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Equivalences between blocks of p -local Mackey algebras

Baptiste Rognerud

May 1, 2013

Abstract

Let G be a finite group and (K, \mathcal{O}, k) be a p -modular system. Let R be \mathcal{O} or k . There is a bijection between the blocks of a group algebra RG and the central primitive idempotents of the p -local Mackey algebra (resp. cohomological Mackey algebra). These idempotents are called blocks. We look at equivalences between these blocks. In the first part, we look at the cohomological case and prove that a splendid derived equivalence between blocks of group algebras can be lifted to an equivalence between the corresponding cohomological blocks. We apply this to nilpotent blocks. In the last part we look at the p -local case. For a block b of kG with cyclic defect group P of order p , we see that the p -local Mackey algebra of this block is derived equivalent to the p -local Mackey algebra of the Brauer correspondent of b in $N_G(P)$. Finally we prove that the principal block of the p -local Mackey algebra of a p -nilpotent group is Morita equivalent to the Mackey algebra of its Sylow p -subgroup.

Key words: Modular representation. Finite group. Mackey functor. Block theory
A.M.S. subject classification: 20C05, 18E30, 16G10.

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1 Introduction and preliminaries

1.1 Introduction

The notion of Mackey functor, introduced by Green in [9], is a generalization of linear representations of a finite group G . A Mackey functor, for Green, is the data of a representation of $N_G(H)$ for every subgroup H of G , together with relations between these representations. A couple of years later, Dress gave a completely different, but equivalent, definition using the formalism of categories. Twenty years later Thévenaz and Webb introduced the Mackey algebra and proved that a Mackey functor is nothing but a module over this algebra. Let R be a commutative ring. The Mackey algebras $\mu_R(G)$ share a lot of properties of group algebras, for example $\mu_R(G)$ is R -free of finite rank and this rank is independent of the ring R . Moreover if R is a field of characteristic which do not divide the order of G , then $\mu_R(G)$ is semi-simple. When (K, \mathcal{O}, k) is a p -modular system, it is possible to define a decomposition theory for $\mu_{\mathcal{O}}(G)$, in particular the Cartan matrix of the Mackey algebra is symmetric. However there are some differences with group algebras: in particular, most of the time the determinant of the Cartan Matrix of $\mu_k(G)$ is not a power of p , and the Mackey algebra over a field of characteristic p is almost never (as soon as $p^2 \mid |G|$) a symmetric algebra.

Let $R = \mathcal{O}$ or k . In their paper, Thévenaz and Webb proved that there is a bijection $b \mapsto b^\mu$ between the blocks of RG and the primitive central idempotents of $\mu_R^1(G)$, called the blocks of $\mu_R^1(G)$, where $\mu_R^1(G)$ is the so called p -local Mackey algebra.

The blocks of RG are in bijection with the blocks of $\mu_R^1(G)$, and this bijection preserves the defect groups. So using the Brauer correspondence, we have the following diagram: Let b a block of RG with defect group D and b' the Brauer correspondent of b in $RN_G(D)$.

$$\begin{array}{ccc} b \in Z(RG) & \longrightarrow & b^\mu \in Z(\mu_R^1(G)) \\ \downarrow & & \downarrow \\ b' \in Z(RN_G(D)) & \longrightarrow & b'^\mu \in Z(\mu_R^1(N_G(D))). \end{array}$$

If D is abelian, it is conjectured by Broué that the block algebras RGb and $RN_G(D)b'$ are deeply connected. It is a very natural question to ask if the same can happen for the corresponding Mackey algebras. However, we should notice that, since the Mackey algebra is (most of the time) not symmetric it is not possible to look at stable equivalences between Mackey algebras.

In this paper we will look at the following situation. Let G and H be two finite groups. let b and c be two block idempotents such that RGb and RHc are Morita or derived equivalent.

Question 1.1. *Let G be a finite group and b be a block of $\mathcal{O}G$ with abelian defect group D . Let b' be the Brauer correspondent of b in $\mathcal{O}N_G(D)$. Is there a derived equivalence $D^b(\mu_{\mathcal{O}}^1(G)b^\mu) \cong D^b(\mu_{\mathcal{O}}^1(N_G(D))b'^\mu)$?*

We will not answer this question in general, but we consider it in the following two cases: first for the cohomological Mackey algebra, which is a quotient $\mu_R^1(G)$. Then we will look at this question for the Mackey algebra of the principal blocks of p -nilpotent groups, and for groups with Sylow p -subgroup of order p .

The main result of this paper is the following theorem which settles the question for the cohomological Mackey algebra in the case of a splendid equivalence (see [15]):

Theorem 1.2. *Let G and H be two finite groups, let b be a block of RG and c be a block of RH . If RGb and RHc are splendidly derived equivalent, then $D^b(\text{co}\mu_R(G)b^\mu) \cong D^b(\text{co}\mu_R(H)c^\mu)$.*

where we abuse notation and denote also by b^μ the image of the block idempotent b^u in the center of the cohomological Mackey algebra.

The first part of this paper is devoted to the definitions and basic results on Mackey functors, and blocks of the Mackey algebra. We will see how the decomposition

matrix of the Mackey algebra can be computed from the knowledge of some information on the p -blocks of the group algebras of some p -local subgroup of G .

In the second section we will look at the cohomological case, using the Yoshida equivalence for cohomological Mackey functors, we will see that a derived equivalence between blocks of group algebras can be lifted to a derived equivalence between the blocks of the corresponding cohomological Mackey algebras as soon as this equivalence sends p -permutation modules to p -permutation modules. For example splendid Morita equivalences, and splendid derived equivalences can be lifted.

The last section of this paper deals about the non cohomological case. The first example will be about principal blocks of p -nilpotent groups. Then we will see that in the case of groups with cyclic Sylow p -subgroup of order p it is possible to answer the question, using the fact the the Mackey algebras are Brauer tree algebras in this situation.

N.B. We will denote by the same letter the block idempotents for the ring \mathcal{O} and the field k .

Notation: Let R be a ring. We denote by $R\text{-Mod}$ the category of (all) R -modules and by $R\text{-mod}$ the category consisting of the finitely generated R -modules. Let G be a finite group and p a prime number. We denote by (K, \mathcal{O}, k) a p -modular system, i-e \mathcal{O} is a complete discrete valuation ring with maximal ideal \mathfrak{p} , such that $\mathcal{O}/\mathfrak{p} = k$ is a field of characteristic p and $\text{Frac}(\mathcal{O}) = K$ a field characteristic zero. We denote by $G\text{-set}$ the category of finite G -sets.

1.2 Basic results on Mackey functors.

Let G be a finite group, and R be a commutative ring. There are several definitions of Mackey functors for G over a ring R , the first one was introduced by Green in [9]:

Definition 1.3. A Mackey functor for G over R consists of the following data:

- For every subgroup H of G , an R -module $M(H)$.
- For subgroups $H \subseteq K$ of G , a morphism of R -modules $t_H^K : M(H) \rightarrow M(K)$ called transfert, or induction, and a morphism of R -modules $r_H^K : M(K) \rightarrow M(H)$ called restriction.
- For every subgroup H of G , and each element x of G , a morphism of R -modules $c_{x,H} : M(H) \rightarrow M({}^xH)$ called conjugacy map.

such that:

1. *Triviality axiom:* For each subgroup H of G , and each element $h \in H$, the morphisms r_H^H , t_H^H et $c_{h,H}$ are the identity morphism of $M(H)$.
2. *Transitivity axiom:* If $H \subseteq K \subseteq L$ are subgroups of G , then $t_K^L \circ t_H^K = t_H^L$ and $r_H^K \circ r_K^L = r_H^L$. Moreover if x and y are elements of G , then $c_{y,xH} \circ c_{x,H} = c_{yx,H}$.
3. *Compatibility axioms:* If $H \subseteq K$ are subgroups of G , and if x is an element of G , then $c_{x,K} \circ t_H^K = t_{xH}^K c_{x,H}$ et $c_{x,H} \circ r_H^K = r_{xH}^K \circ c_{x,K}$.
4. *Mackey axiom* If $H \subseteq K \supseteq L$ are subgroups of G , then

$$r_H^K \circ t_L^K = \sum_{x \in [H \backslash K / L]} t_{H \cap xL}^H \circ c_{x, H^x \cap L} \circ r_{H^x \cap L}^L.$$

where $[H \backslash K / L]$ is a set of representative of the double cosets $H \backslash K / L$.

In particular, for each subgroup H of G , the R -module $M(H)$ is an $N_G(H)/H$ -module.

A morphism f between two Mackey functors M and N is the data of a R -linear morphism $f(H) : M(H) \rightarrow N(H)$ for every subgroup H of G . These morphisms are compatible with transfer, restriction and conjugacy maps (see [9]). We denote by $Mack_R(G)$ the category of Mackey functors for G over R . The following example is fundamental for the second section of this paper.

Example 1. Let V be an RG -module, the fixed point functor FP_V is the Mackey functor for G over R defined as follows:

For $H \leq G$, then $FP_V(H) = V^H := \{ v \in V ; hv = v \forall h \in H \}$. If $H \leq K \leq G$, we have $V^K \subseteq V^H$, so the restriction map r_H^K is the inclusion map. The transfert map $t_H^K : V^H \rightarrow V^K$ is defined by $t_H^K(v) = \sum_{k \in [K/H]} k.v$ where $[K/H]$ is a set of representative of K/H . The conjugacy maps are induced by the action of G on V .

It is not hard to see that the construction $V \mapsto FP_V$ is a functor from $RG\text{-Mod}$ to $Mack_R(G)$.

Conversely we have an obvious functor $ev_1 : Mack_R(G) \rightarrow RG\text{-Mod}$ given by the evaluation at the subgroup $\{1\}$.

Proposition 1.4. [18] *The functors (ev_1, FP_-) are adjoint i-e:*

$$Hom_{Mack_R(G)}(M, FP_V) \cong Hom_{RG}(M(1), V)$$

for a Mackey functor M and an RG -module V .

An other definition of Mackey functors was given by Dress in [8]:

Definition 1.5. A bivariate functor $M = (M^*, M_*)$ from G -set to R -Mod is a pair of functors from G -set $\rightarrow R$ -Mod such that M^* is a contravariant functor, and M_* is a covariant functor. If X is a G -set, then the image by the covariant and by the contravariant part coincide. We denote by $M(X)$ this image. A Mackey functor for G over R is a bivariate functor from G -set to R -Mod such that:

- Let X and Y two finite G -sets, i_X et i_Y the canonical injection of X (resp. Y) in $X \sqcup Y$, then $(M^*(i_X), M^*(i_Y))$ et $(M_*(i_X), M_*(i_Y))$ are inverse isomorphisms.

$$M(X) \oplus M(Y) \cong M(X \sqcup Y).$$

- If

$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ \downarrow b & & \downarrow c \\ Z & \xrightarrow{d} & T \end{array}$$

is a pull back diagram of G -sets, then the diagram

$$\begin{array}{ccc} M(X) & \xleftarrow{M^*(a)} & M(Y) \\ M_*(b) \downarrow & & \downarrow M_*(c) \\ M(Z) & \xleftarrow{M^*(d)} & M(T) \end{array}$$

is commutative.

