Directional metric regularity of multifunctions

Huynh van Ngai, Michel Théra

MATHEMATICS OF OPERATIONS RESEARCH
Vol. 00, No. 0, Xxxxx 0000, pp. 000–000
ISSN 0364-765X | EISSN 1526-5471 | 00 | 0000 | 0001

INFORMS
DOI 10.1287/xxxx.0000.0000

Directional metric regularity of multifunctions

Huynh Van Ngai
Department of Mathematics, University of Quy Nhon, 170 An Duong Vuong, Quy Nhon, Viet Nam
nghiakhiem@yahoo.com

Michel Théra
Laboratoire XLIM, Université de Limoges
michel.thera@unilim.fr

In this paper, we study relative metric regularity of set-valued mappings with emphasis on directional metric regularity. We establish characterizations of relative metric regularity without assuming the completeness of the image spaces, by using the relative lower semicontinuous envelopes of the distance functions to set-valued mappings. We then apply these characterizations to establish a coderivative type criterion for directional metric regularity as well as for the robustness of metric regularity.

Key words: Error bound, Perturbation stability, relative/directional metric regularity, Semicontinuous envelope, Generalized equations.
MSC2000 subject classification: 49J52, 49J53, 90C30.
OR/MS subject classification: Primary: Variational Analysis ; secondary: Optimisation

1. Introduction Set-valued mappings represent the most developed class of objects studied in the framework of variational analysis. Various types of set-valued mappings arise in a considerable number of models ranging from mathematical programs, through game theory and to control and design problems. The most well known and widely used regularity property of set-valued mappings is that of *metric regularity* [6, 17, 36, 45, 55, 59, 49, 20, 16, 33, 24, 47]. The term "metric regularity" was coined by Borwein & Zhuang [7]. Metric regularity or its equivalent notions (openness or covering at a linear rate or Aubin property of the inverse) is a central concept in modern variational analysis. In particular, this property is used as a key ingredient in investigating the behavior of the solution set of generalized equations associated to set-valued mappings.

According to the long history of metric regularity there is an abundant literature on conditions ensuring this property. The roots of this notion go back to the classical *Banach Open Mapping Theorem* (see, for instance [13], Theorem III.12.1) and its subsequent generalization to nonlinear mappings known as *Lyusternik and Graves Theorem*([48, 25], see also [15, 18]). For a detailed account on results on metric regularity as well as on its various applications, we refer the reader to basic monographs and references, [2, 3, 5, 8, 6, 9, 7, 11, 12, 16, 19, 35, 36, 37, 40, 42, 43, 48, 49, 50, 52, 53, 30, 31, 54, 55, 56, 58], as well as to the references given therein.

Let X and Y be metric spaces endowed with metrics both denoted by $d(\cdot,\cdot)$. The open ball with center x and radius $r>0$ is denoted by $B(x,r)$. For a given subset Ω of X, we denote by $\operatorname{conv}\Omega$ and $\operatorname{cone}\Omega$ the convex hull of Ω and the conical convex hull of Ω, respectively. We also use the symbols w and $w^\star$ to indicate the weak and the weak* topology, respectively, and w- lim and w*-

lim represent the weak and the weak* topological limits, respectively. $\operatorname{Int}\Omega$ and $\operatorname{cl}\Omega$ are the interior and the closure of Ω with respect to the norm topology, respectively; $\operatorname{cl}^w\Omega$ stands for the closure in the weak topology and $\operatorname{cl}^{w^\star}\Omega$ is the closure in the weak* topology of a given subset $\Omega\subset X^\star$ in the dual space. We also make use of the property, that for convex sets, the norm and the weak closures coincide. Finally given a mapping $f:X\to Y$ we note $\operatorname{Im} f$ for the range f.

Recall that a set-valued (multivalued) mappping $F:X\rightrightarrows Y$ is a mapping which assigns to every $x\in X$ a subset (possibly empty) $F(x)$ of Y. As usual, we use the notation $\operatorname{gph} F:=\{(x,y)\in X\times Y\ :\ y\in F(x)\}$ for the graph of F, $\operatorname{Dom} F:=\{x\in X:F(x)\neq\emptyset\}$ for the domain of F and $F^{-1}:Y\rightrightarrows X$ for the inverse of F. This inverse (which always exists) is defined by $F^{-1}(y):=\{x\in X\ :\ y\in F(x)\}$, $y\in Y$ and satisfies

$$(x,y)\in\operatorname{gph} F\iff(y,x)\in\operatorname{gph} F^{-1}.$$

If C is a subset of X, we use the standard notation $d(x,C)=\inf_{z\in C}d(x,z)$, with the convention that $d(x,S)=+\infty$ whenever C is empty. We recall that a multifunction F is *metrically regular* at $(x_0,y_0)\in\operatorname{gph} F$ with modulus $\tau>0$ if there exists a neighborhood $B((x_0,y_0),\varepsilon)$ of (x_0,y_0) such that

$$d(x,F^{-1}(y))\leq\tau d(y,F(x))\quad\text{for all}\quad(x,y)\in B((x_0,y_0),\varepsilon).\tag{1}$$

The infimum of all moduli τ satisfying relation (1) is denoted by $\operatorname{reg}F(\bar{x},\bar{y})$ ([19]). In the case for example of a set-valued mapping F with a closed and convex graph, the Robinson-Ursescu Theorem ([57] and [60]), says that F is metrically regular at (x_0,y_0), if and only if y_0 is an interior point to the range of F, i.e., to $\operatorname{Dom}F^{-1}$.

Recently, several generalized or weaker versions of metric regularity (restricted metric regularity [53], calmness, subregularity) have been considered. Especially, Ioffe ([38]) introduced and studied a natural extension of metric regularity called "*relative metric regularity*" which covers almost every notions of metric regularity given in the literature. Roughly speaking, a mapping F is relatively metrically regular relative to some subset $V\subseteq X\times Y$ if the metric regularity property is satisfied at points belonging to V and near the reference point. An important special case of this relative metric regularity concept is the notion of directional metric regularity introduced and studied by Arutyunov, Avakov and Izmailov in [1]. This directional metric regularity is an extension of an earlier concept used by Bonnans & Shapiro ([5]) to study sensitivity analysis.

Our main objective in this paper is to use the theory of error bounds to study directional metric regularity of multifunctions. We develop the method used by the authors in [31, 33, 29] to characterize relative metric regularity by using global/local slopes of a suitable lower semicontinuous envelope type of the distance function to the images of set-valued mappings. A particular advantage of this approach is to avoid the completeness of the image space that is not really necessary in some important situations. These established characterizations permit to derive coderivative conditions as well as stability results for directional metric regularity.

The remainder of this paper is organized as follows. In Section 2, we prove characterizations of relative metric regularity of closed multifunctions on metric spaces by using global/local strong slopes of suitable relative semicontinuous envelope of distance functions to the images of set-valued mappings. Based on these characterizations, we derive in Section 3 coderivative criteria ensuring directional metric regularity. In the final section, results on the perturbation stability of directional metric regularity are reported.

2. Characterizations of relative metric regularity Let X be a metric space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function. As usual, $\mathrm{dom} f := \{x \in X : f(x) < +\infty\}$ denotes the domain of f. We set

$$S := \{x \in X : \quad f(x) \leq 0\}. \tag{2}$$

We use the symbol $[f(x)]_+$ to denote $\max(f(x), 0)$. We shall say that the system (2) admits *a global error bound* if there exists a real $c > 0$ such that

$$d(x, S) \leq c\big[f(x)\big]_+ \quad \text{for all} \quad x \in X. \tag{3}$$

For $x_0 \in S$, we shall say that the system (2) has a *local error bound* at x_0, when there exist reals $c > 0$ and $\varepsilon > 0$ such that relation (3) is satisfied for all x around x_0, i.e., in an open ball $B(x_0, \varepsilon)$ with center x_0 and radius ε.

Since the first error bound result due to Hoffman ([26]), numerous characterizations and criteria for error bounds in terms of various derivative-like objects have been established. [22, 23]. Stability and some other properties of error bounds are examined in [27, 32, 33, 34, 46]. Several conditions using subdifferential operators or directional derivatives and ensuring the error bound property in Banach spaces have been established, for example, in [10, 41, 31]. Error bounds have been also used in sensitivity analysis of linear programming/linear complementarity problem and also as termination criteria for descent algorithms. Recently, Azé [2], Azé & Corvellec [4] have used the so-called strong slope introduced by De Giorgi, Marino & Tosques in [14] to prove criteria for error bounds in complete metric spaces.

Recall from [36][1] that the local and global strong slopes $|\nabla f|(x)$; $|\Gamma f|(x)$ of a function f at $x \in \mathrm{dom} f$ are the quantities defined by

$$|\nabla f|(x) = \limsup_{y \to x,\, y \neq x} \frac{[f(x) - f(y)]_+}{d(x, y)}; \quad |\Gamma f|(x) = \sup_{y \neq x} \frac{[f(x) - f(y)]_+}{d(x, y)} \tag{4}$$

For $x \notin \mathrm{dom} f$, we set $|\nabla f|(x) = |\Gamma f|(x) = +\infty$. When f takes only negative values it coincides with

$$|\nabla f|^{\diamond}(x) := \sup_{y \neq x} \frac{[f(x) - f_+(y)]_+}{d(y, x)}$$

as defined in [28]. As pointed out by Ioffe [39, Poposition 3.8], when f is a convex function defined on a Banach space, then

$$|\nabla f|(x) = \sup_{\|h\| \leq 1} (-|f'(x; h)) = d(0, \partial f(x).$$

Trivially, one has $|\nabla f|(x) \leq |\Gamma f|(x)$, for all $x \in X$.

In the sequel, we will need the following result established by Ngai & Théra ([33]), which gives an estimation via the global strong slope for the distance $d(\bar{x}, S)$ from a given point $\bar{x}$ outside of S to the set S in complete metric spaces.

