

# ELLIS ENVELOPING SEMIGROUP FOR ALMOST CANONICAL MODEL SETS

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ABSTRACT. This work deals with certain point patterns of an Euclidean space, for which the calculation of the Ellis enveloping semigroup of their associated dynamical systems is performed. The algebraic structure and the topology of the Ellis semigroup, as well as its action on the underlying space, are explicitly described. The present work is illustrated with the treatment of the vertex pattern of the so-called Amman-Beenker tiling of the plane.

## INTRODUCTION

This article proposes to study certain aspects of dynamical systems which are associated with *point patterns* of an Euclidean space. The topic of point patterns takes its origins in symbolic dynamic, and concerns also aperiodic tillings. It has however been increasingly addressed by numerous authors since these past thirty years after the discovery by Schetchmann et al. of physical materials now commonly called *Quasicrystals*. In this context a point pattern of an Euclidean space  $\mathbb{R}^d$  is thought as an alloy, where points are understood as positions of atoms, or molecules or electrons, and the quasicrystalline structure then comes when a certain *long range order* is observed on the disposition of points within the pattern.

A great success in the topic of point patterns is the possibility to handle a pattern  $\Lambda_0$  of  $\mathbb{R}^d$  by considering the *dynamical system* associated to it. It consists of a space  $\mathbb{X}_{\Lambda_0}$  called the *hull* of  $\Lambda_0$ , which is formed of all other point patterns locally looking as  $\Lambda_0$  and endowed with a suitable compact topology, together with an action of the space  $\mathbb{R}^d$  by homeomorphisms. Natural properties of a pattern which are of geometric, combinatoric and/or statistical nature are then displayed by topological, dynamical and/or ergodic features on this dynamical system. This is particularly true for long range order on point patterns, where the counterpart seems to rely on the existence of *eigenfunctions* for the associated dynamical system. For instance, within the class of substitutive point patterns the *Meyer property*, which is a strong form of internal order [23], is equivalent to the existence of a non-trivial eigenfunction for the associated dynamical system [22]. This type of statement also exists outside the realm of substitution patterns [17]. Another example concerns the subclass of *regular model sets*, which can be viewed as the most ordered aperiodic patterns, where the property to be a regular model set is equivalent, among patterns admitting the Meyer property, with the property that continuous eigenfunctions forms a separating family for a full measure subset of the associated hull [4] [21]. A third striking result is that *pure point diffractivity* of a pattern [15], with its truly of statistical nature [25], is known to be equivalent to the existence of a basis of eigenfunctions for the Hilbert space provided by the hull together with a certain ergodic measure (there

is a widely developed literature about this aspect of patterns, see for instance [20] [3] and references therein). These statements have been shown under various mild assumptions on the considered pattern.

A certain form of this eigenvalue problem for a point pattern can be addressed, on a topological point of view, by the knowledge of the *Ellis enveloping semigroup* of its dynamical system  $(\mathbb{X}, \mathbb{R}^d)$ . This semigroup has been introduced for dynamical systems by Ellis and Gottschalk [7] as a way to study actions of a group on a compact space from an algebraic point of view. In a series of papers, Glasner investigated this semigroup for fairly general dynamical systems (see the review [11] and references therein), and him together with Megrelishvili showed in [13] that a dichotomy occurs on the Ellis semigroup  $E(\mathbb{X}, \mathbb{R}^d)$ : It is either sequentially compact or contains a topological copy of  $\beta\mathbb{N}$  the *Stone – Čech* compactification of the integers. The former situation admits several equivalent formulations and when it occurs the underlying dynamical system is called *tame* [10]. Tame systems are dynamically simple: Indeed it is proved in [12] that they are uniquely ergodic, almost automorphic and measurably conjugated with a Kronecker system. In the vocabulary of eigenfunctions this means that continuous eigenfunctions of the system separates a residual subset of full measure in the underlying compact space. In the topic of point patterns this just asserts that, following the characterization of model sets provided in [4], a point pattern admitting the Meyer property and with a tame dynamical system must be a regular model set.

In this work we propose to provide a qualitative description of the Ellis semigroup of dynamical systems associated with particular point patterns, the *almost canonical model sets*. These particular patterns are relevant in the crystallographic sense, as well as very accessible mathematically: One can get a complete picture of the hull  $\mathbb{X}_{\Lambda_0}$  of such patterns [19], as well as there associated  $C^*$ -Algebras (a recent source is [27], see references therein), and also perform the computation of their cohomology and K-theory groups [8] [9] [27] as well as the asymptotic exponent of their complexity function [16]. We show that in our situation it is possible to completely describe elements of the Ellis semigroup, their action onto the underlying space, as well as the algebraic and topological structure of this semigroup. The type of calculation made here can be compared with the calculation performed in [26] about Sturmian and Sturmian-like systems (see also example 4.5 of [11]). We also show that for those dynamical systems the Ellis semigroup is of first class on the sense of the dichotomy of [13], that is, almost canonical model sets have tame systems.

## PRESENTATION OF THE CONTENT

In order to construct a model set of  $\mathbb{R}^d$ , one begin by considering a higher dimensional Euclidean space  $\mathbb{R}^{n+d}$  together with a lattice  $\Sigma$  in it, a well as an embedded  $d$ -dimensional slope, usually placed in an 'irrational' manner, which is thought as the space  $\mathbb{R}^d$  itself. Such an environment used to construct a model set is called a *cut & project scheme*. Then a second step is to consider a suitable region  $W$  of the Euclidean subspace  $\mathbb{R}^n$  orthogonal to  $\mathbb{R}^d$ . The model set in question thereby emerges as the orthogonal projection in  $\mathbb{R}^d$  of certain points of  $\Sigma$ , namely those with orthogonal projection in  $\mathbb{R}^n$  falling into the region  $W$ . Mathematically a model set thus writes

$$\Lambda_0 := \left\{ \gamma^\parallel \mid \gamma \in \Sigma \text{ and } \gamma^\perp \in W \right\}$$

where  $\parallel$  and  $\perp$  denotes the orthogonal projections onto  $\mathbb{R}^d$  and  $\mathbb{R}^n$  respectively. In the above context we will speak about a *real model set* (see the discussion of section 1), the word "real" resulting from the fact that the summand  $\mathbb{R}^n$  used here to form the cut & project scheme is an Euclidean space.

The dynamical system  $(\mathbb{X}, \mathbb{R}^d)$  associated with a real model set is of very particular form: It is an *almost automorphic* extension over a torus  $\mathbb{T}^{n+d} := \mathbb{R}^{n+d} / \Sigma$  (see the material of section 2). This property will prove to be central in our task, and shows up with the consideration of a certain factor map, also known as the *parametrization map* [2] [30]

$$\mathbb{X} \xrightarrow{\pi} \mathbb{T}^{n+d}$$

This mapping also demonstrate that any pattern  $\Lambda$  in the hull  $\mathbb{X}$  of a model set  $\Lambda_0$  is in also a model set, such that if it admits  $[w, t]_{\Sigma} \in \mathbb{R}^{n+d} / \Sigma = \mathbb{T}^{n+d}$  as image then  $\Lambda$  is determined, as model set, by the region  $W + w$  in  $\mathbb{R}^n$ , next translated by the vector  $t$  in  $\mathbb{R}^d$ . This is described in better details in first section of the main text.

The first step in determining the Ellis semigroup  $E(\mathbb{X}, \mathbb{R}^d)$  is to describe it as a *suspension* of another (simplier) semigroup (see section 3). To that end we let  $\Gamma$  be the subgroup of  $\mathbb{R}^d$  obtained as orthogonal projection of the lattice  $\Sigma$  used to construct  $\Lambda_0$  as a model set.  $\Gamma$  is not a lattice of  $\mathbb{R}^d$  unless  $\Lambda_0$  itself is a lattice, and will often be dense in  $\mathbb{R}^d$ , although it is always finitely generated. We now consider the collection  $\Xi^{\Gamma}$  of point patterns in  $\mathbb{X}$  which are contained, as subsets of  $\mathbb{R}^d$ , into  $\Gamma$ . This subset of  $\mathbb{X}$  remains stable under the action of any vector of  $\mathbb{R}^d$  which lies in  $\Gamma$ , and when endowed with a suitable topology it gives rise to a new dynamical system  $(\Xi^{\Gamma}, \Gamma)$ . We call this latter the *subsystem* associated with  $\Lambda_0$ . The space  $\Xi^{\Gamma}$  will have a locally compact totally disconnected topology, and as a result its Ellis enveloping semigroup  $E(\Xi^{\Gamma}, \Gamma)$  will be a locally compact totally disconnected topological space (for Ellis semigroup of dynamical systems over locally compact space see section 2). The importance of this semigroup is our setting justified by theorem 3.6, which yields a algebraic isomorphism and homeomorphism

$$E(\mathbb{X}, \mathbb{R}^d) \simeq E(\Xi^{\Gamma}, \Gamma) \times_{\Gamma} \mathbb{R}^d$$

where the right hand term is understood as the quotient of  $E(\Xi^{\Gamma}, \Gamma) \times \mathbb{R}^d$  under a natural diagonal action of  $\Gamma$ . This theorem shows in particular that the Ellis semigroup  $E(\mathbb{X}, \mathbb{R}^d)$  is in our context a *matchbox manifold*: It is locally the product of an Euclidean open subset with a totally disconnected space. It also asserts that the non-trivial (and in particular the non-commutative) part of  $E(\mathbb{X}, \mathbb{R}^d)$  is displayed by the semigroup  $E(\Xi^{\Gamma}, \Gamma)$ .

We will thus from now on focus on the calculation of  $E(\Xi^{\Gamma}, \Gamma)$ . At first, we show the existence of an onto continuous semigroup morphism

$$E(\Xi^{\Gamma}, \Gamma) \xrightarrow{\Pi^*} \mathbb{R}^n$$

This morphism is closely related with the parametrization map presented above, and will allows us to understand the convergence of a net in  $E(\Xi^{\Gamma}, \Gamma)$  by how the corresponding net, through this morphism, converge in  $\mathbb{R}^n$ .

Our wish know is to find a certain semigroup  $S$ , together with a certain semigroup morphism from  $E(\Xi^{\Gamma}, \Gamma)$  into  $S$ , such that the Ellis semigroup  $E(\Xi^{\Gamma}, \Gamma)$  embeds in the direct product  $S \times \mathbb{R}^n$ . In order to simplify the problem we let the *almost canonical*

*property* enter the game. This property consist of a condition on the region  $W$  used to obtain  $\Lambda_0$  as model set, that is,  $W$  must be a polytope of  $\mathbb{R}^n$  satisfying a particular condition (see section 4). With the assumption of almost canonicity for the region  $W$  together with the almost automorphic property observed on the dynamical system  $(\mathbb{X}_{\Lambda_0}, \mathbb{R}^d)$ , we are able to identify the correct semigroup  $S$  as the *face semigroup* associated with the polytope  $W$  in  $\mathbb{R}^n$  (see sections 5 and 6 for presentation and results).

We may shortly present the face semigroup  $\mathfrak{T}_W$  associated with the polytope  $W$  in  $\mathbb{R}^n$  as follows: The polytope  $W$  determines a finite collection of linear hyperplanes  $\mathfrak{H}_W$  in  $\mathbb{R}^n$ , namely the ones which are parallel to at least one face of  $W$ . This collection in turns determine a stratification of  $\mathbb{R}^n$  by cones, all being, for each hyperplane  $H \in \mathfrak{H}_W$ , included into  $H$  or integrally part of one of the two possible complementary half spaces. An illustration of this construction is provided in section 7, where  $W$  is there a regular octagon of  $\mathbb{R}^2$ . Now the face semigroup  $\mathfrak{T}_W$  is set-theoretically the finite collection of cones resulting from this stratification process, together with a (non-commutative) semigroup product stating that the product  $C.C'$  of two cones is the cone where the head of  $C$  enters after being translated by small vectors of  $C'$ . The elements of  $\mathfrak{T}_W$  are more conveniently described as "side maps", which consist of mappings from  $\mathfrak{H}_W$  to the three symbols set  $\{-, 0, +\}$ , giving the relative position of any cone with respect to each hyperplane. This formalism has the advantage to allows for a concise and handy formulation of the product law on this semigroup (see section 6).

The embedding morphism

$$E(\Xi^\Gamma, \Gamma) \longrightarrow \mathfrak{H}_W \times \mathbb{R}^n$$

is made from the observation that a neighborhood basis of any transformation  $g \in E(\Xi^\Gamma, \Gamma)$  is provided by the vector  $w_g := \Pi^*(g)$  of  $\mathbb{R}^n$  together with a certain cone  $C_g \in \mathfrak{T}_W$ , in the sense that a net in  $\Gamma$  converges to  $g$  in the Ellis semigroup  $E(\Xi^\Gamma, \Gamma)$  (such a net exists by construction) if and only if the corresponding net in  $\mathbb{R}^n$  converges to  $w_g$  and eventually lies into  $C_g + w_g$ . In this sense the cone  $C_g$  provides the direction a net must follow in order to converge to the transformation  $g$ . This allows us to calculate the corresponding image subsemigroup in  $\mathfrak{H}_W \times \mathbb{R}^n$ , which is the aim of theorem 6.3, proved to be a finite disjoint union of *subgroups* of  $\mathbb{R}^n$ . Moreover the topology of  $E(\Xi^\Gamma, \Gamma)$  is here completely described by a geometric criterion of convergence for nets.

Finally, we fusion theorems 3.6 and 6.3 to formulate our main theorem (see the statement 7.1), setting the existence of an embedding semigroup morphism

$$E(\mathbb{X}, \mathbb{R}^d) \longrightarrow \mathfrak{H}_W \times \mathbb{T}^{n+d}$$

for which the image subsemigroup together with its topology are identified. Interestingly, this semigroup remains exactly the same for model sets issued after translating, dilating, or deforming the region  $W$  as long as the hyperplanes determined by the faces are unchanged. As a byproduct of the previous analysis we show that the topology of the Ellis semigroup  $E(\mathbb{X}, \mathbb{R}^d)$  admits a first countable topology, and thus is sequentially compact. We conclude this work by determining some algebraic features of this Ellis semigroup, as well as a picture of its underlying action on the space  $\mathbb{X}$ .

## 1. MODEL SETS AND ASSOCIATED DYNAMICAL SYSTEMS

1.1. **General definition of inter model set.** In order to define what a (almost canonical) model set is in  $\mathbb{R}^d$  (see [30] as well as [24] for a more detailed exposition), we consider first an environment used to construct it, namely a *cut & project scheme*. It consists of a data  $(\mathbf{H}, \Sigma, \mathbb{R}^d)$  with associated diagram

$$\begin{array}{ccccc} \mathbf{H} & \longleftarrow & \mathbf{H} \times \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ \cup & & \cup & & \cup \\ \Gamma^* & \longleftarrow & \Sigma & \longleftrightarrow & \Gamma \end{array}$$

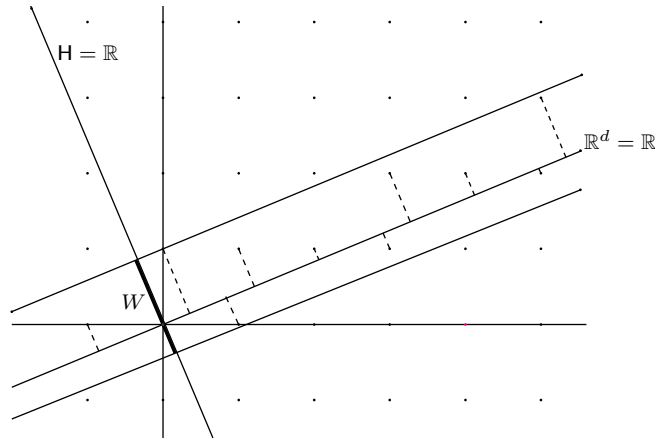
where  $\mathbf{H}$  is a locally compact Abelian group, with:

- $\Sigma$  is a countable lattice of  $\mathbf{H} \times \mathbb{R}^d$ , that is, a countable discrete and co-compact subgroup.
- the canonical projection  $\pi_{\mathbb{R}^d}$  onto  $\mathbb{R}^d$  is bijective from  $\Sigma$  onto its image  $\Gamma$ .
- the image  $\Gamma^*$  of  $\Sigma$  under the canonical projection  $\pi_{\mathbf{H}}$  is a dense subgroup of  $\mathbf{H}$ .

Hence such an environment consists of an Euclidean space  $\mathbb{R}^d$  embedded into  $\mathbf{H} \times \mathbb{R}^d$  in an 'irrational position' with respect to the lattice  $\Sigma$ . There is a well established formalism for these different ingredients: the space  $\mathbb{R}^d$  is often called the *physical space* whereas the space  $\mathbf{H}$  is called the *internal space*. Moreover the morphism  $\Gamma \rightarrow \mathbf{H}$  which maps any  $\gamma$  onto  $\gamma^* := \pi_{\mathbf{H}}(\pi_{\mathbb{R}^d}^{-1}(\gamma)) \in \Gamma^*$  is the *\*-map* of the cut & project scheme, whose graph is the lattice  $\Sigma$ . We will say that a cut & project scheme is *real* whenever the internal space  $\mathbf{H}$  is a finite dimensional real vector space  $\mathbb{R}^n$ .

We shall in addition consider a certain type of subset in the internal space  $\mathbf{H}$ , usually called a *window*, which consists of a compact and topologically regular subset  $W$ , supposed *irredundant* in the sense that the compact subgroup of elements of  $w \in \mathbf{H}$  which satisfy  $W + w = W$  is trivial.

Now if we are given a cut & project scheme together with a window  $W$  in its internal space, we may form a certain point pattern  $\mathfrak{P}(W)$  of  $\mathbb{R}^d$  by projecting into the physical space the subset of points of the lattice  $\Sigma$  lying within the *strip*  $W \times \mathbb{R}^d$ . To illustrate this we consider the following picture (see also [2])



The above picture presents the most simple real cut & project scheme one may consider, that is, with physical and internal spaces being 1-dimensional, and with a

lattice  $\Sigma = \mathbb{Z}^2$  not crossing these spaces except at the origin. As window we consider the projection into the internal space of the unit square in  $\mathbb{R}^2$ .