A morphism between to Mackey functors is a natural transformation of bivariate functors.

Example 2. [1] If X is a finite G -set, the category of G -sets over X is the category with objects (Y, ϕ) where Y is a finite G -set and ϕ is a morphism from Y to X . A morphism f from (Y, ϕ) to (Z, ψ) is a morphism of G -sets $f : Y \rightarrow Z$ such that $\psi \circ f = \phi$.

The Burnside functor at X is the Grothendieck group of the category of G -sets over X , for relations given by disjoint union. This is a Mackey functor for G over R by extending the scalars from \mathbb{Z} to R . If no confusion is possible, we still denote by B the functor after scalar extension.

If X is a G -set, the Burnside group $B(X^2)$ has a ring structure. The product of $(X \xleftarrow{\alpha} Y \xrightarrow{\beta} X)$ and $(X \xleftarrow{\gamma} Z \xrightarrow{\delta} X)$ is given by pullback along β and γ .

$$\begin{array}{ccccc} & & P & & \\ & & \swarrow \cdots & \searrow \cdots & \\ & Y & & & Z \\ \alpha \swarrow & & & & \searrow \delta \\ X & & & & X \\ & \searrow \beta & \swarrow \gamma & & \\ & X & & & \end{array}$$

The identity of this ring is

$$\begin{array}{ccc} & X & \\ // & & // \\ X & & X \end{array}$$

We need a last definition of Mackey functors which was given by Thévenaz and Webb in [18], and uses the Mackey algebra.

Definition 1.6. The Mackey algebra $\mu_R(G)$ for G over R is the unital associative algebra with generators t_H^K , r_H^K and $c_{g,H}$ for $H \leq K \leq G$ and $g \in G$, with the following relations:

- $\sum_{H \leq G} t_H^H = 1_{\mu_R(G)}$.
- $t_H^H = r_H^H = c_{h,H}$ for $H \leq G$ and $h \in H$.
- $t_K^L t_H^K = t_H^L$, $r_H^K r_K^L = r_H^L$ for $H \subseteq K \subseteq L$.
- $c_{g',gH} c_{g,H} = c_{g',g,H}$, for $H \leq G$ and $g, g' \in G$.
- $t_{gH}^{gK} c_{g,H} = c_{g,K} t_H^K$ and $r_{gH}^{gK} c_{g,K} = c_{g,H} r_H^K$, $H \leq K$, $g \in G$.
- $r_L^H t_K^H = \sum_{h \in [L \setminus H/K]} t_{L \cap^h K}^L c_{h, L^h \cap H} r_{L^h \cap H}^K$ for $L \leq H \geq K$.
- All the other products of generators are zero.

Definition 1.7. A Mackey functor for G over R is a left $\mu_R(G)$ -module.

Proposition 1.8. [1] *The Mackey algebra $\mu_R(G)$ is isomorphic to $B(\Omega_G^2)$, where $\Omega_G = \sqcup_{L \leq G} G/L$.*

We will make an intensive use of the connection between the different categories of Mackey functors, so let us recall the following well known result:

Proposition 1.9. [18] *The different definitions of Mackey functors for G over R are equivalent.*

Sketch of proof. • If M is a Mackey functor for G in the sense of Green. The corresponding $\mu_R(G)$ -module is $\tilde{M} := \bigoplus_{H \leq G} M(H)$, the action of the generators is given by applying the corresponding map.

- Conversely if N is a $\mu_R(G)$ -module, then one can define a Mackey functor \hat{N} for G for the Green definition by: $\hat{N}(H) = t_H^H N$ for all subgroups $H \leq G$. If $H \leq K$ are subgroups of G and $x \in G$, $n \in \hat{N}(H)$ we define the transfert (resp. restriction, resp. conjugacy) by multiplying n by t_H^K (resp. r_H^K , resp. $c_{x,H}$).

- If M is a Mackey functor for G for the Dress definition, one can define a Mackey functor M_1 for G for the Green definition by: $M_1(H) = M(G/H)$ for all $H \leq G$. Let $H \leq K$ subgroups of G , let $\pi_H^K : G/H \rightarrow G/K$ the canonical map, let $x \in G$, $\gamma_{x,H} : G/xH \rightarrow G/H$ defined by $\gamma_{x,H}(g^xH) = gxH$. $r_H^K := M^*(\pi_H^K)$, $t_H^K := M_*(\pi_H^K)$ and $c_{x,H} := M^*(\gamma_{x,H})$.
- Conversely if M is a Mackey functor for the Green definition one can define a Mackey functor M_2 for the Dress definition as follows: let X be a finite G -set, then $M_2(X) := (\bigoplus_{x \in X} M(G_x))^G$, where G_x is the stabilizer in G of the element x . Let $f : X \rightarrow Y$ be a morphism of G -sets.
 - Let $u \in (\bigoplus_{x \in X} M(G_x))^G$, then $(M_2(f)_*(u))_{y \in Y} := \sum_{x \in [G_y \setminus f^{-1}(y)]} t_{G_x}^{G_y} u_x$.
 - Let $v \in (\bigoplus_{y \in Y} M(G_y))^G$, then $(M_2(f)^*(v))_{x \in X} := r_{G_x}^{G_{f(x)}} v_{f(x)}$.

One can check that this construction is well defined, and that it gives a Mackey functor for G over R for the Dress definition. \square

In the rest of the paper, if no confusion is possible, we denote by $Mack_R(G)$ the category of Mackey functors for G over R for one of these three definitions.

1.3 Blocks of Mackey algebras

Let G be a finite group and (K, \mathcal{O}, k) be a p -modular system for G which is “big enough” for all the $N_G(H)/H$ for $H \leq G$. In [18] Thévenaz and Webb proved that there is a bijection between the blocks of the group algebra $\mathcal{O}G$ and the blocks of $Mack_{\mathcal{O}}(G, 1)$, where $Mack_{\mathcal{O}}(G, 1)$ is the full subcategory of $Mack_{\mathcal{O}}(G)$ consisting of Mackey functors which are projective relatively to the p -subgroups of G .

The category $Mack_{\mathcal{O}}(G, 1)$ is equivalent to the category of $\mu_{\mathcal{O}}^1(G)$ -modules where $\mu_{\mathcal{O}}^1(G)$ is the subalgebra of $\mu_{\mathcal{O}}(G)$ generated by the r_Q^H , t_Q^H , $c_{Q,x}$ where $Q \leq H \leq G$, $x \in G$ and Q is a p -group. This subalgebra is called the p -local Mackey algebra of G over \mathcal{O} . The same definitions hold for K or k .

Theorem 1.10. *The set of central primitive idempotents of the p -local Mackey algebra $\mu_{\mathcal{O}}^1(G)$ is in bijection with the set of the blocks of $\mathcal{O}G$, the bijection is moreover explicit. If b is a block of $\mathcal{O}G$ then Bouc (Theorem 4.5.2 of [3]) gave an explicit formula for the corresponding central idempotent of $\mu_{\mathcal{O}}^1(G)$ denoted by b^μ .*

Using the equivalence of categories $Mack_{\mathcal{O}}(G, 1) \cong \mu_{\mathcal{O}}^1(G)\text{-Mod}$, we have a decomposition of $Mack_{\mathcal{O}}(G, 1)$ into a product of categories, which were called the blocks of $Mack_{\mathcal{O}}(G, 1)$ by Thévenaz and Webb in [18] Section 17. The formula in 1.10 is rather technical but the action of a block idempotent b^μ on the evaluation

at $\{1\}$ of a Mackey functor M is given as follows: let $m \in M(1)$, writing the idempotent $b = \sum_{x \in G} b(x)x$ where $b(x) \in \mathcal{O}$, we have

$$b^\mu . m = \sum_{x \in G} b(x)c_{x,1}(m). \quad (1)$$

Proposition 1.11. *Let $R = k$ or \mathcal{O} . The isomorphism classes of projective indecomposable modules in a block b^μ of $\text{Mack}_R(G, 1)$ are in bijection with the set of isomorphism classes of indecomposable p -permutation modules contained in the block RGb .*

Proof. By Corollary 12.8 of [18], we know that the projective indecomposable Mackey functors of $\text{Mack}_k(G, 1)$ are in bijection with the indecomposable p -permutation kG -modules: if P is an indecomposable projective Mackey functor, then $P(1)$ is a indecomposable p -permutation module. Let Q be an other indecomposable projective Mackey functor. Then $P \cong Q$ if and only if $P(1) \cong Q(1)$. The same holds for the projective Mackey functors of $\text{Mack}_{\mathcal{O}}(G, 1)$. A projective indecomposable Mackey functor P is in the block b^μ if and only if $b^\mu P \neq 0$, but $b^\mu P$ is projective so:

$$\begin{aligned} b^\mu . P \neq 0 &\Leftrightarrow (b^\mu . P)(1) \neq 0 \\ &\Leftrightarrow b.(P(1)) \neq 0 \\ &\Leftrightarrow P(1) \text{ is in the block } b \text{ of } RG \text{ by (1)}. \end{aligned}$$

□

We will use the following notation: $\text{Mack}_R(b)$ (resp. $\mu_R^1(b)$) for the category of Mackey functors which belong to the blocks b^μ (resp. the algebra $b^\mu \mu_R^1(G)$) for $R = \mathcal{O}$ or k .

1.4 Brauer construction for Mackey functors and decomposition matrices.

Let R be a commutative ring. Let Q be a p -subgroup of G . The Brauer construction for Mackey functors is a functor $\text{Mack}_R(G) \rightarrow \text{Mack}_R(\overline{N}_G(Q))$ denoted by $M \mapsto M^Q$. If $M \in \text{Mack}_R(G)$, then for N/Q a subgroup of $\overline{N}_G(Q)$,

$$M^Q(N/Q) = M(N) / \sum_{Q \not\leq R < N} t_R^N(M(R)).$$

This functor generalizes the Brauer construction for modules since the evaluation at the subgroup $\{1\}$ of $\overline{N}_G(Q)$ is

$$M^Q(Q/Q) = \overline{M}(Q) := M(Q) / \sum_{R < Q} t_R^Q(M(R)).$$

Moreover, when $R = k$ is a field and V is a kG -module, then $\overline{FP}_V(Q) \cong V[Q]$, where $V[Q]$ is the Brauer construction for modules.

Let us recall four classical functors between categories of Mackey functors (see [19] for Green's point of view). Let H be a subgroup of G , there is an induction functor: $Ind_H^G : Mack_R(H) \rightarrow Mack_R(G)$. Let M be a Mackey functor for H over R in the sense of Dress. Let X be a G -set, then

$$(Ind_H^G M)(X) = M(Res_H^G X).$$

The restriction functor: $Res_H^G : Mack_R(G) \rightarrow Mack_R(H)$. Let M be a Mackey functor for G over R , and let X be a H -set, then

$$(Res_H^G M)(X) = M(Ind_H^G X).$$

Let N be a normal subgroup of G , there is an inflation functor: $Inf_{G/N}^G : Mack_R(G/N) \rightarrow Mack_R(G)$, which is defined by:

$$(Inf_{G/N}^G M)(X) = M(X^N),$$

for $M \in Mack_R(G/N)$ and for a finite G -set X .

