG with defect group D. Then $\mu_R^1(b)$ is a symmetric algebra if and only if $|D| < p^2$.*

Proof. By Corollary 4.8 of [20] then Mackey algebra $\mu_R(D)$ is symmetric if and only if $|D| < p^2$. Now, we just have to prove that $\mu_R^1(b)$ is a symmetric algebra if and only if $\mu_R(D)$ is a symmetric algebra. Recall that a finite dimensional R-algebra A is symmetric if and only if for every finitely generate projective A-module P and for every finitely generated A-module which is R-projective, we have an isomorphism of R-modules:

$$Hom_A(P, M) \cong Hom_R(Hom_A(M, P), R),$$

and this isomorphism is natural in P and M (see Proposition 2.7 [8]).
Suppose that $\mu_R^1(b)$ is symmetric. Let L be a finitely generated projective $\mu_R^1(b)$-module i.e. by using the usual equivalence of categories L is a finitely generated projective Mackey functor. Let M be an R-projective finitely generated Mackey functors for D. Then

$$
\begin{aligned}
Hom_{Mack_R(D)}(Res_D^G(L), M) &\cong Hom_{Mack_R(G)}(L, b^\mu Ind_D^G(M)) \\
&\cong Hom_R(Hom_{Mack_R(G)}(L, b^\mu Ind_P^G(M)), R) \\
&\cong Hom_R(Hom_{Mack_R(D)}(Res_D^G(L), M), R).
\end{aligned}
$$

But every finitely projective Mackey functor for the group D is direct summand of the restriction of a finitely generated Mackey functor for G. Since the previous isomorphism is natural, it implies an isomorphism between the corresponding direct summands. Conversely, since every projective Mackey functor for G is direct summand of the induction of a projective Mackey functor for D, by similar arguments, we have the result. $\square$

Theorem 4.11. *Let G and H be two finite groups. Let b be a block of kG with cyclic defect group of order p, and c be a block of kH with cyclic defect group of order p. Then, $D^b(kGb\text{-}Mod) \cong D^b(kHc\text{-}Mod)$ if and only if $D^b(\mu_k^1(b)\text{-}Mod) \cong D^b(\mu_k^1(b')\text{-}Mod)$.*

Proof. By Theorem 20.10 of [22], in this situation, the blocks of Mackey algebras are Brauer tree algebras. Let T_{Mack} be the tree of this algebra. And T_{Mod} be the tree of the corresponding block of the group algebra. The tree T_{Mod} is isomorphic to a subtree of T_{Mack}, still denoted by T_{Mod}. Some properties of the tree T_{Mack} are determined by the knowledge of the tree T_{Mod}. If e is the number of edges of T_{Mod}, then the number of edges of T_{Mack} is $2e$. The exceptional vertex of T_{Mack} is the same as the exceptional vertex of the tree T_{Mod}. Each edge of T_{Mack} which is not in T_{Mod} is a twig. By general results on derived equivalences for Brauer tree algebras, two Brauer tree algebras with same exceptional multiplicity, over the same field, are derived equivalent if and only if they have the same number of edges. $\square$

Even if the tree T_{Mack} seems to be determined by the group algebra kG, if two blocks of group algebras are Morita equivalent, it is not always true that the corresponding

blocks for the Mackey algebras are Morita equivalent (see Example 4.9). The tree T_{Mack} is in fact determined by the corresponding block of kG and its Brauer correspondent in $N_G(P)$ where P is the defect of the block.

Remark 4.12. In [22], all the results are given for blocks of groups with a Sylow p-subgroup of order p. One can check that these results can be generalized quite easily to general blocks with defect group of order p (see Chapter 6 of [19] for the proof).