THEOREM 1. *Let X be a complete metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $\bar{x} \notin S$. Then, setting*

$$m(\bar{x}) := \inf \{|\Gamma f|(x) : \quad d(x, \bar{x}) < d(\bar{x}, S),\ f(x) \leq f(\bar{x})\}, \tag{5}$$

one has

$$m(\bar{x}) d(\bar{x}, S) \leq f(\bar{x}). \tag{6}$$

[1] A. Ioffe was at your knowledge the first one to start using slopes and to advertise them in the optimization community.

Let X be a metric space and let Y be a normed linear space. Consider a multifunction $F : X \rightrightarrows Y$. Let us recall from ([1]) the definition of directional metric regularity.

DEFINITION 1. Let $F : X \rightrightarrows Y$ be a multifunction. Let $(x_0, y_0) \in \operatorname{gph} F$ and $\bar{y} \in Y$ be given. F is said to be directionally metrically regular at (x_0, y_0) in the direction $\bar{y}$ with a modulus $\tau > 0$ if there exist $\varepsilon > 0$, $\delta > 0$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)), \tag{7}$$

for all $(x, y) \in B(x_0, \varepsilon) \times B(y_0, \varepsilon)$ satisfying

$$d(y, F(x)) < \varepsilon \quad \text{and} \quad y \in F(x) + \operatorname{cone} B(\bar{y}, \delta).$$

Here, $\operatorname{cone} B(\bar{y}, \delta)$ stands for the conic hull of $B(\bar{y}, \delta)$, i.e., $\operatorname{cone} B(\bar{y}, \delta) = \bigcup_{\lambda \geq 0} \lambda B(\bar{y}, \delta)$. The infinum of all moduli τ in relation (7) is called the *exact modulus* of the metric regularity at (x_0, y_0) in direction $\bar{y}$, and is denoted by $\operatorname{reg}_{\bar{y}} F(x_0, y_0)$.

Note that if F is metrically regular at $(x_0, y_0) \in \operatorname{gph} F$, then F is directionally metrically regular in all directions $\bar{y} \in Y$. When $\|\bar{y}\| < \delta$, then directionally metric regularity coincides with the usual metric regularity. The notion of directionally metric regularity is a special case of *metric regularity relative to a set* V with $\operatorname{gph} F \subseteq V \subseteq X \times Y$, introduced by Ioffe ([38]). For $y \in Y$, $x \in X$, denote by $V_y := \{x \in X : (x, y) \in V\}$, $V_x := \{y \in Y : (x, y) \in V\}$, and $\operatorname{cl} V_y$ for the closure of V_y.

DEFINITION 2. Let X, Y be metric spaces. Let $F : X \rightrightarrows Y$ be a multifunction and let $(x_0, y_0) \in \operatorname{gph} F$ and fix a subset $V \subseteq X \times Y$. F is said to be metrically regular relative to V at (x_0, y_0) with a constant $\tau > 0$ if there exist $\varepsilon > 0$ such that

$$d(x, F^{-1}(y) \cap \operatorname{cl} V_y) \leq \tau d(y, F(x)), \quad \text{for all } (x, y) \in B((x_0, y_0), \varepsilon) \cap V,\ d(y, F(x)) < \varepsilon. \tag{8}$$

The infinum of all moduli τ is called the exact modulus of metric regularity at (x_0, y_0) relative to V, and denoted by $\operatorname{reg}_V F(x_0, y_0)$.

In papers [33, 29], the lower semicontinuous envelope $x \mapsto \varphi(x, y)$ of the function $x \mapsto d(y, F(x))$ for $y \in Y$ i.e.,

$$\varphi(x, y) := \liminf_{u \to x} d(y, F(u)),$$

has been used to characterize metric regularity of F. Along with the relative metric regularity, we define for each $y \in Y$ the lower semicontinous envelope of the functions $x \mapsto d(y, F(x))$ relative to a set $V \subseteq X \times Y$ by setting

$$\varphi_V(x, y) := \begin{cases} \liminf_{V_y \ni u \to x} d(y, F(u)) & \text{if} \quad x \in \operatorname{cl} V_y \\ +\infty & \text{otherwise.} \end{cases} \tag{9}$$

Obviously, for each $y \in Y$, the function $\varphi_V(\cdot, y)$ is lower semicontinuous.

For a given $\bar{y} \in Y$, directionally metric regularity in a given direction $\bar{y}$ is exactly metric regularity relative to $V(\bar{y}, \delta)$ (for some $\delta > 0$):

$$V(\bar{y}, \delta) := \{(x, y) : y \in F(x) + \operatorname{cone} B(\bar{y}, \delta)\}. \tag{10}$$

In this case, the lower semicontinuous envelope function relative to $V(\delta, \bar{y})$ is denoted simply by

$$\varphi_\delta(x, y) := \varphi_{V(\delta, \bar{y})}(x, y). \tag{11}$$

The following proposition permits to transfer equivalently relative metric regularity of F to the error bound property of the function φ_V.

PROPOSITION 1. *Let $F : X \rightrightarrows Y$ be a closed multifunction (i.e., its graph is closed) and let $(x_0, y_0) \in \operatorname{gph} F$. For $V \subseteq X \times Y$, the following statements holds.*
(i) For all $y \in Y$, one has

$$F^{-1}(y) \cap \operatorname{cl} V_y = \{x \in X : \varphi_V(x, y) = 0\};$$

(ii) F is metrically regular relative to V at (x_0, y_0) with a modulus $\tau > 0$ if and only if there exists $\varepsilon > 0$ such that

$$d(x, F^{-1}(y) \cap \operatorname{cl} V_y) \leq \tau \varphi_V(x, y) \quad \text{for all } (x, y) \in B(x_0, \varepsilon) \times B(y_0, \varepsilon) \text{ with } d(y, F(x)) < \varepsilon.$$

Proof. The proof follows straightforwardly from the definition. □

We establish in the next theorem characterizations of relative metric regularity by using the local/global strong slopes.

THEOREM 2. *Let X be a complete metric space and Y be a metric space. Let $F : X \rightrightarrows Y$ is a closed multifunction and let $(x_0, y_0) \in \operatorname{gph} F$ and let $V \subseteq X \times Y$. For a given $\tau \in (0, +\infty)$, consider the following statements.*
(i) F is metrically regular relative to V at (x_0, y_0);
(ii) There exists $\delta > 0$ such that

$$|\Gamma \varphi_V(\cdot, y)|(x) \geq \tau^{-1} \text{ for all } (x, y) \in \big(B(x_0, \delta) \times B(y_0, \delta)\big),\ x \in \operatorname{cl} V_y \text{ with } d(y, F(x)) \in (0, \delta); \quad (12)$$

(iii) There exists $\delta > 0$ such that

$$|\nabla \varphi_V(\cdot, y)|(x) \geq \tau^{-1} \text{ for all } (x, y) \in \big(B(x_0, \delta) \times B(y_0, \delta)\big),\ x \in \operatorname{cl} V_y \text{ with } d(y, F(x)) \in (0, \delta); \quad (13)$$

(iv) There exist $\delta > 0$ such that

$$|\nabla \varphi_V(\cdot, y)|(x) \geq \tau^{-1} \text{ for all } (x, y) \in \big(B(x_0, \delta) \times B(y_0, \delta)\big) \cap V \text{ with } d(y, F(x)) \in (0, \delta). \quad (14)$$

Then, $(i) \Leftrightarrow (ii) \Leftarrow (iii) \Rightarrow (iv)$. In addition, if Y is a normed linear space; $\operatorname{gph} F \subseteq V$; *$V_x$ is convex for any x near x_0 and V_y is open for y near y_0, then $(i) \Rightarrow (iv)$.*

Proof. The implications $(iii) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are obvious. For $(i) \Rightarrow (iii)$, assume that F is metrically regular relative to V at (x_0, y_0) with modulus $\tau > 0$.Then, there is $\delta > 0$ such that

$$d(x, F^{-1}(y)) \leq \tau \varphi_V(x, y)\ \forall (x, y) \in B(x_0, \delta) \times B(y_0, \delta).$$

Let $(x, y) \in B(x_0, \delta) \times B(y_0, \delta)$ with $\varphi_V(x, y) \in (0, +\infty)$ be given. For any $\varepsilon > 0$, we can find $u \in F^{-1}(y)$ satisfying

$$d(x, u) \leq (\tau + \varepsilon) \varphi_V(x, y).$$

Then, $\varphi_V(u, y) = 0$, $u \neq x$ and therefore,

$$|\Gamma \varphi_V(\cdot, y)|(x) \geq \frac{\varphi_V(x, y) - \varphi_V(u, y)}{d(x, u)} = \frac{\varphi_V(x, y)}{d(x, u)} \geq (\tau + \varepsilon)^{-1}.$$

Since $\varepsilon > 0$ is arbitrary, (ii) holds.