The point pattern  $\mathfrak{P}(W)$  may be written using the  $*$ -map in the following form

$$\mathfrak{P}(W) := \{\gamma \in \Gamma \mid \gamma^* \in W\}$$

We may allow ourselves to translate the resulting point pattern by any vector  $t \in \mathbb{R}^d$  site by site in the physical space, which we call here *physical translation*, or translate the window  $W$  by an element  $w \in \mathbf{H}$ , which we call *internal translation*. In both cases this leads to a new point pattern of  $\mathbb{R}^d$ . We now introduce the class of *model sets* of  $\mathbb{R}^d$  as follows:

**Definition 1.1.** *An inter model set  $\Lambda$  associated with a cut & project scheme  $(\mathbf{H}, \Sigma, \mathbb{R}^d)$  together with a window  $W$  is a subset of  $\mathbb{R}^d$  of the form*

$$\mathfrak{P}(w + \overset{\circ}{W}) - t \subseteq \Lambda \subseteq \mathfrak{P}(w + W) - t$$

An inter model set is called *regular* whenever the window  $W$  used to construct it has boundary of Haar measure zero in  $\mathbf{H}$ . Due to the assumptions on the underlying cut & project scheme and on the window  $W$ , any inter model set is a *Delone set*, that is to say a uniformly discrete and relatively dense subset of  $\mathbb{R}^d$ . In fact, it also admits the stronger property of being a *Meyer set*, meaning that any inter model set  $\Lambda$  admits a uniformly discrete difference subset  $\Lambda - \Lambda$  in  $\mathbb{R}^d$ . Most of the content of this article is about real cut & project schemes together with polytopal windows in their internal spaces, hence providing inter model sets which are regular.

**1.2. Non-singular model sets.** An important notion affiliated with a point pattern  $\Lambda$  is its *language*, namely the collection of all 'circular-shaped' patterns appearing at sites of the point pattern:

$$\mathcal{L}_\Lambda := \{(\Lambda - \gamma) \cap B(0, R) \mid \gamma \in \Lambda, R > 0\}$$

Not all inter model sets coming from a common cut & project scheme and window have same language. However, the class of *non-singular model sets*, also often called *generic model sets*, do share their language:

**Definition 1.2.** *A non-singular model set is an inter model set for which we have equalities*

$$\mathfrak{P}(w + \overset{\circ}{W}) - t = \Lambda = \mathfrak{P}(w + W) - t$$

The situation where such equality occurs for a given couple  $(w, t)$  clearly only depends on the choice of  $w \in \mathbf{H}$ . We will then call an element  $w \in \mathbf{H}$  non-singular when the inter model sets  $\mathfrak{P}(w + \overset{\circ}{W}) - t = \Lambda = \mathfrak{P}(w + W) - t$  are non-singular. Such a subset of non-singular elements may easily be described: it consists of all  $w \in \mathbf{H}$  where no point of the subgroup  $\Gamma^*$  of  $\mathbf{H}$  enters the boundary  $w + \partial W$  of the translated window  $w + W$ . It thus consists of the complementary subset

$$NS := [\Gamma^* - \partial W]^c$$

This set is always non-empty from the Baire category theorem, as  $W$  was assumed topologically regular, hence with boundary of empty interior in  $\mathbf{H}$ , and  $\Sigma$  (hence  $\Gamma^*$ ) was supposed to be countable. As already pointed out, the non-singular model sets

arising from a common cut & project scheme with window have a common language, which means that any pattern of some non-singular model set appears elsewhere in all other non-singular model sets. Denoting  $\mathcal{L}_{NS}$  this language, we are led to consider its associated *hull*

**Definition 1.3.** *Given a cut & project scheme and a window,  $\mathcal{L}_{NS}$  the language of any non-singular model set arising from this data, the hull of this data is the collection*

$$\mathbb{X} := \left\{ \Lambda \text{ point pattern of } \mathbb{R}^d \mid \mathcal{L}_\Lambda = \mathcal{L}_{NS} \right\}$$

We call *model set* any point pattern within the hull  $\mathbb{X}$  associated with some cut & project scheme and window.

The hull  $\mathbb{X}$  associated with some cut & project scheme and window is also called the *local isomorphism class* (or simply *LI-class*) of any model set within this hull.

**1.3. The hull as dynamical system.** There is a natural topology on the hull  $\mathbb{X}$ , which is metrizable and may be described by setting a basis of open neighborhoods of any point pattern  $\Lambda \in \mathbb{X}$  (see for instance [24])

$$(1) \quad \mathcal{U}_{K,\varepsilon}(\Lambda) := \left\{ \Lambda' \in \mathbb{X} \mid \exists |t|, |t'| < \varepsilon, (\Lambda - t) \cap K = (\Lambda' - t') \cap K \right\}$$

where  $K$  is any compact set in  $\mathbb{R}^d$  and  $\varepsilon > 0$ . This topology roughly means that two point patterns are close if they agree on a large domain about the origin up to small shifts. Then the hull  $\mathbb{X}$  endowed with this topology is a compact metrizable space, and is endowed with a natural action of  $\mathbb{R}^d$  given by  $\Lambda.t := \Lambda - t$ , that is, by shifting any point pattern site by site. This provides so a dynamical system  $(\mathbb{X}, \mathbb{R}^d)$ . In order to figure out what exactly consists this space, we invoke the following beautiful result:

**Theorem 1.4.** [30] *Let  $\mathbb{X}$  be the hull associated with a cut & project scheme  $(\mathbf{H}, \Sigma, \mathbb{R}^d)$  and some window. Then  $\mathbb{X}$  is compact and the dynamical system  $(\mathbb{X}, \mathbb{R}^d)$  is minimal. Each  $\Lambda \in \mathbb{X}$  satisfy inclusions of the form*

$$\mathfrak{P}(w_\Lambda + \mathring{W}) - t_\Lambda \subseteq \Lambda \subseteq \mathfrak{P}(w_\Lambda + W) - t_\Lambda$$

where  $(w_\Lambda, t_\Lambda) \in \mathbf{H} \times \mathbb{R}^d$  is unique up to an element of  $\Sigma$ , thus defining a factor map

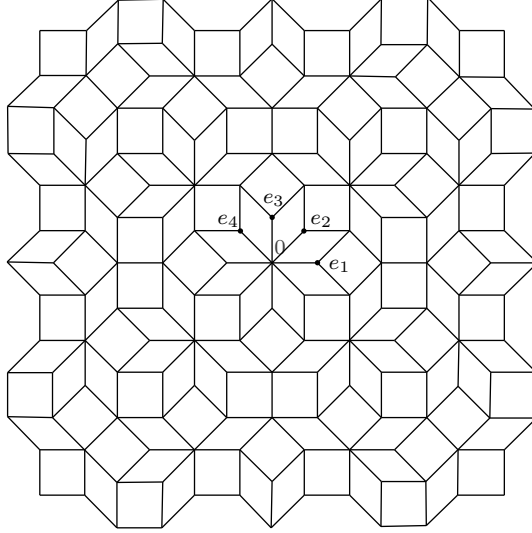
$$\pi : \mathbb{X} \longrightarrow \mathbf{H} \times_\Sigma \mathbb{R}^d$$

with  $\mathbf{H} \times_\Sigma \mathbb{R}^d$  the compact Abelian group quotient of  $\mathbf{H} \times \mathbb{R}^d$  by the lattice  $\Sigma$ .

The map  $\pi$  is injective precisely on the collection of non-singular model sets of  $\mathbb{X}$ .

By factor map we mean here a continuous, onto and  $\mathbb{R}^d$ -equivariant map, where on the compact Abelian group  $\mathbf{H} \times_\Sigma \mathbb{R}^d$  the space  $\mathbb{R}^d$  acts through  $[w, t]_{\Sigma}.s := [w, t + s]_{\Sigma}$ . In the context of real cut & project schemes the compact Abelian group is given by  $[\mathbb{R}^{n+d}]_{\Sigma}$ , that is, a  $n + d$ -torus. In the topic of point patterns the above factor map is called the *parametrization map*, and shows in particular that any model set of  $\mathbb{X}$  is an inter model set in the sense of definition 1.1. In fact, the collection  $\mathbb{X}$  of model sets of a given cut & project scheme and window is precisely the collection of *repetitive* inter model sets arising from this data (see for instance [30]).

1.4. **An explicit example.** A well-known example of model set is the vertex point pattern of the famous *Ammann-Beenker tiling*, from which an uncolored local pattern about the origin shows up as



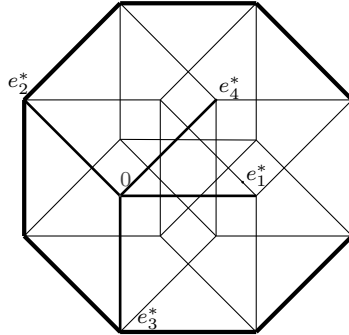
We can set a cut & project scheme and window giving rise to the desired point pattern as follows: We set in a physical space  $\mathbb{R}^2$  the group  $\Gamma$  algebraically generated by four vectors whose coordinates in an orthonormal basis read

$$e_1 = (1, 0) \quad e_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad e_3 = (0, 1) \quad e_4 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

These four vectors are algebraically independent, and thus  $\Gamma$  is isomorphic with  $\mathbb{Z}^4$ . Next we set an internal space  $\mathbb{R}_{int}^2$  to be a 2-dimensional real vector space, into which we define a  $*$ -map through the images of the four above vectors, reading in some orthogonal basis of  $\mathbb{R}_{int}^2$

$$e_1^* = (1, 0) \quad e_2^* = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad e_3^* = (0, -1) \quad e_4^* = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

The four vectors  $\tilde{e}_i := (e_i, e_i^*)$ ,  $i = 1, 2, 3, 4$ , are linearly independant in  $\mathbb{R}_{int}^2 \times \mathbb{R}^2$  and thus form a lattice  $\Sigma$ , which project into  $\mathbb{R}_{int}^2$  into a dense subgroup  $\Gamma^*$ . This sets a real cut & project scheme. We chose the window to be canonical, that is, to be the projection into the internal space of the unit cube of  $\mathbb{R}_{int}^2 \times \mathbb{R}^2$  with respect to the basis  $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)$ . Hence we get a regular octagonal window  $W_{oct}$  of the form



Then the vertex point pattern appearing in the Ammann-Beenker tiling is given by the non-singular model set  $\mathfrak{P} \left( W_{oct} - \frac{e_1^* + e_2^* + e_3^* + e_4^*}{2} \right)$ .



## 2. ELLIS SEMIGROUPS OF DYNAMICAL SYSTEMS

**2.1. Ellis semigroup and equicontinuity.** Let us consider a compact dynamical system, that is, a compact (Hausdorff) space  $\mathbb{X}$  together with an action of a group  $T$  by homeomorphisms. Following [1] we let

**Definition 2.1.** *The Ellis semigroup  $E(\mathbb{X}, T)$  is the pointwise closure of the group of homeomorphisms given by the  $T$ -action in the space  $\mathbb{X}^{\mathbb{X}}$  of self-mappings on  $\mathbb{X}$ .*

The Ellis semigroup  $E(\mathbb{X}, T)$  is then a family of transformations on the space  $\mathbb{X}$  which are pointwise limits of homeomorphisms coming from the  $T$ -action, and is stable under composition. Moreover it is a compact (Hausdorff) space when endowed with the pointwise convergence topology coming from  $\mathbb{X}^{\mathbb{X}}$ . In case the acting group is Abelian then, although the Ellis semigroup may not be itself Abelian, all of its transformations commutes with any homeomorphism coming from the action.

The Ellis semigroup construction is functorial (covariant) in the sense that any onto continuous and  $T$ -equivariant mapping  $\pi : \mathbb{X} \rightarrow \mathbb{Y}$  gives rise to an onto continuous semigroup morphism  $\pi^* : E(\mathbb{X}, T) \rightarrow E(\mathbb{Y}, T)$ , satisfying  $\pi(x.g) = \pi(x).\pi^*(g)$  for any  $x \in \mathbb{X}$  and any transformation  $g \in E(\mathbb{X}, T)$ . Here we have written  $x.g$  for the evaluation of a mapping  $g$  at a point  $x$ . With this convention the Ellis semigroup is always a compact right-topological semigroup, that is, if some net  $(h_\lambda)$  converges pointwise to  $h$  then the net  $(g.h_\lambda)$  converges pointwise to  $g.h$  for any  $g$ , where  $g.h$  stands for the composition map which at each point  $x$  reads  $(x.g).h$ .

Among the whole category of dynamical systems, the certainly most simple objects are the *equicontinuous dynamical systems*. These are the dynamical systems such that the family of homeomorphisms coming from the group action is equicontinuous, and within the more specific class of compact minimal dynamical systems they exactly shows up as the well known class of *Kronecker systems*. About these particular dynamical systems one has the following:

**Theorem 2.2.** [1] [14] *Let  $(\mathbb{X}, T)$  be a minimal dynamical system over a compact metric space, with Abelian acting group. Then the following assertions are equivalents:*

- (1) *the dynamical system  $(\mathbb{X}, T)$  is equicontinuous.*
- (2)  *$E(\mathbb{X}, T)$  is a compact group acting by homeomorphisms on  $\mathbb{X}$ .*
- (3)  *$E(\mathbb{X}, T)$  is metrisable.*
- (4)  *$E(\mathbb{X}, T)$  has left-continuous product.*
- (5)  *$E(\mathbb{X}, T)$  is Abelian.*
- (6)  *$E(\mathbb{X}, T)$  is made of continuous transformations.*

*In this case one has  $E(\mathbb{X}, T) = \mathbb{X}$  as compact Abelian group.*

Here the compact Abelian group structure of a compact minimal equicontinuous system  $(\mathbb{X}, T)$  with Abelian acting group is only determined by the choice (which is arbitrary) of one element  $\mathfrak{e} \in \mathbb{X}$  which plays the role of unit, from which the group structure extends that of  $T$  mapped on the dense orbit  $\mathfrak{e}.T$ . In this case the equality  $E(\mathbb{X}, T) = \mathbb{X}$  is performed by identifying a transformation  $g \in E(\mathbb{X}, T)$  with  $\mathfrak{e}.g$  in  $\mathbb{X}$ .

Outside the scope of equicontinuous systems, The Ellis semigroup is a quite complicated object as it is formed of mappings neither necessarily continuous nor invertible, and is not commutative. However a general construction allows to attach to any compact dynamical system a particular factor:

**Theorem 2.3.** *Let  $(\mathbb{X}, T)$  be a compact dynamical system. There exist an equicontinuous dynamical system  $(\mathbb{X}_{eq}, T)$  together with a factor map  $\pi : \mathbb{X} \rightarrow \mathbb{X}_{eq}$  such that any equicontinuous factor of  $(\mathbb{X}, T)$  factors through  $\pi$ .*

The space  $\mathbb{X}_{eq}$  with  $T$ -action is called the *maximal equicontinuous factor* of  $(\mathbb{X}, T)$ , and is a Kronecker system whenever  $(\mathbb{X}, T)$  is topologically transitive. From theorem 2.2 one has  $E(\mathbb{X}, T) = \mathbb{X}_{eq}$  as compact groups, and from the functorial property of the Ellis semigroup the quotient factor map  $\pi$  from  $\mathbb{X}$  onto its maximal equicontinuous factor gives rise to an onto and continuous semigroup morphism

$$\pi^* : E(\mathbb{X}, T) \rightarrow \mathbb{X}_{eq}$$

**2.2. The tame property.** There exists among dynamical systems over compact metric spaces with jointly continuous action of a topological group, a dichotomy about their Ellis semigroups:

**Theorem 2.4.** [11] *The Ellis semigroup  $E(\mathbb{X}, T)$  of a dynamical system over a compact metric space is either sequentially compact or contains a topological copy of  $\beta\mathbb{N}$  the Stone – Čech compactification of the integers.*

The first alternative admits several different formulations (see [11] [14] and references therein) and whenever it occurs then the underlying dynamical system is called *tame*. Obviously if a compact metric dynamical system admits an Ellis semigroup with first countable topology then it is automatically a tame system. The tameness property is in fact very strong, as the next result shows:

**Theorem 2.5.** [12] *If a compact metric minimal dynamical system  $(\mathbb{X}, T)$  with  $T$  Abelian is tame then its factor map  $\pi : \mathbb{X} \rightarrow \mathbb{X}_{eq}$  is 1-to-1 above a full Haar measure subset  $\mathbb{X}_{eq}^0$  of the compact Abelian group  $\mathbb{X}_{eq}$ .*

Hence on the measure side these dynamical systems are endowed with a unique ergodic probability measure  $\mathfrak{m}$ , such that  $(\mathbb{X}, \mathfrak{m})$  identifies, through  $\pi$ , as measure space with  $\mathbb{X}_{eq}$  endowed with its Haar probability measure. On the topological side a tame dynamical system is more generally an *almost automorphic* dynamical system:

**Definition 2.6.** *A compact dynamical system  $(\mathbb{X}, T)$  is almost automorphic if the factor map  $\pi : \mathbb{X} \rightarrow \mathbb{X}_{eq}$  possess a of one-point fiber.*

In case of metrisability of the space  $\mathbb{X}$  a result of Veech [32] shows that any almost automorphic system has in fact a residual subset of one-point fibers with respect to the mapping  $\pi$ . In the situation of a hull  $\mathbb{X}$  of model sets the factor map  $\pi$  onto the maximal equicontinuous factor is precisely the parametrization map of theorem 1.4. This theorem also asserts that, as  $\pi$  is 1-to-1 on a non-empty subclass of  $\mathbb{X}$  (the non-singular model sets), the dynamical system  $(\mathbb{X}, \mathbb{R}^d)$  is almost automorphic. It

can even be shown [21] that a hull of model sets consists of regular model sets (meaning that the region  $W$  as its boundary of null Haar measure in  $\mathbf{H}$ ) if and only if the map  $\pi$  is 1-to-1 above a full Haar measure subset of  $\mathbb{X}_{eq}$ .