Let D a finite G -set. The Dress construction (see [8] or [1]) at D is an endo-functor of the Mackey functors category: Let $M \in Mack_R(G)$ a Mackey functor for G in the sense of Dress. Let X be a finite G -set, the Dress construction of M , denoted by M_D is:

$$M_D(X) = M(X \times D).$$

Lemma 1.12. 1. The functor $M \mapsto M^Q$ is left adjoint to the functor

$$Ind_{N_G(Q)}^G Inf_{N_G(Q)}^{N_G(Q)} : Mack_R(\overline{N}_G(Q)) \rightarrow Mack_R(G).$$

2. The functor $M \mapsto M^Q$ sends projective functors to projective functors.

3. Let H be a subgroup of G , then $(Ind_H^G(M))^Q = 0$ if Q is not conjugate to a subgroup of H .

Let (K, \mathcal{O}, k) be a p -modular system, and $R = \mathcal{O}$ or k ,

4. Let $M \in Mack_R(G, 1)$, then $M^Q \in Mack_R(\overline{N}_G(Q), 1)$.

5. Let P be a projective Mackey functor of $Mack_k(G, 1)$ and L be a projective functor of $Mack_{\mathcal{O}}(G, 1)$ which lifts P . Then L^Q is a projective Mackey functor of $Mack_{\mathcal{O}}(G, 1)$ which lifts P^Q .

6. Let P be an indecomposable projective Mackey functor of $Mack_k(G, 1)$, then $P^Q(Q/Q) \cong P(1)[Q]$.

7. Let P be an indecomposable projective Mackey functor of $\text{Mack}_R(G, 1)$, then the vertices of P are the maximal p -subgroups Q of G such that $P^Q \neq 0$.

Sketch of proof. 1. Theorem 5.1 of [19] with a different notation.

2. Since $M \mapsto M^Q$ is left adjoint to an exact functor, it sends projective objects to projective objects.
3. By successive adjunction: for $L \in \text{Mack}_R(\overline{N}_G(Q))$ and $M \in \text{Mack}_R(H)$,

$$\text{Hom}_{\text{Mack}_R(\overline{N}_G(Q))}((\text{Ind}_H^G M)^Q, L) \cong \text{Hom}_{\text{Mack}_R(H)}(M, \text{Res}_H^G \text{Ind}_{N_G(Q)}^G \text{Inf}_{N_G(Q)}^{N_G(Q)} L).$$

The result now follows from the Mackey formula.

4. The proof can be deduced from Section 8 and 9 of [18]: More precisely, with notation of [18]. Let f_1^G be the primitive idempotent of the Burnside ring indexed by the trivial subgroup. Then $(f_1^G)^Q = f_1^{\overline{N}_G(Q)}$. It can be viewed by looking at f_1^G as linear combination of primitive idempotents of $\mathbb{Q}B(G)$ denoted by e_P^G for a p -subgroup P of G . Then,

$$(f_1^G)^Q = (\text{Res}_{N_G(Q)}^G f_1^G)^Q = \sum_{P \in [s_p(G)]} (\text{Res}_{N_G(Q)}^G e_P^G)^Q,$$

where $[s_p(G)]$ denote a set of representative of the p -subgroups of G .

Then $\text{Res}_{N_G(Q)}^G e_P^G = \sum_{P_i} e_{P_i}^{N_G(Q)}$, where P_i run through the set of subgroups P' of $N_G(Q)$ up to $N_G(Q)$ -conjugacy, such that P' is G -conjugate to P . Moreover, if P' is a subgroup of $N_G(Q)$, then

$$(e_{P'}^{N_G(Q)})^Q = \begin{cases} 0 & \text{if } Q \not\leq P' \\ e_{P'/Q}^{\overline{N}_G(Q)} & \text{if } Q \leq P'. \end{cases}$$

So $(f_1^G)^Q = f_1^{\overline{N}_G(Q)}$. Now, if M is a Mackey functor for G , and $z \in B(G)$, then

$$(z.M)^Q = z^Q.M^Q.$$

Let M be an indecomposable Mackey functor in $\text{Mack}_R(G, 1)$, we have:

$$M^Q = (f_1^G.M)^Q = (f_1^G)^Q.M^Q = f_1^{\overline{N}_G(Q)}.M^Q.$$

5. Using successive adjunctions,

$$\begin{aligned} \text{Hom}_{\text{Mack}_k(\overline{N}_G(Q))}(L^Q/\mathfrak{p}(L^Q), M) &\cong \text{Hom}_{\text{Mack}_k(\overline{N}_G(Q))}(k \otimes_{\mathcal{O}} L^Q, M) \\ &\cong \text{Hom}_{\text{Mack}_{\mathcal{O}}(\overline{N}_G(Q))}(L^Q, \text{Hom}_k(k, M)) \\ &\cong \text{Hom}_{\text{Mack}_{\mathcal{O}}(G)}(L, \text{Ind}_{N_G(Q)}^G \text{Inf}_{N_G(Q)}^{N_G(Q)} \text{Hom}_k(k, M)). \end{aligned}$$

However, $Ind_{N_G(Q)}^G Inf_{N_G(Q)}^{N_G(Q)} Hom_k(k, M) \cong Hom_k(k, Ind_{N_G(Q)}^G Inf_{N_G(Q)}^{N_G(Q)}(M))$,
so

$$\begin{aligned} Hom_{Mack_k(\overline{N}_G(Q))}(L^Q/\mathfrak{p}(L^Q), M) &\cong Hom_{Mack_{\mathcal{O}}(G)}(L, Hom_k(k, Ind_{N_G(Q)}^G Inf_{N_G(Q)}^{N_G(Q)}(M))) \\ &\cong Hom_{Mack_k(G)}(L/\mathfrak{p}L, Ind_{N_G(Q)}^G Inf_{N_G(Q)}^{N_G(Q)}(M)) \\ &\cong Hom_{Mack_k(\overline{N}_G(Q))}((L/\mathfrak{p}L)^Q, M). \end{aligned}$$

6. Lemme 5.10 of [2].

7. This is the first assertion of Theorem 3.2 of [6]. □

Proposition 1.13 (Decomposition matrix of $\mu_{\mathcal{O}}^1(G)$). *Let G be a finite group, and (K, \mathcal{O}, k) be a p -modular system which is big enough for the groups $\overline{N}_G(Q)$ where Q runs through the p -subgroups of G .*

The decomposition matrix of $\mu_{\mathcal{O}}^1(G)$ has rows indexed by the isomorphism classes of indecomposable p -permutation modules of kG , the columns are indexed by the ordinary irreducible characters of all the $\overline{N}_G(Q)$ where Q runs the p -subgroup of G up to conjugacy.

Let χ be an ordinary character of the group $\overline{N}_G(Q)$ and W be an indecomposable p -permutation of $\mathcal{O}G$, then the decomposition number $d_{\chi, W}$ is equal to

$$d_{\chi, W} = \dim_K Hom_{KG}(K \otimes_{\mathcal{O}} \widehat{W[Q]}, \chi).$$

where $\widehat{W[Q]}$ is the (unique) p -permutation $\mathcal{O}G$ -module which lifts $W[Q]$.

Proof. Since (K, \mathcal{O}, k) is a splitting system for $\mu_{\mathcal{O}}^1(G)$, and since $\mu_K^1(G)$ is a semi-simple algebra, the cartan matrix is symmetric. Let S_{L, χ_i} be a simple $\mu_K^1(G)$ -module and P_{H, V_j} be a projective $\mu_k^1(G)$ Mackey functor where L and H are p -subgroups of G . Then V_j is a simple $k\overline{N}_G(H)$ -module and χ_i is a simple $K\overline{N}_G(L)$ -module. Moreover, since K is a field of characteristic zero, we have $S_{L, \chi_i} = Ind_{N_G(L)}^G Inf_{N_G(L)}^{N_G(L)} FP_{\chi_i}$.

We will denote by Q_{H, V_j} the projective indecomposable $\mu_{\mathcal{O}}^1(G)$ -module which lifts P_{H, V_j} , and by M an $\mu_{\mathcal{O}}^1(G)$ -lattice such that $K \otimes_{\mathcal{O}} M = S_{L, \chi_i}$. then

$$\begin{aligned} d_{S_{L, \chi_i}, P_{H, V_j}} &= \dim_k Hom_{\mu_k^1(G)}(P_{H, V_j}, k \otimes_{\mathcal{O}} M) \\ &= rank_{\mathcal{O}} Hom_{\mu_{\mathcal{O}}^1(G)}(Q_{H, V_j}, M) \\ &= \dim_K Hom_{\mu_K^1(G)}(K \otimes_{\mathcal{O}} Q_{H, V_j}, S_{L, \chi_i}) \\ &= \dim_K Hom_{\mu_K^1(G)}(K \otimes_{\mathcal{O}} Q_{H, V_j}, Ind_{N_G(L)}^G Inf_{N_G(L)}^{N_G(L)} FP_{\chi_i}) \\ &= \dim_K Hom_{\mu_K^1(G)}(K \otimes_{\mathcal{O}} (Q_{H, V_j})^L(L/L), \chi_i). \end{aligned}$$

The last equality comes from two successive adjunctions: (ev_1, FP_-) and $(-^L, Ind_{N_G(L)}^G Inf_{\overline{N}_G(L)}^{N_G(L)})$.

Moreover, $(K \otimes_{\mathcal{O}} Q_{H,V_j})^L(L/L) \cong K \otimes_{\mathcal{O}} ((Q_{H,V_j})^L(L/L))$. By Lemma 1.12, the $\mathcal{O}\overline{N}_G(L)$ -module $(Q_{H,V_j})^L(L/L)$ is the unique, up to isomorphism, p -permutation module which lift $(P_{H,V_j})^L(L/L) \cong P_{H,V_j}(1)[L]$. \square

Remark 1. By Section 4.4 of [5], the sub-matrix indexed by the ordinary characters of G , and the (isomorphism classes of) indecomposable p -permutation kG -module is the decomposition matrix of the cohomological Mackey algebra $co\mu_{\mathcal{O}}(G)$.

2 Equivalences between blocks of cohomological Mackey algebras

In the first part of this section, R denotes an arbitrary commutative unital ring.

For basic results about cohomological Mackey functors see Section 16 of [18]. A Mackey functor M for G is cohomological if whenever $K \leq H \leq G$, one has $t_K^H r_K^H = [H : K]$. Let us denote by $Comack_R(G)$ the full subcategory consisting of cohomological Mackey functors. The category $Comack_R(G)$ is equivalent to the category of modules over the cohomological Mackey algebra, denoted by $co\mu_R(G)$. It is easy to check that the fixed point functors are cohomological.

If $R = \mathcal{O}$ or k , the cohomological Mackey functors form a full subcategory of the category $Mack_R(G, 1)$ and $co\mu_R(G)$ is a quotient of $\mu_R^1(G)$. The block decomposition of Theorem 1.10 is compatible with the cohomological structure. We will denote by $Comack_R(b)$ the category of cohomological Mackey functors which belong to the block b^μ , and $co\mu_R(b)$ the corresponding direct summand of the cohomological algebra.

2.1 Yoshida equivalence

One of the main results about the cohomological Mackey algebra is the Yoshida equivalence (see [20]), which linearizes the definition of Mackey functors. Recall that the center of a category \mathcal{C} is given by the natural transformation of the identity functor.