Let us prove $(ii) \Rightarrow (i)$. Suppose that (12) is satisfied for $\delta > 0$. Let $\varepsilon \in (0, \tau/2)$ be given, and let

$$\alpha := \min\Big\{\delta/2, \frac{\delta}{2(\tau + \varepsilon)}, \delta\tau\Big\}.$$

Let $(x,y)\in B((x_0,y_0),\alpha)$ with $x\in \operatorname{cl} V_y$, $d(y,F(x))<\alpha$ be given. Then,

$$\varphi_V(x,y)<\inf_{u\in X}\varphi_V(u,y)+\alpha.$$

By virtue of the Ekeland variational principle [21] applied to the function $x\mapsto\varphi_V(x,y)$ on X, we can find $z\in X$ satisfying $d(x,z)\le\alpha(\tau+2\varepsilon)$ and $\varphi_V(z,y)\le\varphi_V(x,y)(<\alpha)$ such that

$$\varphi_V(z,y)\le\varphi_V(u,y)+\frac{1}{\tau+2\varepsilon}d(u,z)\quad\text{for all } u\in X.$$

Consequently, $z\in B(x_0,\delta)$ and

$$\varphi_V(z,y)-\varphi_V(u,y)\le\frac{d(z,u)}{\tau+2\varepsilon}\le\frac{d(z,u)}{\tau+\varepsilon}\quad\text{for all } u\in X.$$

Therefore, by relation (12), we must have $z\in F^{-1}(y)\cap\operatorname{cl} V_y$. Consequently,

$$B(x_0,2\alpha\tau)\cap F^{-1}(y)\cap\operatorname{cl} V_y\neq\emptyset. \tag{15}$$

Then for any $z\in X$ with $d(x,z)<d(x,F^{-1}(y))$; $\varphi_V(z,y)\le\varphi_V(x,y)$, one has:

$$d(z,x_0)\le d(z,x)+d(x,x_0)\le d(x_0,F^{-1}(y))+2d(x,x_0)<2\alpha\tau+2\alpha\le\delta.$$

Thus, $z\in B(x_0,\delta)$ and $z\notin F^{-1}(y)$. Therefore, according to (12), one has

$$m(x):=\inf\left\{|\Gamma\varphi_{(}\cdot,y)|(z):\quad\begin{array}{l}d(z,x)<d(x,F^{-1}(y))\\ \varphi_V(z,y)\le\varphi_V(x,y)\end{array}\right\}>\frac{1}{\tau+\varepsilon}.$$

By virtue of Theorem 1 and as $\varepsilon>0$ is arbitrarily small, we obtain

$$d(x,F^{-1}(y)\cap V_y)\le\tau\varphi_V(x,y),$$

which proves $(ii)\Rightarrow(i)$.

To conclude the proof of the theorem, we need to show $(i)\Rightarrow(iv)$ provided Y is a normed linear space; V_y is open for y near y_0 and V_x is a convex set for x near x_0. Let $\delta\in(0,1)$ be such that V_y is open for all $y\in B(y_0,\delta)$; V_x is convex for all $x\in B(x_0,\delta)$ and that

$$d(x,F^{-1}(y))\le\tau d(y,F(x))\quad\forall(x,y)\in\big(B(\bar{x},2\delta)\times B(\bar{y},2\delta)\big)\cap V.$$

Let $(x,y)\in(B(\bar{x},\delta)\times B(\bar{y},\delta))\cap V$ be given with $F(x)\neq\emptyset$; $y\notin F(x)$; $d(y,F(x))<\delta$. For any $\varepsilon\in(0,\delta/2)$, pick $u_\varepsilon\in B(x_0,\varepsilon)\cap V_y$ such that

$$d(u_\varepsilon,x)<\varepsilon^2\varphi_V(x,y);\ d(y,F(u_\varepsilon))\le(1+\varepsilon^2)^{1/2}\varphi_V(x,y). \tag{16}$$

Take $y_\varepsilon\in F(u_\varepsilon)$ such that

$$d(y,F(u_\varepsilon))\le\|y-y_\varepsilon\|<(1+\varepsilon^2)^{1/2}d(y,F(u_\varepsilon)).$$

Then, since $u\in B(x_0,\delta)$, by the convexity of V_u,

$$(u,z_\varepsilon)\in V\quad\text{with } z_\varepsilon:=\varepsilon y+(1-\varepsilon)y_\varepsilon.$$

Furthemore,

$$\|y-z_\varepsilon\|=(1-\varepsilon)\|y-y_\varepsilon\|<(1-\varepsilon)(1+\varepsilon^2)^{1/2}d(y,F(u_\varepsilon))<d(y,F(u_\varepsilon))<(1+\varepsilon)\varphi_V(x,y).$$

Therefore, $z_\varepsilon \notin F(u_\varepsilon)$ and $\|z_\varepsilon - y_0\| \le \|y - y_0\| + \|y - z_\varepsilon\| < 2\delta$. Hence, we can select $x_\varepsilon \in F^{-1}(z_\varepsilon)$ such that

$$\begin{aligned} d(u_\varepsilon, x_\varepsilon) < (1+\varepsilon) d(u_\varepsilon, F^{-1}(z_\varepsilon)) &\le (1+\varepsilon)\tau d(z_\varepsilon, F(u_\varepsilon)) \\ &\le (1+\varepsilon)\varepsilon\tau \|y - y_\varepsilon\|. \end{aligned} \tag{17}$$

Consequently, $\lim_{\varepsilon\to 0^+} d(x, x_\varepsilon) = 0$. Hence, for $\varepsilon > 0$ sufficiently small, $x_\varepsilon \in V_y$, and one has the following estimation

$$\begin{aligned} \varphi_V(x,y) - \varphi_V(x_\varepsilon, y) &\ge \frac{1}{(1+\varepsilon^2)^{1/2}} d(y, F(u_\varepsilon)) - d(y, F(x_\varepsilon)) \\ &> \left(\frac{1}{1+\varepsilon^2} - (1-\varepsilon) \right) \|y - y_\varepsilon\| \\ &= \frac{\varepsilon - \varepsilon^2 + \varepsilon^3}{1+\varepsilon^2} \|y - y_\varepsilon\|. \end{aligned} \tag{18}$$

By combining this relation and relations (16), (17), one obtains

$$\frac{\varphi_V(x,y) - \varphi_V(x_\varepsilon, y)}{d(x, x_\varepsilon)} \ge \frac{\varphi_V(x,y) - \varphi_V(x_\varepsilon, y)}{d(x, u_\varepsilon) + d(u_\varepsilon, x_\varepsilon)} > \frac{(\varepsilon - \varepsilon^2 + \varepsilon^3)\|y - y_\varepsilon\|}{(1+\varepsilon^2)(\varepsilon^2 \varphi_V(x,y) + (1+\varepsilon)\varepsilon\tau\|y - y_\varepsilon\|)}.$$

Since $\lim_{\varepsilon\to 0^+} \|y - y_\varepsilon\| = \lim_{\varepsilon\to 0^+} d(y, F(u_\varepsilon)) = \varphi_V(x,y) > 0$, then

$$|\nabla \varphi_V(\cdot, y)|(x) \ge \liminf_{\varepsilon\to 0^+} \frac{\varphi_V(x,y) - \varphi_V(x_\varepsilon, y)}{d(x, x_\varepsilon)} \ge \tau^{-1},$$

which completes the proof. □

Theorem 2 yields the following exact formula for the relative metric regularity.

COROLLARY 1. *Let X be a complete metric space and let Y be a metric space and let $V \subseteq X \times Y$ with* gph $F \subseteq V$. *Suppose that the multifunction $F : X \rightrightarrows Y$ is closed and $(x_0, y_0) \in gphF$. Then, one has*

$$1/reg_V F(x_0, y_0) = \liminf_{\substack{(x,y) \overset{\varphi}{\to} (x_0,y_0) \\ y \notin F(x) \\ x \in \mathrm{cl}\, V_y}} |\Gamma \varphi_V(\cdot, y)|(x).$$

Moreover, if in addition, Y is a normed linear space; V_x is convex for any x near x_0 and V_y is open for y near y_0, then

$$1/reg_V F(x_0, y_0) \le \liminf_{\substack{(x,y) \overset{\varphi,V}{\to} (x_0,y_0) \\ y \notin F(x)}} |\nabla \varphi_V(\cdot, y)|(x).$$

The notation $(x,y) \overset{\varphi,V}{\to} (x_0, y_0)$ means that $(x,y) \to (x_0, y_0)$ with $\varphi(x,y) \to 0$ and $(x,y) \in V$.

Proof. It follows directly from Theorem 2. □

We next introduce the partial notion of relative metric regularity for a parametric set-valued mapping. Let X, Y be metric spaces and let P be a topological space. Given a set-valued mapping $F : X \times P \rightrightarrows Y$, we consider the implicit multifunction: $S : Y \times P \rightrightarrows Y$ defined by

$$S(y,p) := \{x \in X : \ y \in F(x,p)\}. \tag{19}$$

Let $x_0 \in S(y_0, p_0)$ and a set V: gph $F \subseteq V \subseteq X \times P \times Y$ be given.

DEFINITION 3. The set-valued mapping F is said to be metrically regular uniformly in p relatively to V at $((x_0,p_0),y_0)$ with a modulus $\tau>0$, if there exist $\varepsilon>0$ and a neighborhood W of p_0 such that

$$d(x,S(y,p))\leq \tau d(y,F(x,p)),\quad \text{for all } (x,y,p)\in \big(B((x_0,y_0)\varepsilon)\times W\big)\cap V;\ d(y,F(x,p))<\varepsilon. \tag{20}$$

The infinum of all moduli τ is called the exact modulus of the metric regularity of F uniformly in p at (x_0,y_0) relative to V and is denoted by $\operatorname{reg}_V F(x_0,p_0,y_0)$.

