ELLIS ENVELOPING SEMIGROUP FOR ALMOST CANONICAL MODEL SETS

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ENVELOPING SEMI-GROUP FOR MINIMAL ROTATIONS ON CUT UP TORI

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Abstract. In this paper we give an explicit computation of the Ellis enveloping semigroup associated to particular dynamical systems, which we call rotations on cut up tori. The considered systems are almost one-to-one extensions over rotations on tori, and as a byproduct we prove that a rotation on a cut up torus is a tame system. Our setting covers the case of the discrete dynamical system of an almost canonical cut & project pattern, and is illustrated with the treatement of the octagonal tiling discrete dynamical system.

Outline

The motivation of this work comes from the study of aperiodic point sets, or patterns, of $\mathbb{R}^d$. The intensive study done so far permits to describe important properties of a considered point set in terms of its associated dynamical system. In particular, a striking result in this direction is that the locations of the Bragg peaks in the diffraction spectrum of a physical material modeled by the point set generates the group of eigenvalues of the dynamical system given by the hull with $\mathbb{R}^d$-action. The eigenvalues are in general hard to identify, but the special class of topological eigenvalues (the ones associated with a continuous eigenfunction) is conveniently obtained as the character group of a factor of the system, called the *maximal equicontinuous factor*. Roughly speaking, having plaint of continuous eigenvalues is equivalent to have the system close to its equicontinuous factor. The enveloping semigroup (or Ellis semigroup after its inventor) associated to the dynamical system is a quite sensitive tool regarding this connection. In particular, it has been proven that if the dynamical system is *tame*, a property involving its Ellis semigroup, then (among other consequencies) the system is measure conjugated with its maximal equicontinuous factor, and this latter fact implies pure point diffractivity of the point set. However in general the Ellis semigroup is hard to compute and the tame property for the system not so easily detectable. Thus we concentrate our point of view on a particular class of point sets, called *almost canonical cut & project pattern*, for which we show that the Ellis semigroup is computable. It appears that for this class of point sets the system is tame (implying thus pure point diffraction spectra for these sets, although it was a well known fact already).

A detailed exposition on almost canonical cut & project patterns can be found in the literature (see for instance [3], [7]). The class of such point sets is defined by the method to construct them: as projections of the points of a rank N Bravais lattice lying in a certain neighborhood of an irrationally placed d-dimensionnal subspace E. The neighborhood is determined by a window W (the atomic surface) which is a polyhedron in the $n := N - d$ dimensional subspace $E^\perp$ perpendicular to E. The cut & project pattern is the point set obtained by projecting the lattice points lying

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in $E+W$ onto E along $E^\perp$. The parameters are chosen such that the projection of the Bravais lattice onto $E^\perp$ along E gives rise to a dense subgroup Γ of rank N in $\mathbb{R}^n = E^\perp$, and that the set of points $\partial W + \Gamma$ coincides with the set of points $\bigcup_{i\in I} A_i + \Gamma$, where the A_i are the affine hyperspaces of $\mathbb{R}^n$ containing the faces of W. When splitting $\Gamma = \mathbb{Z}^n \oplus \mathbb{Z}^d$ such that the first summand $\mathbb{Z}^n$ is co-compact in $\mathbb{R}^n$ we obtain by moding it out a n-torus, together with dense set of points $(\bigcup_{i\in I} A_i + \Gamma)/\mathbb{Z}^n$ at which we will cut the torus open to totally disconnect it. This is the *cut up torus* of our dynamical system. The remaining summand $\mathbb{Z}^d$ acts on it by rotation.

We point out a strong link between our analysis and the work of Pikula [8] on the Ellis semigroup of Sturmian-like systems. Here we obtain these results as examples of minimal rotations on cut up circles. The most simple case of the coding of an irrationnal rotation on the cirle was already given by Glasner and Megrelishvili in the last section of [6]. We discuss these at the end of section 4.4.
After having completed this work we became aware of the recent preprint by Pikula [9], in which he considers the enveloping semigroup of almost 1-to-1 extensions of certain minimal group rotations. In the case where the acting group is $\mathbb{Z}$ the two works can be compared. Our results are much more concrete, whereas he considers a more general situation.

1. Overview on the results

The dynamical system. We consider the data $(\mathbb{R}^n, \Gamma, \{A_i\}_{i\in I})$ to be an Euclidean space of dimension n together with a dense subgroup Γ and a finite family of affine hyperspaces $\{A_i\}_{i\in I}$. For each $i \in I$ there are $a_i \in \mathbb{R}^n$ and a codimension 1 linear subspace H_i^0 with $A_i = H_i^0 + a_i$. We require $\bigcap_{i\in I} H_i^0 = \{0\}$. We choose a normal for each H_i^0 and define H_i^+, H_i^- to be the open half spaces in the direction of the normal or against it, respectively. We also set $H_i^\infty = A_i^\infty = \mathbb{R}^n$. We consider the group Γ to be finitely generated, that is, $\Gamma = \mathbb{Z}^n \oplus \mathbb{Z}^d$ with the $\mathbb{Z}^n$ summand a co-compact subgroup of $\mathbb{R}^n$, and keep this decomposition fixed throughout.
We consider the countable family of all hyperspaces obtained as the translation of some A_i by some $\underline{n} \in \Gamma$. The **cut type** I_z of a point $z \in \mathbb{R}^n$ is the (possibly empty) subset of indices

$$I_z := \{i \in I \mid z \in A_i + \Gamma\}$$

According to its cut type, each $z \in \mathbb{R}^n$ admits a finite collection of hyperspaces passing through it. The complementary set of these hyperspaces is a disjoint union of convex open sets, which are affine cones pointed at z. Each cone is determined according to its orientation with respect to the cuts along z. Hence we define a **point type** to be a map $\mathfrak{p} : I \longrightarrow \{-, +, \infty\}$ such that the associated open cone $C_\mathfrak{p} := \bigcap_{i\in I} H_i^{\mathfrak{p}(i)}$ is non-empty. It follows that each affine cone corresponds to at least a point type, and once the domain of a point type $\mathfrak{p}$, defined to be $dom(\mathfrak{p}) := \{i \in I \mid \mathfrak{p}(i) \neq \infty\}$, is considered, an affine cone pointed at z can be written as $C_\mathfrak{p} + z$ for a unique point type $\mathfrak{p}$ with domain $dom(\mathfrak{p}) = I_z$. We denote the finite collection of point types by $\mathfrak{P}$. By construction $I_{z+\underline{n}} = I_z$ for each $\underline{n} \in \Gamma$, so the cut type is also defined for the points of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
The dynamical system, which we call rotation on a cut up torus, can now be described as $(\mathbb{T}_c^n, \mathbb{Z}^d)$ with

$$\mathbb{T}_c^n = \{(z, \mathfrak{p}) \in \mathbb{T}^n \times \mathfrak{P} \mid dom(\mathfrak{p}) = I_z\}$$

and $\mathbb{Z}^d$-action given by $(z,\mathfrak{p}).\underline{n} = (z.\underline{n},\mathfrak{p})$. The map

$$\pi_{\mathbb{T}} : \mathbb{T}^n_c \twoheadrightarrow \mathbb{T}^n$$

forgetting the point type of any element is an onto continuous $\mathbb{Z}^d$-equivariant map. We may describe the topology of $\mathbb{T}^n_c$ as follows. Let

$$NS = \{z \in \mathbb{T}^n \mid I_z = \emptyset\}$$

This is a dense set of $\mathbb{T}^n$. We identify it with the subset of $\mathbb{T}^n_c$ of points (z,∞) (∞ being the constant point type equal to ∞). This latter set is precisely the set of points where $\pi_{\mathbb{T}}$ is one-to-one. The topology of $\mathbb{T}^n_c$ can now be described by saying that NS is dense in $\mathbb{T}^n_c$, and a sequence $(z_n)_{n\in\mathbb{N}} \subset NS$ converges to a point $(z,\mathfrak{p}) \in \mathbb{T}^n_c$ if and only if $z_n \longrightarrow z$ in the Euclidean topology and eventually $z_n - z \in C_{\mathfrak{p}}$. The latter statement should be understood in the following way: if $z_n \longrightarrow z$ then we can eventually lift $z_n - z$ into a sequence converging to 0 in $\mathbb{R}^n$, the space which contains the cone $C_{\mathfrak{p}}$.

The Ellis semigroup. To any compact space with group action by homeomorphisms is associated its Ellis semigroup, consisting of tranformations of the space, obtained as pointwise limits of homeomorphisms given by the group action.
In order to describe the Ellis semigroup of $(\mathbb{T}^n_c,\mathbb{Z}^d)$, we define a transformation type to be a map $\mathfrak{t} : I \longrightarrow \{-,+,0\}$ such that its associated cone $C_{\mathfrak{t}} := \bigcap_{i\in I} H_i^{\mathfrak{t}(i)}$ is non-empty. We denote the set of transformation types by $\mathfrak{T}$. The cones of the transformation types are the constituents of a stratification of $\mathbb{R}^n$ by cones of dimension between 0 and n.
Any cone $C_{\mathfrak{t}}$ associated with a transformation type $\mathfrak{t}$ generates a subspace $< C_{\mathfrak{t}} >$, which possesses a unique maximal summand $W_{\mathfrak{t}}$, with the property that $W_{\mathfrak{t}} \cap \Gamma$ is dense in $W_{\mathfrak{t}}$. We then say that a transformation type is non-trivial if the intersection $C'_{\mathfrak{t}} := C_{\mathfrak{t}} \cap W_{\mathfrak{t}}$ is non-empty, and denote by $\mathfrak{T}_0$ the collection of non-trivial transformation types.
The Ellis semigroup is isomorphic to the disjoint union of groups

$$E(\mathbb{T}^n_c,\mathbb{Z}^d) \simeq \bigsqcup_{\mathfrak{t}\in\mathfrak{T}_0} [W_{\mathfrak{t}} + \Gamma]_{/\mathbb{Z}^n} \times \{\mathfrak{t}\}$$

with **semigroup stucture** given by $(z,\mathfrak{t}).(z',\mathfrak{t}') = (z+z',\mathfrak{t}.\mathfrak{t}')$, where the semigroup law on $\mathfrak{T}_0$ writes:

$$\mathfrak{t}.\mathfrak{t}'(i) := \begin{cases} \mathfrak{t}'(i) \text{ if } \mathfrak{t}'(i) \neq 0 \\ \mathfrak{t}(i) \text{ if } \mathfrak{t}'(i) = 0 \end{cases}$$

We may describe the **topology** of $E(\mathbb{T}^n_c,\mathbb{Z}^d)$ as follows. The inclusion of $\mathbb{Z}^d$ as a dense subset of $E(\mathbb{T}^n_c,\mathbb{Z}^d)$ is given by $\underline{n} \mapsto (\underline{n},\mathfrak{o})$ ($\mathfrak{o}$ being the constant transformation type equals to 0). A sequence$(\underline{n}_k)_{k\in\mathbb{N}} \subset \mathbb{Z}^d$ converges to a transformation $(z,\mathfrak{t}) \in E(\mathbb{T}^n_c,\mathbb{Z}^d)$ if and only if $\underline{n}_k - z \in \mathbb{T}^n$ lifts into a sequence of $\mathbb{R}^n$ converging to 0 and eventually lying into $C_{\mathfrak{t}}$. A general sequence $(z_k,\mathfrak{t}_k)$ converges to a transformation $(z,\mathfrak{t}) \in E(\mathbb{T}^n_c,\mathbb{Z}^d)$ if and only if $\forall\varepsilon > 0, \exists\delta_n > 0$ such that $C'_{\mathfrak{t}_k}(z_k,\delta_k) \subset C'_{\mathfrak{t}}(z,\varepsilon)$for large enough k. The Ellis semigroup for the system $(\mathbb{T}^n_c,\mathbb{Z}^d)$ has a first countable topology, and is consequently **tame** in the sense of [5] (see theorem 6.3 there).
The previous algebraic description shows in particular that:
- Each $[W_{\mathfrak{t}} + \Gamma]_{/\mathbb{Z}^n} \times \{\mathfrak{t}\}$ is a subgroup of $E(\mathbb{T}^n_c,\mathbb{Z}^d)$ with identity $(0,\mathfrak{t})$.

- the semigroup $\mathfrak{T}_0$ is isomorphic to the collection of idempotent transformations in $E(\mathbb{T}_c^n, \mathbb{Z}^d)$, where each $\mathfrak{t} \in \mathfrak{T}_0$ corresponds to $(0, \mathfrak{t})$.
- The general theory of Ellis semigroups endows the collection of idempotent transformations with an pre-order. It is here an order, given on $\mathfrak{T}_0$ by $\mathfrak{t} \leqslant \mathfrak{t}'$ if and only the cone $C_{\mathfrak{t}'}$ is equal or a lower dimensional facet of the cone $C_{\mathfrak{t}}$.
- $E(\mathbb{T}_c^n, \mathbb{Z}^d)$ has a unique minimal ideal, given by $\mathbb{T}^n \times \mathfrak{T}_{min}$ where $\mathfrak{T}_{min}$ denotes the subsemigroup of minimal idempotents.

We may write the Ellis **action** $\mathbb{T}_c^n \times E(\mathbb{T}_c^n, \mathbb{Z}^d) \longrightarrow \mathbb{T}_c^n$ as

$$(z, \mathfrak{p}).(z', \mathfrak{t}) = (z + z', \mathfrak{p}')$$

where

$$\mathfrak{p}'(i) := \begin{cases} \mathfrak{p}.\mathfrak{t}(i) \text{ if } i \in I_{z+z'} \\ \infty \text{ else} \end{cases} \qquad \mathfrak{p}.\mathfrak{t}(i) := \begin{cases} \mathfrak{t}(i) \text{ if } \mathfrak{t}(i) \neq 0 \\ \mathfrak{p}(i) \text{ if } \mathfrak{t}(i) = 0 \end{cases}$$

The richness of the Ellis semigroup is related to the dimension of the set of points of $\mathbb{R}^n$ which can be approximated by elements of Γ inside an intersection $\cap_{i \in I'} H_i$, $I' \subset I$. Generically, the orientation of the hyperplanes H_i is such that any hyperplane contains only one point of Γ, and so we are in the extreme case where $\mathfrak{T}_0 = \mathfrak{T}_{min} \cup \{\mathfrak{o}\}$, that is, the Ellis semigroup consists only of the original group Γ together with its minimal ideal. In this case the constituents of the Ellis semigroup depends on the hyperplanes H_i, but not on the position of these hyperplanes with respect to Γ. When dealing with point patterns this generic case corresponds to the case of largest complexity function (see [7] for definition and results).
At the opposite end, in the case where the group Γ has dense intersection with any finite intersection of hyperplans, then the Ellis semigroup is the biggest in the sense that $\mathfrak{T} = \mathfrak{T}_0$ (that is, there is no non-trivial transformation type). This latter case has been proven to be equivalent with the condition of minimal complexity for the cut & project pattern (see [7]). Hence smaller complexity seems to make the Ellis semigroup richer.

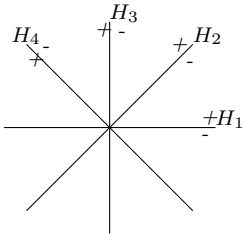
2. Cut up torus

2.1. Cut up Euclidean spaces.

We explain in details the construction of the dynamical system defined by the data $(\mathbb{R}^n, \Gamma, \{A_i\}_{i \in I})$. We wish to illustrate how we procceed, by considering the following example:

Example 2.1. *(Octagonal tiling case) Consider the example of the cut up data on $\mathbb{R}^2$ given, in an orthonormal basis (e_1, e_2), by:*

$$\begin{array}{lll} A_1 = H_1 := < v_1 > = < v_2 - v_4 > & & v_1 := e_1 \\ A_2 = H_2 := < v_2 > = < v_1 + v_3 > & \textit{with} & v_2 := \frac{e_1 + e_2}{\sqrt{2}} \\ A_3 = H_3 := < v_3 > = < v_2 + v_4 > & & v_3 := e_2 \\ A_4 = H_4 := < v_4 > = < v_1 - v_3 > & & v_4 := \frac{e_2 - e_1}{\sqrt{2}} \end{array}$$



These four vectors are rationally independent so the sum $\sum_{i=1}^{4}\mathbb{Z}v_i$ *is the free Abelian group* $\mathbb{Z}^4$. *We write* $\mathbb{Z}^{n=2} := \mathbb{Z}v_1 + \mathbb{Z}v_3$ *and* $\mathbb{Z}^{d=2} := \mathbb{Z}v_2 + \mathbb{Z}v_4$.

As we can see by comparing the following construction with the cut and project scheme given in section 3.1 of [7], this corresponds to the octagonal tiling case.
The subgroup $\Gamma = \mathbb{Z}^{n+d}$ of $\mathbb{R}^n$ has its elements usually denoted by $\underline{n}$, and acts on $\mathbb{R}^n$ by addition: $v.\underline{n} := v + \underline{n}$. We fix an algebraic basis $\{e_i\}$ of $\mathbb{Z}^{n+d}$ with $n+d$ elements: this defines a norm on the group by $|\sum_{i=1}^{n+d} n_i e_i| := max_{1\leqslant i\leqslant n+d}|n_i|$.
The cut up space is constructed as follows:
For a non-negative integer k, write $\mathbb{R}^n \setminus \bigcup_{|\underline{n}|\leqslant k}\bigcup_{i\in I}(A_i + \underline{n}) = \bigsqcup_j D_k^j$ as the disjoint union of its finite collection of connected components D_k^j, and define the k^{th} partial cut up space to be the finite disjoint union

$$\mathbb{R}_k^n := \bigsqcup_j \overline{D_k^j}^{|.|}$$

Every such $\overline{D_k^j}^{|.|}$ forms then a clopen connected component of $\mathbb{R}_k^n$. This definition also holds for $k = -1$, and in this case $\mathbb{R}_{-1}^n$ is the full Euclidian space $\mathbb{R}^n$.
From now on the notation D_k stands for a generic open connected component involved in the construction of $\mathbb{R}_k^n$.
There is a tower of such partial cut up spaces:
by letting the number of cuts increase we reduce the size of each open component, so for for $l > k \geqslant -1$ each D_l lies in some unique D_k, giving by completion a natural continuous and surjective map $\pi_{k,l} : \mathbb{R}_l^n \twoheadrightarrow \mathbb{R}_k^n$.
Of special interest are the mappings $\pi_{-1,k}$: These send the partial cut up spaces $\mathbb{R}_k^n$ onto $\mathbb{R}^n$, and the number of points in a fiber above a point x in $\mathbb{R}^n$ is the number of connected components forming $\mathbb{R}_k^n$ and having x in their Euclidean closure.