Let us denote by $perm_R(G)$ the full subcategory of RG -Mod consisting of the permutation RG -modules, and by $Fun_R(G)$ the category of contravariant functors from $perm_R(G)$ to R -Mod.

Lemma 2.1. *The idempotent completion of $perm_R(G)$ is equivalent to the category of permutation projective RG -modules.*

Proof. Let us denote temporarily by \mathcal{A} the category of permutation projective RG -modules, and let $perm_R^+(G)$ be the idempotent completion of $perm_R(G)$. The objects of this category are the pairs (V, π) where V is a permutation module and $\pi \in Hom_{perm_R(G)}(V, V)$ an idempotent. There is a natural functor F from $perm_R^+(G)$ to \mathcal{A} defined by $F(V, \pi) = \pi(V)$. This functor is dense and fully faithful. \square

We denote by $perm_R^+(G)$ the category of permutation projective RG -modules and by $Fun_R^+(G)$ the category consisting of contravariant functors from $perm_R^+(G)$ to $R\text{-Mod}$. By general properties of the idempotent completion, the categories $Fun_R^+(G)$ and $Fun_R(G)$ are equivalent.

Recall that the Yoshida equivalence is given by two functors: Y and Γ . The first one is the Yoneda functor $Y : Comack_k(G) \rightarrow Fun_R(G)$. The second one is the linearization functor. More precisely:

if M is a Mackey functor, then $Y(M)(RX) := Hom_{Mack_R(G)}(FP_{RX}, M)$.

Conversely if $F \in Fun_R(G)$, then $\Gamma(F)(X) := F(RX)$. If $f : X \rightarrow Y$ is a morphism of G -sets, we have two morphisms between RX and RY . Let us denote by $\pi^*(f) : RY \rightarrow RX$ the morphism defined by $\pi(\sum_{y \in Y} \lambda_y y) = \sum_{x \in X} \lambda_{f(x)} x$, and by $\pi_*(f) : RX \rightarrow RY$ the morphism defined by $\pi_*(f)(\sum_{x \in X} \lambda_x x) = \sum_{x \in X} \lambda_x f(x)$, with $x \in X$, $y \in Y$ and $\lambda_x \in k$, $\forall x \in X$. One can check that $(\Gamma(F), F \circ \pi^*, F \circ \pi_*)$ is a cohomological Mackey functor for G .

Then using the idempotent completion of $perm_R(G)$, we have the following equivalence:

$$Comack_R(G) \cong Fun_R^+(G).$$

We still denote by Y and Γ the functors which give the equivalence after idempotent completion.

Lemma 2.2. *Let V be a permutation projective RG -module, and let U be an RG -module, then $Y(FP_U)(V) = Hom_{Comack_R(G)}(FP_V, FP_U)$.*

Proof. Since V is a permutation projective module, there is a G -set X and a map $\pi : RX \rightarrow RX$ such that $\pi^2 = \pi$ and, $\pi(RX) = V$. By definition of the idempotent completion:

$$Y(FP_U)(V) = Y(\pi)(Y(FP_U)(RX)) = Y(\pi)(Hom_{Comack_R(G)}(FP_{RX}, FP_U)).$$

The elements in $Y(\pi)(Hom_{Comack_R(G)}(FP_{RX}, FP_U))$ are the morphisms: $FP_{RX} \rightarrow FP_U$ of the form $\alpha \circ FP_\pi$ where FP_π is the endomorphism of the Mackey functor FP_{RX} induced by π .

However, $FP_{RX} = FP_{\pi(RX)} \oplus FP_{(Id-\pi)(RX)}$, so the morphisms of the form $\alpha \circ \pi$ are exactly the morphisms from $FP_{\pi(RX)}$ to FP_U . \square

The Yoshida equivalence is compatible with the action of central idempotents:

Theorem 2.3 (Yoshida Equivalence, block version). *There is a commutative diagram:*

$$\begin{array}{ccc}
 & & Z(\text{Fun}_R^+(G)) \\
 & \nearrow \cong \eta & \downarrow \Gamma \cong \\
 Z(RG) & & Z(\text{Comack}_R(G)). \\
 & \searrow \cong \zeta &
 \end{array} \tag{2}$$

Let $1 = e + f \in Z(RG)$ be a decomposition of 1 in sum of two orthogonal idempotents. Then

$$\text{Comack}_R(G) \cong \zeta(e)(\text{Comack}_R(G)) \oplus \zeta(f)(\text{Comack}_R(G)).$$

and

$$\text{Fun}_R^+(G) = \eta(e)(\text{Fun}_R^+(G)) \oplus \eta(f)(\text{Fun}_R^+(G)).$$

The corresponding sub-categories are equivalent.

Sketch of proof. We see $Z(RG)$ as the center of the category $RG\text{-Mod}$.

1. Let α be an endomorphism of the identity functor of $RG\text{-Mod}$. The first map η is defined as follow: let $F \in \text{Fun}_R^+(G)$ and V be a permutation projective RG -module. Then

$$\begin{aligned}
 \eta_F(\alpha)(V) : F(V) &\rightarrow F(V) \\
 x &\mapsto F(\alpha_V)(x).
 \end{aligned}$$

One can check that this gives a natural transformation of the identity functor of $\text{Fun}_R^+(G)$. The map η is a unital ring homomorphism.

Conversely, if ν is a natural transformation of the identity functor of $\text{Fun}_R^+(G)$, let Y_{RG} be the Yoneda functor at the free RG -module of rank 1. It is clear that $\nu_{Y_{RG}}(Id_{RG})(1)$ is an element of the center of RG . The map $\nu \mapsto \nu(Y_{RG})(Id_{RG})(1)$ is a ring homomorphism from $Z(\text{Fun}_R^+(G)) \rightarrow Z(RG)$.

One can check that the above two maps are inverse isomorphisms.

2. It is well known that $Z(RG) \cong Z(\text{Comack}_R(G))$ see [1], however this isomorphism is only explicit when $\text{Comack}_R(G)$ is viewed as the category $\text{co}\mu_R(G)\text{-Mod}$, it is not so easy to have an explicit formula when we use the Dress definition of cohomological Mackey functors. However, by the following lemma, we have a natural map from $Z(\text{Comack}_R(G)) \rightarrow Z(RG\text{-Mod})$ denoted by ev_1 .

Lemma 2.4 ([18]). *Let G be a finite group, then the full subcategory of $Comack_R(G)$ consisting of the fixed point functors is equivalent to $RG\text{-Mod}$.*

Conversely, if α is an endomorphism of the identity functor of $RG\text{-Mod}$, by Lemma 2.4, we have an endomorphism $\zeta(\alpha) = FP_\alpha$ of the identity functor of the full subcategory of $Comack_R(G)$ consisting of the fixed point functors. The cohomological projective Mackey functors are in this subcategory (see Theorem 16.5 in [18]). So if M is a cohomological Mackey functor, choose a projective resolution P_\bullet of M . Denote by $\zeta_M(\alpha)$ the map $M \rightarrow M$ induced by $\zeta_{P_\bullet}(\alpha)$, on the cohomology of P_\bullet . It is straightforward that this map doesn't depend of the choice of the resolution. Moreover $\zeta(\alpha)$ is an endomorphism of the identity functor of $Comack_R(G)$, and ζ is a ring homomorphism.

Let α be an endomorphism of the identity functor of $RG\text{-Mod}$. If V is an RG -module, then $ev_1\zeta_{FP_V}(\alpha_V) = V$. Conversely, if ϕ is an endomorphism of the identity functor of $Comack_R(G)$, let M be a cohomological Mackey functor, let P_\bullet be a projective resolution of M , we have the following commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M \longrightarrow 0 \\ & & \phi_{P_1} \downarrow & & \phi_{P_0} \downarrow & & \phi_M \downarrow \zeta_M(ev_1(\phi_M)) \\ \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M \longrightarrow 0 \end{array}$$

So the two morphisms ϕ_M and $\zeta_M(ev_1(\phi_M))$ are equals.

3. The equivalence between $Z(Fun_R^+(G))$ and $Z(Comack_R(G))$ comes from the fact that the two corresponding categories are equivalent.
4. The triangle (2) is commutative: let α be an endomorphism of the identity functor of $RG\text{-Mod}$, let V be an RG -module, then the map $\zeta^{-1} \circ \Gamma \circ \eta(\alpha)_V : V \rightarrow V$ is the map:

$$\begin{aligned} Hom_{RG}(RG, V) &\rightarrow Hom_{RG}(RG, V) \\ f &\mapsto \alpha_V \circ f. \end{aligned}$$

□

Let R be \mathcal{O} or k , where \mathcal{O} is a complete discrete valuation ring and k is the residue field. Let $1 = b_1 + b_2 + \cdots + b_s$ be a decomposition of 1 in orthogonal sum of central primitive idempotent of RG . This decomposition induces a decomposition of $Comack_R(G) = \bigoplus_{i=1}^s \zeta(b_i)Comack_R(G)$ and $Fun_R^+(G) = \bigoplus_{i=1}^s \eta(b_i)Fun_R^+(G)$. We have the following straightforward lemma:

Lemma 2.5. *Let b be a block idempotent of RG . The category $\eta(b)(\text{Fun}_R^+(b))$ is equivalent to the category denoted by $\text{Fun}_R^+(b)$, consisting of contravariant functors from $\text{perm}_R^+(b)$ to $R\text{-Mod}$, where $\text{perm}_R^+(b)$ is the category consisting of the p -permutation RG -modules which are in the block RGb .*

Corollary 2.6. *Let b be a block of RG , the Yoshida equivalence restricts to an equivalence $\text{Comack}_R(b) \cong \text{Fun}_R^+(b)$.*

2.2 Morita equivalences

Let $R = \mathcal{O}$ or k as above. With the version of Yoshida's equivalence of Corollary 2.6 it is not difficult to lift an equivalence between blocks of group algebras to an equivalence of the corresponding blocks of the cohomological Mackey algebras.

Lemma 2.7. *Let G and H be two finite groups, b be a block of RG , c be a block of RH . Let X be an RH - RG -bimodule such that*

- $X \otimes_{RH} -$ is a functor from $\text{perm}_R^+(c)$ to $\text{perm}_R^+(b)$.

Then X induces a functor, denoted by $L_X : \text{Comack}_R(b) \rightarrow \text{Comack}_R(c)$. Moreover this functor sends an arbitrary fixed point functor to a fixed point functor.