Denote by

$$V_{(x,p)}:=\{y\in Y:\ (x,p,y)\in V\},\ \text{for } (x,p)\in X\times P;$$
$$V_{(y,p)}:=\{x\in X:\ (x,p,y)\in V\},\ \text{for } (y,p)\in Y\times P.$$

For each $(y,p)\in Y\times P$, the lower semicontinuous envelope relative to V of the function: $x\mapsto d(y,F(x,p))$ is defined by

$$\varphi_V(x,y,p):=\begin{cases}\liminf_{u\to x,u\in V_{(y,p)}} d(y,F(u,p)) & \text{if}\quad x\in \operatorname{cl} V_{(y,p)}\\ +\infty & \text{otherwise.}\end{cases} \tag{21}$$

Similarly to Theorem 2, one has

THEOREM 3. *Let X be a complete metric space, Y be a metric space and P be a topological space. Let $F:X\times P\rightrightarrows Y$ be a set-valued mapping and let $((x_0,p_0),y_0)\in \operatorname{gph} F$; $V\subseteq X\times P\times Y$; $\tau\in(0,+\infty)$ be given. Suppose that for any p near $\bar p$, the set-valued mapping $x\rightrightarrows F(x,p)$ is a closed multifunction. Then, among the following statements, one has $(i)\Leftrightarrow(ii)\Leftarrow(iii)\Rightarrow(iv)$. Moreover, if Y is a normed linear space; $V_{(x,p)}$ is convex for any (x,p) near (x_0,p_0) and $V_{(y,p)}$ is open for (y,p) near (y_0,p_0), then $(i)\Rightarrow(iv)$.*
(i) F is metrically regular relative to V uniformly in p at $((x_0,p_0),y_0)$;
(ii) There exist $\delta,\gamma>0$ and a neighborhood W of p_0 such that

$$|\Gamma\varphi_V(\cdot,y,p)|(x)\geq\tau^{-1}\ \textit{for all}\ (x,p,y)\in\big(B(x_0,\delta)\times W\times B(y_0,\delta)\big),\ x\in\operatorname{cl} V(y,p)\ \textit{with}\ d(y,F(x,p))\in(0,\gamma); \tag{22}$$

(iii) There exist $\delta,\gamma>0$ and a neighborhood W of p_0 such that

$$|\nabla\varphi_V(\cdot,y,p)|(x)\geq\tau^{-1}\ \textit{for all}\ (x,p,y)\in\big(B(x_0,\delta)\times W\times B(y_0,\delta)\big),\ x\in\operatorname{cl} V(y,p)\ \textit{with}\ d(y,F(x,p))\in(0,\gamma); \tag{23}$$

(iv) There exist $\delta,\gamma>0$ and a neighborhood W of p_0 such that

$$|\nabla\varphi_V(\cdot,y,p)|(x)\geq\tau^{-1}\ \textit{for all}\ (x,p,y)\in\big(B(x_0,\delta)\times W\times B(y_0,\delta)\big)\cap V\ \textit{with}\ d(y,F(x,p))\in(0,\gamma). \tag{24}$$

Proof. The proof being similar to the one of Theorem 2, we omit it. □

3. Coderivative characterizations of directional metric regularity For the usual metric regularity, sufficient conditions in terms of coderivatives have been given by various authors, for instance, in [3, 43, 49, 31]. In this section, we establish a characterization of directional metric regularity using the Fréchet subdifferential in Asplund spaces, i.e., Banach spaces for which every convex continuous function is generically Fréchet differentiable. There are many equivalent descriptions of Asplund spaces, which can be found, e.g., in [49] and its bibliography. In particular, any

reflexive space is Asplund, as well as each Banach space such that if each of its separable subspaces has a separable dual.

In order to formulate in this section some coderivative characterizations of directional metric regularity, we require some more definitions. Let X be a Banach space. Consider now an extended-real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$. The *Fréchet subdifferential* of f at $\bar{x} \in \operatorname{Dom} f$ is given as

$$\partial f(\bar{x}) = \left\{ x^* \in X^* : \liminf_{x \to \bar{x},\ x \neq \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

For convenience of the reader, we would like to mention that the terminology *regular subdifferential* instead of Fréchet subdifferential is also popular due to its use in Rockafellar and Wets [59]. Every element of the Fréchet subdifferential is termed as a Fréchet (regular) subgradient. If $\bar{x}$ is a point where $f(\bar{x}) = \infty$, then we set $\partial f(\bar{x}) = \emptyset$. In fact one can show that an element x^* is a Fréchet subgradient of f at $\bar{x}$ iff

$$f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \quad \text{where} \quad \lim_{x \to \bar{x}} \frac{o(\|x - \bar{x}\|)}{\|x - \bar{x}\|} = 0.$$

It is well-known that the Fréchet subdifferential satisfies a fuzzy sum rule on Asplund spaces (see [49, Theorem 2.33]). More precisely, if X is an Asplund space and $f_1, f_2 : X \to \mathbb{R} \cup \{\infty\}$ are such that f_1 is Lipschitz continuous around $\overline{x} \in \operatorname{Dom} f_1 \cap \operatorname{Dom} f_2$ and f_2 is lower semicontinuous around $\overline{x}$, then for any $\gamma > 0$ one has

$$\partial(f_1 + f_2)(\overline{x}) \subset \bigcup \{ \partial f_1(x_1) + \partial f_2(x_2) \mid x_i \in \overline{x} + \gamma \overline{B}_X, |f_i(x_i) - f_i(\overline{x})| \leq \gamma, i = 1, 2 \} + \gamma B_{X^*}. \quad (25)$$

For a nonempty closed set $C \subseteq X$, denote by δ_C the *indicator function* associatedwith C (i.e. $\delta_C(x) = 0$, when $x \in C$ and $\delta_C(x) = \infty$ otherwise). The *Fréchet normal cone* to C at $\bar{x}$ is denoted by $N(C, \bar{x})$. It is a closed and convex object in X^* which is defined as $\partial \delta_C(\bar{x})$. Equivalently a vector $x^* \in X^*$ is a Fréchet normal to C at $\bar{x}$ if

$$\langle x^*, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|), \quad \forall x \in C,$$

where $\lim_{x \to \bar{x}} \dfrac{o(\|x - \bar{x}\|)}{\|x - \bar{x}\|} = 0$. Let $F : X \rightrightarrows Y$ be a set-valued map and $(x, y) \in \operatorname{gph} F$. Then the *Fréchet coderivative* at (x, y) is the set-valued map $D^*F(x, y) : Y^* \rightrightarrows X^*$ given by

$$D^*F(x, y)(y^*) := \{ x^* \in X^* \mid (x^*, -y^*) \in N(\operatorname{gph} F, (x, y)) \}.$$

This notion is recognized as a powerful tool of variational analysis when applied to problems of optimization and control (see [49, 51, 44], and the references therein).

In the proof of the main result, we will use the following particular version of Theorem 2 for directional metric regularity.

THEOREM 4. *Let X be a complete metric space and Y be a normed space. Let $F : X \rightrightarrows Y$ be a closed multifunction (i.e., its graph is closed) and fix $(x_0, y_0) \in \operatorname{gph} F$ and $V \subseteq X \times Y$. For a given $\tau \in (0, +\infty)$, then among the following statements, one has $(i) \Leftrightarrow (ii) \Leftarrow (iii)$.*
(i) F is metrically regular in the direction $\bar{y}$ at (x_0, y_0);
(ii) There exists $\delta > 0$ such that

$$|\Gamma \varphi_{V(\bar{y}, \delta)}(\cdot, y)|(x) \geq \tau^{-1} \text{ for all } (x, y) \in \big(B(x_0, \delta) \times B(y_0, \delta)\big), \quad x \in \operatorname{cl} V(\bar{y}, \delta) \text{ with } d(y, F(x)) \in (0, \delta); \quad (26)$$

(iii) There exist $\delta>0$ such that

$$|\nabla\varphi_{V(\bar{y},\delta)}(\cdot,y)|(x)\geq\tau^{-1} \text{ for all } (x,y)\in\big(B(x_0,\delta)\times B(y_0,\delta)\big)\, x\in \operatorname{cl} V(\bar{y},\delta) \text{ with } d(y,F(x))\in(0,\delta). \tag{27}$$

Denote by S_{Y^*} the unit sphere in the dual space Y^* of Y, and by d_* the metric associated with the dual norm on X^*. For given $\bar{y}\in Y$ and $\delta>0$, denote by

$$C_{Y^*}(\bar{y},\delta):=\{y^*\in Y^*:\ |\langle y^*,\bar{y}\rangle|\leq\delta\} \text{ and } S_{Y^*}(\bar{y},\delta):=\{y^*\in Y^*:\ \|y^*\|\leq 1+\delta,\ \langle y^*,\bar{y}\rangle\leq\delta\}, \tag{28}$$

and

$$T(\bar{y},\delta):=\{(y_1^*,y_2^*)\in S_{Y^*}(\bar{y},\delta)\times C_{Y^*}(\bar{y},\delta):\ \|y_1^*+y_2^*\|=1\}. \tag{29}$$

For a given multifunction $F:X\rightrightarrows Y$, we associate the multifunction $G:X\rightrightarrows Y\times Y$ defined by

$$G(x)=F(x)\times F(x),\quad x\in X.$$

Recall also that a multifunction $F:X\rightrightarrows Y$ is said to be *pseudo-Lipschitz* (or *Lipschitz-like* or satisfying the Aubin property) around $(x_0,y_0)\in\operatorname{gph}F$ if there exist constants $L,\delta>0$ such that

$$F(x)\cap B(y_0,\delta)\subseteq F(x')+L\|x-x'\|B_Y,\quad \text{for all } x,x'\in B(x_0,\delta).$$

It is well known that F is pseudo-Lipschitz around (x_0,y_0) if and only if the function $d(\cdot,F(\cdot)):X\times Y\to\mathbb{R}$ is Lipschitz near (x_0,y_0), (see for instance [56, Theorem 1.142]).

A coderivative characterization of directional metric regularity is initiated in the following theorem, which is the main result of this section.

THEOREM 5. *Let X,Y be Asplund spaces. Let $F:X\rightrightarrows Y$ be a closed multifunction and $(x_0,y_0)\in\operatorname{gph}F$ be given. Let F be pseudo-Lipschitz around (x_0,y_0). Suppose that F has convex values around x_0, i.e., $F(x)$ is convex for all x near x_0. If*

$$\liminf_{\substack{(x,y_1,y_2)\overset{G}{\to}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0,D^*G(x,y_1,y_2)(T(\bar{y},\delta)))>m>0, \tag{30}$$

then F is directionally metrically regular in the direction $\bar{y}$ with modulus $\tau\leq m^{-1}$ at (x_0,y_0). The notation $(x,y_1,y_2)\overset{G}{\to}(x_0,y_0,y_0)$ means that $(x,y_1,y_2)\to(x_0,y_0,y_0)$ with $(x,y_1,y_2)\in\operatorname{gph}G$.