**2.3. Ellis semigroup for locally compact dynamical systems.** We wish to include here two elementary results about the Ellis semigroups one may define for dynamical systems over locally compact spaces. Let  $\mathbb{X}$  be a locally compact space together with an action of a group  $T$  by homeomorphisms, and as in the compact case, set the Ellis semigroup  $E(\mathbb{X}, T)$  to be the pointwise closure in the product space  $\mathbb{X}^{\mathbb{X}}$  of the group of homeomorphisms coming from the  $T$ -action. In order to extend some results available in the compact case to this setting we consider the one-point compactification  $\hat{\mathbb{X}}$  of  $\mathbb{X}$ , endowed with a  $T$ -action by homeomorphism so that the infinite point remains fixed through any such homeomorphism. Let us denote by  $\mathcal{F}_{\mathbb{X}}$  the subset of  $\hat{\mathbb{X}}^{\hat{\mathbb{X}}}$  of transformations mapping  $\mathbb{X}$  into itself and keep the point at infinity fixed, endowed with relative topology. Then  $\mathcal{F}_{\mathbb{X}}$  is a semigroup which is isomorphic and homeomorphic with the product space  $\mathbb{X}^{\mathbb{X}}$ , and under this identification

$$E(\mathbb{X}, T) = E(\hat{\mathbb{X}}, T) \cap \mathcal{F}_{\mathbb{X}}$$

Observe that  $E(\mathbb{X}, T)$  is, as in the compact flow case, a right-topological semigroup containing  $T$  as a dense subgroup (or rather the subsequent group of homeomorphisms). The following is a general fact, whose proof for compact dynamical systems can be found in [1]:

**Proposition 2.7.** *Let  $\pi : \mathbb{X} \rightarrow \mathbb{Y}$  be a continuous, proper, onto, and  $T$ -equivariant map between locally compact spaces. Then there exist a continuous, proper, and onto morphism  $\pi^* : E(\mathbb{X}, T) \rightarrow E(\mathbb{Y}, T)$  satisfying the equivariance condition:  $\pi(x.g) = \pi(x).\pi^*(g)$  for any  $x \in \mathbb{X}$  and  $g \in E(\mathbb{X}, T)$ .*

*Proof.* Denote by  $\star_{\mathbb{X}}$  and  $\star_{\mathbb{Y}}$  the respective points at infinity in the compactified spaces. Since  $\pi$  is continuous and proper, it extends to a continuous and onto map  $\hat{\pi} : \hat{\mathbb{X}} \rightarrow \hat{\mathbb{Y}}$ , such that  $\hat{\pi}^{-1}(\star_{\mathbb{Y}}) = \{\star_{\mathbb{X}}\}$ . Obviously  $\hat{\pi}$  is  $T$ -equivariant with respect to the extended  $T$ -actions. There exist then a continuous and onto morphism  $\hat{\pi}^* : E(\hat{\mathbb{X}}, T) \rightarrow E(\hat{\mathbb{Y}}, T)$ , satisfying the equivariance equality for any  $x \in \hat{\mathbb{X}}$  and  $g \in E(\hat{\mathbb{X}}, T)$ :  $\hat{\pi}(x.g) = \hat{\pi}(x).\hat{\pi}^*(g)$ . The later equivariance condition implies that a transformation  $g$  of  $E(\hat{\mathbb{X}}, T)$  lies into  $\mathcal{F}_{\mathbb{X}}$  if and only if  $\hat{\pi}^*(g)$  lies in  $\mathcal{F}_{\mathbb{Y}}$ : it follows that  $E(\mathbb{X}, T) = (\hat{\pi}^*)^{-1}(E(\mathbb{Y}, T))$ . Restricting the morphism on  $E(\mathbb{X}, T)$  gives the map, together with the onto property. Finally a compact set of  $E(\mathbb{Y}, T)$  has to be compact in  $E(\hat{\mathbb{Y}}, T)$  as it is easy to check, so have a compact inverse image in  $E(\hat{\mathbb{X}}, T)$  under  $\hat{\pi}^*$ . This latter is entirely included in  $E(\mathbb{X}, T)$ , so is compact for the relative topology on  $E(\mathbb{X}, T)$ . This gives the properness.  $\square$

Observe that  $\pi^*(t) = t$  holds for any  $t \in T$ . As in the compact setting, if the acting group  $T$  is Abelian then any induced homeomorphism commutes with any mapping in  $E(\mathbb{X}, T)$ . To end this paragraph we set without proof the following easy property on locally compact Kronecker systems:

**Proposition 2.8.** *If  $T$  is a dense subgroup of a locally compact Abelian group  $\mathbb{G}$ ,  $T$  acting by translation, then  $E(\mathbb{G}, T)$  is topologically isomorphic with  $\mathbb{G}$  for which any  $g \in \mathbb{G}$  is identified with its translation map in  $E(\mathbb{G}, T)$ .*

### 3. THE INTERNAL SYSTEM OF A HULL OF MODEL SETS

**3.1. Internal system.** What we introduce here is an analog, in the hull  $\mathbb{X}$ , of the internal space  $\mathbf{H}$  we may find in the compact Abelian group  $\mathbf{H} \times_{\Sigma} \mathbb{R}^d$  (the torus  $[\mathbb{R}^{n+d}]_{\Sigma}$  in case of a real cut & project scheme). We call this analog the *internal system* of a hull of model sets. The consideration of this particular space is not new (it appeared in [8] as well as in the formalism of  $C^*$ -algebras in [27]), although it is often not explicitly mentioned, and we set here the main aspects about this space.

**Definition 3.1.** *Let  $\mathbb{X}$  be the hull of model sets associated with a cut & project scheme  $(\mathbf{H}, \Sigma, \mathbb{R}^d)$  and a window  $W$ . Then its internal system is the subclass  $\Xi^{\Gamma}$  of point patterns which are supported on the structure group  $\Gamma$  in  $\mathbb{R}^d$ , that is,*

$$\Xi^{\Gamma} := \{\Lambda \in \mathbb{X} \mid \Lambda \subset \Gamma\}$$

According to theorem 1.4, any model set admits inclusions of the form stated in definition 1.1, and here as we can see the subclass  $\Xi^{\Gamma}$  exactly consists of the model sets for which these inclusions write

$$\mathfrak{P}(w + \overset{\circ}{W}) \subseteq \Lambda \subseteq \mathfrak{P}(w + W)$$

Equivalently,  $\Xi^{\Gamma}$  is the subclass of model sets in  $\mathbb{X}$  whose image under the parametrization map  $\pi$  of theorem 1.4 is of the form  $[w, 0]_{\Sigma}$  in the compact Abelian group  $\mathbf{H} \times_{\Sigma} \mathbb{R}^d$ . On the other hand, there exists a natural morphism mapping any element  $w$  of the internal space  $\mathbf{H}$  onto  $[w, 0]_{\Sigma}$  in  $\mathbf{H} \times_{\Sigma} \mathbb{R}^d$ , which is 1-to-1 and continuous. This suggests the existence of a mapping from the internal system  $\Xi^{\Gamma}$  onto the internal space  $\mathbf{H}$  of the cut & project scheme. However, similar to the fact that  $\mathbf{H}$  is in general not topologically conjugated with its image in  $\mathbf{H} \times_{\Sigma} \mathbb{R}^d$ , one cannot just consider the topology of  $\mathbb{X}$  induced on  $\Xi^{\Gamma}$ . Rather, we consider on the internal system the topology whose open neighborhood basis at any  $\Lambda \in \Xi^{\Gamma}$  is specified as

$$(2) \quad \mathcal{U}_K(\Lambda) := \{\Lambda' \in \Xi^{\Gamma} \mid \Lambda \cap K = \Lambda' \cap K\}$$

where  $K$  is any compact set in  $\mathbb{R}^d$ . This means that two point patterns are close in  $\Xi^{\Gamma}$  if they exactly match on a large domain about the origin. On the internal system equipped with the above topology, we consider the action of the group  $\Gamma$  by homeomorphisms given by translation site by site on each model set, so that one obtains a dynamical system  $(\Xi^{\Gamma}, \Gamma)$ . From the minimality of the dynamical system  $(\mathbb{X}, \mathbb{R}^d)$  by theorem 1.4, this dynamical system is also minimal. Of particular importance is the sub-collection, often called the *transversal*, of point patterns containing the origin

$$\Xi := \{\Lambda \in \mathbb{X} \mid 0 \in \Lambda \subset \Gamma\}$$

Any model set containing the origin must be entirely included in the structure group  $\Gamma$ , that is,  $\Xi$  is a subset of the internal system  $\Xi^{\Gamma}$ . A fact of fundamental importance is that  $\Xi$  is in fact a *clopen set*, that is, a subset which is both open and closed in the internal system  $\Xi^{\Gamma}$ : Indeed any accumulation point pattern of  $\Xi$  must possess the origin in its support, and thus is actually an element of  $\Xi$ , and on the other hand for each  $\Lambda \in \Xi$  and any radius  $R > 0$  the collection  $\mathcal{U}_{B(0,R)}(\Lambda)$  is an open neighborhood of  $\Lambda$  in  $\Xi^{\Gamma}$  that is clearly contained into the transversal  $\Xi$ .

About the topology of the transversal one may observe that there is a one to one correspondence between circular-shaped local configuration of radius  $R$  in the language  $\mathcal{L}_{NS}$  and subsets in  $\Xi$  of the form  $\mathcal{U}_{B(0,R)}(\Lambda)$ , for the same radius  $R$  and

$\Lambda$  chosen in  $\Xi$ . Thus compactness of  $\Xi$  is rephrased by the existence of only a finite number of such circular-shaped patterns for any radius  $R$ , a property called *finite local complexity* for the underlying point patterns in  $\mathbb{X}$ . This property holds in our context [30] [24], so that  $\Xi$  is a compact open subset of the internal system  $\Xi^\Gamma$ .

**Proposition 3.2.** *The internal system  $\Xi^\Gamma$  of a hull of model sets is a totally disconnected locally compact topological space, and a sub-basis of its topology is formed of all  $\Gamma$ -translates of  $\Xi$  and its complementary set  $\Xi^c$ .*

*Proof.* Any point pattern  $\Lambda \in \Xi^\Gamma$  is uniquely determined by the knowledge of whether a point  $\gamma \in \Gamma$  lies in  $\Lambda$  or not, for each  $\gamma \in \Gamma$ . Thus a sub-basis for the topology of the internal system is given by subsets

$$\Xi_\gamma := \{\Lambda \in \Xi^\Gamma \mid \gamma \in \Lambda\}$$

or their complementary sets  $\Xi_\gamma^c$ , for  $\gamma \in \Gamma$ . Since they are both open they are thus both closed as well, giving that the internal system is totally disconnected. Now any  $\Xi_\gamma$  or  $\Xi_\gamma^c$  is nothing but the  $-\gamma$  translate of  $\Xi$  or the complementary set  $\Xi^c$ , as

$$\Lambda \in \Xi_\gamma \iff \Lambda - \gamma \in \Xi \iff 0 \in (\Lambda - \gamma) \iff \gamma \in \Lambda \iff \Lambda \in \Xi_\gamma$$

As any point pattern  $\Lambda \in \Xi^\Gamma$  must contain at least one element  $\gamma \in \Gamma$  one gets that the compact open subsets  $\Xi_\gamma$  form a covering of  $\Xi^\Gamma$ , giving the local compactness.  $\square$

**Proposition 3.3.** [30] *Let  $\Xi^\Gamma$  be the internal system associated with a cut & project scheme  $(\mathbb{H}, \Sigma, \mathbb{R}^d)$  and some window. Then there exists a factor map*

$$\Pi : \Xi^\Gamma \longrightarrow \mathbb{H}$$

*mapping a point pattern  $\Lambda$  onto the unique element  $\Pi(\Lambda) = w_\Lambda$  of  $\mathbb{H}$  satisfying*

$$\mathfrak{P}(w_\Lambda + \mathring{W}) \subseteq \Lambda \subseteq \mathfrak{P}(w_\Lambda + W)$$

*Moreover, the map  $\Pi$  satisfies  $\Pi(\Xi) = -W$ , and is injective precisely on the dense family of non-singular model sets of  $\Xi^\Gamma$ , whose image is the dense subset  $NS$  of  $\mathbb{H}$ .*

From the above proposition we thus have a correspondence between any  $w \in NS$  with a unique non-singular model set  $\mathfrak{P}(w + W) \in \Xi^\Gamma$ , and we may also write  $NS$  for the dense subclass of non-singular model sets of  $\Xi^\Gamma$ . Thus the internal system  $\Xi^\Gamma$  and the internal space  $\mathbb{H}$  as different completions of a single set  $NS$ . This observation allows us to set, for any subset  $A$  of  $\mathbb{H}$ , a corresponding subset of  $\Xi^\Gamma$  of the form

$$(3) \quad [A]_\Xi := \overline{A \cap NS}^{\Xi^\Gamma}$$

Such a  $[A]_\Xi$  will be non-empty if and only if  $A$  intersect  $NS$ . In particular  $[A]_\Xi$  will have non-empty interior (and hence will be non-empty) whenever  $A$  has non-empty interior. We will have use of the following lemma, which we state without proof:

**Lemma 3.4.** *Let  $X$  be a topological space, and  $Y$  a dense subset. Then each clopen subset  $V$  of  $X$  is equal to the closure of  $V \cap Y$ . Moreover, if two clopen subsets coincides on  $Y$  then they are equal.*

For instance one is able to show  $\Xi = [-W]_\Xi = [-\mathring{W}]_\Xi$ , underlying a link between the topology of the internal system and the geometry of  $W$  in the internal space.

**Proposition 3.5.** *There exists an onto and proper continuous morphism*

$$\Pi^* : E(\Xi^\Gamma, \Gamma) \longrightarrow \mathbf{H}$$

which satisfies the equivariance relation  $\Pi(\Lambda.g) = \Pi(\Lambda) + \Pi^*(g)$  for any model set  $\Lambda \in \Xi^\Gamma$  and any mapping  $g \in E(\Xi^\Gamma, \Gamma)$ .

*Proof.* Let us show that the the map  $\Pi$  of proposition 3.3 is proper, that is, the inverse image of any compact set of  $\mathbf{H}$  is compact in  $\Xi^\Gamma$ : Let  $K$  be a compact subset of  $\mathbf{H}$  and pick up a model set  $\Lambda$  in its inverse image  $\Pi^{-1}(K)$ . Since  $\Gamma^*$  is dense in  $\mathbf{H}$  there exists  $\gamma_1, \dots, \gamma_l$  in  $\Gamma$  such that  $K \subset \bigcup_{k=1}^l \gamma_k^* - \mathring{W}$ . Thus  $w_\Lambda \in K$  falls into some  $\gamma_k^* - \mathring{W}$ , which implies that  $\gamma_k$  lies into  $\mathfrak{P}(w_\Lambda + \mathring{W}) \subset \Lambda$ . This in turns means that  $0 \in \Lambda - \gamma_k$ , or so  $\Lambda - \gamma_k \in \Xi$  and thus  $\Lambda \in \Xi.(-\gamma_k)$ . Hence the closed set  $\Pi^{-1}(K)$  is entirely included into a finite union of translates of  $\Xi$ , each being compact, and so is a compact set of  $\Xi^\Gamma$ . Now we have that proposition 2.7 applies, giving after invoking the statement of proposition 2.8 the desired morphism  $\Pi^*$ .  $\square$

**3.2. Hull and internal system Ellis semigroups.** We wish to rely here the Ellis semigroup of the dynamical systems  $(\mathbb{X}, \mathbb{R}^d)$  with that of  $(\Xi^\Gamma, \Gamma)$ . To this end, let  $g$  be any mapping in the Ellis semigroup  $E(\Xi^\Gamma, \Gamma)$ . Using theorem 1.4 together with the definition of the internal system, one sees that any point patterns  $\Lambda$  in  $\mathbb{X}$  can be written as  $\Lambda_0 = \Lambda - t$  for some model set  $\Lambda \in \Xi^\Gamma$  and some vector  $t \in \mathbb{R}^d$ . The mapping  $g$  is well defined on each  $\Lambda \in \Xi^\Gamma$ , and we may thus extend it into a self-map  $\tilde{g}$  of  $\mathbb{X}$  by setting:

$$(4) \quad \Lambda_0.\tilde{g} = (\Lambda - t).\tilde{g} := \Lambda.g - t$$

This is well defined since if one has  $\Lambda - t = \Lambda' - t'$  with  $\Lambda$  and  $\Lambda' \in \Xi^\Gamma$  then necessarily  $\Gamma - t = \Gamma - t'$ , which means that  $t - t' \in \Gamma$ , and since  $g$  commutes with the  $\Gamma$ -action on  $\Xi^\Gamma$  then applying (4) gives the same result. Let us now consider the semigroup  $E(\Xi^\Gamma, \Gamma) \times_\Gamma \mathbb{R}^d$  to be the (topological) quotient of the direct product semigroup  $E(\Xi^\Gamma, \Gamma) \times \mathbb{R}^d$  by the normal sub-semigroup formed of elements  $(\gamma, \gamma)$  with  $\gamma \in \Gamma$ .

**Theorem 3.6.** *Let  $\mathbb{X}$  and  $\Xi^\Gamma$  be the hull and internal system generated by a cut & project scheme  $(\mathbf{H}, \Sigma, \mathbb{R}^d)$  and some window. Then there is a homeomorphism and semigroup isomorphism*

$$E(\Xi^\Gamma, \Gamma) \times_\Gamma \mathbb{R}^d \simeq E(\mathbb{X}, \mathbb{R}^d)$$

mapping each element  $[g, t]_\Gamma$  of  $E(\Xi^\Gamma, \Gamma) \times_\Gamma \mathbb{R}^d$  onto  $\tilde{g} - t$ .