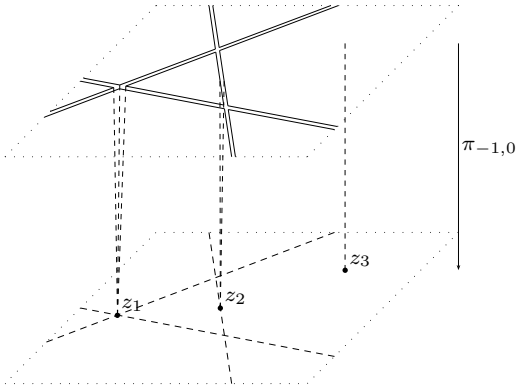


Figure 1. *The map* $\pi_{-1,0} : \mathbb{R}_0^n \twoheadrightarrow \mathbb{R}^n$ *collapses the components along the cuts. The number of points in a fiber above* z *depends on the cuts through* z*: here above* z_1, z_2 *and* z_3 *we have respectively 4, 2 and 1 point.*

One can check the composition property $\pi_{m,l} \circ \pi_{l,k} = \pi_{m,k}$ holding for any $k > l > m \geqslant -1$. Then:

Definition 2.2. *The cut up Euclidian space* $\mathbb{R}_c^n$ *is the inverse limit of the tower of maps* $\{\pi_{l,k} : \mathbb{R}_k^n \twoheadrightarrow \mathbb{R}_l^n\}_{k>l\geqslant -1}$, *that is the subspace* $\{(x_k)_{k\geqslant -1} \,|\, x_l = \pi_{l,k}(x_k)\}$ *of the product space* $\prod_{k\geqslant -1}\mathbb{R}_k^n$, *equiped with product topology.*

Each $\mathbb{R}^n_l$ is then the quotient of $\mathbb{R}^n_c$ through a map $\pi_{l,\infty}$ defined by $x_l = \pi_{l,\infty}((x_k)_k)$, and these maps satisfies the equalities $\pi_{m,l} \circ \pi_{l,\infty} = \pi_{m,\infty}$ for all $l > m \geqslant -1$. Of particular importance is the factor map

$$\pi := \pi_{-1,\infty} : \mathbb{R}^n_c \longrightarrow \mathbb{R}^n$$

As a consequence of the very construction of $\mathbb{R}^n_c$ we have:

Proposition 2.3. *The space $\mathbb{R}^n_c$ is locally compact, σ-compact and has a countable basis for its topology. The map $\pi : \mathbb{R}^n_c \longrightarrow \mathbb{R}^n$ is continuous, onto, and is a proper map, that is the pre-image of any compact set is compact.*

2.2. **The topology of $\mathbb{R}^n_c$.** The Euclidean topology is generated by convex sets obtained by intersecting a finite family of half spaces. This paragraph shows an analog for the topology of the cut up space, expressing the topology of $\mathbb{R}^n_c$ through a family of (cut up) half spaces.
Denote by $\mathfrak{D}$ for the collection of all the generic open connected components D_k, with k running in $\{-1, 0, 1, 2...\}$. Relatively compact elements of $\mathfrak{D}$ are sometimes called $\mathcal{C}$-topes (see [3]).

Definition 2.4. *We call the set*

$$NS := \mathbb{R}^n \setminus \bigcup_{\underline{n} \in \mathbb{Z}^{n+d}} \bigcup_{i \in I} (A_i + \underline{n}) = \mathbb{R}^n \setminus \bigcup_{D \in \mathfrak{D}} \partial D$$

the set of non-singular points in $\mathbb{R}^n$.

By the cut up construction the set of points $z \in \mathbb{R}^n$ above which the fiber in $\mathbb{R}^n_c$ for the map π consists of a unique point is precisely the set of non-singular points. From its very definition, NS is a dense residual subset of $\mathbb{R}^n$. From now on we identifie the embeddings of NS both in $\mathbb{R}^n_c$ and in $\mathbb{R}^n$, the map π sending the first onto the latter.

Lemma 2.5. *The set NS is a dense subset of $\mathbb{R}^n_c$.*

Proof. Let $\pi^{-1}_{k,\infty}(U)$ be an open set of $\mathbb{R}^n_c$, with U an open set of a clopen set $\overline{D_k}$. Then $U \cap D_k$ is a non-void open set so meets NS in at least one point by density, and this point lies then in $\pi^{-1}_{k,\infty}(U)$. Such open sets form a basis for the topology of $\mathbb{R}^n_c$, giving the density.

□

Define the family of *admissible* half spaces, as

$$\mathfrak{A} := \left\{ A_i^{\pm} + \underline{n} \mid i \in I,\ \underline{n} \in \mathbb{Z}^{n+d} \right\}$$

Definition 2.6. *Let P be a subset of $\mathbb{R}^n$. The cut up set P_c (or $[P]_c$) is the closure of $P \cap NS$ in $\mathbb{R}^n_c$.*

Of course P_c may be empty even if P does not. We are particularly interested in the cut up half space A_c, the closure of $A \cap NS$ in $\mathbb{R}^n_c$, for each $A \in \mathfrak{A}$. Clearly we have $\pi(A_c) = \overline{A}$, the Euclidean closure of A. This new class of sets is made of *clopen*

sets of $\mathbb{R}^n_c$ as we will see. For now we state a general tool, ensuring us to describe clopen sets of $\mathbb{R}^n_c$ uniquely through the non-singular points they contain:

Lemma 2.7. *Let X be topological space, and Y a dense subset. For any clopen set V of X one has $V = \overline{V \cap Y}$. If two clopen sets coincide on Y, then they are equal.*

Proof. Let V be clopen in X. We have the inclusion $\overline{V \cap Y} \subseteq V$ since V is closed. If x lies in V, Y being dense one can find a net of points in Y converging to x. Since V contains x and is open, the net eventually lies in V, showing that $x \in \overline{V \cap Y}$.
If V and W are two clopen sets with $V \cap Y = W \cap Y$, then $V = \overline{V \cap Y} = \overline{W \cap Y} = W$. □

Proposition 2.8. *The cut up space $\mathbb{R}^n_c$ is totally disconnected. The family*

$$\mathfrak{A}_c := \{A_c \,|\, A \in \mathfrak{A}\}$$

of cut up half spaces is a pre-basis of clopen sets.
For any pair A, A' in $\mathfrak{A}$ the following Boolean rules are true:

$$[A \cup A']_c = A_c \cup A'_c \qquad (A_c)^c = [A^c]_c \qquad [A \cap A']_c = A_c \cap A'_c$$

Proof. The space $\mathbb{R}^n_c$ has its topology formed by pre-images of open sets of the partial cut up spaces. Let $\overline{D_k}$ be a clopen part of $\mathbb{R}^n_k$. Then we have $[D_k]_c = \pi^{-1}_{k,\infty}(\overline{D_k})$: Indeed $\pi^{-1}_{k,\infty}(\overline{D_k})$ is clopen so by lemma 2.7 rewrites as $\overline{\pi^{-1}_{k,\infty}(\overline{D_k}) \cap NS}$, with closure in $\mathbb{R}^n_c$. The set $\pi^{-1}_{k,\infty}(\overline{D_k}) \cap NS$ is equal to $\overline{D_k} \cap NS$ in $\mathbb{R}^n_k$ (in fact under bijective correspondence through $\pi_{k,\infty}$), in turns equal to $D_k \cap NS$ as subset of $\mathbb{R}^n_k$ since the boundary ∂D_k do not cross the set NS. The latter is equal to $D_k \cap NS$ as a subset of $\mathbb{R}^n$, leading by taking the closure in $\mathbb{R}^n_c$ to the desired equality.
In particular we have shown the equality $[D]_c \cap NS = D \cap NS$ holding for any element D of the family $\mathfrak{D}$. Then any pair D and D' of such components satisfies the Boolean rules of the statement:
For, first observe that $[D \cup D']_c = \overline{(D \cup D') \cap NS} = \overline{(D \cap NS) \cup (D' \cap NS)} = \overline{D \cap NS} \cup \overline{D' \cap NS} = D_c \cup D'_c$, where all closures are with respect to the topology of $\mathbb{R}^n_c$. Then having the equality $D_c \cap NS = D \cap NS$ holding, we have $(D_c)^c \cap NS = D^c \cap NS$ which implies $(D_c)^c = \overline{(D_c)^c \cap NS} = \overline{D^c \cap NS} = [D^c]_c$. The third equality follows from the two others.
It is easy to prove the equality

$$[A^{\pm} + \underline{n}]_c = \bigsqcup_{D_{|\underline{n}|} \subset [A^{\pm} + \underline{n}]} [D_{|\underline{n}|}]_c$$

holding for any affine space A and element $\underline{n} \in \mathbb{Z}^{n+d}$, with disjoint and finite union. This shows that any of the sets given in the statement is clopen, and any two of them satisfies the Boolean rules.
It remains to show that it constitutes a pre-basis for the cut up topology (so far we made use of neither the density of $\mathbb{Z}^{n+d}$ nor the requirement $\bigcap_{i=1}^{l} H_i^0 = \{0\}$: their role is for this step). First we can recover the clopen components through the equality

$$[D_k]_c = \bigcap_{D_k \subset [A^{\pm} + \underline{n}],\, |\underline{n}| \leqslant k} \left[A^{\pm} + \underline{n}\right]_c$$

Choose an open set in some $\mathbb{R}^n_k$: it is the finite disjoint union of open sets, each entirely contained in some clopen component $\overline{D_k}$, so without loss of generality we can choose such open set as the intersection of $\overline{D_k}$ with some open Euclidian set U. Pick a point x inside $\pi^{-1}_{k,\infty}(\overline{D_k} \cap U)$, so that $v := \mu_{k,\infty}(x)$ lies in $\overline{D_k} \cap U$. Due to the requirement $\bigcap_{i=1}^{l} H_i^0 = \{0\}$, if we denote v_i to be a normal to each H_i^0, the linear forms $< ., v_i >$ with kernel H_i^0 linearly generates the dual space $\mathbb{R}^{n*}$, so the open polytopes with boudary faces parallel to H_i^0, hence parallel to the affine spaces A_i, generates the Euclidean topology. Thus we may choose such an open polytopal neighbourhood of v lying into U. Because $\mathbb{Z}^{n+d}$ is dense in $\mathbb{R}^n$, this choice can be done such that any of the hyperplans determining the faces are of the form $A_i + \underline{n}$ for some $\underline{n} \in \mathbb{Z}^{n+d}$. Hence the polytopal neighbourhood is given by the finite intersection of open half spaces of the form $A_i^{\pm} + \underline{n}$ containing it. Taking the finite intersection of the clopens $[A_i^{\pm} + \underline{n}]_c$ and again intersecting with $[D_k]_c$ leads to a clopen neibourhood of x in $\mathbb{R}^n_c$, which is the finite intersection of sets in $\mathfrak{A}_c$, and itself contained into $\pi^{-1}_{(k,\infty)}(\overline{D_k} \cap U)$. This ensure the pre-basis property. □

2.3. **Neighborhood basis.** We derive here from the previous paragraph a description of $\mathbb{R}^n_c$ by means of well chosen neighbourhoods for each point.

Proposition 2.9. *Let $\pi : X \longrightarrow Y$ be a continuous proper map between locally compact spaces. Given a point $x \in X$, denote by F_x the fiber with respect to π containing x. Then if V_x is a clopen neighborhood of x satisfying $V_x \cap F_x = \{x\}$, the family*

$$\{V_x \cap \pi^{-1}(U) \mid U \text{ neighborhood of } \pi(x)\}$$

is a neighborhood basis for x.

Proof. Supose for a contradiction that the stated family is not a neighborhood basis for x. Then there exists an open neighbourhood V of x such that $V_x \cap \pi^{-1}(U)$ intersect V^c for any neighborhood U of $\pi(x)$. Let Λ be the directed set of open neighborhoods of $\pi(x)$. We can select a net $\{x_U\}_{U \in \Lambda}$ in the closed set V^c with $x_U \in V_x \cap \pi^{-1}(U)$ for all U. Hence by choosing U_0 to be a compact neighbourhood of $\pi(x)$ we have a subnet $\{x_U\}_{U_0 \supset U \in \Lambda}$ in the compact neighbourhood $V_x \cap \pi^{-1}(U_0)$ of x. This subnet accumulate to at least one point x' in V^c, with $x' \in F_x$. Since the subnet also sits in V_x and this later is closed, we get $x' \in V_x \cap F_x$, giving $x = x'$, a contradiction to the fact that x is in V and x' does not. □

It remains to find such a V_x. To that end the key result is proposition 2.12, which may be figured out as follows: each point of $\mathbb{R}^n_c$ above a fixed $z \in \mathbb{R}^n$ can be approximated by non-singular points, that is, elements in $\mathbb{R}^n$, so the difference between two points lies into the *way* each is approximated. So it is necessary to, roughly speaking, find the direction in $\mathbb{R}^n$ which lead to a given point x in $\mathbb{R}^n_c$ above z. This direction will be a cone C_x pointed at 0, and the path of convergence to x will be given by $C_x + z$. Any two different cones will lead to two differents points above z. Finally, the objects separating any two cones will simply be an affine cut passing through z.
Now, define for any $z \in \mathbb{R}^n$ the **cut type** of z to be the set of indices

$$I_z := \left\{ i \in I \mid z \in A_i + \mathbb{Z}^{n+d} \right\}$$

Observe that a position $z \in \mathbb{R}^n$ is non-singular precisely when $I_z = \varnothing$.

Definition 2.10. *A point type is a map $\mathfrak{p} : I \longrightarrow \{-, +, \infty\}$ such that the associated open cone $C_\mathfrak{p} := \bigcap_{i\in I} H_i^{\mathfrak{p}(i)}$ is non-empty. Its domain is $dom(\mathfrak{p}) := \{i \in I \mid \mathfrak{p}(i) \neq \infty\}$. We denote the finite collection of point types by $\mathfrak{P}$.*

Observe that to obtain the cone associated to a point type $\mathfrak{p}$, only the indices in the domain of $\mathfrak{p}$ counts, as H_i^∞ is nothing but $\mathbb{R}^n$. Also, different point types may corresponds to a common cone, unless we put some conditions on the domain.

Definition 2.11. *The position of $x \in \mathbb{R}^n_c$ is the image point $\pi(x) \in \mathbb{R}^n$.*
The point type $\mathfrak{p}_x$ of $x \in \mathbb{R}^n_c$ is the element of $\mathfrak{P}$ uniquely defined through:

$$dom(\mathfrak{p}_x) = I_{\pi(x)} \qquad\qquad x \in [H_i^{\mathfrak{p}_x(i)} + \pi(x)]_c \ \forall i \in I_{\pi(x)}$$

We simply write C_x for the cone $C_{\mathfrak{p}_x}$ previously defined.

At this point it is unclear that $\mathfrak{p}_x$ is a point type, that is, according to definition 2.10 $\mathfrak{p}_x$ possess a non-empty associated cone. This fact is checked it the first part of the proof of proposition 2.12.
Observe that for any non-singular point $x \in NS \subset \mathbb{R}^n_c$, as the cut type $I_{\pi(x)}$ is empty, we have $\mathfrak{p}_x = \infty$ on I and $C_x = \mathbb{R}^n$. This means that a non-singular point is well approximated without regards on the direction we follow.