Proof. We use the equivalence $\text{Comack}_R(b) \cong \text{Fun}_R^+(b)$ of Corollary 2.6. The functor $L_X : \text{Fun}_R^+(b) \rightarrow \text{Fun}_R^+(c)$ is defined by $L_X(F)(V) := F(X \otimes_{RH} V)$. Clearly this construction gives a functor from $\text{Fun}_R^+(b) \rightarrow \text{Fun}_R^+(c)$. We will denote by L_X the composite functor,

$$\begin{array}{ccc} \text{Fun}_R^+(b) & \xrightarrow{L_X} & \text{Fun}_R^+(c) \\ \uparrow Y & & \downarrow \Gamma \\ \text{Comack}_R(b) & \xrightarrow{\quad\quad\quad} & \text{Comack}_R(c) \end{array}$$

so if V is a RG -module, and Z is a H -set, then

$$\begin{aligned} L_X(FP_V)(Z) &= \Gamma(L_X(Y(FP_V)))(Z) \\ &= Y(FP_V)(X \otimes_{RH} RZ) \\ &\cong \text{Hom}_{\text{Comack}_R(G)}(FP_{X \otimes_{RH} RZ}, FP_V) \text{ by Lemma 2.2} \\ &\cong \text{Hom}_{RG}(X \otimes_{RH} RZ, V) \\ &\cong \text{Hom}_{RH}(RZ, \text{Hom}_{RG}(X, V)) \\ &\cong FP_{\text{Hom}_{RG}(X, V)}(RZ). \end{aligned}$$

This isomorphism is functorial in Z , so $\phi(FP_V) = FP_{\text{Hom}_{RG}(X, V)}$. □

Proposition 2.8. *Let G and H be two finite groups, let b be a block of RG and c be a block of RH . We suppose that:*

1. *There is an RH - RG -bimodule X such that:*

$$X \otimes_{RG} - : RGb\text{-Mod} \rightarrow RHc\text{-Mod} \text{ is an equivalence.}$$

2. *$X \otimes_{RG} -$ induces a functor: $\text{perm}_R^+(b) \rightarrow \text{perm}_R^+(c)$.*

Then $\text{Comack}_R(b) \cong \text{Comack}_R(c)$.

Proof. By Lemma 2.7 we have a functor $L_X : \text{Fun}_R^+(c) \rightarrow \text{Fun}_R^+(b)$. By general results about Morita equivalences, a quasi-inverse equivalence of $X \otimes_{RG} -$ is given by $X^* \otimes_{RH} -$ where $X^* = \text{Hom}_R(X, R)$. This functor satisfies the Condition of Lemma 2.7: since $X \otimes_{RG} -$ is a dense functor, if V is a RH -module, then V is isomorphic to $X \otimes_{kRG} W$ for some RG -module W . Moreover if V is a p -permutation RH -module, the module W has to be a p -permutation RG -module, so

$$X^* \otimes_{RH} W \cong X^* \otimes X \otimes V \cong V.$$

This gives a functor $L_{X^*} : \text{Fun}_R^+(b) \rightarrow \text{Fun}_R^+(c)$, which is obviously a quasi-inverse equivalence to L_X . \square

Remark 2. The second hypothesis of this proposition is a technical property, one may ask if there exist Morita equivalences with this property. Let G be a finite group, and P be a Sylow p -subgroup of G and $H = N_G(P)$ be the normalizer of the Sylow. Let b a block of kG and c the Brauer correspondent of this block in $N_G(P)$. If $kGb\text{-Mod}$ is splendidly Morita equivalent to $kHc\text{-Mod}$, then the two conditions are satisfied. For example if G is a p -nilpotent group and P is the Sylow p -subgroup of G , let b_0 be the principal block of kG . Then $\text{Comack}_k(b_0) \cong \text{Comack}_k(P)$.

The equivalence L_X between blocks of cohomological Mackey algebras generalizes the equivalence $X \otimes_{RG} -$ since the restriction of L_X to the subcategories of fixed point functors is the functor $X^* \otimes_{RH} -$.

2.3 Derived equivalences

Let $R = \mathcal{O}$ or k . A Morita equivalence between group algebras, with an extra property, can be lifted to a Morita equivalence between blocks of cohomological Mackey algebras. The next theorem will show that a derived equivalence between group algebras, with some additional properties, can be lifted to a derived equivalence for the cohomological Mackey algebras.

Lemma 2.9. *Let G and H be finite groups, b be a block of RG and c be a block of RH . Suppose that there is a complex X_\bullet of RG - RH -bimodules such that:*

\mathcal{P} : For each term X_i of X , the functor $X_i \otimes_{RH} -$ sends p -permutation RHc -modules to p -permutation RGb -modules.

Then there is an additive functor between the categories of complexes:

$$L_{X_\bullet} : Ch^-(co\mu_R(b)\text{-Mod}) \rightarrow Ch^-(co\mu_R(c)\text{-Mod}),$$

which induces a triangulated functor at the level of derived categories:

$$L_{X_\bullet} : D^-(co\mu_R(b)\text{-Mod}) \rightarrow D^-(co\mu_R(c)\text{-Mod}).$$

Proof. We will work with the categories $Fun_R^+(-)$. Let X_\bullet the bounded (upper and lower bounded) two sided complex as in the hypothesis:

$$\cdots \longrightarrow 0 \longrightarrow X_s \xrightarrow{d_s} X_{s-1} \xrightarrow{d_{s-1}} \cdots \longrightarrow X_n \longrightarrow 0 \longrightarrow \cdots$$

and F_\bullet a right bounded complex of functors $\in Fun_R^+(b)$:

$$\cdots \longrightarrow F_i \xrightarrow{\eta_i} F_{i-1} \xrightarrow{\eta_{i-1}} \cdots \longrightarrow F_m \longrightarrow 0.$$

By pre-composition of F_\bullet by X_\bullet , we have a double complex:

$$\begin{array}{ccccc} & & \vdots & & \vdots \\ & & \uparrow & & \uparrow \\ & & (-1)^i F_i(d_{j+1}) & & (-1)^{i-1} F_{i-1}(d_{j+1}) \\ \cdots & \longrightarrow & F_i(X_j) & \xrightarrow{\eta_i(X_j)} & F_{i-1}(X_j) & \longrightarrow \cdots \\ & & \uparrow & & \uparrow \\ & & (-1)^i F_i(d_j) & & (-1)^{i-1} F_{i-1}(d_j) \\ \cdots & \longrightarrow & F_i(X_{j-1}) & \xrightarrow{\eta_i(X_{j-1})} & F_{i-1}(X_{j-1}) & \longrightarrow \cdots \\ & & \uparrow & & \uparrow \\ & & \vdots & & \vdots \end{array}$$

Where we denote by $F_i(X_j)$ the functor $F_i(X_j \otimes_{RG} -)$, by $F_i(d_j)$ the morphism $F_i(d_j \otimes_{RG} Id_-)$ and by $\eta_i(X_j)$ the natural transformation $\eta_i(X_k \otimes_{RG} -)$. Then we take the total complex of this double complex, which we denote by $L_X(F)$:

$$L_X(F)_k := \bigoplus_{i-j=k} F_i(X_j).$$

If $w_{i,j} \in F_i(X_j)$, with $i + j = k$

$$\delta_k(w_{i,j}) = (-1)^i F_i(d_{j+1})(w_{i,j}) + \eta_i(X_j)(w_{i,j}) \in F_i(X_{j+1}) \oplus F_{i-1}(X_j).$$

We will prove the following:

1. $L_X(F)$ is a complex.
 2. $L_X : Ch^-(Fun_k^+(b)) \rightarrow Ch^-(Fun_k^+(c))$ is an additive functor.
 3. $L_X : D^-(Fun_k^+(b)) \rightarrow D^-(Fun_k^+(c))$ is a well defined triangulated functor.
1. We have to check that $\delta_k := L_X(F)_k \rightarrow L_X(F)_{k-1}$ is actually a differential. This is formal: let $w \in L_X(F)$, $(\delta_{k-1} \circ \delta_k(w))_{i-1, j+1}$ the component in $F_{i-1}(X_{j+1})$. A computation give:

$$\begin{aligned} (\delta_{k-1} \circ \delta_k(w))_{i-1, j+1} &= (-1)^{i-1} F_{i-1}(d_{j+1}) \eta_i(X_j)(w_{i, j}) + (-1)^i \eta_i(X_{j-1})(F_i(d_{j+1})) \\ &= 0, \text{ since } \eta_i \text{ is a natural transformation.} \end{aligned}$$

So $L_X(F)$ is a complex.

2. Let $(F_\bullet, \eta_\bullet)$ and $(G_\bullet, \gamma_\bullet)$ two complexes of $Ch^-(Fun_k^+(b))$, and Φ a morphism form F_\bullet to G_\bullet . For each i, j , $\phi_i(X_j)$ is a morphism $F_i(X_j) \rightarrow G_i(X_j)$. So it is clear that we have a morphism $\Phi_k : L_X(F)_k \rightarrow L_X(G)_k$ for all $k \in \mathbb{Z}$. We just need to check that these morphisms commute with the differentials. We will prove that all the following diagrams commute:

$$\begin{array}{ccc} \bigoplus_{i-j=k} F_i(X_j) & \xrightarrow{\alpha} & F_{i-1}(X_j) \\ \downarrow \Phi & & \downarrow \Phi_{i-1, j} \\ \bigoplus_{i-j=k} G_i(X_j) & \xrightarrow{\beta} & G_{i-1}(X_j) \end{array}$$

where the horizontal arrow are just the restrictions of the differential maps to $F_{i-1}(X_j)$ and $G_{i-1}(X_j)$ respectively. For $w = (w_{i, j})$, one can compute that

$$\Phi_{i-1, j} \circ \alpha(w) = \Phi_{i-1, j}(\eta_i(X_j)(w_{i, j})) + (-1)^{i-1} \Phi_{i-1, j} F_{i-1}(d_j)(w_{i-1, j-1}),$$

and

$$\beta \circ \Phi(w) = \gamma_i(X_j)(\Phi_{i, j}(w_{i, j})) + (-1)^{i-1} G_{i-1}(d_j)(\Phi_{i-1, j-1}(w_{i-1, j-1})).$$

Equality holds because Φ is a morphism of complexes, and for each i, j , $\Phi_{i, j}$ is a natural transformation of functors. It is then obvious that L_X is an additive functor from $Ch^-(Fun_R^+(b)) \rightarrow Ch^-(Fun_R^+(c))$.