*Proof.* First we show that the quotient semigroup  $E(\Xi^\Gamma, \Gamma) \times_\Gamma \mathbb{R}^d$  is compact (Hausdorff): From the existence of the morphism  $\Pi^*$  one then gets a natural onto semigroup morphism  $\Pi^* \times id : E(\Xi^\Gamma, \Gamma) \times \mathbb{R}^d \twoheadrightarrow \mathbf{H} \times \mathbb{R}^d$ , which maps the normal sub-semigroup formed by elements  $(\gamma, \gamma)$  with  $\gamma \in \Gamma$  onto the lattice  $\Sigma$ . Since  $\mathbf{H} \times_\Sigma \mathbb{R}^d$  is compact (Hausdorff), and since  $\Pi^* \times id$  is continuous and proper, we deduce that  $E(\Xi^\Gamma, \Gamma) \times \mathbb{R}^d$  must itself be compact (Hausdorff).

Now it is clear that the mapping associating  $g \in E(\Xi^\Gamma, \Gamma)$  with  $\tilde{g} \in \mathbb{X}^\mathbb{X}$  is a semi-group morphism for the composition laws of mappings. This association is moreover continuous: For, it suffices to check the continuity of each evaluation map  $g \mapsto \Lambda_0.\tilde{g}$ , with  $\Lambda_0 \in \mathbb{X}$ . Write  $\Lambda_0$  as  $\Lambda - t$  with some  $\Lambda \in \Xi^\Gamma$ . If  $g_\lambda$  is a net in  $E(\Xi^\Gamma, \Gamma)$  pointwise

converging on  $\Xi^\Gamma$  to a mapping  $g$ , then one has convergence of the net  $\Lambda.g_\lambda$  to  $\Lambda.g$  in the internal system  $\Xi^\Gamma$ . Comparing the topologies coming from the topologies (1) on  $\mathbb{X}$  and (2) on  $\Xi^\Gamma$  one sees that the embedding of  $\Xi^\Gamma$  into the hull  $\mathbb{X}$  is continuous, so that the net  $\Lambda.g_\lambda$  also converges to  $\Lambda.g$  in the hull  $\mathbb{X}$ . Hence the net  $\Lambda_0.\tilde{g}_\lambda = \Lambda.g_\lambda - t$  converges to  $\Lambda.g - t = \Lambda_0.\tilde{g}$ , as desired.

From this we can set a continuous semigroup morphism from  $E(\Xi^\Gamma, \Gamma) \times \mathbb{R}^d$  into  $\mathbb{X}^\mathbb{X}$  associating any pair  $(g, t)$  with  $\tilde{g} - t$ . Clearly any pair of the form  $(\gamma, \gamma)$  with  $\gamma \in \Gamma$  is mapped onto the identity map, thus giving a continuous semigroup morphism

$$\begin{aligned} E(\Xi^\Gamma, \Gamma) \times_\Gamma \mathbb{R}^d &\longrightarrow \mathbb{X}^\mathbb{X} \\ [g, t]_\Gamma &\longrightarrow \tilde{g} - t \end{aligned}$$

This map is 1-to-1: If  $[g, t]_\Gamma$  and  $[g', t']_\Gamma$  are such that  $\tilde{g} - t \equiv \tilde{g}' - t'$  then they must in particular coincide at any model set  $\Lambda \in \Xi^\Gamma$ , thus giving for each such point pattern  $\Lambda.g - t = \Lambda.g' - t'$ . As  $\Lambda.g$  and  $\Lambda.g'$  are supported on  $\Gamma$  we deduce that  $t' - t =: \gamma \in \Gamma$ , and that  $g'$  coincides with  $g + \gamma$  everywhere on  $\Xi^\Gamma$ . It follows that  $[g', t']_\Gamma = [g + \gamma, t + \gamma]_\Gamma = [g, t]_\Gamma$ , hence giving injectivity. Now the stated morphism conjugates, both topologically and algebraically, the semigroup  $E(\Xi^\Gamma, \Gamma) \times \mathbb{R}^d$  with its image in  $\mathbb{X}^\mathbb{X}$ . To conclude it suffices then to show that this image densely contains the group of homeomorphisms coming from the  $\mathbb{R}^d$ -action on  $\mathbb{X}$ . Obviously this group is contained into the image in question, appearing as  $[0, t]_\Gamma$  where 0 stands for the identity mapping on  $\Xi^\Gamma$ , lying into  $\Gamma$  and thus into  $E(\Xi^\Gamma, \Gamma)$ . Let then  $\tilde{g} - t$  be some mapping in this image. A neighborhood basis for this latter in  $\mathbb{X}^\mathbb{X}$  may be stated as finite intersections of sets

$$V_{\mathbb{X}}(\Lambda, U) := \left\{ f \in \mathbb{X}^\mathbb{X} \mid \Lambda.f \in U \right\}$$

containing  $\tilde{g} - t$ . Let then  $\Lambda_1, \dots, \Lambda_k$  be model sets and  $U_1, \dots, U_k$  be open subsets of  $\mathbb{X}$  such that  $\tilde{g} - t$  lies into  $V(\Lambda_j, U_j)$  for each  $j$ . Then to get density it suffices to show the existence of some element of  $\mathbb{R}^d$  also contained into  $V(\Lambda_j, U_j)$  for each  $j$ . Let us write  $\Lambda_j$  as a sum  $\Lambda'_j - t_j$  with  $\Lambda'_j \in \Xi^\Gamma$ . Hence the mapping  $g$ , being the restriction of  $\tilde{g}$  on  $\Xi^\Gamma$ , lies into each subset

$$(5) \quad V_{\Xi^\Gamma}(\Lambda'_j, \Xi^\Gamma \cap (U_j + t + t_j)) := \left\{ f \in (\Xi^\Gamma)^{\Xi^\Gamma} \mid \Lambda'_j.f \in U_j + t + t_j \right\}$$

The embedding of  $\Xi^\Gamma$  into the hull  $\mathbb{X}$  is clearly continuous, so that  $\Xi^\Gamma \cap (U_j + t + t_j)$  are open sets of the internal system and consequently the sets (5) are open in  $(\Xi^\Gamma)^{\Xi^\Gamma}$ . As  $E(\Xi^\Gamma, \Gamma)$  is the closure of  $\Gamma$  into  $(\Xi^\Gamma)^{\Xi^\Gamma}$  one may thus find some  $\gamma \in \Gamma$  within each set (5), giving that  $\gamma - t \in \mathbb{R}^d$  lies into each  $V(\Lambda_j, U_j)$ , as desired.  $\square$

Appart from this, the parametrization map  $\pi$  of theorem 1.4 also implies the existence of an onto continuous semigroup morphism

$$\pi^* : E(\mathbb{X}, \mathbb{R}^d) \longrightarrow \mathbb{H} \times_\Sigma \mathbb{R}^d$$

which satisfies the equivariance relation  $\pi(\Lambda.g) = \pi(\Lambda) + \pi^*(g)$  for any model set  $\Lambda \in \mathbb{X}$  and any Ellis transformation  $g \in E(\mathbb{X}, \mathbb{R}^d)$ . Then the morphism  $\pi^*$  extends the morphism  $\Pi^*$  in the sense that for any transformation  $g$  in  $E(\Xi^\Gamma, \Gamma)$  and  $t \in \mathbb{R}^d$  one has the equality

$$\pi^*(\tilde{g} - t) = [\Pi^*(g), t]_\Sigma$$

#### 4. THE ALMOST CANONICAL PROPERTY FOR MODEL SETS

We wish to define here the almost canonical property on a model set. To this end we restrict ourselves to real cut & project scheme  $(\mathbb{R}^n, \Sigma, \mathbb{R}^d)$ , and we ask the window  $W$  to be a  $n$ -dimensional *compact convex polytope* of the internal space  $\mathbb{R}^n$ . The definition of almost canonical model sets will be derived from a corresponding notion on  $W$ , which consists of a pair of assumptions we will now present.

In fact, it will be much more convenient to consider the *reversed window*  $M := -W$  in the internal space. It thus as well consists of a  $n$ -dimensional compact convex polytope in  $\mathbb{R}^n$ , whose boundary is given by  $\partial M = -\partial W$ . Now if we let  $f$  to be any  $n - 1$  dimensional face of  $M$  we then set

- $A_f$  or  $A_f^0$  to be the affine hyperplane generated by  $f$ .
- $H_f$  or  $H_f^0$  be the corresponding linear hyperplane in  $\mathbb{R}^n$ .
- $Stab_\Gamma(A_f)$  to be the subgroup of  $\gamma \in \Gamma$  with  $\gamma^* \in H_f$ .

We remark that  $Stab_\Gamma(A_f)$  is precisely the subgroup of elements  $\gamma \in \Gamma$  such that  $A_f + \gamma^* = A_f$ , whence the notation. We may also denote  $Stab_\Gamma(A_f)^*$  for its image in the internal space under the  $*$ -map.

**Assumption 1.** *For each face  $f$  of  $M$ , the sum  $Stab_\Gamma(A_f)^* + f$  covers  $A_f$  in  $\mathbb{R}^n$ .*

The above assumption implies in particular that  $Stab_\Gamma(A_f)$  has a relatively dense image in  $H_f$  under the  $*$ -map, and thus must be of rank at least  $n - 1$ . Under the above assumption we get a nice description of the subset of non-singular vectors

$$NS := [\Gamma^* - \partial W]^c = [\Gamma^* + \partial M]^c = \left[ \bigcup_{f \text{ face of } M} \Gamma^* + A_f \right]^c$$

As we see the above subset of non-singular vectors arise as the complementary subsets of all the  $\Gamma$ -translates of *singular hyperplanes*, namely the affine hyperplanes  $A_f$  with  $f$  a face of the reversed window  $M$ . Let us in addition set for each  $n - 1$  dimensional face  $f$  of  $M$

- $H_f^-$  and  $H_f^+$  to be the open half spaces with boundary  $H_f$ .
- $H_f^{-0}$  and  $H_f^{+0}$  to be the closed half spaces with boundary  $H_f$ .
- $A_f^-, A_f^+, A_f^{-0}$  and  $A_f^{+0}$  be the corresponding objects with respect to  $A_f$ .

The choice of orientation on each linear hyperplane provided by the above notation is not relevant, but will be remained fixed from now on. Observe that a hyperplane  $H$  may be associated with two different faces, which in this case leaves a common orientation on the corresponding affine spaces.

Recall that to any Euclidean subset  $A$  may be associated a corresponding subset  $[A]_\Xi$  of the internal system according to (3). We will be specially interested here in a certain collection of Euclidean subsets which we call the family of *admissible half spaces*

$$\mathfrak{A} = \left\{ \gamma^* + A_f^\pm \mid \gamma \in \Gamma, f \text{ face of } M \right\}$$



**Assumption 2.** Any set  $[A]_{\Xi}$  where  $A \in \mathfrak{A}$  is a clopen set.

It may be in fact shown that assumption 2 implies assumption 1, but as we don't really need to prove this fact here we assume both independently. We wish to illustrate what type of polytope could satisfies assumptions 1 and 2 by setting situations where this holds, but first let us define what an *almost canonical model set* is:

**Definition 4.1.** A model set is *almost canonical* when it may be constructed with a real cut & project scheme and a compact convex polytopal window in its internal space satisfying assumptions 1 and 2.

The term *almost canonical* makes reference to the first point patterns defined as model sets, the *canonical* model sets, constructed via a real cut & project scheme  $(\mathbb{R}^n, \Sigma, \mathbb{R}^d)$  together with a window being the orthogonal projection of a unit cube, with respect to the lattice  $\Sigma$ , in the internal space. Our example in section 1 is of this form. The terminology *almost canonical* has been introduced by Julien in [16] in order to set slight generalisations of these model sets. However, our definition doesn't in fact fits exactly the one given in [16] (it can be shown that this one implies the one of Julien), but as we don't want to introduce another definition for something which remains highly close to the one of [16] we allows ourselves to call it almost canonical. As shown in [8], a canonical window always satisfies assumptions 1 and 2 and is thus almost canonical in our sense.

A condition which makes assumptions 1 and 2 holding is the requirement that any stabilizer  $Stab_{\Gamma}(A_f)$  is dense in the corresponding linear hyperplane  $H_f$ , for any face  $f$  of the window  $W$  (or its reversed window  $M$ , which remains the same). A lighten condition which also implies assumptions 1 and 2 is a slight strengthening of assumption 1: If  $\overset{\circ}{f}$  denotes the relative interior of any face  $f$  then

**Assumption 1'.** For each face  $f$  of  $M$ , the sum  $Stab_{\Gamma}(A_f)^* + \overset{\circ}{f}$  covers  $A_f$ .

## 5. PREPARATORY RESULTS ON THE ELLIS SEMIGROUP OF THE INTERNAL SYSTEM

**5.1. Internal system topology.** The family of clopen set  $[A]_{\Xi}$  where  $A$  is an admissible half space may serve to form a basis for the topology of the internal system:

**Proposition 5.1.** The collection of sets  $[A]_{\Xi}$  where  $A \in \mathfrak{A}$  forms a sub-basis for the topology of the the internal system. Moreover, for any pair  $A, A'$  in  $\mathfrak{A}$  the following Boolean rules are true:

$$[A \cup A']_{\Xi} = [A]_{\Xi} \cup [A']_{\Xi} \quad [A]_{\Xi}^c = [A^c]_{\Xi} \quad [A \cap A']_{\Xi} = [A]_{\Xi} \cap [A']_{\Xi}$$

*Proof.* Whenever  $w$  is a non-singular element of  $NS \subset \mathbb{R}^n$  one has for any  $\gamma \in \Gamma$

$$\mathfrak{P}(W + w).\gamma := \mathfrak{P}(W + w) - \gamma = \mathfrak{P}(W + w + \gamma^*)$$

This is the argument which allows to write, for any  $\gamma \in \Gamma$ , the equalities

$$(6) \quad [A_f^+]_{\Xi}.\gamma = [A_f^+ + \gamma^*]_{\Xi}$$

This observation being made, let us start the proof by showing the Boolean equalities. At first, the equality stated on the left is a simple consequence of closure operation. In turns, the equalities at the middle are equivalent to have disjoint decompositions

$$(7) \quad [A_f^+ + \gamma^*]_{\Xi} \sqcup [A_f^- + \gamma^*]_{\Xi} = \Xi^{\Gamma}$$

which reduces, due to the equalities provided in (6), to show  $[A_f^+]_{\Xi} \sqcup [A_f^-]_{\Xi} = \Xi^{\Gamma}$ . To that end, note that any element of the internal system  $\Xi^{\Gamma}$  is the limit of a sequence of non-singular elements, a sequence which can be taken after extraction into one of the two open half spaces  $A_f^+$  and  $A_f^-$ . Therefore such element remains into either  $[A_f^+]_{\Xi}$  or  $[A_f^-]_{\Xi}$ , showing that their union covers the internal system. On the other hand, these subsets are by assumption clopen so must have a clopen intersection. Assume for a contradiction that this is not the empty set: it must contains a non-singular model set  $\Lambda$ , with image under  $\Pi$  a non-singular element  $w_{\Lambda} \in NS \subset \mathbb{R}^n$ . However  $\Lambda$  is the limit of two sequences of non-singular model sets, with associated sequences of non-singular elements in  $\mathbb{R}^n$  taken in  $A_f^+$  for the first sequence and in  $A_f^-$  for the second one. Taking limits one must have  $w_{\Lambda} \in A_f$ , and since  $w_{\Lambda}$  has been taken non-singular one has the desired contradiction.

Having proven the left and middle Boolean equalities, then one can directly deduce the validity of the third one, as it is the case in full generality in Boolean algebras, from the two others.

To show that the sets  $[A]_{\Xi}$  where  $A \in \mathfrak{A}$  forms a sub-basis for the topology, observe that the set  $\overset{\circ}{M} = -\overset{\circ}{W}$  is precisely the intersection of admissible half spaces  $A_f^{s_f}$ , where  $s_f$  is the sign among  $-$  or  $+$  such that  $A_f^{s_f}$  contains the interior of  $M$ . From this we deduce

$$\Xi = [-\overset{\circ}{W}]_{\Xi} = [\overset{\circ}{M}]_{\Xi} = \left[ \bigcap_{f \text{ face of } W} A_f^{s_f} \right]_{\Xi} = \bigcap_{f \text{ face of } W} [A_f^{s_f}]_{\Xi}$$

Thus the set  $\Xi$  and its complementary set can be obtained as finite intersections of sets of the statement. Since by proposition 3.2  $\Xi^{\Gamma}$  admits a sub-basis formed by the  $\Gamma$ -translates of  $\Xi$  and its complementary set, the proof is complete.  $\square$

**5.2. Cones associated with model sets.** We set the *cut type* of a vector  $w \in \mathbb{R}^n$  to be the family of linear hyperplanes for which some parallel singular hyperplane passes through  $w$ ,

$$(8) \quad \mathfrak{H}_w := \{H_f \in \mathfrak{H}_W \mid w \in \Gamma^* + A_f\}$$

To each  $w \in \mathbb{R}^n$  is associated a family of *cones* (also called *corners* in [19]), which are open cones with vertex 0 and boundaries formed by hyperplanes in  $\mathfrak{H}_w$ . We may label each of these cones by a *cone type*  $\mathfrak{c} : \mathfrak{H}_W \rightarrow \{-, +, \infty\}$ , so that the labeled cone is obtained, according to the notations of section 4, as

$$C := \bigcap_{H \in \mathfrak{H}_W} H^{\mathfrak{c}(H)}$$

In the above intersection only hyperplanes where  $\mathfrak{c}$  has values not equal to  $\infty$  are consistent, and we may set the *domain* of a cone type  $\mathfrak{c}$  to be the subset  $dom(\mathfrak{c})$  of  $\mathfrak{H}_W$  where it has value different from  $\infty$ . Moreover, a cone determined by say the

cut type  $\mathfrak{H}_w$  has only one cone type whose domain is precisely  $\mathfrak{H}_w$ . Now given a cone  $C$  in  $\mathbb{R}^n$  and  $w$  to be some vector of  $\mathbb{R}^n$ , we set

$$(9) \quad C(w, \varepsilon) := (C \cap B(0, \varepsilon)) + w$$

to be the head of the cone  $C$ , translated at  $w$  and of length  $\varepsilon$ . One can easily verify that a vector  $w \in \mathbb{R}^n$  belongs to the non-singular vectors  $NS$  if and only if its cut type  $\mathfrak{H}_w$  is empty. In this latter case the unique resulting cone  $C$  is the full Euclidean space  $\mathbb{R}^n$ , for which any set of the form (9) is an Euclidean ball.