Proposition 2.12. *Any point $x \in \mathbb{R}^n_c$ has an open neighbourhood basis of the form*

$$\left\{\mathcal{U}(x, \varepsilon) := [C_x + \pi(x)]_c \cap \pi^{-1}(B(\pi(x), \varepsilon))\right\}_{\varepsilon>0}$$

Proof. Fix $x \in \mathbb{R}^n_c$. By construction of $\mathfrak{p}_x$ we have $x \in [H_i^{\mathfrak{p}_x(i)} + \pi(x)]_c$ for all $i \in I$. Moreover $[H_i^{\mathfrak{p}_x(i)} + \pi(x)]_c$ is a clopen set of the family $\mathfrak{A}_c$: for, if $i \notin I_{\pi(x)}$ then $\mathfrak{p}_x(i) = \infty$ so it is nothing but $\mathbb{R}^n_c$, and in case $i \in I_{\pi(x)}$, as therefore $\pi(x) \in A_i + \underline{n}_i$ for some $\underline{n}_i \in \mathbb{Z}^{n+d}$, it writes as $[A_i^{\mathfrak{p}_x(i)} + \underline{n}_i]_c \in \mathfrak{A}_c$.
It follows that the Boolean rules of proposition 2.8 applies to $[H_i^{\mathfrak{p}_x(i)} + \pi(x)]_c$, $i \in I$, giving $x \in \bigcap_{i\in I}[H_i^{\mathfrak{p}_x(i)} + \pi(x)]_c = [\bigcap_{i\in I} H_i^{\mathfrak{p}_x(i)} + \pi(x)]_c = [C_x+\pi(x)]_c$. This ensure in particular that C_x is non-empty, as $(C_x+\pi(x)) \cap NS = [C_x+\pi(x)]_c \cap NS$ is non-empty.
From proposition 2.8, if y is another point there exists an index i and $\underline{n}$ with (up to a switch of + and - signs) $x \in [A_i^+ + \underline{n}]_c$ and $y \in [A_i^- + \underline{n}]_c$. If moreover $\pi(x) = \pi(y)$, it is then easy to check that we must have $\pi(x) = \pi(y) \in A_i + \underline{n}$. In this latter case it is direct, using $A_i^\pm \pm H_i^0 = A_i^\pm$, to show the equality $[A_i^\pm + \underline{n}]_c = [H_i^\pm + \pi(x)]_c$. Thus for any given y with $\pi(y) = \pi(x)$ there is an index i_0 with $x \in [H_{i_0}^+ + \pi(x)]_c$ and $y \in [H_{i_0}^- + \pi(x)]_c$ (up to a swich of signs). Consequently the hyperplan H_i^0 separates C_x and C_y, so these cones are disjoints in $\mathbb{R}^n$. Now $[C_x+\pi(x)]_c$ and $[C_y+\pi(x)]_c$ are also disjoints, which in turns implies that x is the only point among its fiber contained into the clopen set $[C_x+\pi(x)]_c$. By proposition 2.9, we deduce that the stated family is a neighbourhood basis for x in $\mathbb{R}^n_c$. □

Introduce here the following notation: for a position $z \in \mathbb{R}^n$ and a cone C associated to a point type in $\mathfrak{P}$, denote by $C(z,\varepsilon)$ the set $z + (C \cap B(0,\varepsilon))$. This is the head of the cone C of length ε pointed at position z, and is always an open part of the Euclidean ball $B(z,\varepsilon)$. Observe that $z \notin C(z,\varepsilon)$ unless the cone C is all $\mathbb{R}^n$ (in which case we have $C(z,\varepsilon)$ equals to $B(z,\varepsilon)$), although we always have $z \in \overline{C(z,\varepsilon)}$ in $\mathbb{R}^n$.

Proposition 2.12 enable us to present the cut up space $\mathbb{R}^n_c$ in a more elegant way, as it is done in the following theorem 2.14, as well as its topology since we will describe converging sequences only by means of Euclidean sets.

Lemma 2.13. *For any $x \in \mathbb{R}^n_c$ and $0 < \varepsilon_1 < \varepsilon_2$ we have*

$$\mathcal{U}(x,\varepsilon_1) \subset [C_x(\pi(x),\varepsilon_1)]_c \subset \mathcal{U}(x,\varepsilon_2)$$

Proof. Let $y \in \mathcal{U}(x,\varepsilon_1) = [C_x+\pi(x)]_c \cap \pi^{-1}(B(\pi(x),\varepsilon_1))$. Since $y \in [C_x+\pi(x)]_c$ there is a sequence in $NS \cap (C_x+\pi(x))$ converging to y. Since $y \in \pi^{-1}(B(\pi(x),\varepsilon_1))$ we also have $\pi(y) \in B(\pi(x),\varepsilon_1)$, so the sequence lies into $NS \cap (C_x + \pi(x)) \cap B(\pi(x),\varepsilon_1) = NS \cap C_x(\pi(x),\varepsilon_1)$ eventually. It follows that $y \in [C_x(\pi(x),\varepsilon_1)]_c$.
Now if $y \in [C_x(\pi(x),\varepsilon_1)]_c$, by definition there is a sequence in $NS \cap C_x(\pi(x),\varepsilon_1)$ converging to y in $\mathbb{R}^n_c$. Hence this sequence lies into both $NS \cap (C_x + \pi(x)) = NS \cap [C_x + \pi(x)]_c$ and into $B(\pi(x),\varepsilon_1) \subset \overline{B}(\pi(x),\varepsilon_1) \subset B(\pi(x),\varepsilon_2)$, where $\overline{B}(\pi(x),\varepsilon_1)$ is the closed Euclidean ball. It follows by taking limit that $y \in [C_x + \pi(x)]_c \cap \pi^{-1}(B(\pi(x),\varepsilon_2)) = \mathcal{U}(x,\varepsilon_2)$, as desired. □

Theorem 2.14. *The map*

$$\mathbb{R}^n_c \ni x \longmapsto (\pi(x),\mathfrak{p}_x) \in \{(z,\mathfrak{p}) \in \mathbb{R}^n \times \mathfrak{P} \mid dom(\mathfrak{p}) = I_z\}$$

associating to each point its position and point type is a homeomorphism, with right term equiped with the following topology of convergence: $(z_n,\mathfrak{p}_n) \longrightarrow (z,\mathfrak{p})$ if and only if

$$\forall \varepsilon > 0, \exists \delta_n > 0 \text{ such that } C_{\mathfrak{p}_n}(z_n,\delta_n) \subset C_{\mathfrak{p}}(z,\varepsilon) \text{ for large enough } n$$

Proof. The fact that $dom(\mathfrak{p}_x) = I_{\pi(x)}$ holds for any point x comes from the very construction of $\mathfrak{p}_x$, so the stated map is well-defined.
From proposition 2.12, each point $x \in \mathbb{R}^n_c$ is the unique limit point of the filterbase $\{[C_x+\pi(x)]_c \cap \pi^{-1}(B(\pi(x),\varepsilon))\}_{\varepsilon>0}$, with $C_x = C_{\mathfrak{p}_x}$. Since this filterbase only depends upon the pair $(\pi(x),\mathfrak{p}_x)$, the stated map is 1-to-1. The map is also onto: to each pair $(z,\mathfrak{p})$ we shall consider the family $\{[C_{\mathfrak{p}} + z]_c \cap \pi^{-1}(B(z,\varepsilon))\}_{\varepsilon>0}$. It will form a filterbase in $\mathbb{R}^n_c$ provided none of these sets is empty. This follows from the fact that for each $\varepsilon > 0$ we have $NS \cap [C_{\mathfrak{p}} + z]_c \cap \pi^{-1}(B(z,\varepsilon)) = NS \cap C_{\mathfrak{p}}(z,\varepsilon)$, which is non-empty since $C_{\mathfrak{p}}(z,\varepsilon)$ is open and NS is dense. Let $\varepsilon_0 > 0$ be chosen. for any $\varepsilon < \varepsilon_0$ we have $[C_{\mathfrak{p}} + z]_c \cap \pi^{-1}(B(z,\varepsilon)) \subset \pi^{-1}(\overline{B(z,\varepsilon_0)})$, this latter set being compact since $\overline{B(z,\varepsilon_0)}$ is compact and π is proper. This means that for great enough ε the filterbase is contained into a fixed compact set of $\mathbb{R}^n_c$. It then possess an accumulation point x, which has to satisfies $\pi(x) = z$ and $C_x = C_{\mathfrak{p}}$. The fact that $C_x = C_{\mathfrak{p}}$ ensure that $\mathfrak{p}_x$ and $\mathfrak{p}$ coincide on $I_{\pi(x)} = I_z$. But by assumption we also have

$dom(\mathfrak{p}) = I_z = I_{\pi(x)} = dom(\mathfrak{p}_x)$, so in fact $\mathfrak{p}_x = \mathfrak{p}$ on I. Hence $(z, \mathfrak{p}) = (\pi(x), \mathfrak{p}_x)$, and the map is onto.
We next show that convergence of sequences in $\mathbb{R}^n_c$ is given by the criteria of the statement. Let then $x_n \leftrightarrow (z_n, \mathfrak{p}_n)$ and $x \leftrightarrow (z, \mathfrak{p})$. By proposition 2.12, $x_n \longrightarrow x$ if and only if

$$\forall \varepsilon,\ x_n \in \mathcal{U}(x, \varepsilon) \text{ for large enough } n$$

or equivalently,

$$\forall \varepsilon,\ \exists\ \delta_n > 0 \text{ with } \mathcal{U}(x_n, \delta_n) \subset \mathcal{U}(x, \varepsilon) \text{ for large enough } n$$

Now using lemma 2.13 we can easily show that it is equivalent to

$$\forall \varepsilon,\ \exists\ \delta_n > 0 \text{ with } [C_{\mathfrak{p}_n}(z_n, \delta_n)]_c \subset [C_{\mathfrak{p}}(z, \varepsilon)]_c \text{ for large enough } n \tag{1}$$

It is in turn equivalent to the criteria of the statement. For, if (1) holds then by applying π we get $\overline{C_{\mathfrak{p}_n}(z_n, \delta_n)} \subset \overline{C_{\mathfrak{p}}(z, \varepsilon)}$. Since the open sets $C_{\mathfrak{p}_n}(z_n, \delta_n)$ and $C_{\mathfrak{p}}(z, \varepsilon)$ are regular (meaning that they both are the interior of their closure), we obtain the criteria holding. Conversely if the criteria of the statement holds then intersecting with NS and taking closure in $\mathbb{R}^n_c$ ensure that (1) holds.

□

Note that through this picture of the cut up space, NS corresponds to the set $\{(z, \infty) \mid I_z = \varnothing\}$. Moreover, a sequence $(z_n, \mathfrak{p}_n)$ will converges to a non-singular point (z, ∞) if and only if, as $C_\infty(z, \varepsilon)$ is the Euclidean ball $B(z, \varepsilon)$, z_n converges to z in $\mathbb{R}^n$. At the opposite end, a non-singular sequence (z_n, ∞) will converges to $(z, \mathfrak{p})$ if and only if z_n converges to z in $\mathbb{R}^n$ and $z_n - z$ lies into $C_{\mathfrak{p}}$ for great enough n.

2.4. **Dynamics and the cut up torus.** Here we deal with dynamical features: as the next proposition states, the translation action of $\mathbb{Z}^{n+d}$ on $\mathbb{R}^n$ has an extension to an action by homeomorphisms on $\mathbb{R}^n_c$. In fact, so far we could have worked with any countable dense subgroup of $\mathbb{R}^n$ instead of $\mathbb{Z}^{n+d}$.

Proposition 2.15. *There is a unique $\mathbb{Z}^{n+d}$-action on $\mathbb{R}^n_c$ by homeomorphisms such that π becomes equivariant. This action satisfies for each $A \in \mathfrak{A}$*

$$A^{\pm}_c . \underline{n} = [A^{\pm} + \underline{n}]_c$$

Proof. Restrict the $\mathbb{Z}^{n+d}$-action of $\mathbb{R}^n$ on the set NS: this can be done by the very construction of the non-singular points. Choosing a $\underline{n} \in \mathbb{Z}^{n+d}$, it defines a bijection of NS. For any integer k, it extends to a surjective and continuous map $T^{\underline{n}}_k : \mathbb{R}^n_{k+|\underline{n}|} \twoheadrightarrow \mathbb{R}^n_k$: on the Euclidian closure of any connected component this map is well defined and uniformly continuous. Taking inverse limit over k one obtains a continuous map $T^{\underline{n}}$. Then this map coincides with regular translation by $\underline{n}$ on NS, so we get the equivariance rule $\pi(T^{\underline{n}}(x)) = \pi(x) + \underline{n}$ for any $x \in \mathbb{R}^n_c$. To see that it is an action, observe that $T^{\underline{0}}$ is the identity map and that $T^{\underline{n}} \circ T^{\underline{m}}$ and $T^{\underline{n}+\underline{m}}$ coincide on NS, so are equal everywhere.
Finally, since NS is stable under this action the sets $T^{\underline{n}}(A^{\pm}_c)$ and $[A^{\pm} + \underline{n}]_c$ coincide on NS, so are equal by lemma 2.7. Unicity of this action is clear from the density of NS. □

From now on we write $x.\underline{n}$ for a translated image $T^{\underline{n}}(x)$.
Thanks to this action, we can define a cut up version of the n-dimensionnal torus together with an action as:

Definition 2.16. *The cut up n-torus $\mathbb{T}^n_c$ is the quotient $\mathbb{R}^n_c/\mathbb{Z}^n$, a space on which $\mathbb{Z}^d$ acts by homeomorphisms. Identifying $\mathbb{R}^n/\mathbb{Z}^n$ with the n-torus $\mathbb{T}^n$, there is an induced continuous, onto and $\mathbb{Z}^d$-equivariant map*

$$\pi_{\mathbb{T}} : \mathbb{T}^n_c \twoheadrightarrow \mathbb{T}^n$$

We call the image point $\pi_{\mathbb{T}}(x)$ the position of $x \in \mathbb{T}^n_c$.

We denote by $\rho_{\mathbb{Z}^n} : \mathbb{R}^n_c \twoheadrightarrow \mathbb{T}^n_c$ the quotient map. For any point $x \in \mathbb{R}^n_c$ and any element $\underline{n} \in \mathbb{Z}^{n+d}$, from the previous proposition we derive the equality of point types $\mathfrak{p}_{x.\underline{n}} = \mathfrak{p}_x$. Thus we can without misunderstanding associate a point type $\mathfrak{p}_x$ and a cone C_x for any $x \in \mathbb{T}^n_c$, to be the ones associated to any lift of x in $\mathbb{R}^n_c$.

Theorem 2.17. *The space $\mathbb{T}^n_c$ is a Cantor space, endowed with a minimal $\mathbb{Z}^d$-action. The diagramm of continuous, onto and equivariant maps*

$$\begin{array}{ccc} \mathbb{R}^n_c & \xrightarrow{\pi} & \mathbb{R}^n \\ \downarrow{\scriptstyle \rho_{\mathbb{Z}^n}} & & \downarrow{\scriptstyle [.]_{\mathbb{Z}^n}} \\ \mathbb{T}^n_c & \xrightarrow{\pi_{\mathbb{T}}} & \mathbb{T}^n \end{array}$$

is commutative, with vertical arrows being covering maps. Moreover, $\pi_{\mathbb{T}}$ is one-to-one exactly on $NS/\mathbb{Z}^n$.
The map

$$\mathbb{T}^n_c \ni x \longmapsto (\pi_{\mathbb{T}}(x), \mathfrak{p}_x) \in \{(z, \mathfrak{p}) \in \mathbb{T}^n \times \mathfrak{P} \mid dom(\mathfrak{p}) = I_z\}$$

associating to each point its position and point type is a homeomorphism, with topology on the right term determined by the convergence rule: $(z_n, \mathfrak{p}_n) \longrightarrow (z, \mathfrak{p})$ if and only if there are lifts $\underline{z}_n$ and $\underline{z}$ in $\mathbb{R}^n$ of z_n and z such that

$$\forall \varepsilon > 0, \exists \delta_n > 0 \text{ such that } C_{\mathfrak{p}_n}(\underline{z}_n, \delta_n) \subset C_{\mathfrak{p}}(\underline{z}, \varepsilon) \text{ eventually on } n$$

It is the dynamical system $(\mathbb{T}^n_c, \mathbb{Z}^d)$ we are interrested in, but it is more conveniant to work on $\mathbb{R}^n_c$, as we can deal with geometrical objects like cut up half spaces and cones. For that reason, most of the remaining work will be focused on $(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. Nicely, we will recover all the results for $(\mathbb{T}^n_c, \mathbb{Z}^d)$ by simply moding out $\mathbb{Z}^n$ in an appropriate way.

3. The Ellis semigroup

The enveloping semigroup of a compact dynamical system has been introduced by Robert Ellis as a way to study actions of a group T on a compact space X from an algebraic point of view. Here a compact dynamical system, or flow, stands for a compact (Hausdorff) space X together with an action of a group T by homeomorphisms. The simplest example of such a system is a *Kronecker flow*, that is a

compact group G together with a subgroup T acting by right-translation. In this case the group T forms an equicontinuous family of homeomorphisms on G, and in this situation we call the system (G, T) an *equicontinuous flow*. It turns out that among the class of compact minimal flows with T being Abelian, the kronecker flows are exactly the equicontinuous ones ([1] p.53, thm 6), and in this case G is Abelian. A general result relates general flows with Kronecker flows:

Theorem 3.1. [1]. *Let (X, T) be a flow. There exist a closed T-invariant equivalence relation $\sim_{eq}$ on X, such that the quotient space $X_{eq} := X/_{\sim_{eq}}$ with T-action is an equicontinuous flow, which is maximal in the sense that any equicontinuous factor of (X, T) factors through X_{eq}.*
In case (X, T) is minimal and T is Abelian, (X_{eq}, T) is an Abelian Kronecker flow.

The flow (X_{eq}, T) is called the *maximal equicontinuous factor* of (X, T).

Definition 3.2. *A flow (X, T) is an almost automorphic system if the factor map $\pi : X \twoheadrightarrow X_{eq}$ possess a one-point fiber.*

In case the flow is also metric minimal, one may equivalently require that the set of one-point fibers forms a dense residual subset of X (see [11]), from now on denoted by NS. In the case of a minimal rotation by $\mathbb{Z}^d$ on a cut up torus $\mathbb{T}^n_c$, it can easily be checked that the maximal equicontinuous factor is the torus $\mathbb{T}^n$, with factor map $\pi_{\mathbb{T}}$. As $\pi_{\mathbb{T}}$ is 1-to-1 on a non-empty set by theorem 2.17, the flow $(\mathbb{T}^n_c, \mathbb{Z}^d)$ is almost automorphic.

Equicontinuity properties of a group action can be rephrazed in terms of the Ellis enveloping semigroup of a flow. Denote by X^X the space of maps from X into itself with pointwise convergence topology, or equivalently the product space $\prod_X X$ endowed with the (compact Hausdorff) product topology. It naturally contains the group of homeomorphisms of X coming from the T-action, and carries a semigroup law given by composition of maps.