The following lemmata are needed in the proof of Theorem 5.

LEMMA 1. *Let $F:X\rightrightarrows Y$ be a multifunction with convex values for x near x_0 and $(x_0,y_0)\in\operatorname{gph}F$. Then for any $\bar{y}\in Y$, $\delta_1,\delta_2>0$, there exist $\eta,\delta>0$ such that for all $x\in B(x_0,\eta)$, one has*

$$\big(F(x)+\operatorname{cone}B(\bar{y},\delta)\big)\cap B(y_0,\eta)\cap\{y\in Y:\ d(y,F(x))<\eta\}\subseteq F(x)\cap B(y_0,\delta_1)+\operatorname{cone}B(\bar{y},\delta_2). \tag{31}$$

Proof. Let $\bar{y}\in Y$, δ_1,δ_2 be given. If $\|\bar{y}\|<\delta_2$ then the conclusion holds trivially. Suppose $\|\bar{y}\|\geq\delta_2$. Take $\delta=\delta_2/2$ and $\varepsilon\in(0,\delta_1/2)$ sufficiently small such that

$$\frac{\varepsilon(\|\bar{y}\|+\delta_2/2)}{\delta_1-2\varepsilon}<\delta_2/2.$$

Let $\eta\in(0,\varepsilon/2)$ such that $F(x)$ is convex for all $x\in B(x_0,\eta)$ and let now $x\in B(x_0,\eta)$ and $y\in\big(F(x)+\operatorname{cone}B(\bar{y},\delta)\big)\cap B(y_0,\eta)$ with $d(y,F(x))<\varepsilon/2$ be given. Then, there exist $z,v\in F(x)$ such that

$$y=z+\lambda(\bar{y}+\delta u),\ \text{for}\ \lambda\geq 0,\ u\in Y\ \|u\|\leq 1;\ \|y-v\|<\varepsilon/2.$$

If $z \in B(y_0, \delta_1)$ then the proof is over. Otherwise, one has

$$\lambda(\|\bar{y}\| + \delta) \geq \|y - z\| \geq \|z - y_0\| - \|y - y_0\| \geq \delta_1 - \eta > \delta_1 - \varepsilon.$$

By setting

$$t := \frac{\delta_1 - 2\varepsilon}{\delta_1 - \varepsilon}, \;\; w := tz + (1-t)v \in F(x),$$

one has

$$\|w - y_0\| \leq t\|z - v\| + \|v - y_0\| \leq t\lambda\|y + \delta u\| + t\|y - v\| + \|v - y_0\| < \delta_1 - 2\varepsilon/2 + \varepsilon < \delta_1$$

and,

$$\frac{(1-t)\|y - v\|}{t\lambda} < \frac{\varepsilon(\|\bar{y}\| + \delta)}{\delta_1 - 2\varepsilon} < \delta_2/2.$$

Thus,

$$y - w = t\lambda\left(\bar{y} + \delta u + (1-t)\frac{y - v}{t\lambda}\right) \in \operatorname{cone} B(\bar{y}, \delta_2),$$

which implies that $y \in F(x) \cap B(y_0, \delta_1) + \operatorname{cone} B(\bar{y}, \delta_2)$. □

Associated with the multifunction F, for given $\varepsilon > 0$, $(x_0, y_0) \in \operatorname{gph} F$, we define the *localization* of F by

$$F_{(x_0, y_0, \varepsilon)}(x) := \begin{cases} F(x) \cap \bar{B}(y_0, \delta_0) & \text{if} \quad x \in \bar{B}(x_0, \varepsilon) \\ \emptyset & \text{otherwise.} \end{cases} \tag{32}$$

Note that, by definition, one has

$$D^*F(x, y) = D^*F_{(x_0, y_0, \varepsilon)}(x, y) \quad \forall (x, y) \in \operatorname{gph} F \cap (B(x_0, \varepsilon) \times B(y_0, \varepsilon)). \tag{33}$$

The preceding lemma implies obviously the next corollary.

COROLLARY 2. *Let $F : X \rightrightarrows Y$ be a multifunction with convex values for x near x_0 and $(x_0, y_0) \in \operatorname{gph} F$. Then the two following conditions are equivalent:*

1. *F is directionally metrically regular in the direction $\bar{y}$;*
2. *For any $\varepsilon > 0$, $F_{(x_0, y_0, \varepsilon)}$ is directionally metrically in the direction $\bar{y}$.*

LEMMA 2. *Let $C \subseteq Y$ be a nonempty convex cone. One has*

$$N(C, z) \subseteq \{z^* \in Y^* : \langle z^*, z \rangle = 0\}, \quad \textit{for all } z \in C.$$

Proof. Let $z \in C$ be given. Then $\lambda z \in C$ for all $\lambda > 0$. Hence, for $z^* \in N(C, z)$, one has

$$\langle z^*, \lambda z - z \rangle \leq 0 \quad \forall \lambda > 0.$$

Thus, $\langle z^*, z \rangle = 0$. □

Given a multifunction $F : X \rightrightarrows Y$, $\bar{y} \in Y$, for $y \in Y$ and $\delta > 0$, we define the set $\mathcal{V}(y, \delta) \subseteq X \times Y$ by

$$\mathcal{V}(y, \delta) := \{(x, z) \in X \times Y : \;\; y \in F(x) + z, \; z \in \operatorname{cone} \bar{B}(\bar{y}, \delta)\}. \tag{34}$$

LEMMA 3. *Let X, Y be Asplund spaces and let $F : X \rightrightarrows Y$ be a closed multifunction with convex values for x near x_0, and $(x_0, y_0) \in \operatorname{gph} F$, $\bar{y} \in Y$ be given. We suppose by assumption that*

$$\lim_{\substack{(x,y)\overset{F}{\to}(x_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*F(x,y)(C_{Y^*}(\bar{y},\delta)\cap S_{Y^*})) > 0. \tag{35}$$

Then there exist $\kappa > 0, \varepsilon > 0, \delta_0 > 0$ such that for all $\delta \in (0,\delta_0)$, for any $(x,y) \in B((x_0,y_0),\varepsilon), z \in \operatorname{cone}\bar{B}(\bar{y},\delta)$, with $d(y,F(x)) \in (0,\varepsilon_0)$ and $y - z \in F(x)\cap\bar{B}(y_0,\varepsilon)$, we can find $\eta > 0$ such that

$$d((x',z'),\mathcal{V}(y,\delta)) \le \tau d(y, F(x') + z') \text{ for all } (x',z') \in B((x,z),\eta),\ z' \in \operatorname{cone}\bar{B}(\bar{y},\delta). \tag{36}$$

Proof. Since (35), then there exists $\delta_0 \in (0,1)$ such that

$$\inf_{(x,y)\in \operatorname{gph} F\cap B((x_0,y_0),\delta_0)} d_*(0, D^*F(x,y)(C_{Y^*}(\bar{y},\delta_0)\cap S_{Y^*})) := m > 0. \tag{37}$$

Let $\Phi : X \times Y \rightrightarrows Y$ be a multifunction defined by

$$\Phi(x,z) := \begin{cases} F(x) + z & \text{if} \quad z \in \operatorname{cone}\bar{B}(\bar{y},\delta),\\ \emptyset \quad \text{otherwise.}\end{cases}$$

Then Φ is a closed multifunction, and by a direct calculation, for $(x,z,y) \in \operatorname{gph}\Phi$, one has

$$D^*\Phi(x,z,y)(y^*) = \left\{(x^*, y^* + z^*) \in X^* \times Y^* :\ x^* \in D^*F(x, y-z)(y^*),\ z^* \in N(\operatorname{cone}\bar{B}(\bar{y},\delta), z)\right\}. \tag{38}$$

Then, $\mathcal{V}(y,\delta) = \Phi^{-1}(y)$. Let $\varepsilon, \delta \in (0, \delta_0/2)$ with $(1 + \delta + |\bar{y}|)\delta < \delta_0$. Let $(x,y) \in B((x_0,y_0),\varepsilon)$, $z \in \operatorname{cone}\bar{B}(\bar{y},\delta)$ with $d(y,F(x)) \in (0,\varepsilon)$; $y - z \in F(x)\cap\bar{B}(y_0,\varepsilon)$ be given. Take $\eta = \min\{\delta, \|z\|\} > 0$, and $(x',z',y') \in \operatorname{gph}\Phi \cap B((x,z,y),\eta)$, $(x^*,w^*) \in D^*\Phi(x',z')(y^*)$ with $\|y^*\| = 1$. Then $x^* \in D^*F(x', y' - z')(y^*)$; $w^* = y^* + z^*$ with some $z^* \in N(\operatorname{cone}\bar{B}(\bar{y},\delta), z')$. Since $N(\operatorname{cone}\bar{B}(\bar{y},\delta), z') \subseteq \{z^* \in Y^* : \langle z^*, z'\rangle = 0\}$, then $|\langle z^*, \bar{y}\rangle| \le \delta\|z^*\|$. If $\|w^*\| < \delta$, then $\|z^*\| < 1 + \delta$, and moreover,

$$|\langle y^*, \bar{y}\rangle| \le |\langle z^*, \bar{y}\rangle| + \delta\|\bar{y}\| \le (1 + \delta + \|\bar{y}\|)\delta < \delta_0.$$

Hence, $y^* \in C_{Y^*}(\bar{y},\delta_0)\cap S_{Y^*}$. Since $(x, y-z) \in B((x_0,y_0),\varepsilon)$, $(x',z',y') \in \operatorname{gph}\Phi\cap B((x,z,y),\eta)$, then $(x', y' - z') \in B((x_0,y_0),\delta_0)$ Therefore, from (37), we obtain $\|x^*\| \ge m$. Therefore,

$$\liminf_{(x',z',y')\to\Phi(x,z,y)} d_*(0, D^*\Phi(x',z',y')(S_{Y^*})) \ge \min\{m,\delta\}.$$

Thanks to the standard coderivative characterization of metric regularity for closed multifunctions (see, e.g., [3, 36]), we conclude that Φ is metrically regular around (x,z,y). Thus, there exists $\eta > 0$ such that

$$d((x',z'),\mathcal{V}(y,\delta)) \le \tau d(y,\Phi(x',z')) = \tau d(y, F(x') + z') \text{ for all } (x',z') \in B((x,z),\eta),\ z' \in \operatorname{cone}\bar{B}(\bar{y},\delta).$$

So the lemma is proved . □

The next lemma is a penalty result which is similar to the one by Clarke ([11]).