**Proposition 5.2.** *Given a model set  $\Lambda \in \Xi^\Gamma$ , there exists a cone  $C_\Lambda$ , admitting a cone type  $\mathfrak{c}_\Lambda$  with domain  $\mathfrak{H}_{w_\Lambda}$ , such that the following equivalence holds for each admissible half space  $A \in \mathfrak{A}$*

$$\Lambda \in [A]_\Xi \iff C_\Lambda(w_\Lambda, \varepsilon) \subset A \text{ for some } \varepsilon > 0$$

*Proof.* Let  $\Lambda \in \Xi^\Gamma$  be chosen. If  $H$  is a hyperplane of the cut type  $\mathfrak{H}_{w_\Lambda}$ , that is, if one has some  $\gamma \in \Gamma$  and some face  $f$  with  $w_\Lambda \in \gamma^* + A_f$ ,  $A_f$  parallel to  $H$ , then the hyperplane  $H + w_\Lambda$  is equal to  $\gamma^* + A_f$  and the half spaces  $H^\pm + w_\Lambda$  are admissible. Therefore  $[H^+ + w_\Lambda]_\Xi$  and  $[H^- + w_\Lambda]_\Xi$  are clopen complementary sets, and the one containing  $\Lambda$  defines the sign  $\mathfrak{c}_\Lambda(H)$ . This provides  $\mathfrak{c}_\Lambda$  uniquely. From the Boolean rules stated in proposition 5.1 the model set  $\Lambda$  is so that

$$(10) \quad \Lambda \in \bigcap_{H \in \mathfrak{H}_{w_\Lambda}} [H^{\mathfrak{c}_\Lambda(H)} + w_\Lambda]_\Xi = \left[ \bigcap_{H \in \mathfrak{H}_{w_\Lambda}} H^{\mathfrak{c}_\Lambda(H)} + w_\Lambda \right]_\Xi = [C_\Lambda + w_\Lambda]_\Xi$$

with  $C_\Lambda$  the unique cone with cone type  $\mathfrak{c}_\Lambda$ , in particular non-empty. We now show that a model set  $\Lambda$  has a neighborhood basis in the internal system obtained as

$$(11) \quad [C_\Lambda + w_\Lambda]_\Xi \cap \Pi^{-1}(B(w_\Lambda, \varepsilon))$$

From the inclusion of  $\Lambda$  stated in (10) it is clear that (11) is a family of open neighborhoods of  $\Lambda$  in the internal system. We will make use of the following lemma:

**Lemma 5.3.** *Let  $\pi : X \rightarrow Y$  be a continuous and proper map between locally compact spaces. Let  $X_x$  be the fiber of  $x$  with respect to  $\pi$  for each  $x \in X$ . If there is a clopen neighborhood  $V_x$  of  $x$  satisfying  $V_x \cap X_x = \{x\}$ , then a neighborhood basis of  $x$  is provided by  $V_x \cap \pi^{-1}(U)$  with  $U$  running among the neighborhoods of  $\pi(x)$ .*

*Proof.* Suppose for a contradiction that the stated family is not a neighborhood basis of  $x$ . One may then select an open neighborhood  $V$  of  $x$  such that  $V_x \cap \pi^{-1}(U)$  meets  $V^c$  for each neighborhood  $U$  of  $\pi(x)$ . Let  $\Delta$  be the directed family of open neighborhoods of  $\pi(x)$  falling into some compact neighborhood  $U_0$  of  $\pi(x)$ . One may select a net  $\{x_U\}_{U \in \Delta}$  into  $V^c$  and with each  $x_U$  belonging to  $V_x \cap \pi^{-1}(U)$ . This net falls into the compact set  $V_x \cap \pi^{-1}(U_0)$  and in  $V^c$  as well. Taking some accumulation point  $x'$ , necessarily lying into both  $V_x$  and  $X_x$ , and in the closed set  $V^c$  as well, gives the contradiction as we supposed  $V_x \cap X_x = \{x\}$  contained into  $V$ .  $\square$

We then show that a clopen neighborhood of  $\Lambda$  which fits the condition of the above lemma is provided by  $[C_\Lambda + w_\Lambda]_\Xi$ : For, suppose that  $\Lambda$  and  $\Lambda'$  are such that  $w_\Lambda = w_{\Lambda'} =: w$  in  $\mathbb{R}^n$ . From proposition 5.1 there is a face  $f$  of  $W$  as well as an

element  $\gamma \in \Gamma$  such that (up to a permutation of signs  $+$  and  $-$ )  $\Lambda \in [A_f^+ + \gamma^*]_{\Xi}$  and  $\Lambda' \in [A_f^- + \gamma^*]_{\Xi}$ . Then the vector  $w$  falls into both closed half planes  $A_f^{+0} + \gamma^*$  and  $A_f^{-0} + \gamma^*$ , and thus into  $A_f + \gamma^*$ . The latter hyperplane can consequently also be written  $H_f + w$ , and it follows that  $\Lambda \in [H_f^+ + w]_{\Xi}$  whereas  $\Lambda' \in [H_f^- + w]_{\Xi}$ . This shows that  $\Lambda'$  is outside  $[C_{\Lambda} + w_{\Lambda}]_{\Xi}$ , as desired.

Now, it is clear that  $\Lambda \in [A]_{\Xi}$  if and only if one has a subset of the form  $[C_{\Lambda} + w_{\Lambda}]_{\Xi} \cap \Pi^{-1}(B(w_{\Lambda}, \varepsilon))$  included into  $[A]_{\Xi}$  for some  $\varepsilon > 0$ . Then intersecting with  $NS$  gives that  $C_{\Lambda}(w_{\Lambda}, \varepsilon) \cap NS$  falls into  $A \cap NS$ , and by taking closure and next interior in  $\mathbb{R}^n$  one obtains the right-hand inclusion of the statement. Conversely if the right-hand inclusion of the statement occurs for some  $\Lambda \in \Xi^{\Gamma}$  then we may choose a sequence of non-singular model sets converging to it, in a manner that the associated sequence of non-singular vectors falls into (11), and thus into  $C_{\Lambda}(w_{\Lambda}, \varepsilon)$ . The sequence of non-singular model sets lies then into  $[A]_{\Xi}$ , and since this latter is closed we obtain the result.  $\square$

**5.3. Topology of the internal system Ellis semigroup.** Recall that by construction the Ellis semigroup for the internal system is a closure of the group  $\Gamma$ , or rather the resulting group of homeomorphisms on the internal system. Thus for any Euclidean subset  $A$  one may set a corresponding subset  $[A]_E$  to be the closure of  $\{\gamma \in \Gamma \mid \gamma^* \in A\}$  in the Ellis semigroup of the internal system. We would in fact consider a specific family of Euclidean subsets, namely

$$\mathfrak{A}_{Ellis} := \{H^{\mathfrak{t}} + w \mid H \in \mathfrak{H}_W, \mathfrak{t} \in \{-, 0, +\}, w \in \mathbb{R}^n\}$$

Observe that the above family contains the family  $\mathfrak{A}$  of admissible half spaces, in a strict sense however.

**Proposition 5.4.** *Any set  $[A]_E$  where  $A \in \mathfrak{A}_{Ellis}$  is clopen, and the collection of these sets forms a sub-basis for the topology of the the internal system Ellis semigroup. Moreover, for any pair  $A, A'$  in  $\mathfrak{A}_{Ellis}$  the following Boolean rules are true:*

$$[A \cup A']_E = [A]_E \cup [A']_E \quad [A]_E^c = [A^c]_E \quad [A \cap A']_E = [A]_E \cap [A']_E$$

*Proof.* From proposition 5.1, the sets  $[A]_{\Xi}$  where  $A$  is an admissible half space are clopen subsets of the internal system  $\Xi^{\Gamma}$ , and form a sub-basis for its topology. It thus follows that the sets

$$V(\Lambda, [A]_{\Xi}) := \{g \in E(\Xi^{\Gamma}, \Gamma) \mid \Lambda.g \in [A]_{\Xi}\}$$

where  $\Lambda$  is any model set in the internal system and  $A$  is any admissible half space, are clopen subsets of the Ellis semigroup  $E(\Xi^{\Gamma}, \Gamma)$ , and that they form a sub-basis for its topology. Moreover, using the fact that  $[\gamma^* + A_f^{\pm}]_{\Xi}$  is equal to  $[A_f^{\pm}]_{\Xi}.\gamma$  whatever the element  $\gamma \in \Gamma$  one can directly check that  $V(\Lambda, [\gamma^* + A_f^{\pm}]_{\Xi})$  is equal to  $V(\Lambda.(-\gamma), [A_f^{\pm}]_{\Xi})$ . This shows that a sub-basis for the Ellis semigroup topology is obtained as the collection

$$(12) \quad \left\{ V\left(\Lambda, [A_f^{\pm}]_{\Xi}\right) \mid \Lambda \in \Xi^{\Gamma}, f \text{ face of } M \right\}$$

In order to rely these sets with the ones given in the statement we show here the cornerstone lemma of this proposition:

**Lemma 5.5.** *Let  $\Lambda$  be chosen in the internal system. Then*

$$V\left(\Lambda, [A_f^+]\Xi\right) = \begin{cases} [A_f^{+0} - w_\Lambda]_E & \text{if } \mathbf{c}_\Lambda(H_f) = + \\ [A_f^+ - w_\Lambda]_E & \text{if } \mathbf{c}_\Lambda(H_f) = - \\ [A_f^{+0} - w_\Lambda]_E = [A_f^+ - w_\Lambda]_E & \text{if } \mathbf{c}_\Lambda(H_f) = \infty \end{cases}$$

*The same statement holds with all the + and - signs switched.*

*Proof.* Recall from lemma 3.4 that a clopen set of  $E(\Xi^\Gamma, \Gamma)$  is the closure of its subset of  $\Gamma$ -elements. Now given  $V(\Lambda, [A_f^+]\Xi)$ , an element  $\gamma \in \Gamma$  lies inside if and only if  $\Lambda \cdot \gamma \in [A_f^+]\Xi$ , which happens from proposition 8.1 if and only if  $C_{\Lambda, \gamma}(w_{\Lambda, \gamma}, \varepsilon)$  embeds into  $A_f^+$  for some  $\varepsilon > 0$ . As the cones of  $\Lambda$  and its  $\gamma$ -translate are the same, and because the factor map  $\Pi$  is  $\Gamma$ -equivariant, the previous condition is equivalent to have

$$(13) \quad C_\Lambda(\gamma^*, \varepsilon) \subset A_f^+ - w_\Lambda$$

for some  $\varepsilon > 0$ . It is then obvious that:

- whenever  $\gamma^* \in A_f^+ - w_\Lambda$  this condition is satisfied,
- whenever  $\gamma^* \in A_f^- - w_\Lambda$  this condition is not satisfied.

Now suppose that  $\mathbf{c}_\Lambda(H_f) = \infty$ , so that  $H_f$  doesn't belong to the cut type of  $w_\Lambda$ : Then no elements of  $\Gamma$  has its image under the  $*$ -map entering  $A_f - w_\Lambda$ , and thus by taking closure in the Ellis semigroup one has the desired equality in the case  $\mathbf{c}_\Lambda(H_f) = \infty$ .

Suppose by contrast that  $\mathbf{c}_\Lambda(H_f) \neq \infty$ , so that there exists elements of  $\Gamma$  whose image under the  $*$ -map falls into  $A_f - w_\Lambda$ . Then for each such  $\gamma \in \Gamma$  the hyperplane  $A_f - w_\Lambda$  may also be written  $H_f + \gamma^*$ , giving  $A_f^+ - w_\Lambda = H_f^+ + \gamma^*$ . Hence such a  $\gamma$  satisfies (13) if and only if the cone  $C_\Lambda$  lies into  $H_f^+$ , which rewrites as  $\mathbf{c}_\Lambda(H_f) = +$ . Again by taking closure in the Ellis semigroup, one has the desired equalities in the case  $\mathbf{c}_\Lambda(H_f) \neq \infty$ .

The argument remains valid when interchanging the + and - signs everywhere, completing the proof.  $\square$

**Lemma 5.6.** *For each hyperplane  $H$  and vector  $w \in \mathbb{R}^n$  one has a partition of the Ellis semigroup by clopen sets*

$$(14) \quad E(\Xi^\Gamma, \Gamma) = [H^- + w]_E \sqcup [H + w]_E \sqcup [H^+ + w]_E$$

*Proof.* First observe that by construction the group  $\Gamma$  is dense in the Ellis semigroup, and consequently the union of the three right-hand sets stated in the equality must cover the Ellis semigroup. Now select a face  $f$  with  $H = H_f$  and let  $w' \in \mathbb{R}^n$  be so that  $H^t + w$  restates as  $A_f^t - w'$  for each sign  $t \in \{-, 0, +\}$  (this can always be achieved as  $H$  and  $A_f$  are parallel). This choice of vector  $w'$  will be kept along this proof. It is quite clear that the middle-term  $[H_f + w]_E$  is non-empty if and only if one has elements  $\gamma \in \Gamma$  such that  $\gamma^* \in H_f + w$ , or equivalently into  $A_f - w'$ , which in turns exactly means that  $H$  is a hyperplane of the cut type  $\mathfrak{H}_{w'}$ . Thus we will consider two cases:

Suppose that  $H \in \mathfrak{H}_{w'}$ : we may select two cones, both determined by the cut type  $\mathfrak{H}_{w'}$ , living into opposite sides with respect to  $H$ . Let us pick two model sets  $\Lambda$  and  $\Lambda'$  with common associated vector  $w'$  in  $\mathbb{R}^n$  and associated with these cones, so that  $\mathfrak{c}_\Lambda(H) = +$  and  $\mathfrak{c}_{\Lambda'}(H) = -$  up to a switch of signs (the existence of such model sets is shown in theorem 8.1 appearing further, whose proof is independent of the present statement). Then by the previous lemma the set  $[H_f^- + w]_E$  is the clopen subset  $V(\Lambda, [A_f^-]_\Xi)$ , and is disjoint from the two others since they are both included into  $V(\Lambda, [A_f^+]_\Xi)$ . In the same way the set  $[H_f^+ + w]_E$  is the clopen subset  $V(\Lambda', [A_f^+]_\Xi)$ , and is disjoint from the two others since they are both included into  $V(\Lambda', [A_f^-]_\Xi)$ . As the left-hand term and the right-hand term are clopen and disjoint from the respective two others sets then the stated union must disjoint, and the middle term is clopen as well.

If  $H \notin \mathfrak{H}_{w'}$  then things are even easier: the middle-term becomes empty, and in pretty much the same way as it was done just before, by picking only one model set with associated vector  $w'$  one can show that the two sets of the union are clopen and disjoint.  $\square$

Now the proof of the statement almost immediately follows: from lemma 5.6 the sets of the statement are clopen sets, and form a sub-basis since any subset of the family (12) writes as one of them by lemma 5.5. It remains to show the Boolean rules: the left-hand rule is a direct consequence of the closure operation, whereas the middle-hand rule follows from the family of partitions given by lemma 5.6. The third rule naturally follows from the two others.  $\square$

## 6. MAIN RESULT ON THE INTERNAL SYSTEM ELLIS SEMIGROUP

**6.1. The face semigroup of a convex polytope.** Assuming we are given a real cut & project scheme  $(\mathbb{R}^n, \Sigma, \mathbb{R}^d)$  with an almost canonical window  $W$  in the internal space, we shall define what is called the *face semigroup* of  $W$  ([6], [29]).

Let  $\mathfrak{H}_W$  be the family of linear hyperplanes parallel to the faces of  $W$ . Then it defines a stratification of  $\mathbb{R}^n$  by cones of dimension between 0 and  $n$  (those cones are called *faces* in [29]), that is, by non empty sets of the form

$$(15) \quad \bigcap_{H \in \mathfrak{H}_W} H^{\mathfrak{t}(H)}$$

where  $\mathfrak{t}(H)$  is a symbol among  $\{-, 0, +\}$  for each  $H \in \mathfrak{H}_W$ . Then each such cone  $C$  is determined through a unique map  $\mathfrak{t}_C : \mathfrak{H}_W \rightarrow \{-, 0, +\}$ , which we call here its *cone type*. A special class of cones is that of *chambers*, that is, the cones of maximal dimension  $n$ , which are open in  $\mathbb{R}^n$  and are precisely those with a nowhere-vanishing cone type. On the other extreme is the unique cone of dimension 0, namely the singleton  $\{0\}$ , whose type is entirely vanishing and which we denote by  $\mathfrak{o}$ .

Let us denote by  $\mathfrak{T}_W$  the above set of cones, and define on this set a semigroup law of the following form: if  $C, C' \in \mathfrak{T}_W$  are given, then the product  $C.C'$  is the face whose type is given by

$$\mathfrak{t}_{C.C'}(H) = \mathfrak{t}_C.\mathfrak{t}_{C'}(H) := \begin{cases} \mathfrak{t}_{C'}(H) & \text{if } \mathfrak{t}_C(H) = 0 \\ \mathfrak{t}_C(H) & \text{else} \end{cases}$$

The reading direction is from right to left, as for actions: first looking at the value of  $t_{C'}(H)$ , we keep it when  $t_C(H) = 0$  and replace it by  $t_C(H)$  else, which in this case makes us forgetting the existence of  $t_{C'}$ . It may easily checked that this product law is well defined on  $\mathfrak{T}_W$ , that is, the product of two (non empty) cones is again a (non empty) cone, and is associative.