3. Let $(F_\bullet, \eta_\bullet)$ and $(G_\bullet, \gamma_\bullet)$ two quasi-isomorphic complexes of $Ch^-(Fun_R^+(b))$. We need to check that $L_X(F)$ and $L_X(G)$ are quasi-isomorphic functors. Let $\Phi : F \rightarrow G$ a quasi-isomorphism. We prove that the homology groups of $L_X(F)$ are subgroups of the homology groups of F , so it is clear that a quasi-isomorphism from F to G induce a quasi-isomorphism from $L_X(F) \rightarrow L_X(G)$. If $k \in \mathbb{Z}$,

$$Ker(\delta_k)_{i-1, j} = Ker(\eta_{i-1}(X_j)) \cap Ker(F_{i-1}(d_{j+1})) \subseteq Ker(\eta_{i-1}(X_j)).$$

and

$$\text{Im}(\delta_{k-1})_{i-1,j} \supseteq \text{Im}(\eta_i(X_i)) \cup \text{Im}(F_{i-1}(d_j)) \supseteq \text{Im}(\eta_i(X_j)).$$

so $H^k(L_X(F))_{i-1,j} \subseteq H^{i-1}(F(X_j))$, and $H^k(L_X) \subseteq \bigoplus_{i-j=k} H^i(F(X_j))$. \square

Lemma 2.10. *Let G and H be two finite groups, let b be a block of RG and c be a block of RH . Let X and Y be two complexes of RGb - RHc -bimodule with Property \mathcal{P} , then*

- $L_{X \oplus Y} \cong L_X \oplus L_Y$.
- *If the complex X is a contractible complex, then L_X is contractible in the following sense: the complex $L_X(F)$ is contractible for every complex of functors of $\text{Fun}_R^+(b)$.*

Proof. The first part is clear. For the second part, let (X_\bullet, d_\bullet) be a contractible complex. Let (s_\bullet) a chain homotopy i-e a family of maps $s_i : X_i \rightarrow X_{i-1}$ such that $id_{X_i} = s_{i-1}d_i + d_{i+1}s_i$. Let $(F_\bullet, \eta_\bullet)$ be a complex of $\text{Fun}_R^+(b)$. We will show that $L_X(F)$ is a contractible complex. Let

$$S_{i,j} := (-1)^i F_i(s_j \otimes Id_-) : F_i(X_{j+1} \otimes -) \rightarrow F_i(X_j \otimes -).$$

Then we have:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \bigoplus_{i-j=k} F_i(X_j) & \longrightarrow & \bigoplus_{i-j=k} F_i(X_{j+1}) & \longrightarrow & \cdots \\ & & \searrow^{\oplus S_{i,j-1}} & & \searrow^{\oplus S_{i,j}} & & \\ \bigoplus_{i-j=k} F_i(X_{j-1}) & \longrightarrow & \bigoplus_{i-j=k} F_i(X_j) & \longrightarrow & \bigoplus_{i-j=k} F_i(X_{j+1}) & \longrightarrow & \cdots \end{array}$$

Let denote by $S_k := \bigoplus_{i-j=k} S_{i,j-1}$. We have to check that this is effectively an homotopy. We will show that the component of $S_{k-1}\delta_k + \delta_{k+1}S_k$ which lands in $F_i(X_j)$ is the identity of $F_i(X_j)$. This can be seen in the following diagram:

$$\begin{array}{ccc} \bigoplus_{i-j=k} F_i(X_j) & \xrightarrow{a} & F_i(X_{j+1}) \\ \swarrow^b & & \swarrow^d \\ F_{i+1}(X_j) \oplus F_i(X_{j-1}) & \xrightarrow{c} & F_i(X_j) \end{array}$$

where the maps are:

$$\begin{aligned} a &:= \eta_{i+1}(X_{j+1}) + (-1)^i F_i(d_{j+1}). \\ b &:= (-1)^{i+1} F_{i+1}(s_j) + (-1)^i F_i(s_{j-1}). \\ c &:= \eta_{i+1}(X_j) + (1)^i F_i(d_j). \\ d &:= (-1)^i F_i(s_j). \end{aligned}$$

Then,

$$\begin{aligned} da + cb &= F_i(d_{j+1}s_j + s_{j-1}d_j) + (-1)^i F_i(s_j)\eta_{i+1}(X_{j+1}) + (-1)^{i+1}\eta_{i+1}(X_j)F_{i+1}(s_j) \\ &= Id_{F_i(X_j)}, \end{aligned}$$

since η_{i+1} is a natural transformation from F_{i+1} to F_i . \square

Lemma 2.11. *Let G, H and K three finite groups and let b, c and d blocks of respectively RG, RH and RK . Let (X_\bullet, d_\bullet^X) be a bounded complex of RGb - RKd -bimodules. Let (Y_\bullet, d_\bullet^Y) be a bounded complex of RHc - RGb -bimodules and let $(F_\bullet, \eta_\bullet)$ be a right bounded complex of $\text{Fun}_R^+(c)$. If the two complexes X and Y have Property \mathcal{P} , then,*

$$(L_X \circ L_Y) \cong L_{Y \otimes_{RG} X}.$$

Proof. We choose the following convention for tensor product of complexes :

$$(Y \otimes_{RG} X)_k := \bigoplus_{i+j=k} Y_i \otimes X_j.$$

The differential is:

$$D_k := \bigoplus_{i-j=k} (-1)^i 1 \otimes d_j^X + d_i^Y \otimes 1.$$

we prove that $L_X \circ L_Y(F)$ is equal to the total complex of the following bi-complex:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_m((Y \otimes X)_{m-k}) & \xrightarrow{\eta_m((Y \otimes X)_{m-k})} & F_{m-1}((Y \otimes X)_{m-k}) & \longrightarrow & \cdots \\ & & \downarrow (-1)^m F_m(D_{m-k+1}) & & \downarrow (-1)^{m-1} F_{m-1}(D_{m-k+1}) & & \\ \cdots & \longrightarrow & F_m((Y \otimes X)_{m-k+1}) & \xrightarrow{\eta_m((Y \otimes X)_{m-k+1})} & F_{m-1}((Y \otimes X)_{m-k+1}) & \longrightarrow & \cdots \end{array}$$

To see this, we just need to change the order of summation, and remember that our functors are additive functors and the direct sums over the index n are finite, since our complexes X and Y are right and left bounded. Then we have to check that the differentials are exactly the maps of the previous diagram.

$$\begin{aligned} (L_X \circ L_Y(F))_k &= \bigoplus_{n \in \mathbb{Z}} \bigoplus_{m \in \mathbb{Z}} F_m(Y_{m-n} \otimes_{RG} X_{n-k}) \\ &\cong \bigoplus_{m \in \mathbb{Z}} F_m\left(\bigoplus_{n \in \mathbb{Z}} (Y_{m-n} \otimes_{RG} X_{n-k})\right) \\ &= \bigoplus_{m \in \mathbb{Z}} F_m((Y \otimes_{RG} X)_{m-k}). \end{aligned}$$

Let $m, n \in \mathbb{Z}^2$, the differential on $F_m(Y_{m-n} \otimes X_{n-k})$ is:

$$F_m((-1)^m d_{m-n+1}^Y \otimes 1 + (-1)^n 1 \otimes d_{n-k+1}^X) + \eta_m(Y_{m-n} \otimes X_{n-k}),$$

so the differential on $\bigoplus_{n \in \mathbb{Z}} F_m(Y_{m-n} \otimes X_{n-k})$ is

$$\begin{aligned} & \bigoplus_{n \in \mathbb{Z}} (F_m((-1)^m d_{m-n+1}^Y \otimes 1 + (-1)^n 1 \otimes d_{n-k+1}^X) + \eta_m(Y_{m-n} \otimes X_{k-n})) = \\ & (-1)^m (F_m(\bigoplus_{n \in \mathbb{Z}} d_{m-n+1}^Y \otimes 1) \bigoplus_{n \in \mathbb{Z}} F_m(\bigoplus_{n \in \mathbb{Z}} (-1)^{m-n} 1 \otimes d_{n-k+1}^X)) + \eta_m(Y \otimes X)_{m-k} \\ & = (-1)^m F_m(D_{m-k+1}) + \eta_m((X \otimes X^*)_{m-k}). \end{aligned}$$

since:

$$\begin{aligned} D_{m-k+1} &= \bigoplus_{n \in \mathbb{Z}} d_{m-n+1}^Y \otimes 1 + \bigoplus_{n \in \mathbb{Z}} (-1)^{m-n+1} 1 \otimes d_{n-k}^X \\ &= \bigoplus_{n \in \mathbb{Z}} d_{m-n+1}^Y \otimes 1 + \bigoplus_{n \in \mathbb{Z}} (-1)^{m-n} 1 \otimes d_{n-k+1}^X. \end{aligned}$$

It is not difficult, but rather technical to check that this is functorial in F . \square

We will apply Lemmas 2.9, 2.10 and 2.11 to the following situation. Let RGb and RHc be two blocks of group algebras which are derived equivalent. By standard results about derived equivalences for block algebras, we can suppose that the equivalence $D^b(RGb\text{-Mod}) \cong D^b(RHc\text{-Mod})$ is given by tensor product with a split endomorphism two sided complex X_\bullet , a quasi-inverse equivalence is given by tensor product with the R -linear dual X_\bullet^* .

Theorem 2.12. *Let G and H be finite groups, b be a block of RG and c be a block of RH . Suppose that:*

1. *There is a two sided tilting complex X_\bullet of RH - RG -bimodules, such that $X \otimes_{RH} X^* \cong RGb$ and $X^* \otimes_{RG} X \cong RHc$ in the homotopy categories of corresponding bimodules.*
2. *If V is a p -permutation RGb -module, then $X_i \otimes_{RGb} V$ is a p -permutation RHc -module for all $i \in \mathbb{Z}$.*
3. *If W is a p -permutation RHc -module, then $X_i^* \otimes_{RHc} W$ is a p -permutation RGb -module for all $i \in \mathbb{Z}$.*

Then the functor L_X induces an equivalence of triangulated categories

$$L_X : D^-(\text{Comack}_R(c)) \cong D^-(\text{Comack}_R(b)).$$

Proof. We will prove that $D^-(\text{Fun}_k^+(b)) \cong D^-(\text{Fun}_k^+(c))$. Thanks to Lemma 2.9, we have two triangulated functors:

$$L_X : D^-(\text{Fun}_R^+(c)) \rightarrow D^-(\text{Fun}_R^+(b))$$

and

$$L_{X^*} : D^-(\text{Fun}_R^+(b)) \rightarrow D^-(\text{Fun}_R^+(c)).$$

Thanks to Lemma 2.11, we have $L_{X^*} \circ L_X \cong L_{X \otimes X^*}$.

By hypothesis the equivalence $X \otimes X^* \cong RGb$ is in the homotopy category, then $X \otimes X^* = RGb \oplus C$ where C is a contractible complex. Thanks to Lemma 2.10,

$$L_{X^*} \circ L_X \cong L_{X \otimes X^*} = L_{RGb \oplus C} \cong L_{RGb} \oplus L_C$$

It is clear that $L_{RGb} \cong \text{Id}_{D^-(\text{Fun}_R^+(b))}$ and $L_C \cong 0$. Conversely $L_X L_{X^*} \cong \text{Id}_{D^-(\text{Fun}_R^+(c))}$. \square

As an immediate corollary, we have:

Corollary 2.13. *Let b be a block of RG and c be a block of RH such that RGb and RHc are splendidly derived equivalent, then*

$$D^b(\text{co}\mu_R(b)\text{-Mod}) \cong D^b(\text{co}\mu_R(c)\text{-Mod}).$$

2.4 Application to nilpotent blocks

Although the determinant of the Cartan Matrix of a block b of kG is a power of p , for the corresponding blocks of the Mackey algebra, it is much more complicated, see [5]. By the results of [18] this determinant is non zero. However the determinant of the blocks of cohomological Mackey algebra can be zero. Bouc in [5] proved that this determinant is non zero if and only if the block b is a nilpotent block with cyclic defect group. This proof is based on a combinatorial approach, and it may be surprising that nilpotent blocks appear in that situation. We will apply Theorem 2.12, and show that it is in fact very natural.