LEMMA 4. *Let C be a subset of a metric space X and let $x_0 \in C$ and $\varepsilon > 0$. Then for a function $f : X \to \mathbb{R}\cup\{+\infty\}$ which is Lipschitz on $B(x_0, 2\varepsilon)$ with constant $L > 0$, one has*

$$f^* := \inf\left\{f(x)\ :\ x \in C\cap B(x_0,2\varepsilon)\right\} \le \inf\left\{f(x) + td(x,C)\ :\ x \in B(x_0,\varepsilon)\right\},$$

whenever $t \ge L$.

Proof. For any $x \in B(x_0,\varepsilon)$, pick a sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq C$ with $\lim_{k\to\infty} d(x,z_k) = d(x,C)$. Then $z_k \in B(x_0,2\varepsilon)$ when k is sufficiently large. Therefore,

$$f^* \le f(z_k) \le f(x) + Ld(x,z_k) \to f(x) + Ld(x,C). \qquad \square$$

Proof of Theorem 5. By the assumption, there is $\delta_0 \in (0,1)$ such that

$$\inf_{(x,y_1,y_2)\in \operatorname{gph} G\cap B((x_0,y_0,y_0),2\delta_0)} d_*(0, D^*G(x,y_1,y_2)(T(\bar{y},\delta_0))) \ge m + \delta_0. \tag{39}$$

According to Corollary 2 and relation (33), by considering the localization $F_{(x_0,y_0,\delta_0)}$ instead of F, without any loss of generality, we can assume that

$$F(x) \subseteq \bar{B}(y_0,\delta_0) \quad \text{for all } x \in \bar{B}(x_0,\delta_0).$$

Note that for all $(x,y_1) \in \operatorname{gph} F$, $y_1^* \in Y^*$, one has

$$D^*G(x,y_1,y_1)((y_1^*,0)) = D^*F(x,y_1)(T(y_1^*)).$$

Hence, (30) implies obviously (35). Therefore, according to Lemma 3, there is $\kappa > 0$ such that for all $\delta \in (0,\delta_0)$, for any $(x,y) \in B((x_0,y_0),\delta_0)$, $z \in \operatorname{cone} \bar{B}(\bar{y},\delta)$, with $d(y,F(x)) \in (0,\delta_0)$ and $y - z \in F(x)$ (we may choose the same δ_0 as above), we can find $\gamma \in (0,\delta_0/2)$ such that

$$d((x',z'),\mathcal{V}(y,\delta)) \le \tau d(y,F(x')+z') \text{ for all } (x',z') \in B((x,z),\gamma),\ z' \in \operatorname{cone} \bar{B}(\bar{y},\delta), \tag{40}$$

where $\mathcal{V}(y,\delta)$ is defined by (34). Since F is pseudo-Lipschitz around x_0, there is $\delta_0 > 0$ (we can assume it is the same than the previous one) and $L > 0$ such that

$$F(x') \cap \bar{B}(y_0,\delta_0) \subseteq F(x) + L\|x - x'\|B_Y \quad \forall x,x' \in B((x_0,y_0),\delta_0). \tag{41}$$

Moreover, the function $d(\cdot,F(\cdot)) : X \times Y \to \mathbb{R}$ is Lipschitz around (x_0,y_0) as recalled above, say, on $B((x_0,y_0),\delta_0)$, with a Lipschitz modulus equal to L. By virtue of Theorem 4, it suffices to show that one has $|\nabla\varphi_\delta(\cdot,y)|(x) > m$ for any $(x,y) \in \big(B(x_0,\delta) \times B(y_0,\delta)\big)$ $x \in \operatorname{cl} V_y(\bar{y},\delta)$ with $d(y,F(x)) \in (0,\delta)$. Remind that, $V(\bar{y},\delta)$, $V_y(\bar{y},\delta)$, $\varphi_\delta(\cdot,y)$ are defined by (10), (11), respectively. Indeed, let $(x,y) \in B(x_0,\delta) \times B(y_0,\delta)$, $x \in \operatorname{cl} V(\bar{y},\delta)$ with $d(y,F(x)) \in (0,\delta)$ be given. Set $|\nabla\varphi_\delta(\cdot,y)|(x) := \alpha$. Since $d(y,F(\cdot))$ is Lipschitz on $B(x_0,\delta_0)$, then

$$\varphi_\delta(x',y) = d(y,F(x')) \quad \forall x' \in B(x_0,\delta_0) \cap \operatorname{cl} V_y(\bar{y},\delta).$$

By the definition of the strong slope, for each $\varepsilon \in (0,\delta)$, there is $\eta \in (0,\varepsilon)$ with

$$2\eta + \varepsilon < \min\{\gamma/2, \varepsilon d(y,F(x))\} \text{ and } 1 - (\alpha+\varepsilon+2)\eta > 0$$

such that

$$d(y,F(x')) \ge (1-\varepsilon)d(y,F(x)) \,\forall x' \in B(x,4\eta)$$

and that

$$d(y,F(x)) \le d(y,F(x')) + (m+\varepsilon)\|x' - x\| \quad \text{for all } x' \in \bar{B}(x,3\eta) \cap \operatorname{cl} V_y(\bar{y},\delta).$$

Take $u \in B(x,\eta^2/4) \cap V_y(\bar{y},\delta)$, $v \in F(u)$ such that $\|y - v\| \le d(y,F(x)) + \eta^2/4$. Then,

$$\|y - v\| \le d(y,F(x')) + (\alpha+\varepsilon)\|x' - x\| + \eta^2/4 \,\forall x' \in \bar{B}(u,2\eta) \cap \operatorname{cl} V_y(\bar{y},\delta).$$

Consequently,

$$\|y-v\| \le d(y,F(x')) + (\alpha+\varepsilon)\|x'-u\| + (\alpha+\varepsilon+1)\eta^2/4 \,\forall (x',z') \in \big(\bar{B}(u,2\eta)\times Y\big)\cap \mathcal{V}(\bar{y},\delta). \quad (42)$$

Let $z \in \operatorname{cone} B(\bar{y},\delta)$ such that $y-z \in F(u)$. Then,

$$\|z\| \ge d(y,F(u)) \ge (1-\varepsilon)d(y,F(x)) > 0.$$

Hence, by virtue of relation (41), there exists a neighborhood of u, say $B(u,2\eta)$ such that

$$\begin{aligned} y \in F(u)\cap B(y_0,\delta_0)+z &\subseteq F(u')+\|u-u'\|B_Y + z \\ &\subseteq F(u') + \operatorname{cone} B(\bar{y},\delta_0) \quad \text{for all} \quad u' \in B(u,2\eta). \end{aligned} \quad (43)$$

Since the function $d(y,F(\cdot))$ is Lipschitz on $B(x_0,\delta_0)$, then from relation (42), according to Lemma 4, it follows that there is $t>0$ such that

$$\|y-v\| \le d(y,F(x')) + (\alpha+\varepsilon)\|x'-u\| + td((x',z'),\mathcal{V}(\bar{y},\delta)) + (\alpha+\varepsilon+1)\eta^2/4 \,\forall (x',z') \in \bar{B}(u,\eta)\times\bar{B}(z,\eta).$$

Moreover, by (40), one obtains

$$\begin{aligned} \|y-v\| \le\ & d(y,F(x')) + (\alpha+\varepsilon)\|x'-u\| + t\kappa d(y,F(x')+z') + (\alpha+\varepsilon+1)\eta^2/4 \\ & \text{for all} (x',z') \in \bar{B}(u,\eta)\times\bar{B}(z,\eta),\ z' \in \operatorname{cone}\bar{B}(\bar{y},\delta). \end{aligned}$$

Thus, setting $G(x) := F(x)\times F(x)$, $x\in X$, we derive

$$\begin{aligned} \|y-v\| \le \|y-w_1\| + \ & (\alpha+\varepsilon)\|x'-u\| + t\kappa\|y-w_2-z'\| + \\ & + \delta_{\operatorname{gph} G}(x',w_1,w_2) + \delta_{\operatorname{cone}\bar{B}(\bar{y},\delta)}(z') + (\alpha+\varepsilon+1)\eta^2/4 \\ & \text{for all} (x',w_1,w_2,z') \in \bar{B}(u,\eta)\times Y\times Y\times\bar{B}(z,\eta),\ z'\in\operatorname{cone}\bar{B}(\bar{y},\delta). \end{aligned}$$