**Definition 6.1.** *The face semigroup associated with the polytope  $W$  in  $\mathbb{R}^n$  is the set  $\mathfrak{T}_W$  equiped with the above product law.*

It is clear from the formula that  $\mathfrak{o}$  is an identity for  $\mathfrak{T}_W$ . Moreover, any cone  $C$  satisfies the equality  $C.C = C$ , that is, is *idempotent* in  $\mathfrak{T}_W$ . There moreover exists a natural partial order on the face semigroup which let  $C \leq C'$  when and only when  $C'$  is a lower dimensional facet of  $C$ , or equivalently when the inclusion  $C' \subseteq \overline{C}$  occurs. This may be rephrased by means of the semigroup law on  $\mathfrak{T}_W$ , as we have

$$C \leq C' \iff t_C = t_{C'}.t_C$$

With respect to this order, the chambers are the minimal cones whereas  $\mathfrak{o}$  is the (unique) maximal cone in the face semigroup. Some authors use the reverse order instead, but it appears more convenient for later needs to set the order in the above way.

**6.2. Taking  $\Gamma$  into account.** Here we introduce a modified version of the face semigroup obtained from an almost canonical window  $W$  of the internal space  $\mathbb{R}^n$  of some real cut & project scheme.

Let us call a cone  $C$  of the face semigroup *non-trivial* whenever the origin in  $\mathbb{R}^n$  is an accumulation point of elements of  $C \cap \Gamma^*$ . We moreover denote the family of non-trivial cones of the face semigroup by  $\mathfrak{T}_{W,\Gamma}$ , and refer it as the *non-trivial face semigroup*. It is at this point not clear whether  $\mathfrak{T}_{W,\Gamma}$  is a sub-semigroup of  $\mathfrak{T}_W$ . However to convince ourselves that it is the case, we may observe that the product  $C.C'$  of two cones of the face semigroup is the only cone containing a small head of the cone  $C'$  when this latter is shifted by a small vector of  $C$ , and that this preserves the subset  $\mathfrak{T}_{W,\Gamma}$  in the face semigroup.

Now given a non-trivial cone  $C$ , as  $C \cap \Gamma^*$  accumulates at 0 then the vector space  $\langle C \rangle$  spanned by  $C$  admits a subgroup  $\langle C \rangle \cap \Gamma^*$  which cannot be uniformly discrete, and thus is "dense along some subspace". More precisely we set in our setting a theorem of [31]:

**Theorem 6.2.** *The vector space  $\langle C \rangle$  uniquely writes as a direct sum  $V \oplus D$ , where  $V \cap \Gamma^*$  is dense in  $V$ ,  $D \cap \Gamma^*$  is uniformly discrete in  $D$ , and  $\langle C \rangle \cap \Gamma^* = (V \cap \Gamma^*) \oplus (D \cap \Gamma^*)$ .*

Now given a non-trivial cone  $C$  of the face semigroup with decomposition  $\langle C \rangle = V \oplus D$  provided by the previous theorem, the summand  $V$  is non-trivial and thus one may attach to it another smaller cone

$$\mathbb{C} := C \cap V$$

We call  $\mathbb{C}$  the *plain cone* associated to  $C$ . From its very construction the cone  $\mathbb{C}$  is open in the space  $V$  and span this latter, and  $\mathbb{C} \cap \Gamma^*$  is a dense subset of the plain cone  $\mathbb{C}$ . It is moreover easy to observe that  $\mathbb{C} = C$  when and only when the set  $C \cap \Gamma^*$

is dense in the cone  $C$ . For any non trivial cone type  $\mathfrak{t} \in \mathfrak{T}_{W,\Gamma}$  we may denote  $C_{\mathfrak{t}}$  to be the plain cone associated with  $C_{\mathfrak{t}}$ .

**6.3. The main theorem for internal system Ellis semigroup.** Let us consider an Ellis transformation  $g \in E(\Xi^\Gamma, \Gamma)$  with associated translation vector  $w_g$  in  $\mathbb{R}^n$ . Given a hyperplane  $H \in \mathfrak{H}_W$ , it has been shown in lemma 5.6 that the mapping  $g$  falls into one and only one clopen subset of the form  $[H^{\mathfrak{t}} + w_g]_E$ , whose sign for any hyperplane  $H \in \mathfrak{H}_W$  determines a face type  $\mathfrak{t}_g$  uniquely. To see that  $\mathfrak{t}_g$  is a face type in the above sense, that is, is associated with a non-empty cone  $C_g$  of the stratification obtained from  $\mathfrak{H}_W$ , observe that from the Boolean rules of proposition 5.4 one has

$$(16) \quad g \in \bigcap_{H \in \mathfrak{H}_W} [H^{\mathfrak{t}_g(H)} + w_g]_E = \left[ \bigcap_{H \in \mathfrak{H}_W} H^{\mathfrak{t}_g(H)} + w_g \right]_E = [C_g + w_g]_E$$

which ensure that  $C_g$  must be non-empty. Having related the internal system Ellis semigroup with the face semigroup just defined, we are now able to set our main theorem concerning the internal system Ellis semigroup:

**Theorem 6.3.** *The mapping associating to any transformation  $g$  the couple  $(w_g, \mathfrak{t}_g)$  establish an isomorphism between the Ellis semigroup  $E(\Xi^\Gamma, \Gamma)$  and the sub-semigroup of the direct product  $\mathbb{R}^n \times \mathfrak{T}_{W,\Gamma}$  given by*

$$\bigsqcup_{\mathfrak{t} \in \mathfrak{T}_{W,\Gamma}} [(C_{\mathfrak{t}} + \Gamma^*) \times \{\mathfrak{t}\}]$$

Moreover, this isomorphism becomes a homeomorphism when the above union is equipped with the following convergence class:  $(w_\lambda, \mathfrak{t}_\lambda) \longrightarrow (w, \mathfrak{t})$  iff

$$\forall \varepsilon > 0, \exists \delta_\lambda > 0 \text{ such that } C_{\mathfrak{t}_\lambda}(w_\lambda, \delta_\lambda) \subset C_{\mathfrak{t}}(w, \varepsilon) \text{ for large enough } \lambda$$

Finally, the Ellis semigroup  $E(\Xi^\Gamma, \Gamma)$  has a first countable topology.

The convergence class of the statement is to be more precise the full family of nets and limit points which obeys the above condition. This family completely characterizes the Ellis semigroup topology since, being derived from the topology of the internal system Ellis semigroup, it satisfies a correct set of axioms which permit to recover the closure operator on the Ellis semigroup, and thus its topology (see [18]).

The remaining part of this section is turned on the proof of the above theorem. To this end we decompose the proof into basically three parts: the first one states the existence of a semigroup isomorphism between the internal system Ellis semigroup and a sub-semigroup of the direct product  $\mathbb{R}^n \times \mathfrak{T}_W$ , and the second step sets the proof that the isomorphic image stands into  $\mathbb{R}^n \times \mathfrak{T}_{W,\Gamma}$  and is of the form stated above. In a third part we then show the topological part of the statement.

6.3.1. *Step 1: Existence of the semigroup isomorphism.*

**Proposition 6.4.** *The mapping  $E(\Xi^\Gamma, \Gamma) \longrightarrow \mathfrak{T}_W$  associating to each transformation  $g$  its face type  $\mathfrak{t}_g$  is a semigroup morphism.*



*Proof.* We have to show that given two transformations  $g$  and  $h$  the face types  $\mathfrak{t}_{g,h}$  and  $\mathfrak{t}_g \cdot \mathfrak{t}_h$  are equal. By (16) the transformation  $g.h$  lies into the clopen subset  $[C_{g.h} + w_{g,h}]_E$ . Since by construction  $\Gamma$  is dense in the Ellis semigroup, and since the composition law on this latter is right-continuous, one can find a  $\gamma \in \Gamma$  sufficiently close to  $h$  in the sense that

$$(i) \quad \gamma \in [C_h + w_h]_E \quad (ii) \quad g.\gamma \in [C_{g,h} + w_{g,h}]_E$$

Now from lemma 5.6 together with the Boolean rules of proposition 5.4, one can deduce from (i) that  $\gamma^* \in C_h + w_h$ , or equivalently  $(\gamma^* - w_h) \in C_h$  in the internal space. Moreover, as the transformation  $g.\gamma$  lies both into the clopen subset  $[C_{g,\gamma} + w_{g,\gamma}]_E$  and the open subset  $(\Pi^*)^{-1}(B(w_{g,\gamma}, \varepsilon))$ , again from the density of  $\Gamma$  in the Ellis semigroup together with point (ii) one can find an element  $\gamma_\varepsilon \in \Gamma$  sufficiently close to  $g.\gamma$  so that

$$\gamma_\varepsilon^* \in (C_{g,h} + w_{g,h}) \cap (C_{g,\gamma} + w_{g,\gamma}) \cap B(w_{g,\gamma}, \varepsilon)$$

Since the cone associated with  $g.\gamma$  is equal to the one associated with  $g$ , the previous fact implies that

$$C_g(\gamma^* - w_h, \varepsilon) \cap C_{g,h} \neq \emptyset \quad \forall \varepsilon > 0 \quad \text{with} \quad \gamma^* - w_h \in C_h$$

Let us now consider three cases about a hyperplane  $H \in \mathfrak{H}_W$ :

- $\mathfrak{t}_h(H) = +$ : in that case the vector  $\gamma^* - w_h \in C_h$  falls into the open half space  $H^+$ , and thus one may find a  $\varepsilon_0$  with  $C_g(\gamma^* - w_h, \varepsilon_0)$  included into  $H^+$ , so that  $H^+$  must intersect the cone  $C_{g,h}$ . This force  $C_{g,h} \subset H^+$ , or equivalently  $\mathfrak{t}_{g,h}(H) = +$ .
- $\mathfrak{t}_h(H) = -$ : by the same type of argument one can show that  $\mathfrak{t}_{g,h}(H) = -$ .
- $\mathfrak{t}_h(H) = 0$ : in this latter case one has  $\gamma^* - w_h \in H$  and thus  $C_g(\gamma^* - w_h, \varepsilon) \subset H^{\mathfrak{t}_g(H)}$  whatever the symbol  $\mathfrak{t}_g(H)$ . It thus follows that  $H^{\mathfrak{t}_g(H)} \cap C_{g,h}$  is non-empty, which necessary gives  $C_{g,h} \subset H^{\mathfrak{t}_g(H)}$ , or equivalently  $\mathfrak{t}_{g,h}(H) = \mathfrak{t}_g(H)$ .

The above three cases show that the cone type  $\mathfrak{t}_{g,h}$  is equal to the composition  $\mathfrak{t}_g \cdot \mathfrak{t}_h$ , as desired.  $\square$

Combining the previous proposition together with the existence of the onto morphism of proposition 3.5, we see that the mapping associating to each transformation  $g$  in  $E(\Xi^\Gamma, \Gamma)$  the couple  $(w_g, \mathfrak{t}_g)$  in the product semigroup  $\mathbb{R}^n \times \mathfrak{T}_W$  is a semigroup morphism. Thus in order to settle step 1 what we only need to show is injectivity:

Suppose for that purpose that two transformations  $g$  and  $h$  satisfy  $w_g = w_h =: w$  in the internal space. Then by using the sub-basis of proposition 5.4 one can find a vector  $w_0$  as well as a hyperplane  $H \in \mathfrak{H}_W$  such that  $g$  and  $h$  falls into different clopen subsets among the partition

$$E(\Xi^\Gamma, \Gamma) = [H^- + w_0]_E \sqcup [H + w_0]_E \sqcup [H^+ + w_0]_E$$

Thus one must have that  $w$  and  $w_0$  are equal up to a vector of the hyperplane  $H$ , and it implies that the signs  $\mathfrak{t}_g(H)$  and  $\mathfrak{t}_h(H)$  must be different. This exactly means that the associated cone types  $\mathfrak{t}_g$  and  $\mathfrak{t}_h$  are different, and the proof of step 1 is complete.

### 6.3.2. Step 2: Determination of the isomorphic image.

The question now is to identify the sub-semigroup of  $\mathbb{R}^n \times \mathfrak{T}$  isomorphic with the internal system Ellis semigroup via the previous mapping. To that end, one may set

this sub-semigroup as a disjoint union

$$\bigsqcup_{\mathfrak{t} \in \mathfrak{T}_W} \mathbb{R}_{\mathfrak{t}}^n \times \{\mathfrak{t}\}$$

for some Euclidean subsets  $\mathbb{R}_{\mathfrak{t}}^n$ , the *allowed translations* of a cone type  $\mathfrak{t}$ , which we thus need to identify. A first point about this is the following lemma.

**Lemma 6.5.** *For any cone type  $\mathfrak{t} \in \mathfrak{T}_W$  with associated cone  $C_{\mathfrak{t}}$  one has*

$$\mathbb{R}_{\mathfrak{t}}^n = \{w \in \mathbb{R}^n \mid (C_{\mathfrak{t}} + w) \cap \Gamma^* \text{ accumulates at } w\}$$

*Proof.* Given some  $\mathfrak{t} \in \mathfrak{T}_W$  with associated cone  $C_{\mathfrak{t}}$ , its set of allowed translations  $\mathbb{R}_{\mathfrak{t}}^n$  is by construction  $\mathbb{R}_{\mathfrak{t}}^n = \{w_g \mid g \in E(\Xi^{\Gamma}, \Gamma) \text{ and } \mathfrak{t}_g = \mathfrak{t}\}$ .

Let us show " $\supseteq$ ": If  $w$  is such that  $(C_{\mathfrak{t}} + w) \cap \Gamma^*$  accumulates at  $w$  then the intersection  $(C_{\mathfrak{t}} + w) \cap B(w, \varepsilon) \cap \Gamma^*$  is non-empty for any  $\varepsilon > 0$ , and thus the family  $\{(C_{\mathfrak{t}} + w)_E \cap (\Pi^*)^{-1}(B(w, \varepsilon))\}_{\varepsilon > 0}$  forms a filterbase in the space  $E(\Xi^{\Gamma}, \Gamma)$ . In turns, the morphism  $\Pi^*$  is by proposition 2.7 a proper map so this filterbase, for  $0 < \varepsilon < \varepsilon_0$ , lies into the fixed compact subset  $(\Pi^*)^{-1}(\overline{B}(w, \varepsilon_0))$  and thus possess an accumulation point  $g$ . This Ellis transformation necessarily satisfies  $w_g = w$ , and because the set  $[C_{\mathfrak{t}} + w]_E = [C_{\mathfrak{t}} + w_g]_E$  is closed, containing the above filterbase, it thus contains  $g$ . We deduce that  $C_g = C_{\mathfrak{t}}$ , or equivalently  $\mathfrak{t}_g = \mathfrak{t}$ , giving that  $w = w_g \in \mathbb{R}_{\mathfrak{t}}^n$ .

Conversely we show " $\subseteq$ ": Given some cone type  $\mathfrak{t}$  and some Ellis transformation  $g$  with  $\mathfrak{t} = \mathfrak{t}_g$ , then as  $g$  lies into  $[C_g + w_g]_E$  one can select a net of elements of  $(C_g + w_g) \cap \Gamma^* = (C_{\mathfrak{t}} + w_g) \cap \Gamma^*$  converging to  $g$  in the internal system Ellis semigroup. Thus applying  $\Pi^*$  we obtain a net of  $(C_{\mathfrak{t}} + w_g) \cap \Gamma^*$  converging to  $w_g$  in the Euclidean space  $\mathbb{R}^n$ , so that  $(C_{\mathfrak{t}} + w_g) \cap \Gamma^*$  accumulates at  $w_g$ .  $\square$

Let now  $\mathfrak{T}_{W,0}$  be the homomorphic image of the internal system Ellis semigroup into the face semigroup  $\mathfrak{T}_W$  through the morphism of proposition 6.4. Then it precisely consists of those cone types which have a non-empty associated subset  $\mathbb{R}_{\mathfrak{t}}^n$  of allowed translations. From the very definition of the plain face semigroup  $\mathfrak{T}_{W,\Gamma}$ , a face type  $\mathfrak{t}$  is non-trivial when and only when  $0$  lies into  $\mathbb{R}_{\mathfrak{t}}^n$ , which shows in particular that  $\mathfrak{T}_{W,0}$  contains the plain face semigroup  $\mathfrak{T}_{W,\Gamma}$ .

We will now set any Euclidean subset  $\mathbb{R}_{\mathfrak{t}}^n$  in a more suitable form. Obviously it is sufficient to consider cone types of the homomorphic image  $\mathfrak{T}_{W,0}$ . Observe that for any such cone type their associated Euclidean subset of allowed translations is stable under  $\Gamma^*$ -translation.

**Proposition 6.6.** *Let  $\mathfrak{t} \in \mathfrak{T}_{W,0}$ , with  $\langle C_{\mathfrak{t}} \rangle = V_{\mathfrak{t}} \oplus D_{\mathfrak{t}}$  its associated direct sum decomposition of theorem 6.2. Then one has*

$$\mathbb{R}_{\mathfrak{t}}^n = V_{\mathfrak{t}} + \Gamma^*$$

*Proof.* For  $\mathfrak{t} \in \mathfrak{T}_{W,0}$  and  $\langle C_{\mathfrak{t}} \rangle = V_{\mathfrak{t}} \oplus D_{\mathfrak{t}}$  its associated direct sum decomposition, denote by  $P^V$  (resp.  $P^D$ ) the skew projection of  $\langle C_{\mathfrak{t}} \rangle$  with range  $V_{\mathfrak{t}}$  and kernel  $D_{\mathfrak{t}}$  (resp. the skew projection with range  $D_{\mathfrak{t}}$  and kernel  $V_{\mathfrak{t}}$ ). Then from the particular form of the decomposition one has  $P^V(\langle C_{\mathfrak{t}} \rangle \cap \Gamma^*) = V_{\mathfrak{t}} \cap \Gamma^*$  and  $P^D(\langle C_{\mathfrak{t}} \rangle \cap \Gamma^*) = D_{\mathfrak{t}} \cap \Gamma^*$ .

Let us show first that  $\mathbb{R}_t^n$  lies into  $\langle C_t \rangle + \Gamma^*$ : Any vector  $w \in \mathbb{R}_t^n$  admits some  $\gamma^*$  into  $(C_t + w) \cap \Gamma^*$ , so that  $\gamma^* - w$  lies into  $C_t$  and thus into  $\langle C_t \rangle$ . So does the vector  $w - \gamma^*$ , giving that  $w$  lies into  $\langle C_t \rangle + \Gamma^*$ .