Definition 3.3. *Let (X, T) be a flow. Denote by $\tilde{T}$ the group of maps in X^X induced by the T-action. The associated Ellis semigroup $E(X, T)$ is the closure of $\tilde{T}$ in X^X.*

$E(X, T)$ is then a set of transformations on X, and we write $x.g$ for the evaluation of the map g at a point x. It is always a compact right-topological semigroup, that is, if $\{h_\lambda\}_\lambda$ is a net converging to h, then the net $\{g.h_\lambda\}_\lambda$ converges to $g.h$ for any g, where $g.h$ stands for the composition which at each point x reads $(x.g).h$.

The Ellis semigroup construction is functorial (covariant) in the sense that any onto continuous T-equivariant map $\pi : X \twoheadrightarrow Y$ gives rise to an onto continuous semigroup morphism $\pi^* : E(X, T) \twoheadrightarrow E(Y, T)$, satisfying $\pi(x.g) = \pi(x).\pi^*(g)$ for any $x \in X$ and any transformation $g \in E(X, T)$.
The philosophy about Ellis semigroup is that the algebraic and topological properties of the Ellis semigroup of a flow are correlated with the nature of the dynamical system. A standard result in this direction is the following:

Theorem 3.4. *For a flow (X, T), the following assertions are equivalents (see* [1]*, theorem 3 p.52)):*
(1) the flow (X, T) is equicontinuous.

(2) The Ellis semigroup $E(X,T)$ is a compact group, acting by homeomorphisms on X.
In addition, if (X,T) is minimal with T Abelian, this is also equivalent to (see [1]*, p.55):*
(3) $E(X,T)$ has left-continuous product.
(4) $E(X,T)$ is Abelian.
(5) $E(X,T)$ is made of continuous transformations.

From the previous theorem we see that in the non-equicontinuous case the Ellis semigroup is quite difficult to understand and to handle, since none of the criteria of theorem 3.4 are satisfied. However, there exists a property on the transformations of $E(X,T)$ one might expect to have. A transformation is Baire class 1 if it is the pointwise limit of a *sequence* (by contrast to a net) of continuous transformations.

Definition 3.5. *(see* [5], [4]*) A metric flow (X,T) is tame if every element of $E(X,T)$ is a Baire class 1 transformation.*

In this later case any transformation of $E(X,T)$ is measurable, but still two distinct transformations can be equal almost everywhere. Observe that if the semigroup $E(X,T)$ is first countable, that is, admits at any point a countable local neighbourhood basis, then the underlying system is tame.
Theorem 3.1 is related to the spectral analysis of (X,T). Suppose here that (X,T) is a metric flow with T Abelian, and is endowed with a T-invariant probability measure μ. There is a natural unitary representation of T on the Hilbert space $L^2(X,\mu)$. Denote by $\hat{T}$ the Pontryagin dual of T. For a character $\omega \in \hat{T}$, let $\mathcal{H}_\omega$ be the subspace of L^2-functions satisfying the equality $U_t(f) = \omega(t)f$ for all $t \in T$. Generically the space $\mathcal{H}_\omega$ is trivial, and a character ω is said to be an *eigenvalue* for (X,T,μ) if the subspace $\mathcal{H}_\omega$ is non-trivial. $\mathcal{H}_\omega$ is then called the *eigenspace* for ω, and any non zero L^2-function in $\mathcal{H}_\omega$ an *eigenfunction* for ω. Moreover an eigenvalue ω is said to be a *topological eigenvalue* for (X,T,μ) if the subspace $\mathcal{H}_\omega$ is non-trivial and consist of (L^2-classes of) continuous functions. The collection $Ev(X,T,\mu)$ of all eigenvalues of (X,T,μ) is always a (possibly empty) countable subgroup of $\hat{T}$, and contains the collection $Ev^{top}(X,T,\mu)$ of topological eigenvalues as a subgroup.

Definition 3.6. *A metric flow with invariant measure (X,T,μ) is said to have pure point dynamical spectrum if $L^2(X,\mu) = \bigoplus_{\omega \in Ev(X,T,\mu)} \mathcal{H}_\omega$ holds, and is said to have topological pure point dynamical spectrum if $L^2(X,\mu) = \bigoplus_{\omega \in Ev^{top}(X,T,\mu)} \mathcal{H}_\omega$ holds.*

It is well known that Abelian Kronecker flows with Haar measure have topological pure point dynamical spectrum. The following theorem enlarges the class of systems where the same conclusion occurs, using the notions of definitions 3.2, 3.5 and 3.6:

Theorem 3.7. [4]. *Let (X,T) be a metric minimal flow with T Abelian. Then (1)⇒(2)⇒(3):*
(1) (X,T) is a tame system.
(2) (X,T) is an almost automorphic system, with unique ergodic probability measure μ satisfying $\mu(NS) = 1$.
(3) (X,T,μ) has topological pure point dynamical spectrum.

3.1. **Ellis semigroup for locally compact flows.** Let X be a locally compact space together with an action of a group T by homeomorphisms. The one-point compactification $\hat{X}$ of X is a compact space, endowed with a T-action by homeomorphism so that the infinite point remains fixed through any homeomorphism $t \in T$. Denote by $\mathcal{F}_X$ the set of maps in $\hat{X}^{\hat{X}}$ which send X into itself and keep the point at infinity fixed. Then:

Definition 3.8. *The Ellis semigroup of (X,T) is defined to be*

$$E(X,T) := E(\hat{X},T) \cap \mathcal{F}_X$$

with topology induced from $\hat{X}^{\hat{X}}$.

Recall that the topology of X^X is generated by sets as follows: For $x \in X$, $U \subset X$ let $V(x,U)$ (or sometimes $V_X(x,U)$ when we want the space X to be specified) be the subset

$$V(x,U) := \left\{ g \in X^X \mid g(x) \in U \right\}$$

It is open if U is open, closed if U is, and with x among X and U among any pre-basis for the topology of X, it forms a pre-basis for the topology.
Observe that $E(X,T)$ is, as in the compact flow case, a right-topological semigroup containing T as a dense subgroup (it rather contains the group of transformations $\tilde{T}$, but keeping this in mind we make in the sequel an abuse of notation and identify $\tilde{T}$ with T)). Although transformations of $E(X,T)$ are defined on $\hat{X}$, they all fix the point at infinity and so may be seen as transformations in X^X (it is quite direct to show that on $E(X,T)$ the topologies coming from $\hat{X}^{\hat{X}}$ and X^X coincide). Although the Ellis semigroup is rich in the compact case it is far from clear how big it is in the non-compact case: it might well only consist of T itself, since we assume the transformations to send **each** point of X anywhere but at the point at infinity. Thus we cannot pretend that this is the optimal way to define an Ellis enveloping semigroup for locally compact flows. However, in our context this construction will be of great use in the description of $E(\mathbb{T}^n_c, \mathbb{Z}^d)$. The following is a general fact, whose proof in the context of compact flows case can be found in [1]:

Proposition 3.9. *Let $\pi : X \twoheadrightarrow Y$ be a continuous, proper, onto, and T-equivariant map between locally compact spaces. Then there exist a continuous, proper, and onto morphism $\pi^* : E(X,T) \twoheadrightarrow E(Y,T)$ satisfying the equivariance condition: $\pi(x.g) = \pi(x).\pi^*(g)$ for any $x \in X$ and $g \in E(X,T)$.*

Proof. Denote by $\star_X$ and $\star_Y$ the respective points at infinity in the compactified spaces. Since π is continuous and proper, it extends to a continuous and onto map $\hat{\pi} : \hat{X} \twoheadrightarrow \hat{Y}$, such that $\hat{\pi}^{-1}(\star_Y) = \{\star_X\}$. Obviously $\hat{\pi}$ is T-equivariant with respect to the extended T-actions. There exist then a continuous and onto morphism $\hat{\pi}^* : E(\hat{X},T) \twoheadrightarrow E(\hat{Y},T)$, satisfying the equivariance equality for any $x \in \hat{X}$ and $g \in E(\hat{X},T)$: $\hat{\pi}(x.g) = \hat{\pi}(x).\hat{\pi}^*(g)$. The later equivariance condition implies that a transformation g of $E(\hat{X},T)$ lies into $\mathcal{F}_X$ if and only if $\hat{\pi}^*(g)$ lies in $\mathcal{F}_Y$: it follows that $E(X,T) = (\hat{\pi}^*)^{-1}(E(Y,T))$. Restricting the morphism on $E(X,T)$ gives the map, together with the onto property. Finally a compact set of $E(Y,T)$ has to be compact in $E(\hat{Y},T)$ as it is easy to check, so have a compact pre-image in $E(\hat{X},T)$ under $\hat{\pi}^*$. This latter is entirely included in $E(X,T)$, so is compact for the relative

topology on $E(X,T)$. This gives the properness.

□

Observe that $\pi^*(t) = t$ holds for any $t \in T$. If the group T is Abelian, then any element of T commutes with any element of the enveloping semigroup $E(X,T)$, for any locally compact space X (for X compact see [1], the locally compact case being a direct consequence).

Proposition 3.10. *If T is a dense subgroup of a locally compact Abelian group G, T acting by translation, then $E(G,T)$ is a topological group topologicaly isomorphic with G, and through this identification $E(G,T)$ acts by translation.*

Proof. First observe that the Ellis semigroup $E(G,G)$ is well defined, and by construction contains G and $E(G,T)$. It is easy to see that on G, the group topology and the pointwise convergence topology inherited from $E(G,G)$ coincides, since an open set $V(x,U)$ in G rewrites as $x^{-1}.U$. Now T being dense in G, we get $G \subset E(G,T)$ and thus $E(G,G) = E(G,T)$. As G is Abelian, any transformation g of $E(G,G)$ commutes with any element of G, and thus is uniquely determined through its value at 0, so coincide with the translation function by $g(0)$ on G. □

As a consequence of this, we may identify $E(\mathbb{R}^n, \mathbb{Z}^{n+d})$ with $\mathbb{R}^n$ and $E(\mathbb{T}^n, \mathbb{Z}^d)$ with $\mathbb{T}^n$.
Propositions 3.9 and 3.10 infer the existence of the onto continuous morphisms

$$\pi^* : E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) \twoheadrightarrow \mathbb{R}^n \qquad\qquad \pi^*_{\mathbb{T}} : E(\mathbb{T}^n_c, \mathbb{Z}^d) \twoheadrightarrow \mathbb{T}^n$$

with the composition properties: $\pi(x.g) = \pi(x) + \pi^*(g)$ for any $x \in \mathbb{R}^n_c$ and $g \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$, and $\pi_{\mathbb{T}}(x.g) = \pi_{\mathbb{T}}(x) + \pi^*_{\mathbb{T}}(g)$ for any $x \in \mathbb{T}^n_c$ and $g \in E(\mathbb{T}^n_c, \mathbb{Z}^d)$. Moreover, as the arrow π^* is proper, $\mathbb{R}^n$ is locally compact implies that $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ is also locally compact.

3.2. **Relation between $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ and $E(\mathbb{T}^n_c, \mathbb{Z}^d)$.** Now we relate the Ellis semigroup $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ with $E(\mathbb{T}^n_c, \mathbb{Z}^d)$ by constructing a morphism $\rho^*_{\mathbb{Z}^n}$ from the first onto the latter. Observe that, although we have an onto continuous and $\mathbb{Z}^{n+d}$-equivariant map $\rho_{\mathbb{Z}^n} : \mathbb{R}^n_c \twoheadrightarrow \mathbb{T}^n_c$ (where on $\mathbb{T}^n_c$ the summand $\mathbb{Z}^n$ acts trivially), since $\rho_{\mathbb{Z}^n}$ is not proper (as $\mathbb{T}^n_c$ is compact and $\mathbb{R}^n_c$ is not) proposition 3.9 do not infer the existence of $\rho^*_{\mathbb{Z}^n}$.
We construct $\rho^*_{\mathbb{Z}^n}$ as follows. Write $[x]_{\mathbb{Z}^n}$ for the $\mathbb{Z}^n$-orbit in $\mathbb{R}^n_c$ of x.
We know that because the group $\mathbb{Z}^{n+d}$ is Abelian and dense in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$, any transformation g of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ commutes with any $\underline{n} \in \mathbb{Z}^{n+d}$ so $[x]_{\mathbb{Z}^n}.g = [x.g]_{\mathbb{Z}^n}$ for any x and any g. Hence any g passes through the quotient map $\rho_{\mathbb{Z}^n}$, defining so an element $\rho^*_{\mathbb{Z}^n}(g)$ in the compact space $(\mathbb{T}^n_c)^{\mathbb{T}^n_c}$. The map $\rho^*_{\mathbb{Z}^n} : E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) \longrightarrow (\mathbb{T}^n_c)^{\mathbb{T}^n_c}$ is obviously a semigroup morphism, and is continuous with respect to the pointwise convergence topologies: For, we have the equality

$$(\rho^*_{\mathbb{Z}^n})^{-1}(V_{\mathbb{T}^n_c}(x,U)) = V_{\mathbb{R}^n_c}(\tilde{x}, \rho^{-1}_{\mathbb{Z}^n}(U)) \bigcap E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$$

holding for any x and U, where $\tilde{x}$ is any lift of x. Then by continuity of $\rho_{\mathbb{Z}^n}$ and because $\mathbb{Z}^{n+d}$ is dense in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$, we get a continuous morphism

$$\rho^*_{\mathbb{Z}^n} : E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) \twoheadrightarrow E(\mathbb{T}^n_c, \mathbb{Z}^d)$$

which, from its very construction, satisfies the equivariance condition $\rho_{\mathbb{Z}^n}(x.g) = \rho_{\mathbb{Z}^n}(x).\rho^*_{\mathbb{Z}^n}(g)$ for any $x \in \mathbb{R}^n_c$ and $g \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$.
It is direct to check that the diagramm

$$\begin{array}{ccc} E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) & \xrightarrow{\pi^*} & \mathbb{R}^n \\ \downarrow{\scriptstyle \rho^*_{\mathbb{Z}^n}} & & \downarrow{\scriptstyle [.]_{\mathbb{Z}^n}} \\ E(\mathbb{T}^n_c, \mathbb{Z}^d) & \xrightarrow{\pi^*_{\mathbb{T}}} & \mathbb{T}^n \end{array}$$

is commutative.

Lemma 3.11. *The morphism $\rho^*_{\mathbb{Z}^n}$ is onto.*

Proof. Because the map $\rho_{\mathbb{Z}^n} : \mathbb{R}^n_c \twoheadrightarrow \mathbb{T}^n_c$ is a covering map and $\mathbb{T}^n_c$ is compact, there exist a compact subset K' of $\mathbb{R}^n_c$ such that $\rho_{\mathbb{Z}^n}(K') = \mathbb{T}^n_c$, that is we have $\mathbb{R}^n_c = \bigcup_{\underline{n} \in \mathbb{Z}^n} K'.\underline{n}$. Take any point x in $\mathbb{R}^n_c$: The set $K := V(x, K')$ is closed and $\mathbb{Z}^n$-generating, as it is easy to see thanks to the equality $V(x, K').\underline{n} = V(x, K'.(-\underline{n}))$. To show that K is compact, observe that $\pi^*(K) \subset \pi(K') - \pi(x)$, which is a compact Euclidean set. Then $K \subset (\pi^*)^{-1}(\pi(K') - \pi(x))$, in turn compact since according to proposition 3.9 π^* is proper. K being closed, it is then compact. Finally, $\rho^*_{\mathbb{Z}^n}(E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}))$ contains $\mathbb{Z}^n$ and is equal to $\rho^*_{\mathbb{Z}^n}(K)$ which is compact, so is the full semigroup $E(\mathbb{T}^n_c, \mathbb{Z}^d)$. □

Lemma 3.12. $Ker(\rho^*_{\mathbb{Z}^n}) = \mathbb{Z}^n$.

Proof. Obviously the kernel of $\rho^*_{\mathbb{Z}^n}$ contains $\mathbb{Z}^n$. Conversely, If g and h are transformations in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ with same image in $E(\mathbb{T}^n_c, \mathbb{Z}^d)$, then $\pi^*(g)$ and $\pi^*(h)$ have same image on the torus, so differ by an element $\underline{n} \in \mathbb{Z}^n$, say $\pi^*(h) + \underline{n} = \pi^*(g)$. Then $h.\underline{n}$ and g have same images under π^* and $\rho^*_{\mathbb{Z}^n}$. If x is a point in $\mathbb{R}^n_c$, $\rho_{\mathbb{Z}^n}(x.h.\underline{n}) = \rho_{\mathbb{Z}^n}(x.g)$ so there is an $\underline{m} \in \mathbb{Z}^n$ with $x.h.(\underline{n} + \underline{m}) = x.g$. It follows that $\pi(x.h.\underline{n}) + \underline{m} = \pi(x.g) = \pi(x) + \pi^*(g) = \pi(x) + \pi^*(h.\underline{n}) = \pi(x.h.\underline{n})$, giving $\underline{m} = 0$. This shows that $h.\underline{n}$ and g coincides at every point of $\mathbb{R}^n_c$, ans thus are equal. □

In addition to these facts, we prove here that $\rho^*_{\mathbb{Z}^n}$ is a covering map, so that any converging net of $E(\mathbb{T}^n_c, \mathbb{Z}^d)$ can be lifted into a converging net of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$.