Next, applying the Ekeland variational principle to the function

$$\begin{aligned} (x',w_1,w_2,z') \mapsto \psi(x',w_1,w_2,z') := \ & \|y-w_1\| + (\alpha+\varepsilon)\|x'-u\| + t\kappa\|y-w_2-z'\| + \\ & + \delta_{\operatorname{gph} G}(x',w_1,w_2) + \delta_{\operatorname{cone}\bar{B}(\bar{y},\delta)}(z') \end{aligned}$$

on $\bar{B}(u,\eta)\times Y\times Y\times\bar{B}(z,\eta)$, we can select $(u_1,v_1,v_2,z_1) \in (u,v,y-z,z) + \frac{\eta}{4}B_{X\times Y\times Y\times Y}$ with $(u_1,v_1,v_2)\in \operatorname{gph} G$, $z_1 \in \operatorname{cone}\bar{B}(\bar{y},\delta)$ such that

$$\|y-v_1\| + \tau\kappa\|y-v_2-z_1\| \le \|y-v\| (\le d(y,F(x)) + \eta^2/4); \quad (44)$$

and

$$\psi(u_1,v_1,v_2,z_1) \le \psi(x',w_1,w_2,z') + (\alpha+\varepsilon+1)\eta\|(x',w_1,w_2,z') - (u_1,v_1,v_2,z_1)\|$$

for all $(x',w_1,w_2,z')\in\bar{B}(u,\eta)\times Y\times Y\times\bar{B}(z,\eta)$. Thus,

$$0 \in \partial(\psi + (\alpha+\varepsilon+1)\eta\|\cdot - (u_1,v_1,v_2,z_1)\|)(u_1,v_1,v_2,z_1).$$

According to the fuzzy sum rule, we can find

$$v_3 \in B(v_1,\eta);\ v_4 \in B(v_2,\eta);$$

$$(u_2,w_1,w_2) \in B(u_1,\eta)\times B(v_1,\eta\times B(v_2,\eta)\cap \operatorname{gph} G;\ z_2,\ z_3 \in B(z,\eta);$$

$$v_3^* \in \partial\|y-\cdot\|(v_3);\ (u_2^*,-w_1^*,-w_2^*) \in N(\operatorname{gph} G,(u_2,w_1,w_2));$$

$$(v_4^*,z_3^*) \in t\kappa\|y-\cdot-\cdot\|(v_4,z_3);\ \ z_2^* \in N(\operatorname{cone}\bar{B}(\bar{y},\delta),z_2),$$

satisfying

$$\begin{array}{l} \|v_3^*-w_1^*\|<(\alpha+\varepsilon+2)\eta;\ \ \|v_4^*-w_2^*\|<(m+\varepsilon+2)\eta;\\ \|z_3^*+z_2^*\|<(m+\varepsilon+2)\eta;\ \ \|u_2^*\|\le\alpha+\varepsilon+(\alpha+\varepsilon+2)\eta. \end{array} \tag{45}$$

Since $v_3^*\in\partial\|y-\cdot\|(v_3)$ (note that $\|y-v_3\|\ge\|y-v\|-\|v_3-v\|\ge d(y,F(x))-\varepsilon-2\eta>0$), then $\|v_3^*\|=1$ and $\langle v_3^*,v_3-y\rangle=\|y-v_3\|$. Thus, $\|w_1^*\|\le 1+(\alpha+\varepsilon+2)\eta$, and the first relation of (45) follows that

$$\langle w_1^*,w_1-y\rangle\ge\langle v_3^*,v_3-y\rangle-(\alpha+\varepsilon+2)\eta\|v_3-y\|-2\eta=(1-(\alpha+\varepsilon+2)\eta)\|v_3-y\|-2\eta.$$

As $\eta\le\varepsilon d(y,F(x))\le\varepsilon d(y,F(u))/(1-\varepsilon)$ for all $u\in B(x,\eta)$, one obtains

$$\langle w_1^*,w_1-y\rangle\ge(1-\varepsilon_1)\|w_1-y\|, \tag{46}$$

where

$$\varepsilon_1:=(\alpha+\varepsilon+2)\eta-2(\alpha+\varepsilon+2)\eta\varepsilon(1-\varepsilon)^{-1}-2\varepsilon(1-\varepsilon)^{-1}.$$

On the other hand, since $F(u_2)$ is convex and $w_1^*\in -N(F(u_2),w_1)$, then by relation (63), there is $w_1'\in F(u_2)$ such that $y-w_1'\in\operatorname{cone}B(\bar y,\delta_0)$. Therefore

$$\langle w_1^*,y-w_1'\rangle=\langle w_1^*,y-w_1'\rangle+\langle w_1^*,w_1-w_1'\rangle<0.$$

Consequently,

$$\langle w_1^*,\bar y\rangle\le\delta_0\|w_1^*\|/2\le\delta_0(1+(\alpha+\varepsilon+2)\eta)/2. \tag{47}$$

Next, since $(v_4^*,z_3^*)\in t\kappa\|y-\cdot-\cdot\|(v_4,z_3)$, then $v_4^*=z_3^*$ and $\|z_3^*\|\le t\kappa$. Hence from (45), one has

$$\|w_2^*-z_2^*\|\le\|w_2^*-v_4^*\|+\|z_2^*-z_3^*\|\le 2(\alpha+\varepsilon+2)\eta.$$

As $z_2^*\in N(\operatorname{cone}\bar B(\bar y,\delta),z_2)$, with $z_2\ne 0$, then $\langle z_2^*,z_2\rangle=0$. Therefore,

$$|\langle w_2^*,z_2\rangle|\le 2(\alpha+\varepsilon+2)\eta\|z_2\|<.$$

As $z_2\in\operatorname{cone}\bar B(\bar y,\delta)$, one obtains

$$|\langle w_2^*,\bar y\rangle|\le 2(\alpha+\varepsilon+2)\eta+\delta. \tag{48}$$

Moreover,

$$|\langle w_2^*,w_2-y\rangle|\le|\langle z_2^*,w_2-y-z_2\rangle|+|\langle z_2^*-w_2^*,w_2-y\rangle|\le\varepsilon_2\|w_1-y\|,$$

where

$$\varepsilon_2=\big((t\kappa+2(\alpha+\varepsilon+2)+2(\alpha+\varepsilon+2)(\|y_0\|+2\delta_0+2\eta)\big)\varepsilon(1-\varepsilon)^{-1}.$$

The second inequality of the preceding relation follows from

$$\|z_2^*\|\le\|z_3^*\|+\|z_2^*-z_3^*\|\le t\kappa+(\alpha+\varepsilon+2),$$

and

$$\|w_2-y\|\le\|w_2-v_2\|+\|v_2-(y-z)\|+\|z\|<2\eta+2\delta_0+\|y_0\|.$$

Hence, by using the convexity of $F(u_2)$, and $w_2^*\in -N(F(u_2),w_2)$

$$\langle w_2^*,w_1-y\rangle=\langle w_2^*,w_1-w_2\rangle+\langle w_2^*,w_2-y\rangle\ge-\varepsilon_2\|w_1-y\|. \tag{49}$$

From relations (51) and (49), one derives that

$$\langle w_1^*+w_2^*,w_1-y\rangle\ge(1-\varepsilon_1-\varepsilon_2)\|w_1-y\|. \tag{50}$$

Consequently, $\|w_1^*+w_2^*\| \geq 1-\varepsilon_1-\varepsilon_2$.

Set

$$y_1^* = \frac{w_1^*}{\|w_1^*+w_2^*\|};\ y_2^* = \frac{w_2^*}{\|w_1^*+w_2^*\|} \text{ and } x^* = \frac{u_2^*}{\|w_1^*+w_2^*\|}.$$

From relations (47), (48), (50), one has

$$\langle y_1^*, \bar{y}\rangle \leq \delta_0\|y_1^*\| \leq \frac{\delta_0(1+(\alpha+\varepsilon+2)\eta)}{2(1-\varepsilon_1-\varepsilon_2)};$$

$$\langle y_2^*, \bar{y}\rangle \leq \frac{2(\alpha+\varepsilon+2)\eta+\delta}{1-\varepsilon_1-\varepsilon_2};$$

$$x^* \in D^*G(u_2,w_1,w_2)(y_1^*,y_2^*);\ \|y_1^*+y_2^*\|=1.$$

As $\varepsilon_1,\varepsilon_2$ go to 0 as $\varepsilon,\eta\to 0$, then $y_1^* \in S_{Y^*}(\bar{y},\delta)$ and $y_2^* \in C_{Y^*}(\bar{y},\delta)$. Thus $(y_1^*,y_2^*) \in T(\bar{y},\delta)$. As $(u_2,w_1,w_2) \in B((x_0,y_0,y_0),\delta_0)$, according to (45), one obtains

$$\alpha+\delta_0 \leq \|x^*\| = \|u_2^*\|/\|w_1^*+w_2^*\| \leq \frac{\alpha+\varepsilon+(\alpha+\varepsilon+2)\eta}{1-\varepsilon_1-\varepsilon_2}. \tag{51}$$

As $\varepsilon,\eta,\varepsilon_1,\varepsilon_2$ are arbitrary small, we obtain $m+\delta_0 \leq \alpha$ and the proof is complete. □

Condition (30) is also a necessary condition for directional metric regularity in Banach spaces as showed in the next proposition.

PROPOSITION 2. *Let X,Y be Banach spaces and let $F: X \rightrightarrows Y$ be a closed multifunction, $(x_0,y_0) \in \operatorname{gph} F$ and $\bar{y} \in Y$. Suppose that F has convex values for x near x_0. If F is metrically regular in the direction $\bar{y} \in Y$ at (x_0,y_0), then*

$$\liminf_{\substack{(x,y_1,y_2)\overset{G}{\to}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*G(x,y_1,y_2)(T(\bar{y},\delta))) > 0..$$

Proof. Assume that F is metrically regular in the direction $\bar{y} \in Y$, i.e., there exist $\tau>0,\delta>0,\varepsilon>0$ such that

$$d(x,F^{-1}(y)) \leq \tau d(y,F(x)) \quad \text{for all } (x,y) \in B(x_0,\varepsilon)\times B(y_0,\varepsilon);\ y \in F(x)+\operatorname{cone} B(\bar{y},\delta). \tag{52}$$

For $\gamma \in (0,\delta)$, let $(x,y_1,y_2) \in \operatorname{gph} G \cap B(x_0,\varepsilon/2)\times B(y_0,\varepsilon/2)\times B(y_0,\varepsilon/2)$; $(y_1^*,y_2^*) \in T(\bar{y},\gamma)$ and $x^* \in D^*G(x,y_1,y_2)(y_1^*,y_2)$. For any $\alpha \in (0,1)$, there exists $\beta \in (0,\varepsilon/2)$ such that

$$\langle x^*, u-x\rangle - \langle y_1^*, v_1-y_1\rangle + \langle y_2^*, v_2-y_2\rangle \leq \varepsilon(\|u-x\|+\|v_1-y_1\|+\|v_2-y_2\|), \tag{53}$$

for all $(u,v_1,v_2) \in \operatorname{gph} G \cap B((x,y_1,y_2),\beta)$.