Let B be a block of kG , for an arbitrary finite group G . If B is a nilpotent block with defect group P , then by (see [14] or [12]), there is an isomorphism of k -algebras,

$$B \cong \text{Mat}(m, kP),$$

for some $m \in \mathbb{N}$. This result is not true for the corresponding blocks of the Mackey algebra (see remark 4). We will discuss the existence of Morita equivalence between $\mu_k^1(b)$ and $\mu_k(P)$ in the next section. For the cohomological Mackey algebra, we can lift an equivalence between blocks of group algebras, but for this we need that the equivalence sends p -permutation modules to p -permutation modules. Unfortunately it is not always the case:

Example 3. Let $G = SL_2(\mathbb{F}_3) \cong Q_8 \rtimes C_3$, and k be a field of characteristic 3. The group G is 3-nilpotent, so the blocks of this group algebra are nilpotent. Let b the block idempotent such that the block $kGb = B$ contains the simple kG -module

W , where W is the simple kQ_8 -module of dimension 2 which is extended to kG by Clifford theory. Then $kGb\text{-mod} \cong kC_3\text{-mod}$ and a functor $\Phi : kC_3\text{-mod} \rightarrow kGb\text{-mod}$ can be given by:

$$X \rightarrow Iso(\phi)Inf_{G/Q_8}^G(X) \otimes_k W,$$

where the action of G is the diagonal action, and $\phi : G/Q_8 \cong C_3$. In particular the trivial p -permutation module is sent to W which is not a p -permutation module. So we cannot apply Proposition 2.8. Moreover one can check that the Cartan matrix of $co\mu(b)$ is $\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$, and the Cartan matrix of $co\mu_k(C_3)$ is $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$.

By the results of sections 7.3 and 7.4 of [15] and results of [4] and [13], if $p > 2$, or P is abelian (N.B. in fact one can ask weaker condition in case of $p = 2$), we can replace the bimodule which gives the Morita equivalence between B and kP by a splendid tilting complex of B - kP -bimodule.

Corollary 2.14. *Let $B = kGb$ a nilpotent block of defect p -group P . If $p = 2$ assume that P is abelian. Then*

$$D^b(co\mu_k(b)\text{-Mod}) \cong D^b(co\mu_k(P)\text{-Mod}) \text{ as triangulated categories.}$$

Since the determinant of Cartan matrices is invariant under derived equivalences, the determinant of the cohomological Mackey algebra is non zero if and only if the determinant of the cohomological Mackey algebra of the defect p -group is non zero. However it is well know that this is the case if and only if the p -group is cyclic: indeed the projective indecomposable cohomological Mackey functors for a p -group P are $FP_{Ind_Q^P(k)}$ for $Q \leq P$. By adjunction, the coefficient of the Cartan matrix indexed by two projective $FP_{Ind_Q^P(k)}$ and $FP_{Ind_{Q'}^P(k)}$ is:

$$\begin{aligned} C_{Q,Q'} &= dim_k Hom_{kP}(Ind_Q^P(k), Ind_{Q'}^P(k)) \\ &= dim_k Hom_{kP}(k, Res_Q^P Ind_{Q'}^P k) \\ &= Card([Q \setminus P / Q']). \end{aligned}$$

By the main result of [17] this matrix is non degenerate if and only if P is cyclic.

3 Example of equivalences between blocks of p -local Mackey algebras.

In this section we will give some examples of equivalences of blocks of p -local algebras. We first look at the case of p -nilpotent groups. We will prove that

the p -local algebra of the principal block of such a group is Morita equivalent to the Mackey algebra of its Sylow p -subgroup. But as in [15] the case of the non-principal block seems unexpectedly much more difficult. We will give an example which proves that in general there is no Morita equivalence between the p -local Mackey algebra of the block and the Mackey algebra of the defect. Then we will look at the case of a finite group with p -Sylow subgroups of order p . In that case the p -local Mackey algebra looks like a group algebra. It is a Brauer tree algebra, so we can use the background which was developed for Broué's conjecture on blocks of algebras of groups with cyclic Sylow p -subgroups.

Remark 3. Since the p -local Mackey algebra and the cohomological Mackey algebra share a lot of properties, for example, they have the same number of simple modules in each block and the projective cohomological Mackey functors are the biggest cohomological quotients of the p -local projective Mackey functors, one may ask if an equivalence between blocks of p -local Mackey algebras induces in some sense, an equivalence between the corresponding blocks of the cohomological Mackey algebras. The following example shows that the situation is not that simple.

Example 4. Let k be a field of characteristic 2 and $G = C_2$ be the (cyclic) group of order 2. Then a basis of $\mu_k(C_2)$ is given by: $t_{C_2}^{C_2}$, $t_1^{C_2}r_1^{C_2}$, $t_1^{C_2}$, $r_1^{C_2}$, t_1^1 and t_1^1x where $x \in C_2$ and t_1^1x means $t_1^1c_{1,x}$. Then there is an automorphism ϕ of $\mu_k(C_2)$ where ϕ is defined over the basis elements by: $\phi(t_{C_2}^{C_2}) = t_1^1$, $\phi(t_1^{C_2}r_1^{C_2}) = t_1^1 + t_1^1x$, $\phi(t_1^{C_2}) = r_1^{C_2}$, $\phi(r_1^{C_2}) = t_1^{C_2}$, $\phi(t_1^1) = t_{C_2}^{C_2}$ and $\phi(t_1^1x) = t_{C_2}^{C_2} + t_1^{C_2}r_1^{C_2}$. This gives an unitary automorphism of $\mu_k(C_2)$. By general results of Morita theory, the bimodule $\mu_k(C_2)_\phi$ induces a Morita self-equivalence of $\mu_k(C_2)$.

The projective indecomposable Mackey functors for C_2 are $B_{C_2/1}$ and B_{C_2/C_2} . As a module over the Mackey algebra, $B_{C_2/1}$ as basis: $t_1^{C_2}$, t_1^1 and t_1^1x , and B_{C_2/C_2} as basis: $t_{C_2}^{C_2}$, $t_1^{C_2}r_1^{C_2}$, $r_1^{C_2}$. So the Morita equivalence induced by ϕ exchanges the projective $B_{C_2/1}$ and B_{C_2/C_2} .

Since the Cartan matrix of $co\mu_k(C_2)$ is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, there is no self-Morita equivalence which exchanges the two projective indecomposable cohomological Mackey functors.

3.1 Principal block of p -nilpotent groups.

Lemma 3.1. *Let M and M' two be projective Mackey functors of $Mack_k(G, 1)$, and $f : M \rightarrow M'$ be a morphism. The morphism f is an isomorphism if and only if $f(1) : M(1) \rightarrow M'(1)$ is an isomorphism of kG -modules.*

Proof. Lemme 6.2 in [2] (in french). □

Theorem 3.2. *Let $G = N \rtimes P$ a p -nilpotent group, where P is a Sylow p -subgroup of G . Let b_0 be the principal block of kG . Then $\mu_k^1(b_0)$ -Mod is Morita equivalent to $\mu_k(P)$ -Mod.*

Remark 4. If b is a nilpotent block, for some $m \in \mathbb{N}$, we have an isomorphism $kGb \cong \text{Mat}(m, kP)$, where P is a defect group of the block b . This is not the case for the Mackey algebras. Example if $G = S_3$ and k is a field of characteristic 2. Let b be the principal block of kS_3 , one can check that $\dim_k(\mu_k(C_2)) = 6$ and $\dim_k(\mu_k^1(b)) = 56$.

Proof. Recall that the principal block idempotent of kG is $b_0 = \frac{1}{N} \sum_{n \in N} n$, and $kGb_0 \cong kP$. So there is a Morita equivalence $kGb_0\text{-Mod} \cong kP\text{-Mod}$. This equivalence is given by the following adjoint pair $(\text{Res}_P^G, b_0 \text{Ind}_P^G)$.

For the proof, we will work with the categories of Mackey functors with Dress definition and Green definition, since the following adjunction is easier to understand with the Dress definition, but the action of the block idempotent is easier to understand with the Green definition. We have two functors:

$$\text{Res}_P^G : \text{Mack}_k(b_0) \rightarrow \text{Mack}_k(P),$$

and

$$\text{Ind}_P^G : \text{Mack}_k(P) \rightarrow \text{Mack}_k(G).$$

Applying the block idempotent b_0^μ , we have a functor

$$b_0^\mu \text{Ind}_P^G : \text{Mack}_k(P) \rightarrow \text{Mack}_k(b_0).$$

1. The functor Res_P^G is left and right adjoint to the functor $b_0 \text{Ind}_P^G$, since it is the case for Ind_P^G and Res_P^G . Recall that the unit and co-unit of the above adjunction are given by:

$$\begin{aligned} N_M : M &\rightarrow b_0 \text{Ind}_P^G \text{Res}_P^G M \text{ is defined by } N_M = (b_0 M)^*(\epsilon_-) \\ E'_M : b_0 \text{Ind}_P^G \text{Res}_P^G(M) &\rightarrow M \text{ is defined by } E'_M = (b_0 M)_*(\epsilon_-) \\ N'_{M'} : M' &\rightarrow \text{Res}_P^G b_0 \text{Ind}_P^G M' \text{ is defined by } N'_{M'} = M'_*(\eta_-) \\ E_{M'} : \text{Res}_P^G b_0 \text{Ind}_P^G M' &\rightarrow M' \text{ is defined by } E_{M'} = M'^*(\eta_-). \end{aligned}$$

where ϵ and η are the unit and co-unit of the usual adjunction $(\text{Ind}_P^G, \text{Res}_P^G)$ for $\text{Res}_P^G : G\text{-set} \rightarrow P\text{-set}$, and $\text{Ind}_P^G : P\text{-set} \rightarrow G\text{-set}$.

2. Let M be a projective Mackey functor in $\text{Mack}_k(b_0)$, and let M' be a projective Mackey functor of $\text{Mack}_k(P)$. We need to check that N_M and E'_M above are inverse isomorphisms. Similarly, we have to check that $E_{M'}$ and $N'_{M'}$ are inverse isomorphisms.

By Lemma 3.1 it is enough to check that this is true after evaluation at the trivial subgroup. However, we have a natural isomorphism of kG -modules $(\text{Ind}_P^G M)(1) \cong \text{Ind}_P^G(M(1))$, so

$$\begin{aligned} N_M(1) &: M(1) \rightarrow (b_0 \text{Ind}_P^G \text{Res}_P^G(M))(1) \cong b_0 \text{Ind}_P^G \text{Res}_P^G(M(1)) \\ E'_M(1) &: (b_0 \text{Ind}_P^G \text{Res}_P^G(M))(1) \cong b_0 \text{Ind}_P^G \text{Res}_P^G(M(1)) \rightarrow M(1) \\ N'_{M'}(1) &: M'(1) \rightarrow (\text{Res}_P^G b_0 \text{Ind}_P^G M')(1) \cong \text{Res}_P^G b_0 \text{Ind}_P^G M'(1). \\ E_{M'}(1) &: (\text{Res}_P^G b_0 \text{Ind}_P^G M')(1) \cong \text{Res}_P^G b_0 \text{Ind}_P^G(M'(1)) \rightarrow M'(1) \end{aligned}$$

are units and co-units of the adjunction $(b_0 \text{Ind}_P^G, \text{Res}_P^G)$ for modules over the group algebras. We have the required isomorphisms.