Proposition 3.13. *$\rho^*_{\mathbb{Z}^n}$ is a covering map.*

Proof. Given $g \in E(\mathbb{T}^n_c, \mathbb{Z}^d)$, take U to be an open neighbourhood of $\pi^*(g)$ which is trivializing for $[.]_{\mathbb{Z}^n}$ and select an open set V of $\mathbb{R}^n$ with $[.]_{\mathbb{Z}^n} : V \approx U$. Hence $[.]^{-1}_{\mathbb{Z}^n}(U) = V.\mathbb{Z}^n \approx U \times \mathbb{Z}^n$. Then $U' := (\pi^*_{\mathbb{T}})^{-1}(U)$ is an open neighbourhood of g such that $V' := (\pi^*)^{-1}(V)$ and U', as it is quite direct to show using commutativity of the diagramm, onto properties of the maps and the equalities $Ker(\rho^*_{\mathbb{Z}^n}) = Ker([.]_{\mathbb{Z}^n}) = \mathbb{Z}^n$, are homeomorphic through $\rho^*_{\mathbb{Z}^n}$. Consequently $(\rho^*_{\mathbb{Z}^n})^{-1}(U') = V'.\mathbb{Z}^n \approx U' \times \mathbb{Z}^n$. □

3.3. **The topology of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$.** In this paragraph we show that $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ has its topology generated by a family of what may be seen as half spaces, an uncountable family though. Remarkably, the Ellis semigroup ony depends on the relative position of the group $\mathbb{Z}^{n+d}$ with respect to the collection of linear hyperspaces H_i^0, $i \in I$, so we could moove freely the affine spaces A_i along their orthogonal direction without altering the resulting Ellis semigroup.
By proposition 2.8, the pointwise convergence topology on $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ has a pre-basis of clopen sets given by the

$$\{V(x, A_c) \,|\, x \in \mathbb{R}^n_c,\, A \in \mathfrak{A}\}$$

Consider the family

$$\mathfrak{H} := \{H_i^t + z \,|\, i \in I,\, t \in \{-, 0, +\},\, z \in \mathbb{R}^n\}$$

This family if made up open half spaces and affine hyperspaces parallel to the H_i's.

Definition 3.14. *Let P be a subset of $\mathbb{R}^n$. The set P_E (or $[P]_E$) stands for the closure of $P \cap \mathbb{Z}^{n+d}$ in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$.*

Proposition 3.15. *The space $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ is totally disconnected, and the family*

$$\mathfrak{H}_E := \{H_E \,|\, H \in \mathfrak{H}\}$$

is a pre-basis of clopen sets for the topology of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. Moreover, any $H \in \mathfrak{H}$ satisfies $\mathbb{Z}^{n+d} \cap [H]_E = \mathbb{Z}^{n+d} \cap H$, and for any pair H, H' of sets in $\mathfrak{H}$, the following Boolean rules are true:

$$[H \cup H']_E = H_E \cup H'_E \qquad (H_E)^c = [H^c]_E \qquad [H \cap H']_E = H_E \cap H'_E$$

Observe that the equality $[H \cup H']_E = H_E \cup H'_E$ is a straightforward property of the closing operation.
The following lemma is a cornerstone for the sequel since it will gives us a dichotomy for what a set $V(x, A_c)$ looks like: it corresponds to an open half space in $\mathbb{R}^n$ or a closed one, only depending in some sense on the point x. For convenience we denote A_i^{+0} (resp. A_i^{-0}) to be the closed half space $A_i^+ \cup A_i$ (resp. $A_i^- \cup A_i$) for each $i \in I$.

Lemma 3.16. *Consider the clopen set $V(x, [A_i^+]_c)$. Then*

$$V(x, [A_i^+]_c) = \begin{cases} [A_i^{+0} - \pi(x)]_E & \text{if } \mathfrak{p}_x(i) = + \\ [A_i^+ - \pi(x)]_E & \text{if } \mathfrak{p}_x(i) = - \\ [A_i^{+0} - \pi(x)]_E = [A_i^+ - \pi(x)]_E & \text{if } \mathfrak{p}_x(i) = \infty \end{cases}$$

The same holds with the $+$ and $-$ signs switched.

Proof. Recall from lemma 2.7 that two clopen sets are equal in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ if and only if they coincide on the dense subset $\mathbb{Z}^{n+d}$. Given $V(x, [A_i^+]_c)$, an element $\underline{m} \in \mathbb{Z}^{n+d}$ lies inside if and only if $x.\underline{m} \in [A_i^+]_c$. Because $[A_i^+]_c$ is open, using proposition 2.12 and lemma 2.13 we have that $x.\underline{m} \in [A_i^+]_c$ if and only if

$$[C_x(\pi(x) + \underline{m}, \varepsilon_0)]_c \subset [A_i^+]_c \tag{2}$$

for some ε_0. This is in turn equivalent to

$$C_x(\pi(x) + \underline{m}, \varepsilon_0) \subset A_i^+ \tag{3}$$

For, if (2) holds then applying π we get $\overline{C_x(\pi(x)+\underline{m},\varepsilon_0)} \subset \overline{A_i^+}$, so taking interiors gives (3). Conversely if (3) holds then intersecting with NS and next taking closure in $\mathbb{R}^n_c$ gives (2).
Now we can deal with the more comfortable condition (3), as its statement only involves Euclidean sets. A rearrangement of (3) gives the equivalent condition

$$C_x(\underline{m},\varepsilon_0) \subset A_i^+ - \pi(x) \tag{4}$$

for some ε_0, that is, the head of length ϵ_0 of the cone C_x pointed at position $\underline{m}$ has to be included into the open half space $A_i^+ - \pi(x)$.
For $\underline{m} \in A_i^+ - \pi(x)$ the inclusion (4) is automatically satisfied;
For $\underline{m} \in A_i^- - \pi(x)$ the inclusion (4) is never satisfied;
If $i \in I_{\pi(x)}$, so that $\mathfrak{p}_x(i) = \pm$, we have $\mathbb{Z}^{n+d} \cap (A_i - \pi(x))$ non-void, so for $\underline{m}$ selected inside we have $A_i - \pi(x) = H_i^0 + \underline{m}$, giving $A_i^+ - \pi(x) = H_i^+ + \underline{m}$. In turns, the inclusion (4) is satisfied if and only if $C_x \subset H_i^+$ (this latter property depending only on x rather than on the chosen $\underline{m}$), that is $\mathfrak{p}_x(i) = +$. Taking closure in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ leads to the stated dichotomy.
If $i \notin I_{\pi(x)}$, so that $\mathfrak{p}_x(i) = \infty$, the intersection $\mathbb{Z}^{n+d} \cap (A_i - \pi(x))$ is void and the three sets become equal.
The same argument hold when switching the $+$ and $-$ signs, giving the statement. □

Lemma 3.17. *The family $\mathfrak{H}_E$ is made of clopen sets of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$, and each $H \in \mathfrak{H}$ satisfy the equality*

$$\mathbb{Z}^{n+d} \cap [H]_E = \mathbb{Z}^{n+d} \cap H$$

Moreover the Boolean rules of proposition 3.15 hold.

Proof. Any set of the form $[H_i^+ + z]_E$, with $i \in I$, is clopen in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. For, using the previous lemma $[H_i^+ + z]_E$ is equal to $V(x, [A_i^+]_c)$ if we choose x in the fiber above $a_i - z$, for any fixed $a_i \in A_i$, and with extra condition $\mathfrak{p}_x(i) = -$ in case $i \in I_{a_i - z}$. In addition we have $\mathbb{Z}^{n+d} \cap [H_i^+ + z]_E = \mathbb{Z}^{n+d} \cap V(x, [A_i^+]_c) = \mathbb{Z}^{n+d} \cap (A_i^+ - \pi(x)) = \mathbb{Z}^{n+d} \cap (H_i^+ + z)$.
Similarly $[H_i^- + z]_E$ is clopen and satisfy $\mathbb{Z}^{n+d} \cap [H_i^- + z]_E = \mathbb{Z}^{n+d} \cap (H_i^- + z)$.
Moreover the clopen sets $[H_i^- + z]_E$ and $[H_i^+ + z]_E$ are disjoint by lemma 2.7, since their intersection is clopen and contains no point of $\mathbb{Z}^{n+d}$.
It remains the case of $[H_i^0 + z]_E$. As $\mathbb{Z}^{n+d}$ is dense in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ we have $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ equal to $[H_i^- + z]_E \cup [H_i^0 + z]_E \cup [H_i^+ + z]_E$, so $([H_i^0 + z]_E)^c \subseteq [H_i^- + z]_E \cup [H_i^+ + z]_E$. This inclusion is in fact an equality. For, if we can find $g \in [H_i^0 + z]_E \cap ([H_i^- + z]_E \cup [H_i^+ + z]_E)$, then there is a net in $\mathbb{Z}^{n+d} \cap (H_i^0 + z)$ converging to g, which, as $([H_i^- + z]_E \cup [H_i^+ + z]_E)$ is clopen, has to eventually lie into $\mathbb{Z}^{n+d} \cap ([H_i^- + z]_E \cup [H_i^+ + z]_E) = \mathbb{Z}^{n+d} \cap (H_i^- \cup H_i^+ + z)$, giving a contradiction. It follows that $[H_i^0 + z]_E$ has a clopen complementary set, so is clopen. Moreover we have found

$$E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) = [H_i^- + z]_E \sqcup [H_i^0 + z]_E \sqcup [H_i^+ + z]_E \tag{5}$$

($\sqcup$ means disjoint union). Intersecting (5) with $\mathbb{Z}^{n+d}$ gives $\mathbb{Z}^{n+d} \cap [H_i^0 + z]_E = \mathbb{Z}^{n+d} \cap (H_i^0 + z)$.
Finally, each $H \in \mathfrak{H}$ is either $H_i^- + z$, $H_i^0 + z$ or $H_i^+ + z$. From (5) it follows that

$(H_E)^c = [H^c]_E$ holds in each case. As the first and second Boolean rules hold, the third holds as well. □

Lemma 3.18. *The family $\mathfrak{H}_E$ is a pre-basis for the topology of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$.*

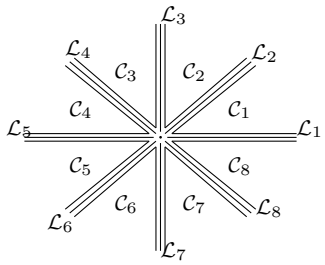
Proof. As noted, a pre-basis is obtained by $V(x, A_c)$ for x running over $\mathbb{R}^n_c$ and A running over $\mathfrak{A}$. A writes as $A_i^{\pm} + \underline{n}$, so $A_c = [A_i^{\pm}]_c.\underline{n}$, and consequently $V(x, A_c) = V(x.(-\underline{n}), [A_i^{\pm}]_c)$. By lemma 3.16 this is either $[H_i^+ - \pi(x) + a_i + \underline{n}]_E$ or $[H_i^+ - \pi(x) + a_i + \underline{n}]_E \sqcup [H_i^0 - \pi(x) + a_i + \underline{n}]_E$ where a_i is any vector of A_i. Consequently $V(x, A_c)$ is either an element of $\mathfrak{H}_E$ or a finite union of elements of $\mathfrak{H}_E$. This implies that $\mathfrak{H}_E$ is a pre-basis for the topology of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. □

3.4. **Neighborhood basis for $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ and transformation types.** Here we want to exibit a neighborhood basis for any transformation $g \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. By proposition 2.9 it suffice to find a clopen neighborhood of g isolating it among its fiber with respect to π^*.

Definition 3.19. *A transformation type is a map $\mathfrak{t} : I \longrightarrow \{-, 0, +\}$ such that its associated cone $C_{\mathfrak{t}} := \bigcap_{i \in I} H_i^{\mathfrak{t}(i)}$ is non-empty. We denote the set of transformation types by $\mathfrak{T}$.*

We point out that $\mathfrak{t}$ and $C_{\mathfrak{t}}$ uniquely determine each other, and that the family of cones associated to each element of $\mathfrak{T}$ partition $\mathbb{R}^n$ into cones pointed at 0, of dimension between 0 and n.

Example 3.20.
In the octagonal tiling case the family $\mathfrak{T}$ partition $\mathbb{R}^2$ into 17 different cones: one being $\{0\}$, eight open half lines $\{L_1, ..., L_8\}$, pointed at 0, with labels such that L_i, $L_{i+4} \subset H_i^0$ for all $1 \leqslant i \leqslant 4$, and eight open 1/8th spaces $\{C_1, ..., C_8\}$ pointed at 0. With respect to the orientation endowed on each hyperplan in 2.1 we see for instance that $\mathfrak{t}_{L_1}(1) = 0$, $\mathfrak{t}_{L_1}(2) = -$, $\mathfrak{t}_{L_1}(3) = -$ and $\mathfrak{t}_{L_1}(4) = -$. In the same way each cone determine each transformation type in $\mathfrak{T}$ uniquely.



Recall that we have the decomposition (5)

$$E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) = [H_i^- + z]_E \sqcup [H_i^0 + z]_E \sqcup [H_i^+ + z]_E$$

into three disjoint clopen sets, which holds for each index $i \in I$ and any $z \in \mathbb{R}^n$.

Definition 3.21. *The translation part of $g \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ is the image point $\pi^*(g) \in \mathbb{R}^n$.*
The transformation type $\mathfrak{t}_g$ of g is the element of $\mathfrak{T}$ uniquely defined through the inclusion $g \in [H_i^{\mathfrak{t}_g(i)} + \pi^*(g)]_E$ for all $i \in I$. We denote $C_g := C_{\mathfrak{t}_g}$.

The fact that the cones C_g previously associated are non-empty is shown in the proof of the next proposition.

Proposition 3.22. *Any transformation $g \in \mathbb{E}_c$ has an open neighbourhood basis of the form*

$$\{\mathcal{U}(g,\varepsilon) := [C_g + \pi^*(g)]_E \cap (\pi^*)^{-1}(B(\pi^*(g),\varepsilon))\}_{\varepsilon>0}$$

Proof. Fix $g \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. From the very definition of $\mathfrak{t}_g$ we have $g \in [H_i^{\mathfrak{t}_g(i)} + \pi^*(g)]_E$ for each $i \in I$. On the other hand the Boolean rules of proposition 3.15 yield $\bigcap_{i\in I}[H_i^{\mathfrak{t}_g(i)} + \pi^*(g)]_E = [\bigcap_{i\in I} H_i^{\mathfrak{t}_g(i)} + \pi^*(g)]_E = [C_g + \pi^*(g)]_E$, so we deduce that $g \in [C_g + \pi^*(g)]_E$, ensuring also that C_g is non-empty.
If $h \neq g$ is such that $\pi^*(h) = \pi^*(g)$ then by proposition 3.15 there exists $z \in \mathbb{R}^n$ and $i_0 \in I$ such that g and h lie into different clopen sets in the disjoint decomposition

$$E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) = [H^-_{i_0} + z]_E \sqcup [H^0_{i_0} + z]_E \sqcup [H^+_{i_0} + z]_E$$

This already implies that z is equal to $\pi^*(g) = \pi^*(h)$ up to a vector of $H^0_{i_0}$, and it follows that $\mathfrak{t}_g(i_0) \neq \mathfrak{t}_h(i_0)$, so the cones C_g and C_h are disjoint in $\mathbb{R}^n$. Consequently $[C_g + \pi^*(g)]_E$ and $[C_h + \pi^*(h)]_E$ are disjoint, and this implies that g is the unique transformation, among its fiber with respect to π^*, contained into $[C_g + \pi^*(g)]_E$. As this latter is clopen in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$, we can conclude by proposition 2.9 that the stated family is a neighbourhood basis for g in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. □

Corollary 3.23. *The spaces $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ and $E(\mathbb{T}^n_c, \mathbb{Z}^d)$ are first countable, and the flow $(\mathbb{T}^n_c, \mathbb{Z}^d)$ is tame.*

We may introduce a notation: for a point $z \in \mathbb{R}^n$ and a cone C associated to a transformation type in $\mathfrak{T}$, denote by $C(z,\varepsilon)$ the set $z + C \cap B(0,\varepsilon)$. This is geometrically the head of the cone C of length ε pointed at position z, and is a part of the ball $B(z,\varepsilon)$. Observe that $z \notin C(z,\varepsilon)$ unless the cone C is the singleton $\{0\}$ (in which case $C(z,\varepsilon) = \{z\}$ for all ε), and that we always have $z \in \overline{C(z,\varepsilon)}$ in $\mathbb{R}^n$.

Remark 3.24. *Using proposition 3.15 we have*

$$\mathcal{U}(g,\varepsilon) \cap \mathbb{Z}^{n+d} = C_g(\pi^*(g),\varepsilon) \cap \mathbb{Z}^{n+d}$$

holding for each transformation $g \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. Thus a sequence of elements of $\mathbb{Z}^{n+d}$ converges to g in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ if and only if it converges to $\pi^*(g)$ in $\mathbb{R}^n$ and eventually lies into $\pi^*(g) + C_g$.

4. The structure of $E(\mathbb{T}^n_c, \mathbb{Z}^d)$

In this last section we make use of the onto morphism π^* together with the the collection of transformation types to derive a picture of the Ellis semigroup $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. We describe the topology as well as the algebraic structure only by means of thanslation parts and transformation types associated to each transformation $g \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. Moreover, using the picture of $\mathbb{R}^n_c$ stated in theorem 2.14 we refine the action of the Ellis semigroup onto the cut up space. These results are finally applied to give a complete description of $E(\mathbb{T}^n_c, \mathbb{Z}^d)$, as well as its action on the cut up torus $\mathbb{T}^n_c$.

4.1. **The main structure theorem.**

Proposition 4.1. *The map*

$$E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) \ni g \longmapsto (\pi^*(g), \mathfrak{t}_g) \in \left\{(z, \mathfrak{t}) \in \mathbb{R}^n \times \mathfrak{T} \mid z \in \overline{(C_\mathfrak{t} + z) \cap \mathbb{Z}^{n+d}}\right\}$$

associating to each transformation its translation and transformation type is well-defined and bijective.