Now we show that  $\mathbb{R}_t^n$  lies into  $V_t + \Gamma$ : Given  $w \in \mathbb{R}_t^n$ , one may write  $w = w' + \gamma^*$  with  $w' \in \langle C_t \rangle$  and  $\gamma \in \Gamma^*$ ,  $w'$  itself being into  $\mathbb{R}_t^n$  as this latter is stable under  $\Gamma^*$ -translation. It thus suffices to prove that  $w'$  lies into  $V_t + \Gamma$  to get the point. From the previous lemma  $w'$  is the limit point of a sequence  $(\gamma_k^*)$  of elements in  $(C_t + w') \cap \Gamma^*$ , in turns included into  $\langle C_t \rangle \cap \Gamma^*$ . Thus  $P^D(\gamma_k^*)$  converges to  $P^D(w')$  and  $P^V(\gamma_k^*)$  converges to  $P^V(w')$ . But has the sequence  $(P^D(\gamma_k^*))$  lies into the uniformly discrete subset  $D_t \cap \Gamma^*$  of  $D_t$  it must be eventually constant, equal to  $P^D(w')$  for great enough  $k$ . Hence  $P^D(w')$  lies into  $\Gamma^*$ , which gives  $w' = P^W(w') + P^D(w') \in V_t + \Gamma^*$ , as desired.

We wish to observe that the sequence eventually satisfies  $P^V(\gamma_k^*) = \gamma_k^* - P^D(w') \in (C_t + w') \cap \Gamma^* - P^D(w')$ , with  $P^D(w') \in \Gamma^*$ , and thus  $P^V(\gamma_k^*) \in (C_t + P^V(w')) \cap \Gamma^*$ . Hence  $P^V(\gamma_k^*) - P^V(w') = P^V(\gamma_k^* - w')$  lies into both  $V_t$  and  $C_t$  eventually, which ensure that the intersection  $C_t := C_t \cap V_t$  is non-empty.

Now we show that  $\mathbb{R}_t^n$  contains  $V_t + \Gamma$ : to that end it suffice from  $\Gamma^*$ -invariance to show that it contains  $V_t$ . First it is clear that the subset  $C_t$  is a (non-empty) open cone of the space  $V_t$ , since is the intersection of  $C_t$  which is open in its own spanned space  $\langle C_t \rangle$  with the subspace  $V_t$ . Let now  $w \in V_t$  be given. Then  $C_t$  is open in  $W_t$  and is a cone pointed at 0, so that  $C_t + w$  is an open cone of  $V_t$  pointed at  $w$ . But from the density of  $V_t \cap \Gamma^*$  in  $V_t$  one can obtain  $w$  as an accumulation point of  $C_t + w \cap \Gamma^*$  and thus of  $(C_t + w) \cap \Gamma^*$ , showing that  $w \in \mathbb{R}_t^n$ , as desired.  $\square$

From the previous proposition one gets that any cone type  $\mathfrak{t}$  of  $\mathfrak{T}_{W,0}$  has the origin 0 as allowed translation, and thus is an element of  $\mathfrak{T}_{W,\Gamma}$ . This shows that the internal system Ellis semigroup is isomorphic with a sub-semigroup of the direct product  $\mathbb{R}^n \times \mathfrak{T}_{W,\Gamma}$ , and that its isomorphic image is of the form stated in theorem 6.3, reminding that  $V_t$  is spanned by the pain cone  $C_t$  for any  $\mathfrak{t} \in \mathfrak{T}_{W,\Gamma}$ . This completes step 2.

### 6.3.3. Step 3: The topology of convergence.

Let us first show the first countability property of the internal system Ellis semigroup: From the injectivity of the mapping associating to any transformation  $g$  the couple  $(w_g, \mathfrak{t}_g)$ , one can deduce that  $g$  is the only transformation in its fiber with respect to  $\Pi^*$  falling into the clopen subset  $[C_g + w_g]_E$  of the Ellis semigroup. It follows by lemma 5.3 that a neighborhood basis of  $g$  is provided by the intersections

$$(17) \quad [C_g + w_g]_E \cap (\Pi^*)^{-1}(B(w_g, \varepsilon))$$

It is then clear that one can extract a countable sub-basis of this family, completing the argument. Now we wish to show the bi-continuity of the stated isomorphism, and to that end we let  $(g_\lambda)$  be a net of the Ellis semigroup with associated net  $(w_\lambda, \mathfrak{t}_\lambda)$  in the direct product  $\mathbb{R}^n \times \mathfrak{T}_{W,\Gamma}$ , and  $g$  be some Ellis transformation with associated couple  $(w, \mathfrak{t})$ . Let us first set a useful lemma:

**Lemma 6.7.** *There exists an  $\varepsilon_0 > 0$  such that, for any  $\mathfrak{t} \in \mathfrak{T}_{W,\Gamma}$  and  $w \in \mathbb{R}^n = V_{\mathfrak{t}} + \Gamma^*$ , we have*

$$C_{\mathfrak{t}}(w, \varepsilon) \cap \Gamma^* = \mathbf{C}_{\mathfrak{t}}(w, \varepsilon) \cap \Gamma^* \quad \forall 0 < \varepsilon \leq \varepsilon_0$$

*Proof.* Clearly the cone  $C_{\mathfrak{t}}(w, \varepsilon)$  contains  $\mathbf{C}_{\mathfrak{t}}(w, \varepsilon)$  for all  $\varepsilon > 0$ . Conversely let  $\mathfrak{t} \in \mathfrak{T}_{W,\Gamma}$  be chosen, with associated cone  $C_{\mathfrak{t}}$  in  $\mathbb{R}^n$  and the direct sum decomposition  $\langle C_{\mathfrak{t}} \rangle = V_{\mathfrak{t}} \oplus D_{\mathfrak{t}}$  provided by theorem 6.2. As  $D_{\mathfrak{t}} \cap \Gamma^*$  is uniformly discrete in  $D_{\mathfrak{t}}$ , with  $\varepsilon_{\mathfrak{t}} > 0$  some radius of discreteness, we must have

$$\langle C_{\mathfrak{t}} \rangle \cap B(w, \varepsilon_{\mathfrak{t}}) \cap \Gamma^* = (V_{\mathfrak{t}} + w) \cap B(w, \varepsilon_{\mathfrak{t}}) \cap \Gamma^*$$

for any  $w \in V_{\mathfrak{t}} + \Gamma^*$ . Hence by intersecting with  $C_{\mathfrak{t}} + w$  we obtain

$$C_{\mathfrak{t}}(w, \varepsilon_{\mathfrak{t}}) \cap \Gamma^* = (C_{\mathfrak{t}} + w) \cap (V_{\mathfrak{t}} + w) \cap B(w, \varepsilon_{\mathfrak{t}}) \cap \Gamma^* = \mathbf{C}_{\mathfrak{t}}(w, \varepsilon_{\mathfrak{t}}) \cap \Gamma^*$$

Finally, taking  $\varepsilon_0$  to be the minimum over  $\varepsilon_{\mathfrak{t}}$ ,  $\mathfrak{t} \in \mathfrak{T}_{W,\Gamma}$ , gives the statement.  $\square$

Then  $g_{\lambda}$  converges to  $g$  if and only if for any  $\varepsilon > 0$ , which can be chosen less than the constant  $\varepsilon_0$  of lemma 6.7, there is some net of positive real numbers  $(\delta_{\lambda})$ , which can be chosen less than the constant  $\varepsilon_0$  as well, such that one has for great enough  $\lambda$

$$[C_{\mathfrak{t}_{\lambda}} + w_{\lambda}]_E \cap (\Pi^*)^{-1}(B(w_{\lambda}, \delta_{\lambda})) \subset [C_{\mathfrak{t}} + w]_E \cap (\Pi^*)^{-1}(B(w, \varepsilon))$$

By lemma 6.7, intersecting with  $\Gamma^*$  leads for great enough  $\lambda$

$$\mathbf{C}_{\mathfrak{t}_{\lambda}}(w_{\lambda}, \delta_{\lambda}) \cap \Gamma^* \subset \mathbf{C}_{\mathfrak{t}}(w, \varepsilon) \cap \Gamma^*$$

Now the affine space generated by  $\mathbf{C}_{\mathfrak{t}_{\lambda}}(w_{\lambda}, \delta_{\lambda})$  is precisely  $V_{\mathfrak{t}_{\lambda}} + w_{\lambda}$ , which contains, since  $w_{\lambda}$  is an allowed translation for  $\mathfrak{t}_{\lambda}$ , a dense subset of elements of  $\Gamma^*$ . The same occurs about  $w$  with respect to  $\mathfrak{t}$ , and thus we get for great enough  $\lambda$  the inclusions

$$V_{\mathfrak{t}_{\lambda}} + w_{\lambda} \subset V_{\mathfrak{t}} + w$$

As  $\mathbf{C}_{\mathfrak{t}}(w, \varepsilon)$  is a topologically regular open subset of  $V_{\mathfrak{t}} + w$ , its intersection with  $V_{\mathfrak{t}_{\lambda}} + w_{\lambda}$  forms an open topologically regular subset of this latter affine space, containing  $\mathbf{C}_{\mathfrak{t}_{\lambda}}(w_{\lambda}, \delta_{\lambda}) \cap \Gamma^*$ . As  $\mathbf{C}_{\mathfrak{t}_{\lambda}}(w_{\lambda}, \delta_{\lambda})$  is a topologically regular open subset of  $V_{\mathfrak{t}_{\lambda}} + w_{\lambda}$  as well, taking closure and next interior in  $V_{\mathfrak{t}_{\lambda}} + w_{\lambda}$  provides for great enough  $\lambda$

$$\mathbf{C}_{\mathfrak{t}_{\lambda}}(w_{\lambda}, \delta_{\lambda}) \subset \mathbf{C}_{\mathfrak{t}}(w, \varepsilon)$$

thus giving the " $\Rightarrow$ " part of the statement.

Conversely, let us suppose that for any  $\varepsilon > 0$ , which can be chosen less than the constant  $\varepsilon_0$  of lemma 6.7, there is some net of positive real numbers  $(\delta_{\lambda})$ , which can be chosen less than the constant  $\varepsilon_0$  as well, such that one has  $\mathbf{C}_{\mathfrak{t}_{\lambda}}(w_{\lambda}, \delta_{\lambda}) \subset \mathbf{C}_{\mathfrak{t}}(w, \varepsilon) \subset C_{\mathfrak{t}} + w$  for great enough  $\lambda$ . Now the first point is that the net  $(w_{\lambda})$  converges to  $w$  in  $\mathbb{R}^n$ , and so  $g_{\lambda}$  falls into the inverse image of any ball  $B(w, \varepsilon)$  for great enough  $\lambda$ . Secondly, any  $g_{\lambda}$  has a neighborhood of the form  $[C_{\mathfrak{t}_{\lambda}} + w_{\lambda}]_E \cap (\Pi^*)^{-1}(B(w_{\lambda}, \delta_{\lambda}))$ , which is contained into the subset  $[C_{\mathfrak{t}_{\lambda}}(w_{\lambda}, \delta_{\lambda})]_E = [\mathbf{C}_{\mathfrak{t}_{\lambda}}(w_{\lambda}, \delta_{\lambda})]_E$  and thus into  $[C_{\mathfrak{t}} + w]_E$  for great enough  $\lambda$ . Combining these two arguments we deduce from the neighborhood basis formula (17) that  $g_{\lambda}$  converges to  $g$  in the internal system Ellis semigroup.

This completes the proof of theorem 6.3.  $\square$

## 7. RESULTS ON THE HULL ELLIS SEMIGROUP AND ADDITIONAL ALGEBRAIC FEATURES

In this section we arrive at our main result, namely, the algebraic and topological description of the Ellis semigroup for a hull  $\mathbb{X}$  of almost canonical model sets together with its  $\mathbb{R}^d$ -action.

**7.1. The main result.** From theorem 3.6, any transformation  $\mathbf{g}$  in the semigroup  $E(\mathbb{X}, \mathbb{R}^d)$  may be written as  $\tilde{g} - s$  where  $g$  is a transformation in  $E(\Xi^\Gamma, \Gamma)$  and  $s$  a vector of  $\mathbb{R}^d$ , and with  $g$  uniquely defined up to an element of  $\Gamma$ . Thus we may associate to any transformation  $\mathbf{g} = \tilde{g} - s$  the cone type of any underlying transformation  $g \in E(\Xi^\Gamma, \Gamma)$ , which we write  $\mathbf{t}_g$ , thus providing a semigroup morphism from the hull Ellis semigroup  $E(\mathbb{X}, \mathbb{R}^d)$  into the non-trivial face semigroup  $\mathfrak{T}_{W, \Gamma}$ . We are now able to formulate the main result of this work, which is completely deduced from theorems 3.6 and 6.3:

**Theorem 7.1.** *The mapping associating to any transformation  $\mathbf{g}$  the couple  $(z_{\mathbf{g}}, \mathbf{t}_{\mathbf{g}})$  establish an isomorphism between the Ellis semigroup  $E(\mathbb{X}, \mathbb{R}^d)$  and the sub-semigroup of the direct product  $[\mathbb{R}^{n+d}]_\Sigma \times \mathfrak{T}_{W, \Gamma}$  given by*

$$\bigsqcup_{\mathbf{t} \in \mathfrak{T}_{W, \Gamma}} \left[ \langle \mathbf{C}_{\mathbf{t}} \rangle \times \mathbb{R}^d \right]_\Sigma \times \{\mathbf{t}\}$$

Moreover, this isomorphism becomes a homeomorphism when the above union is equipped with the following convergence class:  $(z_\lambda, \mathbf{t}_\lambda) \rightarrow (z, \mathbf{t})$  if and only if one can write  $z_\lambda = [w_\lambda, s_\lambda]_\Sigma$  and  $z = [w, s]_\Sigma$  such that

- (i)  $s_\lambda \rightarrow s$  in  $\mathbb{R}^d$
- (ii)  $\forall \varepsilon > 0, \exists \delta_\lambda > 0$  such that  $\mathbf{C}_{\mathbf{t}_\lambda}(w_\lambda, \delta_\lambda) \subset \mathbf{C}_{\mathbf{t}}(w, \varepsilon)$  for large enough  $\lambda$  in  $\mathbb{R}^n$

Finally, the Ellis semigroup  $E(\mathbb{X}, \mathbb{R}^d)$  has a first countable topology, and the dynamical system  $(\mathbb{X}, \mathbb{R}^d)$  is tame.

### 7.2. Additional algebraic features.

**7.2.1. Invertible Ellis transformations.** One can naturally ask whether there are invertible transformations in the hull Ellis semigroup which are invertible, though not being a homeomorphism provided by the  $\mathbb{R}^d$ -action. It turns out that the answer is no. For, it has been seen that any cone type  $\mathbf{t} \in \mathfrak{T}_{W, \Gamma}$  is idempotent, and thus an invertible transformation must correspond to a couple of the form  $(z, \mathbf{o})$  where  $\mathbf{o}$  is the identity cone type in  $\mathbf{t} \in \mathfrak{T}_{W, \Gamma}$ . Since the cone with cone type  $\mathbf{o}$  is precisely the trivial cone  $\{0\}$ , its associated plain cone  $\mathbf{C}_{\mathbf{o}}$  is nothing but  $\{0\}$  and theorem 7.1 ensures that  $z$  must be an element of the form  $[0, s]_\Sigma$  in  $[\{0\} \times \mathbb{R}^d]_\Sigma$ . It follows that the underlying transformation is the homeomorphism arising from the translation by the vector  $s \in \mathbb{R}^d$ .

**7.2.2. Range of Ellis transformations.** It is natural to set on the Ellis semigroup  $E(\mathbb{X}, \mathbb{R}^d)$  a pre-order by letting  $\mathbf{g} \leq \mathbf{g}'$  whenever the *range* of the mapping  $\mathbf{g}$  is contained into that of  $\mathbf{g}'$ . By range we mean here the subset  $r(\mathbf{g}) := \mathbb{X} \cdot \mathbf{g}$  of the hull  $\mathbb{X}$ . When one considers idempotent transformations  $\mathbf{q}$  and  $\mathbf{q}'$  then it is direct to show that  $\mathbf{q} \leq \mathbf{q}'$  when and only when one has  $\mathbf{q} = \mathbf{q} \cdot \mathbf{q}'$ , thus turning this pre-order into

algebraic terms in this particular setting. In the case of an almost canonical hull Ellis semigroup we are able to describe this pre-order in a quite elegant manner:

**Proposition 7.2.** *For any transformations of  $E(\mathbb{X}, \mathbb{R}^d)$  we have the equivalence*

$$\mathfrak{g} \leq \mathfrak{g}' \iff C_{\mathfrak{g}} \leq C_{\mathfrak{g}'}$$

The proposition above asserts that the range of  $\mathfrak{g}$  is contained into the range of  $\mathfrak{g}'$  if and only if the cone  $C_{\mathfrak{g}'}$  is equal or a lower dimensional facet of the cone  $C_{\mathfrak{g}}$ .