3. If $M \in \text{Mack}_k(b_0)$, let P_\bullet be a projective resolution of M in $\text{Mack}_k(b_0)$, then we have the following commutative diagram,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow N_{P_1} & & \downarrow N_{P_0} & & \downarrow N_M & & \\ \cdots & \longrightarrow & b_0 \text{Ind}_P^G \text{Res}_P^G(P_1) & \longrightarrow & b_0 \text{Ind}_P^G \text{Res}_P^G(P_0) & \longrightarrow & b_0 \text{Ind}_P^G \text{Res}_P^G(M) & \longrightarrow & 0 \end{array}$$

Since the N_{P_i} for $i \geq 0$ are isomorphisms N_M is an isomorphism. By the same method, if $M' \in \text{Mack}_k(P)$, $E_{M'}$ is an isomorphism. \square

Corollary 3.3. *There is an isomorphism of algebras, $\mu_k^1(b_0) \cong B(X^2)$ for the P -set $X \cong \text{Iso}_{G/N}^P \text{Def}_{G/N}^G \Omega_G$, and $B(X^2)$ is the evaluation of the Burnside functor at X^2 (see example 2).*

Proof. By Theorem 3.2, we have an equivalence $\mu_k^1(b_0)\text{-Mod} \cong \mu_k(P)\text{-Mod}$, so by Morita Theorem, there is an algebra isomorphism $\mu_k^1(b_0) \cong \text{End}_{\mu_k(P)}(T)$, where T is the bimodule $\text{Res}_P^G(\mu_k^1(b_0))$. We will denote by B_0 the Mackey functor, in the sense of Dress, which corresponds to the $\mu_k^1(b_0)$ -module $\mu_k^1(b_0)$. Since the projective Mackey functors of $\text{Mack}_k(P)$ are exactly the Dress constructions B_X of the Burnside functor where X is a P -set, we have

$$\text{Res}_P^G(B_0) \cong B_X,$$

for some P -set X . In particular $kX \cong \text{Res}_P^G(B_0)(1)$. But $\text{Res}_P^G(B_0)(1) = b_0 t_1^1 \mu_k^1(G)$. A basis of $\mu_k^1(G)$ is given by $t_B^A x R_{B^x}^C$, where A and C are subgroups of G , the elements $x \in [A \backslash G / C]$ and B is a p -subgroup of $A \cap^x C$ up to $A \cap^x C$ -conjugacy. So a basis of $t_1^1 \mu_k^1(G)$ is the set of $t_1^1 x r_1^H$ for $x \in G$ and $H \leq G$. Set $\gamma_{H,x} = t_1^1 b_0 x R_1^H$. We have that $\gamma_{H,nx} = \gamma_{H,x}$ and $\gamma_{H,xh} = \gamma_{H,x}$ for $x \in G$, $n \in N$ and $h \in H$. The set $\{\gamma_{H,x}; H \leq G, x \in G/NH\}$ is a $\mu_k(P)$ -basis of $t_1^1 \mu_k^1(b_0)$. The action of $y \in P$ on a element $\gamma_{H,x}$ is given by $y \cdot \gamma_{H,x} = \gamma_{H,yx}$. So,

$$kX \cong \bigoplus_{H \leq G} \text{Res}_P^G(kG/NH),$$

but for a p -group P , two permutation modules are isomorphic if and only if the corresponding P -sets are isomorphic by [7]. Hence

$$\begin{aligned} X &\cong \sqcup_{H \leq G} \text{Res}_P^G(G/NH) \\ &\cong \sqcup_{H \leq G} \text{Res}_P^G \text{Ind}_{G/N}^G \text{Def}_{G/N}^G(G/H) \\ &\cong \text{Res}_P^G(\text{Inf}_{G/N}^G \text{Def}_{G/N}^G(\Omega_G)) \\ &\cong \text{Iso}_{G/N}^P \text{Def}_{G/N}^G(\Omega_G). \end{aligned}$$

So $\text{Res}_P^G(B_0) \cong B_X$, where $X = \text{Iso}_{G/N}^P \text{Def}_{G/N}^G(\Omega_G)$, and

$$\mu_k^1(b_0) \cong \text{End}_{\text{Mack}_k(P)}(B_X) \cong B(X^2),$$

by adjunction property of B_X see [1]. □

This can be viewed as an analogue of the isomorphism $kGb \cong \text{Mat}(n, kP)$ for nilpotent blocks.

3.2 Groups with Sylow p -subgroup of order p .

For general results about equivalences between blocks of p -local Mackey algebras, we have the following result, which is a direct corollary of Theorem 20.10 of [18].

Theorem 3.4. *Let G and H two finite groups. let b be a block of kG with cyclic defect group of order p , and c be a block of kH with cyclic defect group of order p . Then,*

$$D^b(kGb\text{-Mod}) \cong D^b(kHc\text{-Mod}) \text{ if and only if } D^b(\mu_k^1(b)\text{-Mod}) \cong D^b(\mu_k^1(b')\text{-Mod}).$$

Proof. By Theorem 20.10 of [18], in this situation, the blocks of Mackey algebras are Brauer tree algebras. Let T_{Mack} be the tree of this algebra. And T_{Mod} be the tree of the corresponding block of the group algebra. The tree T_{Mod} is isomorphic to a subtree of T_{Mack} , still denoted by T_{Mod} . The tree T_{Mack} is determined by the knowledge of the tree T_{Mod} . If e is the number of edges of T_{Mod} , then the number of edges of T_{Mack} is $2e$. The exceptional vertex of T_{Mack} is the same as the exceptional vertex of the tree of the block, and with same multiplicity. Each edge of T_{Mack} which is not in T_{Mod} is a twig. By general results of derived equivalences for Brauer tree algebras, two Brauer trees algebras, over the same field, are derived equivalent if and only if they have the same number of edges. □

Even if the tree T_{Mack} seems to be determined by the group algebra kG , if two blocks of group algebras are Morita equivalent, it is not always true that the corresponding blocks for the Mackey algebras are Morita equivalent (see Example 5). The tree T_{Mack} is in fact determined by the corresponding block of kG and the Brauer correspondent of this block in $N_G(P)$ where P is a Sylow p -subgroup of G .

Proposition 3.5. *Let G and H be two finite groups with same p -local structure and common Sylow p -subgroup C of order p . Let b (resp. c) be a block of kG (resp. kH) of defect group C . If $kGb\text{-Mod} \cong kHc\text{-Mod}$ by a splendid bimodule M , then $\mu_k^1(e)\text{-Mod} \cong \mu_k^1(c)\text{-Mod}$.*

Proof. By Theorem 20.10 of [18] and Theorem 3.4, the block algebras $\mu_k^1(e)$ and $\mu_k^1(f)$ are derived equivalent Brauer tree algebras. Since two such algebras are Morita equivalent if and only if they have isomorphic trees and same exceptional multiplicity, it is enough too prove that they have the same Cartan matrices. We will prove that the decomposition matrices of $\mu_{\mathcal{O}}^1(b)$ and $\mu_{\mathcal{O}}^1(c)$ are the same. By Proposition 1.13, the decomposition matrices of $\mu_{\mathcal{O}}^1(b)$ can be computed from the knowledge of the p -block $\mathcal{O}Gb$ and the Brauer correspondent of b in $\mathcal{O}\overline{N}_G(C)$.

Suppose that there are e simple kG -modules in the block b of kG , then there are $e + 1$ simple KG -modules in this block, one of this simple module may be exceptional. The number of simple $k\overline{N}_G(C)$ -modules W in a block b' which is the Brauer correspondent of \bar{b} is e . Since $\overline{N}_G(C)$ is a p' -group, each simple $k\overline{N}_G(C)$ -module gives rise to a unique simple $K\overline{N}_G(C)$ -module. Thus the decomposition matrix of $\mu_{\mathcal{O}}^1(b)$ is the following block matrix with $2e + 1$ columns and $2e$ rows:

$$\begin{pmatrix} & 0_{e \times e} \\ D(\text{co}\mu_{\mathcal{O}}(b)) & \\ & Id_{e \times e} \end{pmatrix}$$

Where $D(\text{co}\mu_{\mathcal{O}}(b))$ is the decomposition matrix of $\text{co}\mu_{\mathcal{O}}(b)$.

So if two blocks kGb and kHc , with cyclic defect group of order p are splendidly Morita equivalent, the blocks $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are splendidly Morita equivalent by Section 5 of [15]. By the results of Section 2, the blocks of cohomological algebras $\text{co}\mu_{\mathcal{O}}(b)$ and $\text{co}\mu_{\mathcal{O}}(c)$ are Morita equivalent, so the Cartan matrices of $\mu_k^1(b)$ and $\mu_k^1(c)$ are the same. \square

Remark 5. Let (K, \mathcal{O}, k) a p -modular system. Let G be a finite group and b be a block of $\mathcal{O}G$ with cyclic defect group. The Mackey algebra of this block is a Brauer tree algebra, so there is a Green walk on this tree. One can lift this Green walk for $\mu_{\mathcal{O}}^1(b)$ exactly as Green did in [10]. This show that the Mackey algebra over \mathcal{O} is a Green order in the sense of [16]. König and Zimmermann in [11] proved that two Green orders with trees having same number of vertices and same exceptional vertex plus some local properties are derived equivalent. However, in this case it doesn't seems easy to check these local conditions.

Example 5. Let $G = SL_2(\mathbb{F}_3) \cong Q_8 \rtimes C_3$, and k be a field of characteristic 3. The group G is 3-nilpotent. Let b the block idempotent such that the block kGb contains the simple kG -module W of dimension 2. Then $kGb\text{-Mod} \cong kC_3\text{-Mod}$.

One can ask if the same happens to the corresponding blocks of the Mackey algebras. But the Cartan matrix of $\mu_k(C_3)$ is $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and the Cartan matrix of $\mu_k^1(b)$ is $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, so there is no Morita equivalence between these two algebras. By Theorem 3.4 they are derived equivalent. In fact in this example it is not difficult to make everything explicit: the projective indecomposable Mackey functors in the block b^μ are in bijection with the indecomposable p -permutation modules of the corresponding blocks. These indecomposable p -permutation modules are: $Ind_{Q_8}^G(W)$ and $Ind_{C_6}^G Inf_{C_2}^{C_6} \epsilon$ where $C_6 \cong Z(Q_8) \times C_3$, and ϵ is the non trivial simple kC_2 -module. So the projective indecomposable Mackey functors are $P = Ind_{Q_8}^G FP_W$ and $Q = Ind_{C_6}^G B_\epsilon$, where B_ϵ is a direct summand of the Dress construction of the Burnside functor for C_6 : B_{C_6/C_3} .

More precisely, for each C_6 -sets, if Y is any finite C_6 -set, then $B_{C_6/C_3}(Y) = B(Y \times C_6/C_3)$ is a right kC_2 -module, since

$$C_6/C_3 \cong End_{C_6\text{-set}}(C_6/C_3) \mapsto End_{C_6\text{-set}}(Y \times C_6/C_3).$$

So $B_\epsilon(Y) = B(Y \times C_6/C_3) \otimes_{k(C_6/C_3)} \epsilon$. With this it is not difficult to compute the Cartan matrix of the block. Using the adjunction property of FP_- , all the coefficients of this matrix, except the coefficient $dim_k Hom_{Mack_k(G)}(Q, Q)$, can be computed at the level of modules for group algebras. The last one can be computed using the formula of Proposition 5.11 in [2].

The derived equivalence can be given by a two terms complex:

$$0 \longrightarrow P^2 \xrightarrow{(\pi, 0)} Q \longrightarrow 0$$

where π is the morphism of maximal rank between P and Q .

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