Proof. We first check that the stated map is well-defined: From proposition 3.22 each transformation $g \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ is the limit point of a sequence in $\mathbb{Z}^{n+d}$, which from remark 3.24, converges to $\pi^*(g)$ in the Euclidean topology and has to lie into $\mathbb{Z}^{n+d} \cap \mathcal{U}(g, \varepsilon) = \mathbb{Z}^{n+d} \cap C_g(\pi^*(g), \varepsilon)$ eventually. We thus obtain in particular $\pi^*(g) \in \overline{(C_g + \pi^*(g)) \cap \mathbb{Z}^{n+d}}$, and the map is well-defined.
The neighborhood basis associated to each transformation g in proposition 3.22 only depends upon the pair $(\pi^*(g), \mathfrak{t}_g)$, so the stated map is 1-to-1.
The map is also onto: to each pair $(z, \mathfrak{t})$ with condition $z \in \overline{(C_\mathfrak{t} + z) \cap \mathbb{Z}^{n+d}}$, we have a non-empty intersection $C_\mathfrak{t}(z, \varepsilon) \cap \mathbb{Z}^{n+d}$ for each $\varepsilon > 0$, and thus the family $\{[C_\mathfrak{t}(z, \varepsilon)]_E\}_{\varepsilon > 0}$ forms a filterbase in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. Let $\varepsilon_0 > 0$ be chosen. for any $\varepsilon < \varepsilon_0$ we have $[C_\mathfrak{t}(z, \varepsilon)]_E \subset (\pi^*)^{-1}(\overline{C_\mathfrak{t}(z, \varepsilon_0)})$, this latter set being compact since $\overline{C_\mathfrak{t}(z, \varepsilon_0)}$ is compact and π^* is proper. This means that eventually on ε the filterbase is contained into a fixed compact set of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$. It then possess an accumulation point g, which has to satisfy $\pi^*(g) = z$ and $C_g = C_\mathfrak{t}$. Thus the map is onto. □

Definition 4.2. *We call the Euclidean subset* $\mathbb{R}^n_\mathfrak{t} := \left\{z \in \mathbb{R}^n \mid z \in \overline{(C_\mathfrak{t} + z) \cap \mathbb{Z}^{n+d}}\right\}$ *the set of allowed translations for* $\mathfrak{t}$.
We call a transformation type $\mathfrak{t}$ *trivial if the set of allowed translations* $\mathbb{R}^n_\mathfrak{t}$ *is empty. We denote by* $\mathfrak{T}_0$ *the family of non-trivial transformation types.*

From proposition 4.1 the transformation type of any $g \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ is always non-trivial (as $\pi^*(g) \in \mathbb{R}^n_{\mathfrak{t}_g}$), so in the sequel we only consider non-trivial transformation types.
Observe that any set $\mathbb{R}^n_\mathfrak{t}$ is stable under translation by $\mathbb{Z}^{n+d}$.

Proposition 4.3. *For each non-trivial transformation type* $\mathfrak{t} \in \mathfrak{T}_0$ *there is a unique Euclidean subspace* $W_\mathfrak{t} \subset \mathbb{R}^n$ *with*

$$\mathbb{R}^n_\mathfrak{t} = W_\mathfrak{t} + \mathbb{Z}^{n+d}$$

Moreover $W_\mathfrak{t} \cap \mathbb{Z}^{n+d}$ *is dense in* $W_\mathfrak{t}$ *for each* $\mathfrak{t}$.

Proof. Consider for any transformation type $\mathfrak{t}$ the vector subspace $< C_\mathfrak{t} >$ of $\mathbb{R}^n$. From theorem 2.3 of [10], It splits as the direct sum decomposition

$$< C_\mathfrak{t} >= W_\mathfrak{t} + D_\mathfrak{t}$$

into vector subspaces such that $\mathbb{Z}^{n+d} \cap W_\mathfrak{t}$ is dense in $W_\mathfrak{t}$, $\mathbb{Z}^{n+d} \cap D_\mathfrak{t}$ is uniformly discrete in $D_\mathfrak{t}$, and $\mathbb{Z}^{n+d} = \left(\mathbb{Z}^{n+d} \cap W_\mathfrak{t}\right) + \left(\mathbb{Z}^{n+d} \cap D_\mathfrak{t}\right)$.
Write P^W (resp. P^D) to be the linear skew projection onto $W_\mathfrak{t}$ along $D_\mathfrak{t}$ (resp. the linear skew projection onto $D_\mathfrak{t}$ along $W_\mathfrak{t}$). Note that $P^W(\mathbb{Z}^{n+d}) = \mathbb{Z}^{n+d} \cap W_\mathfrak{t}$ and $P^D(\mathbb{Z}^{n+d}) = \mathbb{Z}^{n+d} \cap D_\mathfrak{t}$.
We show that $\mathbb{R}^n_\mathfrak{t}$ is contained into $W_\mathfrak{t} + \mathbb{Z}^{n+d}$: Let $z \in \mathbb{R}^n_\mathfrak{t}$, this latter being non-empty

from the requirement $\mathfrak{t} \in \mathfrak{T}_0$. Hence there exist a sequence $(\underline{n}_k)_{k\in\mathbb{N}} \subset \mathbb{Z}^{n+d} \cap (C_\mathfrak{t} + z)$ converging to z in $\mathbb{R}^n$, so that $P^D(\underline{n}_k) \longrightarrow P^D(z)$ and $P^W(\underline{n}_k) \longrightarrow P^W(z)$. The sequence $(P^D(\underline{n}_k))_{k\in\mathbb{N}}$ lies into $\mathbb{Z}^{n+d} \cap D_\mathfrak{t}$ which is uniformly discrete in $D_\mathfrak{t}$, so since it converges we get $P^D(z) = P^D(\underline{n}_k) \in \mathbb{Z}^{n+d}$ for great enough k. It follows that $z = P^W(z) + P^D(z) \in W_\mathfrak{t} + \mathbb{Z}^{n+d}$.
We point out that we eventually have $P^W(\underline{n}_k) = \underline{n}_k - P^D(z) \in \mathbb{Z}^{n+d} \cap (C_\mathfrak{t} + z) - P^D(z)$, with $P^D(z) \in \mathbb{Z}^{n+d}$, so eventually $P^W(\underline{n}_k) \in \mathbb{Z}^{n+d} \cap (C_\mathfrak{t} + P^W(z))$. Then $P^W(\underline{n}_k) - P^W(z) = P^W(\underline{n}_k - z)$ eventually lies into both $W_\mathfrak{t}$ and $C_\mathfrak{t}$, which ensure the intersection $C_\mathfrak{t} \cap W_\mathfrak{t}$ to be non-empty.
Conversely we show that $\mathbb{R}^n_\mathfrak{t}$ contains $W_\mathfrak{t} + \mathbb{Z}^{n+d}$: here it obviously suffice to show that $\mathbb{R}^n_\mathfrak{t}$ contains $W_\mathfrak{t}$.
First observe that $C_\mathfrak{t} \cap W_\mathfrak{t}$ is a non-empty open cone of $W_\mathfrak{t}$: for, $C_\mathfrak{t}$ is an open cone of $< C_\mathfrak{t} >$, as the space $< C_\mathfrak{t} >$ is nothing but $\bigcap_{i\in I, \mathfrak{t}(i)=0} H^0_i$ (which is the whole space $\mathbb{R}^n$ in case $\mathfrak{t}$ is never 0), and $C_\mathfrak{t}$ writes as the intersection of this space with the open cone $\bigcap_{i\in I,\mathfrak{t}(i)\neq 0} H_i^{\mathfrak{t}(i)}$ (in case $\mathfrak{t}$ is always 0 this gives $C_\mathfrak{t} =< C_\mathfrak{t} >= \{0\}$).
Let then z be into $W_\mathfrak{t}$. Because $C_\mathfrak{t} \cap W_\mathfrak{t}$ is a non-empty open cone of $W_\mathfrak{t}$, containing 0 in its closure, $(C_\mathfrak{t} + z) \cap W_\mathfrak{t} = (C_\mathfrak{t} \cap W_\mathfrak{t}) + z$ is an open set of $W_\mathfrak{t}$, containing z in its closure. As $\mathbb{Z}^{n+d} \cap W_\mathfrak{t}$ is dense in $W_\mathfrak{t}$, we may select a sequence in $\mathbb{Z}^{n+d} \cap (C_\mathfrak{t} + z) \cap W_\mathfrak{t}$ converging to z. This ensure that $z \in \mathbb{R}^n_\mathfrak{t}$. □

From proposition 4.1 and 4.3 we have obtained an identification of the transformation semigroup $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ with the disjoint union of Euclidean subsets

$$\bigsqcup_{\mathfrak{t}\in\mathfrak{T}_0} \left[W_\mathfrak{t} + \mathbb{Z}^{n+d}\right] \times \{\mathfrak{t}\}$$

by identifying any transformation with its pair of translation part and transformation type. Our next goal will be to endow this union with the corresponding topology and semigroup law.
Let us introduce the following notation:
for $\mathfrak{t}$ a non-trivial transformation type, write

$$C'_\mathfrak{t} := C_\mathfrak{t} \cap W_\mathfrak{t}$$

As it is pointed out in the proof of proposition 4.3 this set is always non-empty (provided that $\mathfrak{t}$ is non-trivial), and is a cone pointed at 0 in $\mathbb{R}^n$. For each $z \in \mathbb{R}^n$ and each $\varepsilon > 0$ we denote $C'_\mathfrak{t}(z, \varepsilon)$ to be $(B(0, \varepsilon) \cap C'_\mathfrak{t}) + z$, the head of the cone $C'_\mathfrak{t}$ of length ε pointed at position z. Obviously $C'_\mathfrak{t}(z, \varepsilon) \subset B(z, \varepsilon)$ whatever the $\mathfrak{t}$, z and ε.
Here we state our main result:

Theorem 4.4. *Let $\mathfrak{T}_0$ be the collection of non-trivial transformation types, and let $W_\mathfrak{t}$ be the Euclidean space associated with $\mathfrak{t} \in \mathfrak{T}_0$ of proposition 4.3. The map*

$$E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) \ni g \longmapsto (\pi^*(g), \mathfrak{t}_g) \in \bigsqcup_{\mathfrak{t}\in\mathfrak{T}_0} \left[W_\mathfrak{t} + \mathbb{Z}^{n+d}\right] \times \{\mathfrak{t}\}$$

associating to each transformation its translation and transformation type is a topological isomorphism of semigroups, with right term equiped with:
- the semigroup law given by $(z, \mathfrak{t}).(z', \mathfrak{t}') = (z + z', \mathfrak{t}.\mathfrak{t}')$, where the semigroup law on

$\mathfrak{T}_0$ *writes:*

$$\mathfrak{t}.\mathfrak{t}'(i) := \begin{cases} \mathfrak{t}'(i) \text{ if } \mathfrak{t}'(i) \neq 0 \\ \mathfrak{t}(i) \text{ if } \mathfrak{t}'(i) = 0 \end{cases}$$

- the topology of convergence: $(z_n, \mathfrak{t}_n) \longrightarrow (z, \mathfrak{t})$ *iff*

$$\forall \varepsilon > 0, \exists \delta_n > 0 \text{ such that } C'_{\mathfrak{t}_n}(z_n, \delta_n) \subset C'_{\mathfrak{t}}(z, \varepsilon) \text{ for large enough } n$$

We consider first the algebraic part of the statement. Let us endow $\mathfrak{T}_0$ with the composition law stated in theorem 4.4.
Observe that $\mathfrak{T}$ is made up idempotent elements, that is $\mathfrak{t}.\mathfrak{t} = \mathfrak{t}$ for any transformation type $\mathfrak{t}$.

Proposition 4.5. *Any g and h in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ satisfy the equality $\mathfrak{t}_{g.h} = \mathfrak{t}_g.\mathfrak{t}_h$.*

Proof. Because $g.h \in [C_{g.h} + \pi^*(g.h)]_E$ and the product law on $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ is right-continuous, we can choose $\underline{n} \in \mathbb{Z}^{n+d}$ sufficiently close to h in the sense that

$$(i)\ \ g.\underline{n} \in [C_{g.h} + \pi^*(g.h)]_E \qquad (ii)\ \ \underline{n} \in C_h + \pi^*(h)$$

From (i) we have $\mathcal{U}(g.\underline{n}, \varepsilon_0) \subset [C_{g.h} + \pi^*(g.h)]_E$ for some $\varepsilon_0 > 0$, so intersecting with $\mathbb{Z}^{n+d}$ we get that $C_g(\pi^*(g) + \underline{n}, \varepsilon)$ intersect $C_{g.h} + \pi^*(g.h)$ for any $\varepsilon \leqslant \varepsilon_0$. This means that

$$C_g(\underline{n} - \pi^*(h), \varepsilon_0) \cap C_{g.h} \neq \emptyset\ \forall \varepsilon \leqslant \varepsilon_0 \ \text{ with } \ (\underline{n} - \pi^*(h)) \in C_h$$

Now divide the situation into three cases:
(1) $\mathfrak{t}_h(i) = +$: then $(\underline{n} - \pi^*(h)) \in C_h \subset H_i^+$, so there is an $\varepsilon_1 > 0$, which can be chosen $\leqslant \varepsilon_0$, with $B(\underline{n} - \pi^*(h), \varepsilon_1) \subset H_i^+$. As a result, $C_g(\underline{n} - \pi^*(h), \varepsilon_1) \subset B(\underline{n} - \pi^*(h), \varepsilon_1) \subset H_i^+$ so H_i^+ has non-empty intersection with $C_{g.h}$, giving $C_{g.h} \subset H_i^+$, that is $\mathfrak{t}_{g.h}(i) = +$.
(2) $\mathfrak{t}_h(i) = -$: this case rules out as the previous case, giving $\mathfrak{t}_{g.h}(i) = -$.
(3) $\mathfrak{t}_h(i) = 0$: then $(\underline{n} - \pi^*(h)) \in H_i^0$ so $C_g(\underline{n} - \pi^*(h), \varepsilon_0) \subset H_i^{\mathfrak{t}_g(i)}$ whatever $\mathfrak{t}_g(i)$ is. We deduce that $C_g(\underline{n} - \pi^*(h), \varepsilon_0) \cap H_i^{\mathfrak{t}_g(i)} \cap C_{g.h}$ is non-empty, thus giving $C_{g.h} \subset H_i^{\mathfrak{t}_g(i)}$, that is $\mathfrak{t}_{g.h}(i) = \mathfrak{t}_g(i)$.
Considering these three cases we have $\mathfrak{t}_{g.h} = \mathfrak{t}_g.\mathfrak{t}_h$ holding. □

Corollary 4.6. *The law on $\mathfrak{T}_0$ is a semigroup product. Each pair of transformations g and h satisfy*

$$(\pi^*(g.h), \mathfrak{t}_{g.h}) = (\pi^*(g) + \pi^*(h), \mathfrak{t}_g.\mathfrak{t}_h)$$

Moreover, $\mathfrak{T}_0$ is isomorphic with the subsemigroup $\mathfrak{I}$ of idempotent transformations of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$.

Proof. The equality for any pair g and h follows from the very construction of π^* and proposition 4.5. In particular we have, as 0 lies into $\mathbb{R}^n_{\mathfrak{t}}$ for any non-trivial transformation type,

$$(0, \mathfrak{t}.\mathfrak{t}') = (0, \mathfrak{t}).(0, \mathfrak{t}') \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$$

And thus $\mathfrak{t}.\mathfrak{t}' \in \mathfrak{T}_0$, that is, $\mathfrak{T}_0$ is a (associative) semigroup. Now the collection $\mathfrak{I}$ of idempotent transformations has to be contained into $\{(0, \mathfrak{t}) \mid \mathfrak{t} \in \mathfrak{T}_0\}$, isomorphic with $\mathfrak{T}_0$. As this latter is made up idempotent elements, it follows that $\mathfrak{I} = \{(0, \mathfrak{t}) \mid \mathfrak{t} \in \mathfrak{T}_0\} \simeq \mathfrak{T}_0$. □

Remark 4.7. *For any transformation type* $\mathfrak{t}$ *the set* $\left[W_{\mathfrak{t}}+\mathbb{Z}^{n+d}\right]\times\{\mathfrak{t}\}$ *is a subgroup of* $E(\mathbb{R}^n_c,\mathbb{Z}^{n+d})$*, with identity* $(0,\mathfrak{t})$*, isomorphic through* π^* *with a subgroup of* $\mathbb{R}^n$*.*

Next we show the topological part of theorem 4.4.