Proposition 4.13. *Let G and H be two finite groups. Let b (resp. c) be a block of kG (resp. kH) with a defect p-group of order p. If kGb and kHc are splendidly Morita equivalent, then there is an equivalence of categories: $\mu_k^1(e)\text{-}Mod \cong \mu_k^1(c)\text{-}Mod$.*

Proof. By Theorem 20.10 of [22] and Theorem 4.11, the block algebras $\mu_k^1(e)$ and $\mu_k^1(f)$ are derived equivalent Brauer tree algebras. Since two Brauer tree algebras over the same field are Morita equivalent if and only if they have isomorphic trees and same exceptional multiplicity, it is enough too prove that they have the same Cartan matrices. We will prove that the decomposition matrices of $\mu_\mathcal{O}^1(b)$ and $\mu_\mathcal{O}^1(c)$ are the same. By Proposition 3.5, the decomposition matrices of $\mu_\mathcal{O}^1(b)$ can be computed from the knowledge of the p-blocks $\mathcal{O}Gb$ and the Brauer correspondent of b in $\mathcal{O}\overline{N}_G(C)$.

Suppose that there are e simple kG-modules in the block b of kG, then there are $e+1$ simple KG-modules in this block, one of this simple module may be exceptional. The number of simple $k\overline{N}_G(C)$-modules W in a block b' which is the Brauer correspondent of $\overline{b}$ is e. Since $\overline{N}_G(C)$ is a p'-group, each simple $k\overline{N}_G(C)$-module gives rise to a unique simple $K\overline{N}_G(C)$-module. Thus the decomposition matrix of $\mu_\mathcal{O}^1(b)$ is the following block matrix with $2e+1$ columns and $2e$ rows:

$$\begin{pmatrix} & & 0_{e\times e} \\ & D(co\mu_\mathcal{O}(b)) & \\ & & Id_{e\times e} \end{pmatrix}$$

Where $D(co\mu_\mathcal{O}(b))$ is the decomposition matrix of $co\mu_\mathcal{O}(b)$.

So if two blocks kGb and kHc, with cyclic defect group of order p are splendidly Morita equivalent, the blocks $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are splendidly Morita equivalent by Section 5 of [17]. By Proposition 4.4 of [21], the blocks of the cohomological Mackey algebras $co\mu_\mathcal{O}(b)$ and $co\mu_\mathcal{O}(c)$ are Morita equivalent, so the Cartan matrices of $\mu_k^1(b)$ and $\mu_k^1(c)$ are the same. □

Corollary 4.14. *Let G and H be two finite groups. Let b be a block of $\mathcal{O}G$ and let c be a block of $\mathcal{O}H$. Let us suppose that the blocks b and c have a defect group of order p. Then*

1. *$D^b(\mu_\mathcal{O}^1(b)) \cong D^b(\mu_\mathcal{O}^1(c))$ if and only if $D^b(\mathcal{O}Gb) \cong D^b(\mathcal{O}Hc)$.*

2. *If $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are splendidly Morita equivalent, then $\mu_\mathcal{O}^1(b)\text{-}Mod \cong \mu_\mathcal{O}^1(c)\text{-}Mod$.*

Proof. For this proof, we suppose that the reader is familiar with the notion of *Green orders*. For the definition and properties of Green orders see [14] and [13].

The Mackey algebra $\mu_k^1(b)$ over k of a block with defect of order p is a Brauer tree algebra, so there is a Green walk on this tree. One can lifts this Green walk for $\mu_{\mathcal{O}}^1(b)$ exactly as Green did in [12]. This shows that the Mackey algebra over $\mathcal{O}$ is a Green order in the sense of [18]. König and Zimmermann in [13] proved that two *generic* Green orders with trees having same number of vertices and same exceptional vertex plus some local properties (the orders attached to the vertices have to coincide) are derived equivalent. In our situation, that is: group algebras or p-local Mackey algebras over a p-modular system which is 'large enough', one can prove that, the order attached to a non exceptional vertex is $\mathcal{O}$ and the order attached to the exceptional vertex depends only on the exceptional multiplicity (for more details see Lemme 6.3.24 of [19]). The first part of the theorem follows. The second part follows from the fact that two generic Green orders are Morita equivalent if and only if they have isomorphic trees, with same exceptional multiplicity and the local datas of the two trees coincide. Since the exceptional datas of the trees of $\mu_{\mathcal{O}}^1(b)$ and $\mu_{\mathcal{O}}^1(c)$ coincide, the result follows from Proposition 4.13 $\qquad\square$

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