*Proof.* Let  $\mathfrak{g}$  and  $\mathfrak{g}'$  be chosen. Each are element of a sub-group respectively given by  $[\langle C_{\mathfrak{t}_{\mathfrak{g}}} \rangle \times \mathbb{R}^d]_{\Sigma} \times \{\mathfrak{t}_{\mathfrak{g}}\}$  and  $[\langle C_{\mathfrak{t}_{\mathfrak{g}'}} \rangle \times \mathbb{R}^d]_{\Sigma} \times \{\mathfrak{t}_{\mathfrak{g}'}\}$ , and thus one can see that  $r(\mathfrak{g}) = r([0]_{\Sigma} \times \{\mathfrak{t}_{\mathfrak{g}}\})$  and that  $r(\mathfrak{g}') = r([0]_{\Sigma} \times \{\mathfrak{t}_{\mathfrak{g}'}\})$ . From what have been just said it becomes clear that  $\mathfrak{g} \leq \mathfrak{g}'$  if and only if  $\mathfrak{t}_{\mathfrak{g}} = \mathfrak{t}_{\mathfrak{g}} \cdot \mathfrak{t}_{\mathfrak{g}'}$ , which exactly means that the cone  $C_{\mathfrak{g}'}$  is equal or a lower dimensional facet of the cone  $C_{\mathfrak{g}}$ , or equivalently  $C_{\mathfrak{g}} \leq C_{\mathfrak{g}'}$ .  $\square$

**7.2.3. Ideals.** The general theory of Ellis semigroups allows a great importance to the ideal theory of any Ellis semigroup. In the case of a almost canonical hull Ellis semigroup it is easy to prove the proposition stated below, showing that the ideal theory of the hull Ellis semigroup reduces to the ideal theory of the semigroup  $\mathfrak{T}_{W,\Gamma}$ :

**Proposition 7.3.** *Each right ideal  $\mathfrak{M}$  of the non-trivial face semigroup  $\mathfrak{T}_{W,\Gamma}$  defines a right ideal of the hull Ellis semigroup by*

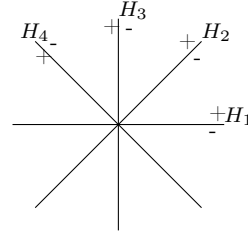
$$\bigsqcup_{\mathfrak{t} \in \mathfrak{M}} [\langle C_{\mathfrak{t}} \rangle \times \mathbb{R}^d]_{\Sigma} \times \{\mathfrak{t}\}$$

and conversely each right ideal of  $E(\mathbb{X}, \mathbb{R}^d)$  arises in this manner.

We can in particular easily identify the unique minimal ideal of  $E(\mathbb{X}, \mathbb{R}^d)$ : This latter is isomorphic with the direct product  $[\mathbb{R}^{n+d}]_{\Sigma} \times \mathfrak{M}^{ch}$  where  $\mathfrak{M}^{ch}$  is the family of cone types associated with the *chambers* of the stratification defined by the collection of hyperplanes used to construct the face semigroup  $\mathfrak{T}_W$ .

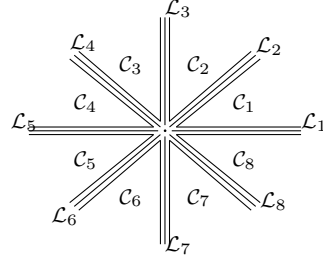
**7.3. An explicit computation.** We consider the hull  $\mathbb{X}_{oct}$  associated with the real cut & project scheme and octagonal window presented in 1.4. The associated family of linear hyperplanes parallel to faces of the window (or its reversed set) is provided, in the orthonormal basis  $(e_1^*, e_2^*)$  of the internal space  $\mathbb{R}_{int}^2$ , by

$$\begin{aligned} H_1 &:= \langle v_1 \rangle = \langle v_2 - v_4 \rangle & v_1 &:= e_1^* \\ H_2 &:= \langle v_2 \rangle = \langle v_1 + v_3 \rangle & v_2 &:= \frac{e_1^* + e_2^*}{\sqrt{2}} \\ H_3 &:= \langle v_3 \rangle = \langle v_2 + v_4 \rangle & v_3 &:= e_2^* \\ H_4 &:= \langle v_4 \rangle = \langle v_1 - v_3 \rangle & v_4 &:= \frac{e_2^* - e_1^*}{\sqrt{2}} \end{aligned}$$



The stratification obtained from these hyperplanes is of the following form:

The internal space  $\mathbb{R}_{int}^2$  is partitionned into 17 different cones: the singleton  $\{0\}$ , eight half-lines  $\{L_1, \dots, L_8\}$  pointed at 0 though not containing it which we label  $L_i, L_{i+4} \subset H_i$  for  $1 \leq i \leq 4$ , and eight chambers  $\{C_1, \dots, C_8\}$ , each consisting of an  $1/8^{th}$  part of the space and being open cones pointed at 0.



Now the stabilizers  $Stab_{\Gamma}(H_i)$  are dense in  $H_i$  for each index  $1 \leq i \leq 4$ , and we deduce that each cone of this stratification is non-trivial, and moreover equal to its associated plain cone. Thus  $\langle C_0 \rangle = \{0\}$  as usual, whereas  $\langle C_{L_i} \rangle = \langle C_{L_{i+4}} \rangle = H_i$  for each value  $1 \leq i \leq 4$ , and  $\langle C_{t_{C_i}} \rangle = \mathbb{R}^2$  for each index  $1 \leq i \leq 8$ .

Consequently, the hull Ellis semigroup  $E(\mathbb{X}_{oct}, \mathbb{R}^2)$  is in this case obtained as

$$\left( \bigsqcup_{i=1}^8 [\mathbb{R}^4]_{\mathbb{Z}^4} \times \{t_{C_i}\} \right) \bigsqcup \left( \bigsqcup_{i=1}^4 [H_i \times \mathbb{R}^2]_{\mathbb{Z}^4} \times \{t_{L_i}, t_{L_{i+4}}\} \right) \bigsqcup \mathbb{R}^2$$

## 8. THE ELLIS ACTION ON THE HULL

**8.1. A further look on cones.** We saw in section 5 that to any model set  $\Lambda$  of the internal system can be associated a cone  $C_{\Lambda}$ , that is, an open connected cones pointed at 0 with boundary delimited by hyperplanes of a sub-family  $\mathfrak{H}_{w_{\Lambda}}$  of  $\mathfrak{H}_W$ . Moreover each such cone admits a unique cone type  $\mathfrak{c}_{\Lambda}$  with domain  $\mathfrak{H}_{w_{\Lambda}}$ , and there can be only finitely many such cone types, whose family is denoted  $\mathfrak{C}$ . Now if one look at some model set  $\Lambda_0$  in the hull  $\mathbb{X}$  then it always can be written  $\Lambda_0 = \Lambda - t$ , where  $\Lambda$  lies in the internal system and  $t$  is a vector of  $\mathbb{R}^d$ . This presentation is unique up to a translation of both the model set  $\Lambda$  and the vector  $t$  by some  $\gamma \in \Gamma$ . Thus one may without misunderstanding define the cut type  $\mathfrak{H}_{z_{\Lambda_0}}$  and the cone type  $\mathfrak{c}_{\Lambda_0}$  with domain  $\mathfrak{H}_{z_{\Lambda_0}}$  to be the ones associated with  $\Lambda \in \Xi^{\Gamma}$  in the decomposition  $\Lambda_0 = \Lambda - t$ . We may then describe the hull, as it was already done by Le [19], as follows:

**Theorem 8.1.** *The mapping associating to any model set  $\Lambda$  the couple  $(z_{\Lambda}, \mathfrak{c}_{\Lambda})$  establish a bijective correspondence between the hull  $\mathbb{X}$  and*

$$\left\{ (z, \mathfrak{c}) \in \left[ \mathbb{R}^{n+d} \right]_{\Sigma} \times \mathfrak{C} \mid \text{dom}(\mathfrak{c}) = \mathfrak{H}_z \right\}$$

*Proof.* From what has been just said it is sufficient to prove that the mapping associating to any model set  $\Lambda$  in  $\Xi^{\Gamma}$  the couple  $(w_{\Lambda}, \mathfrak{c}_{\Lambda})$  establish a bijective correspondence between the internal system  $\Xi^{\Gamma}$  and

$$\{(w, \mathfrak{c}) \in \mathbb{R}^n \times \mathfrak{C} \mid \text{dom}(\mathfrak{c}) = \mathfrak{H}_w\}$$

First from the very construction of the cone type  $\mathfrak{c}_{\Lambda}$  associated with any  $\Lambda \in \Xi^{\Gamma}$  this association is well defined. By the arguments used in the proof of proposition 5.1 each model set  $\Lambda \in \Xi^{\Gamma}$  is the limit of a filterbase (11) which only depends on the couple  $(w_{\Lambda}, \mathfrak{c}_{\Lambda})$ , and thus the association is 1-to-1. Moreover this association is onto: If  $(w, \mathfrak{c})$  is a couple with  $\text{dom}(\mathfrak{c})$  equal to  $\mathfrak{H}_w$ , then consider the family of

subsets  $[C_c + w]_{\Xi} \cap \Pi^{-1}(B(w, \varepsilon))$  of the internal system. Each such set contains some non-singular model sets, and thus forms a filterbase in  $\Xi^{\Gamma}$ . As  $\Pi$  is a proper map this filterbase is eventually contained into a compact subset of the form  $\Pi^{-1}(\overline{B}(w, \varepsilon))$  and thus admits an accumulation element  $\Lambda$ . This latter must satisfies  $\Pi(\Lambda) = w_{\Lambda} = w$  and  $C_{\Lambda} = C_c$  on the other hand. But as the domains of  $\mathfrak{c}$  and  $\mathfrak{c}_{\Lambda}$  are both equal to the cut type of  $w_{\Lambda} = w$  the couple  $(w_{\Lambda}, \mathfrak{c}_{\Lambda})$  is nothing but  $(w, \mathfrak{c})$ , showing that the association is onto.  $\square$

**8.2. The Ellis action.** We wish to use here the descriptions of the hull obtained in the above paragraph and that of its Ellis semigroup performed in theorem 7.1. To this end we set an action of the non-trivial face semigroup  $\mathfrak{T}_{W, \Gamma}$  onto the family  $\mathfrak{C}$  of cone types introduced above:

For  $\mathfrak{c} \in \mathfrak{C}$  and  $\mathfrak{t} \in \mathfrak{T}_{W, \Gamma}$  let us define a map  $\mathfrak{H}_W \rightarrow \{-, +, \infty\}$  as

$$\mathfrak{c}.\mathfrak{t}(H) := \begin{cases} \mathfrak{c}(H) & \text{if } \mathfrak{t}(H) = 0 \\ \mathfrak{t}(H) & \text{else} \end{cases}$$

This definition is not properly an action of  $\mathfrak{T}_{W, \Gamma}$  on  $\mathfrak{C}$  as the resulting map may not be a cone issuing from any model set of the hull  $\mathbb{X}$ . However it allows us to recover the Ellis action as follows:

**Proposition 8.2.** *The Ellis action  $\mathbb{X} \times E(\mathbb{X}, \mathbb{R}^d) \rightarrow \mathbb{X}$  obtains as*

$$(z, \mathfrak{c}).(z', \mathfrak{t}) = (z + z', \mathfrak{c}') \quad \text{where} \quad \mathfrak{c}'(H) := \begin{cases} \mathfrak{c}.\mathfrak{t}(H) & \text{if } H \in \mathfrak{H}_{z+z'} \\ \infty & \text{else} \end{cases}$$

**8.3. An illustration of the Ellis action.** In order to illustrate the Ellis action described as above, we focus here on the example of the hull  $\mathbb{X}_{oct}$  associated with the data given in paragraph 1.4. More precisely we won't describe the action of any transformation but rather the one of the idempotent transformations (as the other part is only a shifting in the parametrization torus  $[\mathbb{R}^4]_{\mathbb{Z}^4}$ ). Moreover it can be checked that the idempotent Ellis transformations are precisely those Ellis transformations mapped onto  $0 \in [\mathbb{R}^4]_{\mathbb{Z}^4}$  under  $\pi^*$ , or equivalently, those which preserve fibers in  $\mathbb{X}_{oct}$  with respect to the parametrization map  $\pi$ . Here we won't describe the Ellis action of these idempotents at any model set, but rather on the single fiber above  $0 \in [\mathbb{R}^4]_{\mathbb{Z}^4}$ , any other fiber can be treated in the same manner.

First we need to know the cut type of 0: it is easily checked that  $\mathfrak{H}_0 = \mathfrak{H}_{W_{oct}} = \{H_1, H_2, H_3, H_4\}$ , so that the fiber above 0 in the hull consists of eight model sets  $\{\Lambda_{C_1}, \dots, \Lambda_{C_8}\}$ , each associated with some cone which is in this particular case a *chamber* among  $\{C_1, \dots, C_8\}$ . Then we can compute the action of any of the 17 idempotent transformations  $[0]_{\mathbb{Z}^4} \times \{\mathfrak{t}\}$ ,  $\mathfrak{t} \in \mathfrak{T}_{W_{oct}}$ :

The identity map, given by  $[0]_{\mathbb{Z}^4} \times \{\mathfrak{o}\}$ , preserves any of the eight model sets, whereas any idempotent map  $[0]_{\mathbb{Z}^4} \times \{\mathfrak{t}_{C_i}\}$  associated with the chamber  $C_i$  maps all of these model sets onto a single one, namely  $\Lambda_{C_i}$ . For an idempotent map of the form  $[0]_{\mathbb{Z}^4} \times \{\mathfrak{t}_{L_i}\}$  with  $L_i$  some half line contained into the hyperplane  $H_i$ , each model set with associated cone belonging in the side  $\pm$  of  $H_i$  is mapped onto the unique model set whose cone belongs into the same side  $\pm$  of  $H_i$  and has  $L_i$  in its boundary.



Therefore these transformations have two distinct model sets of this fiber in their range, namely these which have  $L_i$  in the boundary of their associated cone.

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## REFERENCES

- [1] J. Auslander, *Minimal Flows and Their Extensions*, Notas de Matemática no 153, Elsevier Science, 1988. <http://books.google.fr/books?id=e3wFvPvpWvwC>
- [2] M. Baake, J. Hermisson, P. A. B. Pleasant, *The torus parametrization of quasisperiodic LI-class*, 1997.
- [3] M. Baake, D. Lenz, *Dynamical systems on translation bounded measures: pure point dynamical and diffraction spectra*, Ergodic Theory and Dynamical Systems, Volume 24, Issue 06, December 2004, pp 1867-1893.
- [4] M. Baake, D. Lenz, R.V. Moody, *Characterization of model sets by dynamical systems*, Ergodic Theory and Dynamical Systems 27.02 (2007): 341-382.
- [5] M. Barge, J. Kellendonk, *Proximity and pure point spectrum for tiling dynamical systems*, arXiv preprint arXiv:1108.4065 (2011).
- [6] K.S. Brown, *Semigroups, rings, and Markov chains*, arXiv:math/0006145.
- [7] R. Ellis, W.H. Gottschalk, *Homomorphisms of transformation groups*, Transactions of the American Mathematical Society, Vol. 94, 1960, pp. 258-271.
- [8] A. Forrest, J. Hunton, J. Kellendonk, *Cohomology of canonical projection tilings*, Commun. Math. Phys. 226, 2002, pp 289-322.
- [9] F. Gähler, J. Hunton, J. Kellendonk, *Integral cohomology for rational projection method patterns*, Algebraic and Geometric Topology (2013) 1661-1708.
- [10] E. Glasner, *On tame dynamical systems*, Colloq. Math. 105, pp 283-295, 2006.
- [11] E. Glasner, *Enveloping semigroups in topological dynamics*, Topology and its Applications, Volume 154, Issue 11, 1 June 2007, Pages 2344–2363.
- [12] E. Glasner, *The structure of tame dynamical systems*, Ergodic Theory and Dynamical Systems, vol.27 no 6, 2007, pp 1819-1837.
- [13] E. Glasner, M. Megrelishvili, *Hereditarily non-sensitive dynamical systems and linear representations*, arXiv preprint math/0406192, 2004.
- [14] E. Glasner, M. Megrelishvili, V. Uspenskij, *On metrizable enveloping semigroups*, arXiv:math/0606373v3, 2006.
- [15] A. Hof, *On diffraction by aperiodic structures*, Comm. Math. Phys. Volume 169, Number 1 (1995), p. 25-43.
- [16] A. Julien, *Complexity and cohomology for cut and projection tilings*, arXiv:0804.0145v1, 2008.
- [17] J. Kellendonk, L. Sadun, *Meyer sets, eigenvalues, and Cantor fiber bundles*, arXiv:1211.2250, nov. 2012.
- [18] J.L. Kelley, *General topology*.
- [19] , T.T.Q. Le, *Local rules for quasiperiodic tilings*, in: R.V. Moody (ed.) *The Mathematics of Long Range Aperiodic Order*, 1997, pp. 331–366.
- [20] J-Y. Lee, R.V. Moody, B. Solomyak, *Pure point dynamical and diffraction spectra*, Annales Henri Poincaré. Birkhäuser Verlag, 2002. p. 1003-1018.
- [21] J-Y. Lee, R.V. Moody, *A characterization of model multi-colour sets*, Annales Henri Poincaré. Vol. 7. No. 1. Birkhäuser-Verlag (2006).
- [22] J-Y. Lee, B. Solomyak, *Pisot self-affine tilings, discrete spectrum, and the Meyer property*, arXiv preprint arXiv:1002.0039, 2010.
- [23] R.V. Moody, *Meyer sets and their duals*, NATO ASI Series C Mathematical and Physical Sciences-Advanced Study Institute, 1997, vol. 489, p. 403-442.
- [24] R.V. Moody, *Model sets: a survey*, arXiv:math/0002020 (2000).
- [25] R.V. Moody, *Mathematical quasicrystals: a tale of two topologies*, Proceedings of the International Congress of Mathematical Physics (2003).
- [26] R. Pikula, *Enveloping semigroups of affine skew products and Sturmian-like systems*, Ph.D Thesis, 2009.

- [27] I. Putnam, *Non-Commutative Methods for the K-Theory of  $C^*$ -Algebras of Aperiodic Patterns from Cut-and-Project Systems*, Communications in Mathematical Physics, Volume 294, Issue 3, pp 703-729, March 2010.
- [28] E.A. Robinson Jr, *A Halmos–von Neumann theorem for model sets, and almost automorphic dynamical systems*, Dynamics, ergodic theory, and geometry (2007): 243-272.
- [29] F.V. Saliola, *The face semigroup algebra of a hyperplane arrangement*, arXiv:math/0511717.
- [30] M. Schlottmann, *Generalized model sets and dynamical systems*, CRM Monograph Series (1999).
- [31] M. Senechal, *Quasicrystals and geometry*, Cambridge university press, 1995.
- [32] W.A. Veech, *Point distal flows*, American Journal of Mathematics, vol.92 no 1, 1970, pp. 205-242.

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