Lemma 4.8. *There exists an* $\varepsilon_0>0$ *such that, for any* $\mathfrak{t}\in\mathfrak{T}_0$ *and* $z\in\mathbb{R}^n_{\mathfrak{t}}=W_{\mathfrak{t}}+\mathbb{Z}^{n+d}$*, we have*

$$C_{\mathfrak{t}}(z,\varepsilon)\cap\mathbb{Z}^{n+d}=C'_{\mathfrak{t}}(z,\varepsilon)\cap\mathbb{Z}^{n+d}\quad\forall 0<\varepsilon\leqslant\varepsilon_0$$

Proof. Obviously $C_{\mathfrak{t}}(z,\varepsilon)$ contains $C'_{\mathfrak{t}}(z,\varepsilon)$ whatever the $\varepsilon>0$. Conversely, consider $\mathfrak{t}\in\mathfrak{T}_0$ with associated cone $C_{\mathfrak{t}}$ in $\mathbb{R}^n$. We have the direct sum decomposition

$$<C_{\mathfrak{t}}>=W_{\mathfrak{t}}+D_{\mathfrak{t}}$$

into vector subspaces such that $\mathbb{Z}^{n+d}\cap W_{\mathfrak{t}}$ is dense in $W_{\mathfrak{t}}$, $\mathbb{Z}^{n+d}\cap D_{\mathfrak{t}}$ is uniformly discrete in $D_{\mathfrak{t}}$, and $\mathbb{Z}^{n+d}=\left(\mathbb{Z}^{n+d}\cap W_{\mathfrak{t}}\right)+\left(\mathbb{Z}^{n+d}\cap D_{\mathfrak{t}}\right)$.
If $\varepsilon_{\mathfrak{t}}>0$ denotes the radius of discreteness of $\left(\mathbb{Z}^{n+d}\cap D_{\mathfrak{t}}\right)$ we obtain

$$<C_{\mathfrak{t}}>\cap B(z,\varepsilon_{\mathfrak{t}})\cap\mathbb{Z}^{n+d}=(W_{\mathfrak{t}}+z)\cap B(z,\varepsilon_{\mathfrak{t}})\cap\mathbb{Z}^{n+d}$$

for any $z\in W_{\mathfrak{t}}+\mathbb{Z}^{n+d}$. Hence by intersecting with $C_{\mathfrak{t}}+z$ we obtain, as by construction $C_{\mathfrak{t}}\subset<C_{\mathfrak{t}}>$,

$$C_{\mathfrak{t}}(z,\varepsilon_{\mathfrak{t}})\cap\mathbb{Z}^{n+d}=(W_{\mathfrak{t}}+z)\cap(C_{\mathfrak{t}}+z)\cap B(z,\varepsilon_{\mathfrak{t}})\cap\mathbb{Z}^{n+d}=C'_{\mathfrak{t}}(z,\varepsilon_{\mathfrak{t}})\cap\mathbb{Z}^{n+d}$$

Taking ε_0 to be the minimum over $\varepsilon_{\mathfrak{t}}$, $\mathfrak{t}\in\mathfrak{T}_0$, gives the statement. □

Lemma 4.9. *Any transformation* $g\in E(\mathbb{R}^n_c,\mathbb{Z}^{n+d})$ *and* $\varepsilon_1<\varepsilon_2$ *satisfies*

$$\mathcal{U}(g,\varepsilon_1)\subset[C_g(\pi^*(g),\varepsilon_1)]_E\subset\mathcal{U}(g,\varepsilon_2)$$

Proof. We start with the left hand inclusion. As $\mathcal{U}(g,\varepsilon_1)$ is open in $E(\mathbb{R}^n_c,\mathbb{Z}^{n+d})$ and $\mathbb{Z}^{n+d}$ is dense, using remark 3.24 we have $\overline{C_g(\pi^*(g),\varepsilon)\cap\mathbb{Z}^{n+d}}=\mathcal{U}(g,\varepsilon_1)\cap\mathbb{Z}^{n+d}$ dense in $\mathcal{U}(g,\varepsilon_1)$. It follows that $\mathcal{U}(g,\varepsilon_1)\subset\overline{C_g(\pi^*(g),\varepsilon_1)}$, with closure in $E(\mathbb{R}^n_c,\mathbb{Z}^{n+d})$, which is nothing but $[C_g(\pi^*(g),\varepsilon_1)]_E$.
For the right hand inclusion, consider $h\in[C_g(\pi^*(g),\varepsilon_1)]_E$. By definition there is a net in $\mathbb{Z}^{n+d}\cap C_g(\pi^*(g),\varepsilon_1)$ converging to h in $E(\mathbb{R}^n_c,\mathbb{Z}^{n+d})$. Hence this net lies into both $\mathbb{Z}^{n+d}\cap(C_g+\pi^*(g))=\mathbb{Z}^{n+d}\cap[C_g+\pi^*(g)]_c$ and into $B(\pi^*(g),\varepsilon_1)\subset\overline{B}(\pi^*(g),\varepsilon_1)\subset B(\pi^*(g),\varepsilon_2)$, where $\overline{B}$ denote the closed Euclidean ball. It follows by taking limit that $h\in[C_g+\pi^*(g)]_E\cap(\pi^*)^{-1}(B(\pi^*(g),\varepsilon_2))=\mathcal{U}(g,\varepsilon_2)$, as desired. □

Proposition 4.10. *For* $g_n\leftrightarrow(z_n,\mathfrak{t}_n)$ *and* $g\leftrightarrow(z,\mathfrak{t})$*, we have* $g_n\longrightarrow g$ *if and only if*

$$\forall\varepsilon>0,\exists\delta_n>0\text{ such that }C'_{\mathfrak{t}_n}(z_n,\delta_n)\subset C'_{\mathfrak{t}}(z,\varepsilon)\text{ for large enough }n$$

Proof. Let then $g_n\leftrightarrow(z_n,\mathfrak{t}_n)$ and $g\leftrightarrow(z,\mathfrak{t})$. By proposition 3.22, $g_n\longrightarrow g$ if and only if

$$\forall 0<\varepsilon\leqslant\varepsilon_0,\ g_n\in\mathcal{U}(g,\varepsilon)\text{ for great enough }n$$

for any fixed ε_0, chosen here to be the constant of lemma 4.8. It is equivalent to

$$\forall\ 0<\varepsilon\leqslant\varepsilon_0,\ \exists\ \delta_n>0\ \text{ such that }\ \mathcal{U}(g_n,\delta_n)\subset\mathcal{U}(g,\varepsilon)\text{ for great enough }n$$

Taking $\delta_n \leqslant \varepsilon_0$ and intersecting with $\mathbb{Z}^{n+d}$ leads by remark 3.24 and the previous lemma to the existence for all $0 < \varepsilon \leqslant \varepsilon_0$ of some $0 < \delta_n \leqslant \varepsilon_0$ such that

$$C'_{\mathfrak{t}_n}(z_n, \delta_n) \cap \mathbb{Z}^{n+d} \subset C'_{\mathfrak{t}}(z, \varepsilon) \cap \mathbb{Z}^{n+d} \text{ for great enough } n$$

This implies that $z_n \in \overline{C'_{\mathfrak{t}_n}(z_n, \delta_n)} \subset \overline{C'_{\mathfrak{t}}(z, \varepsilon)}$. Now as z lies into $W_{\mathfrak{t}} + \mathbb{Z}^{n+d}$ there exists a $\underline{m} \in \mathbb{Z}^{n+d}$ such that $z \in W_{\mathfrak{t}} + \underline{m}$. It follows that $z_n \in \overline{C'_{\mathfrak{t}_n}(z_n, \delta_n)} \subset W_{\mathfrak{t}} + \underline{m}$, so that $z_n - \underline{m} \in \overline{C'_{\mathfrak{t}_n}(z_n - \underline{m}, \delta_n)} \subset W_{\mathfrak{t}}$. In turns, the affine space generated by $C'_{\mathfrak{t}_n}(z_n - \underline{m}, \delta_n)$ is contained into $W_{\mathfrak{t}}$, and as it is nothing but $W_{\mathfrak{t}_n} + z_n - \underline{m}$ we obtain

$$W_{\mathfrak{t}_n} + z_n \subset W_{\mathfrak{t}} + \underline{m} = W_{\mathfrak{t}} + z$$

Now $C'_{\mathfrak{t}}(z, \varepsilon)$ is open in $W_{\mathfrak{t}} + z$, so $(W_{\mathfrak{t}_n} + z_n) \cap C'_{\mathfrak{t}}(z, \varepsilon)$ is open in $W_{\mathfrak{t}_n} + z_n$, and non-empty since there is the inclusion of non-empty sets

$$C'_{\mathfrak{t}_n}(z_n, \delta_n) \cap \mathbb{Z}^{n+d} \subset (W_{\mathfrak{t}_n} + z_n) \cap C'_{\mathfrak{t}}(z, \varepsilon) \cap \mathbb{Z}^{n+d}$$

We may easily show that $C'_{\mathfrak{t}_n}(z_n, \delta_n)$ and $(W_{\mathfrak{t}_n} + z_n) \cap C'_{\mathfrak{t}}(z, \varepsilon)$ are regular open sets of $W_{\mathfrak{t}_n} + z_n$, and since $(W_{\mathfrak{t}_n} + z_n) \cap \mathbb{Z}^{n+d}$ is dense in $W_{\mathfrak{t}_n} + z_n$ we can conclude, by taking closure an next interior in $W_{\mathfrak{t}_n} + z_n$, that

$$\forall\, 0 < \varepsilon \leqslant \varepsilon_0,\ \exists\, 0 < \delta_n \leqslant \varepsilon_0 \ \text{ such that } \ C'_{\mathfrak{t}_n}(z_n, \delta_n) \subset C'_{\mathfrak{t}}(z, \varepsilon) \text{ for great enough } n$$

Thus we have the " $\Rightarrow$ " part of the statement.
Conversely if we suppose that

$$\forall \varepsilon > 0, \exists \delta_n > 0 \text{ such that } C'_{\mathfrak{t}_n}(z_n, \delta_n) \subset C'_{\mathfrak{t}}(z, \varepsilon) \text{ for large enough } n$$

then intersecting with $\mathbb{Z}^{n+d}$ and taking closure in $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ gives

$$\forall \varepsilon > 0, \exists \delta_n > 0 \text{ such that } [C'_{\mathfrak{t}_n}(z_n, \delta_n)]_E \subset [C'_{\mathfrak{t}}(z, \varepsilon)]_E \text{ for large enough } n$$

Now by lemma 4.9 we deduce that

$$\forall \varepsilon > 0, \exists \delta'_n > 0 \text{ such that } g_n \in \mathcal{U}(g_n, \delta'_n) \subset \mathcal{U}(g, \varepsilon) \text{ for large enough } n$$

so that $g_n \longrightarrow g$. □

Remark 4.11. *We proved along the preceeding lines that if* $(z_n, \mathfrak{t}_n) \longrightarrow (z, \mathfrak{t})$ *then for great enough* n *we have*

$$W_{\mathfrak{t}_n} + z_n \subset W_{\mathfrak{t}} + z$$

In particular, for any fixed non-trivial transformation type $\mathfrak{t}_0$ *the disjoint union*

$$\bigsqcup_{\mathfrak{t}: W_{\mathfrak{t}_0} \subset W_{\mathfrak{t}}} \left[W_{\mathfrak{t}} + \mathbb{Z}^{n+d} \right] \times \{\mathfrak{t}\}$$

is closed with respect to the topology of theorem 4.4.

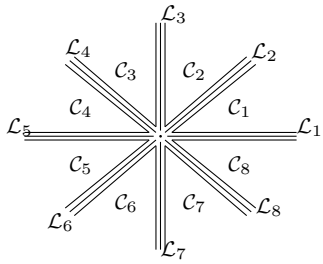
Example 4.12. *In the octagonal example, as* C_j *is open for all* $1 \leqslant j \leqslant 8$*, the generated space* $< C_j >$ *if all* $\mathbb{R}^2$*. As* $\mathbb{Z}^4$ *is dense we have* $W_{\mathfrak{t}_{C_j}} = \mathbb{R}^2$*, and consequently* $\mathbb{R}^2_{\mathfrak{t}_{C_j}} = \mathbb{R}^2$ *for all* $1 \leqslant j \leqslant 8$*. Moreover,* $< L_i >=< L_{i+4} >= H_i$ *for all* $1 \leqslant i \leqslant 4$*, and the group* $H_i \cap \mathbb{Z}^4$ *is dense in* H_i *(being not monogen), so* $W_{\mathfrak{t}_{L_i}} = W_{\mathfrak{t}_{L_{i+4}}} = H_i$

for all $1 \leqslant i \leqslant 4$. *It yields* $\mathbb{R}^n_{\mathfrak{t}_{L_i}} = \mathbb{R}^n_{\mathfrak{t}_{L_{i+4}}} = H_i + \mathbb{Z}^4$ *for all* $1 \leqslant i \leqslant 4$. *Finally we have* $\mathbb{R}^n_{\{0\}} = \mathbb{Z}^4$ *so we get a semigroup isomorphism and homeomorphism*

$$E(\mathbb{R}^2_{oct}, \mathbb{Z}^4) \simeq \left(\bigsqcup_{i=1}^{8} \mathbb{R}^2 \times \{\mathfrak{t}_{C_i}\} \right) \bigsqcup \left(\bigsqcup_{j=1}^{4} [H_i + \mathbb{Z}^4] \times \{\mathfrak{t}_{L_i}, \mathfrak{t}_{L_{i+4}}\} \right) \bigsqcup \mathbb{Z}^4$$

with topology and semigroup law given by theorem 4.4.

We can for instance compute the product $\mathfrak{t}_{C_4}.\mathfrak{t}_{L_1}$*:* $\mathfrak{t}_{L_1}(1) = 0$ *and* $\mathfrak{t}_{L_1}(i) = -$ *for* $i = 2, 3, 4$ *so* $\mathfrak{t}_{C_4}.\mathfrak{t}_{L_1}(1) = \mathfrak{t}_{C_4}(1) = +$ *and* $\mathfrak{t}_{C_4}.\mathfrak{t}_{L_1}(i) = \mathfrak{t}_{L_1}(i) = -$ *for* $i = 2, 3, 4$. *Hence* $\mathfrak{t}_{C_4}.\mathfrak{t}_{L_1} = \mathfrak{t}_{C_1}$.



4.2. **Action formula.** Set for any $\mathfrak{p} \in \mathfrak{P}$ and $\mathfrak{t} \in \mathfrak{T}_0$ a map $\mathfrak{p}.\mathfrak{t} : I \longrightarrow \{-, +, \infty\}$, as

$$\mathfrak{p}.\mathfrak{t}(i) := \begin{cases} \mathfrak{t}(i) \text{ if } \mathfrak{t}(i) \neq 0 \\ \mathfrak{p}(i) \text{ if } \mathfrak{t}(i) = 0 \end{cases}$$

Proposition 4.13. *The Ellis action* $\mathbb{R}^n_c \times E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) \to \mathbb{R}^n_c$ *is recovered as*

$$(z, \mathfrak{p}).(z', \mathfrak{t}) = (z + z', \mathfrak{p}') \quad \textit{where} \quad \mathfrak{p}'(i) := \begin{cases} \mathfrak{p}.\mathfrak{t}(i) \ \textit{if } i \in I_{z+z'} \\ \infty \ \textit{else} \end{cases}$$

Proof. Let x and g be chosen, corresponding to the pairs $(\pi(x), \mathfrak{p}_x)$ and $(\pi^*(g), \mathfrak{t}_g)$. It suffice to show that $\mathfrak{p}_x.\mathfrak{t}_g = \mathfrak{p}_{x.g}$ on the domain $Dom(\mathfrak{p}_{x.g}) = I_{\pi(x.g)} = I_{\pi(x)+\pi^*(g)}$. Since the evaluation map $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) \twoheadrightarrow \mathbb{R}^n_c$ at point x is continuous we can find an $\underline{n} \in \mathbb{Z}^{n+d}$ such that $\underline{n}$ is close to g and $x.\underline{n}$ is close to $x.g$, that is

$$x.\underline{n} \in [C_{x.g} + \pi(x.g)]_c \qquad \underline{n} \in (C_g + \pi^*(g))$$

Applying π we obtain $\pi(x) + \underline{n} \in (\overline{C_{x.g}} + \pi(x) + \pi^*(g))$, giving that $\underline{n}$ is both contained into $\overline{C_{x.g}} + \pi^*(g)$ and $C_g + \pi^*(g)$. It follows that C_g and $\overline{C_{x.g}}$ have non-empty intersection.
Select i in $Dom(\mathfrak{p}_{x.g}) = I_{\pi(x)+\pi^*(g)}$. There is three cases:
(1) $\mathfrak{t}_g(i) = +$: then $\mathfrak{p}_x.\mathfrak{t}_g(i) = +$. Moreover, as $C_g \subset H_i^+$, $\overline{C_{x.g}}$ intersect the open set H_i^+ so the intersection $C_{x.g} \cap H_i^+$ must be non-empty. This force $C_{x.g}$ to be included into H_i^+, giving $\mathfrak{p}_{x.g}(i) = +$.
(2) $\mathfrak{t}_g(i) = -$: this rules out as the previous case, giving $\mathfrak{p}_x.\mathfrak{t}_g(i) = \mathfrak{p}_{x.g}(i) = -$.
(3) $\mathfrak{t}_g(i) = 0$: Thus $\mathfrak{p}_x.\mathfrak{t}_g(i) = \mathfrak{p}_x(i)$. Let us show the equality $\mathfrak{p}_{x.g}(i) = \mathfrak{p}_x(i)$: as $\mathfrak{t}_g(i) = 0$, by remark 3.24 we can choose a sequence of elements $\underline{n}_k \in \mathbb{Z}^{n+d}$, converging to g, with $\underline{n}_k \in C_g + \pi^*(g) \subset H_i^0 + \pi^*(g)$. Consequently $x.\underline{n}_k \in ([H_i^{\mathfrak{p}_x(i)} + \pi(x)]_c).\underline{n}_k = [H_i^{\mathfrak{p}_x(i)} + \pi(x) + \underline{n}_k]_c \subset [H_i^{\mathfrak{p}_x(i)} + \pi(x) + H_i^0 + \pi^*(g)]_c = [H_i^{\mathfrak{p}_x(i)} + \pi(x.g)]_c$. Taking

the limit on k we get $x.g \in [H_i^{\mathfrak{p}_x(i)} + \pi(x.g)]_c$, giving $\mathfrak{p}_{x.g}(i) = \mathfrak{p}_x(i)$.
Consequently, $\mathfrak{p}_{x.g}(i) = \mathfrak{p}_x.\mathfrak{t}_g(i)$ for any $i \in Dom(\mathfrak{p}_{x.g})$, giving the statement. □

The action may be rephrased in terms of cones as follows: if we are given two couples $(z, \mathfrak{p})$ and $(z', \mathfrak{t})$, the corresponding image point is the point above $z + z'$ with cone containing $C_{\mathfrak{p}.\mathfrak{t}}$ (one may show that $C_{\mathfrak{p}.\mathfrak{t}}$ is always non-empty).

4.3. **Order, minimal idempotents and the minimal ideal.**

Definition 4.14. *The algebraic pre-order on $\mathfrak{T}_0$ is defined by:*

$$\mathfrak{t} \leqslant \mathfrak{t}' \text{ if and only if } \mathfrak{t}.\mathfrak{t}' = \mathfrak{t}$$

We denote by $\mathfrak{T}_{min}$ to be the set of minimal transformation types in $(\mathfrak{T}_0, \leqslant)$.

Proposition 4.15. *The algebraic pre-order is an order on $\mathfrak{T}_0$, and $\mathfrak{t} < \mathfrak{t}'$ if and only if the cone $C_{\mathfrak{t}'}$ is a lower dimensionnal facet of the cone $C_{\mathfrak{t}}$.*

Proof. Suppose that $\mathfrak{t}.\mathfrak{t}' = \mathfrak{t}$ and $\mathfrak{t}'.\mathfrak{t} = \mathfrak{t}'$. For $i \in I$, $\mathfrak{t}(i) = \mathfrak{t}.\mathfrak{t}'(i)$ so if $\mathfrak{t}'(i) \neq \mathfrak{t}(i)$ then $\mathfrak{t}'(i)$ must be equal to 0. But we have in turns $0 = \mathfrak{t}'(i) = \mathfrak{t}'.\mathfrak{t}(i)$, which force $\mathfrak{t}(i)$ to be 0 as well. Hence $\mathfrak{t}$ and $\mathfrak{t}'$ coincides on I so are equal. It follows that the pre-order is an order on $\mathfrak{T}_0$.
Now, observe that the cone $C_{\mathfrak{t}'}$ is a lower dimensionnal facet of the cone $C_{\mathfrak{t}}$ if and only if $C_{\mathfrak{t}'} \subset \overline{C_{\mathfrak{t}}} \setminus C_{\mathfrak{t}}$. Then suppose $\mathfrak{t} < \mathfrak{t}'$: we have $\mathfrak{t}.\mathfrak{t}'(i) = \mathfrak{t}(i)$ for all $i \in I$, and because $\mathfrak{t}$ and $\mathfrak{t}'$ are different the subset $J \subset I$ of indices with $\mathfrak{t}'(j) \neq \mathfrak{t}(j)$ is non-empty. This automatically implies that $\mathfrak{t}'(j) = 0$ and $\mathfrak{t}(j) \neq 0$. It follows that for each index $i \in I$, either $C_{\mathfrak{t}}$ and $C_{\mathfrak{t}'}$ are in the same relative position with H_i^0, or $C_{\mathfrak{t}'}$ lies in the boundary space H_i^0 whereas $C_{\mathfrak{t}}$ lies into some open half part. From this it easily follows that $C_{\mathfrak{t}'} \subset \overline{C_{\mathfrak{t}}} \setminus C_{\mathfrak{t}}$.
Conversely, if $C_{\mathfrak{t}'} \subset \overline{C_{\mathfrak{t}}} \setminus C_{\mathfrak{t}}$ then for each index $i \in I$:
- $\mathfrak{t}(i) = 0$ means that $C_{\mathfrak{t}'} \subset \overline{C_{\mathfrak{t}}} \subset H_i^0$, so $\mathfrak{t}'(i) = 0$.
- $\mathfrak{t}(i) = +$ implies that $C_{\mathfrak{t}'} \subset \overline{C_{\mathfrak{t}}} \subset H_i^{+0}$, that is, $\mathfrak{t}'(i) = 0$ or $+$.
- $\mathfrak{t}(i) = -$ implies that $C_{\mathfrak{t}'} \subset \overline{C_{\mathfrak{t}}} \subset H_i^{-0}$, that is, $\mathfrak{t}'(i) = 0$ or $-$.
Considering there three cases we obtain $\mathfrak{t}.\mathfrak{t}'(i) = \mathfrak{t}(i)$ for each i, so that $\mathfrak{t} \leqslant \mathfrak{t}'$. Since we supposed $C_{\mathfrak{t}'}$ different from $C_{\mathfrak{t}}$ we thus obtain $\mathfrak{t} < \mathfrak{t}'$.

□

Example 4.16. *In the octagonal example $\{C_{\mathfrak{t}} \mid \mathfrak{t} \in \mathfrak{T}_0 = \mathfrak{T}\}$ consist of 17 cones. According to the order previously defined, all the cones $C_1, ..., C_8$ are minimal, and each cone $L_1, ..., L_8$ have two inferior cones, namely their respective closest neighbours among $C_1, ..., C_8$. The one point set $\{0\}$ is maximal, that is, all the other cones are inferior.*

Definition 4.17. *a subset $\mathfrak{I}$ of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ is a (right) ideal if for any $g \in \mathfrak{I}$ and $h \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ the composition $g.h$ lies into $\mathfrak{I}$. An ideal is said to be minimal if it does not contain any proper ideal of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$.*

Proposition 4.18. *For each non-trivial transformation type* $\mathfrak{t}$ *the following are equivalent:*

$$(i)\ \mathfrak{t} \text{ is minimal} \qquad (ii)\ \mathfrak{t}(i) \neq 0 \text{ on } I \qquad (iii)\ W_{\mathfrak{t}} = \mathbb{R}^n$$

The set

$$\mathfrak{M} := \mathbb{R}^n \times \mathfrak{T}_{min}$$

is closed, and is the unique minimal ideal of $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$*. Moreover the product on* $\mathfrak{M}$ *writes as*

$$(z, \mathfrak{t}).(z', \mathfrak{t}') = (z + z', \mathfrak{t}')$$

Proof. Suppose that $\mathfrak{t} \in (\mathfrak{T}_0, \leqslant)$ is minimal. If $\mathfrak{t}(i_0) = 0$ for some $i_0 \in I$, consider any non-trivial transformation type $\mathfrak{t}'$ with $\mathfrak{t}'(i_0) \neq 0$ (it suffice to take one with associated cone not included in $H^0_{i_0}$), and put $\mathfrak{t}_0 := \mathfrak{t}'.\mathfrak{t}$. Then $\mathfrak{t}_0(i_0) = \mathfrak{t}'.\mathfrak{t}(i_0) = \mathfrak{t}'(i_0) \neq \mathfrak{t}(i_0)$ so $\mathfrak{t}_0$ and $\mathfrak{t}$ are different transformation types, and nonetheless we have $\mathfrak{t}_0.\mathfrak{t} = (\mathfrak{t}'.\mathfrak{t}).\mathfrak{t} = \mathfrak{t}'.\mathfrak{t} = \mathfrak{t}_0$, as any transformation type is idempotent. Thus $\mathfrak{t}_0 < \mathfrak{t}$, which contradict the fact that $\mathfrak{t}$ was assumed to be minimal. Hence $\mathfrak{t}(i) \neq 0$ on I.
If $\mathfrak{t}(i) \neq 0$ on I, then $C_{\mathfrak{t}}$ is open in $\mathbb{R}^n$, so $< C_{\mathfrak{t}} >= \mathbb{R}^n$ in which $\mathbb{Z}^{n+d}$ is dense, so we get $W_{\mathfrak{t}} = \mathbb{R}^n$.
If $W_{\mathfrak{t}} = \mathbb{R}^n$ then as $C'_{\mathfrak{t}} = C_{\mathfrak{t}} \cap W_{\mathfrak{t}} = C_{\mathfrak{t}}$ is open in $W_{\mathfrak{t}} = \mathbb{R}^n$ we deduce $\mathfrak{t}(i) \neq 0$ for each $i \in I$. It follows that $\mathfrak{t}'.\mathfrak{t}(i) = \mathfrak{t}(i)$ on I for any other transformation type $\mathfrak{t}'$, giving that $\mathfrak{t}$ is minimal.
Consider next the stated set $\mathfrak{M} := \mathbb{R}^n \times \mathfrak{T}_{min}$. It is closed by remark 4.11. By theorem 4.4, $\mathfrak{M}$ lies into the Ellis semigroup $E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$.
Consider then any $(z, \mathfrak{t}) \in \mathfrak{M}$ and any $(z', \mathfrak{t}') \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$, with product $(z, \mathfrak{t}).(z', \mathfrak{t}') = (z + z', \mathfrak{t}.\mathfrak{t}')$. As $\mathfrak{t}$ never vanishes on I, thanks to the product law on $\mathfrak{T}_0$ the product $\mathfrak{t}.\mathfrak{t}'$ will never vanishes on I whatever the value of $\mathfrak{t}'$ is, and so $\mathfrak{t}.\mathfrak{t}'$ is minimal. The product pair is then into $\mathfrak{M}$, so $\mathfrak{M}$ is a right ideal. Also the product law on $\mathfrak{M}$ directly derives from the product law on the transformation types.
we show that $\mathfrak{M}$ is the unique minimal ideal:
Suppose $\mathfrak{M}'$ is another right ideal, containing some $(z, \mathfrak{t})$. Let $(0, \mathfrak{t}_0) \in \mathfrak{M}$, so that the product writes $(z, \mathfrak{t}).(0, \mathfrak{t}_0) = (z, \mathfrak{t}_0)$ (as $\mathfrak{t}_0$ is never 0) and lies into $\mathfrak{M}'$. Then as $\mathfrak{t}_0$ is minimal we have $(-z, \mathfrak{t}_0) \in E(\mathbb{R}^n_c, \mathbb{Z}^{n+d})$ and the product $(z, \mathfrak{t}_0).(-z, \mathfrak{t}_0) = (0, \mathfrak{t}_0)$ lies into $\mathfrak{M}'$. Consequently $\mathfrak{M} = (0, \mathfrak{t}_0).\mathfrak{M} = (0, \mathfrak{t}_0).E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) \subset \mathfrak{M}'.E(\mathbb{R}^n_c, \mathbb{Z}^{n+d}) = \mathfrak{M}'$. It follows that any right ideal contains $\mathfrak{M}$, thus this latter is a minimal ideal, and is the unique one. □

4.4. **Results for the cut up torus.** The analysis made in the two previous sections pass through the $\mathbb{Z}^n$-quotient map on the Ellis semigroup $E(\mathbb{T}^n_c, \mathbb{Z}^d)$, giving:

Theorem 4.19. *Let* $\mathfrak{T}_0$ *be the collection of non-trivial transformation types, and let* $W_{\mathfrak{t}}$ *be the Euclidean space associated with* $\mathfrak{t} \in \mathfrak{T}_0$ *of proposition 4.3. The map*

$$E(\mathbb{T}^n_c, \mathbb{Z}^d) \ni g \longmapsto (\pi^*_{\mathbb{T}}(g), \mathfrak{t}_g) \in \bigsqcup_{\mathfrak{t} \in \mathfrak{T}_0} \left[W_{\mathfrak{t}} + \mathbb{Z}^{n+d} \right] /_{\mathbb{Z}^n} \times \{\mathfrak{t}\}$$

associating to each transformation its translation and transformation type is a topological semigroup isomorphism, with right term equiped with:

- the semigroup law given by $(z,\mathfrak{t}).(z',\mathfrak{t}') = (z+z', \mathfrak{t}.\mathfrak{t}')$*, where the semigroup law on* $\mathfrak{T}_0$ *writes:*

$$\mathfrak{t}.\mathfrak{t}'(i) := \begin{cases} \mathfrak{t}'(i) \text{ if } \mathfrak{t}'(i) \neq 0 \\ \mathfrak{t}(i) \text{ if } \mathfrak{t}'(i) = 0 \end{cases}$$

- the topology of convergence: $(z_n, \mathfrak{t}_n) \longrightarrow (z,\mathfrak{t})$ *if and only if* $z_n - z$ *lifts into a sequence* $\underline{z_n} - \underline{z}$ *of* $\mathbb{R}^n$ *such that*

$$\forall \varepsilon > 0, \exists \delta_n > 0 \text{ such that } C'_{\mathfrak{t}_n}(\underline{z_n}, \delta_n) \subset C'_{\mathfrak{t}}(\underline{z}, \varepsilon) \text{ for large enough } n$$

Moreover,
- the semigroup $E(\mathbb{T}^n_c, \mathbb{Z}^d)$ *acts on* $\mathbb{T}^n_c$ *through* $(z,\mathfrak{p}).(z',\mathfrak{t}) = (z+z', \mathfrak{p}')$*, where*

$$\mathfrak{p}'(i) := \begin{cases} \mathfrak{p}.\mathfrak{t}(i) \text{ if } i \in I_{z+z'} \\ \infty \text{ else} \end{cases} \qquad \mathfrak{p}.\mathfrak{t}(i) := \begin{cases} \mathfrak{t}(i) \text{ if } \mathfrak{t}(i) \neq 0 \\ \mathfrak{p}(i) \text{ if } \mathfrak{t}(i) = 0 \end{cases}$$

- Each $[W_{\mathfrak{t}} + \Gamma]_{/\mathbb{Z}^n} \times \{\mathfrak{t}\}$ *is a subgroup of* $E(\mathbb{T}^n_c, \mathbb{Z}^d)$ *with identity* $(0,\mathfrak{t})$*.*
- the semigroup $\mathfrak{T}_0$ *is isomorphic with the collection of idempotent transformations in* $E(\mathbb{T}^n_c, \mathbb{Z}^d)$*, each writing as* $(0,\mathfrak{t})$ *for some* $\mathfrak{t} \in \mathfrak{T}_0$*.*
- The general theory on Ellis semigroup endows the collection of idempotent transformations with an pre-order. It is here an order, given on $\mathfrak{T}_0$ *by* $\mathfrak{t} \leqslant \mathfrak{t}'$ *if and only the cone* $C_{\mathfrak{t}'}$ *is equal or a lower dimensionnal facet of the cone* $C_{\mathfrak{t}}$*.*
- $E(\mathbb{T}^n_c, \mathbb{Z}^d)$ *has a unique minimal ideal, given by* $\mathbb{T}^n \times \mathfrak{T}_{min}$ *if* $\mathfrak{T}_{min}$ *denotes the subsemigroup of* $\mathfrak{T}_0$ *of minimal idempotents regarding the order.*

Example 4.20. *In the octagonal example we have the semigroup isomorphism and homeomorphism*

$$E(\mathbb{T}^2_{oct}, \mathbb{Z}^2) \simeq \left(\bigsqcup_{i=1}^{8} \mathbb{T}^2 \times \{\mathfrak{t}_{C_i}\} \right) \bigsqcup \left(\bigsqcup_{j=1}^{4} [H_i + \mathbb{Z}^4]_{/\mathbb{Z}^{n=2}} \times \{\mathfrak{t}_{L_i}, \mathfrak{t}_{L_{i+4}}\} \right) \bigsqcup \mathbb{Z}^{d=2}$$

with topology and semigroup law given by theorem 4.19.

The example of a Sturmian system $(\mathbb{X}_{\alpha,\beta}, \mathbb{Z})$ arising as the coding of a rotation by α on the circle by block exchange $[0,\beta) \sqcup [\beta, 1)$ is covered by the previous analysis applied to a 1-dimensionnal Euclidean space, with 'affine hyperspaces' the singletons $\{0\}$ and $\{\beta\}$, and dense subgroup $\mathbb{Z}^{n=1} + \alpha\mathbb{Z}^{d=1}$, α being an irrational number. It leads to the semigroup

$$E(\mathbb{X}_{\alpha,\beta}, \mathbb{Z}) = \mathbb{T} \times \{q^-, q^+\} \bigsqcup \alpha\mathbb{Z}$$

which in particular does not depend upon the value $0 < \beta < 1$ (neither algebraicaly nor topologicaly). The subspace $\mathbb{T} \times \{q^-, q^+\}$ is the unique minimal right ideal of the Ellis semigroup, a compact space which was already computed in [6] in the case $\alpha = \beta$, and called the 'two arrows space' of Alexandroff and Urysohn (see also [2] p.212). In the octagonal tiling case the subspace $\bigsqcup_{i=1}^{8} \mathbb{T}^2 \times \{\mathfrak{t}_{C_i}\}$ is, in the same spirit, the unique minimal right ideal of the Ellis semigroup and a compact subspace.
If we consider a spliting of the circle by $[0,\beta_1) \sqcup [\beta_1, \beta_2) \sqcup ... \sqcup [\beta_n, 1)$ instead of only two blocks for the coding of an irrationnal rotation by α, then the resulting

compact dynamical system $(\mathbb{X}_{\alpha,\beta_1,\beta_2,...,\beta_n},\mathbb{Z})$ is the rotation on the cut up circle obtained from the one-dimensionnal Euclidean space with dense subgroup $\mathbb{Z}^{n=1}+\alpha\mathbb{Z}^{d=1}$ and $\{0\},\{\beta_1\},\{\beta_2\},...,\{\beta_n\}$ as the collection of 'hyperplanes'. The associated Ellis semigroup thus writes also as

$$E(\mathbb{X}_{\alpha,\beta_1,\beta_2,...,\beta_n},\mathbb{Z})=\mathbb{T}\times\left\{q^-,q^+\right\}\bigsqcup\alpha\mathbb{Z}$$

This is the same as the result of Pikula [8], who considers the case of a $\mathbb{N}$-action of this system.

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