For $\delta_1 \in (0,\delta)$, take $w \in B_Y$ such that $\langle y_2^*, \bar{y}+\delta w\rangle \leq \gamma-\delta_1$. Since (52), for all sufficiently small $t>0$, we can find $u \in F^{-1}(y_2+t(\bar{y}+\delta w))$ such that

$$\|x-u\| \leq (1+\alpha)\tau d(y_2+t(\bar{y}+\delta w),F(x)) \leq (1+\alpha)t\|\bar{y}+\delta u\| < \beta.$$

Since $y_1^* \in -N(F(x),y_1)$ and $F(x)$ is convex, then $\langle y_1^*, y_2-y_1\rangle \geq 0$. Therefore, by taking $v_1=v_2=y_2+t(\bar{y}+\delta w)$ into account in (53), one obtains

$$\begin{aligned}(1+\alpha)\tau t\|\bar{y}+\delta u\|\|x^*\| \geq \langle x^*, x-u\rangle &\geq \langle y_1^*+y_2^*, v-y_2\rangle + \langle y_1^*, y_2-y_1\rangle\\ &\geq t(\delta_1-\gamma) - \alpha t\|\bar{y}+\delta u\|((1+\alpha)\tau+1).\end{aligned}$$

As $\alpha > 0$, $\delta_1 \in (0, \delta)$ are arbitrary, one has

$$\|x^*\| \geq \frac{\delta - \gamma}{\tau\|\bar{y} + \delta u\|} \geq \frac{\delta - \gamma}{\tau(\|\bar{y}\| + \delta)}.$$

Thus,

$$\liminf_{\substack{(x,y_1,y_2)\overset{G}{\to}(x_0,y_0,y_0)\\ \gamma\to 0^+}} d_*(0, D^*G(x,y_1,y_2)(T(\bar{y},\gamma))) \geq \frac{\delta}{\tau(\|\bar{y}\| + \delta)} > 0.$$

The proof is complete. □

Combining this proposition and Theorem 5, one has

THEOREM 6. *Let X, Y be Asplund spaces. Suppose $F : X \rightrightarrows Y$ be a closed multifunction and $(x_0, y_0) \in \operatorname{gph} F$ such that F has convex values around x_0. Suppose further that F is pseudo-Lipschitz around (x_0, y_0). Then, F is metrically regular in direction $\bar{y} \in Y$ at (x_0, y_0) if and only if*

$$\liminf_{\substack{(x,y_1,y_2)\overset{G}{\to}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*G(x,y_1,y_2)(T(\bar{y},\delta))) > 0.$$

Recall that the *Mordukhovich limiting coderative* of F denoted by $D^*_M F(x,y) : Y^* \rightrightarrows X^*$ is defined by

$$D^*_M F(x,y)(y^*) := \liminf_{\substack{u,v)\overset{F}{\to}(x,y)\\ v^*\overset{w^*}{\to}y^*}} D^*F(u,v)(v^*) = \left\{ x^* \in X^* : \begin{array}{l} (x_n, y_n) \overset{F}{\to} (x,y) \\ x_n^* \in D^*F(x_n,y_n)(y_n^*) \\ y_n^* \overset{w^*}{\to} y^*,\ x_n^* \overset{w^*}{\to} x^* \end{array} \right\}. \tag{54}$$

Let us now recall the notion of *partial sequential normal compactness* (PSNC, in short, see [49, page 76]). A multifunction $F : X \rightrightarrows Y$ is *partially sequentially normally compact* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, iff, for any sequences $\{(x_k, y_k, x_k^\star, y_k^\star)\}_{n\in\mathbb{N}} \subset \operatorname{gph} F \times X^\star \times Y^\star$ satisfying

$$(x_k, y_k) \to (\bar{x}, \bar{y}), x_k^\star \in D^\star_M F(x_k, y_k)(y_k^\star), x_k^\star \overset{w^\star}{\to} 0, \|y_k^\star\| \to 0,$$

one has $\|x_k^\star\| \to 0$ as $k \to \infty$.

REMARK 1. Condition (PSNC) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is satisfied if X is finite dimensional, or F is pseudo-Lipschitz around that point.

The next corollary that follows directly from the preceding theorem, gives a point-based condition for directional metric regularity.

COROLLARY 3. *Under the assumptions of Theorem 6, suppose further that G^{-1} is PSNC at (x_0, y_0, y_0). Then F is metrically regular in the direction $\bar{y} \in Y$ at (x_0, y_0) if and only if*

$$d_*(0, D^*_M G(x_0, y_0, y_0)(T(\bar{y}, 0))) > 0.$$

With an analogous proof, we obtain the following parametric version of Theorem 5 .

THEOREM 7. *Let X, Y be Asplund spaces and P be a topological space. Let $F : X \times P \rightrightarrows Y$ be a set-valued mapping and let $((x_0, p_0), y_0) \in \operatorname{gph} F$. Suppose the following conditions are satisfied:*
(a) For any p near $\bar{p}$, the set-valued mapping $x \rightrightarrows F(x,p)$ is a closed multifunction;
(b) For (x,p) near (x_0, p_0), $F(x,p)$ is convex;

(c) $F(\cdot,p)$ is pseudo-Lipschitz uniformly in p around (x_0,p_0).

Then, for $\bar{y}\in Y$, F is directionally metrically regular in direction $\bar{y}$ uniformly in p at (x_0,p_0,y_0) if and only if

$$\liminf_{\substack{(x,p,y_1,y_2)\overset{G}{\to}(x_0,p_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0,D^*G_p(x,y_1,y_2)(T(\bar{y},\delta)))>0, \tag{55}$$

where,

$$G(x,p)=G_p(x):=F(x,p)\times F(x,p),\quad (x,p)\in X\times P,$$

We next consider a special case of $F(x,p):=f(x,p)-K:=f_p(x)-K$, here, $K\subseteq Y$ is a nonempty closed convex subset. $f:X\times P\to Y$ is a (locally) continuous mapping around a given point $(x_0,p_0)\in X\times P$ with $f(x_0,p_0)\in K$, and $f(\cdot,p)$ is Lipschitz uniformly in p near (x_0,p_0). Obviously, for this case, assumptions $(a),(b),(c)$ of Theorem 7 are satisfied as well. Moreover, by setting $g_p:=(f_p,f_p):X\to Y\times Y$, one has

$$D^*G_p(x,y_1,y_2)(y^*)=\begin{cases} D^*g_p(x)(y^*) & \text{if } f(x,p)-y_i\in K,\ y_i^*\in N(K,f(x,p)-y_i),\ i=1,2\\ \emptyset & \text{otherwise,}\end{cases}$$

where, we use the usual notations: $f_p(x):=f(x,p)$; $D^*f_p(x)(y^*):=D^*f_p(x,f(x,p))(y^*)$. Hence, Theorem 7 yields the following corollary.

Corollary 4. *Let X,Y be Asplund spaces and P be a topological space. Let $K\subseteq Y$ be a nonempty closed convex subset and let $f:X\times P\to Y$ be a locally continuous mapping around $(x_0,p_0)\in X\times P$ with $k_0:=f(x_0,p_0)\in Q$. Suppose further that $f(\cdot,p)$ is Lipschitz uniformly in p near (x_0,p_0). If for $\bar{y}\in Y$,*

$$\liminf_{\substack{(x,p,k_1,k_2)\to(x_0,p_0,k_0,k_0)\\ \delta\downarrow 0^+}} d_*(0,D^*f_p(x)(T(\bar{y},\delta))\cap(N(K,k_1)\times N(K,k_2)))>m>0, \tag{56}$$

then the mapping $F(x,p):=f(x,p)-K$, $(x,p)\in X\times P$ is directionally metrically regular in direction $\bar{y}$ uniformly in p, with modulus $\tau=m^{-1}$ at (x_0,p_0), i.e., there exist $\varepsilon>0$, $\delta>0$ and a neighborhood W of $p_0$such that

$$d(x,S(y,p))\le\tau d(f(x,p)-y,K)\quad \textit{for all } (x,p,0)\in B(x_0,\varepsilon)\times W\times B(0,\varepsilon),$$

with $y\in f(x,p)-K+\operatorname{cone}B(\bar{y},\delta)$.

In particular, one has

$$d(x,S(p))\le\tau d(f(x,p),K)\quad \textit{for all } (x,p)\in B(x_0,\varepsilon)\times W,$$

with $f(x,p)\in K-\operatorname{cone}B(\bar{y},\delta)$. Here,

$$S(y,p)=\{x\in X:\ f(x,p)-y\in K\},\quad S(p):=\{x\in X:\ f(x,p)\in K\}.$$

Remark 2. Note that if K is *sequentially normally compact at $\bar{k}$*, i.e., for all sequences $(k_n)_{n\in\mathbb{N}}\subseteq K$, $(k_n^*)_{n\in\mathbb{N}}$ with $k_n^*\in N(K,k_n)$,

$$k_n\to\bar{k}\quad k_n^*\overset{w^*}{\to}0\quad\Longleftrightarrow\ \|k_n^*\|\to 0,$$

and P is a **metric space**, then instead of (56), the following point-based condition

$$d_*(0,D^*_{\lim}g_{p_0}(x_0)(T(\bar{y},0)\cap(N(K,k_0)\times N(K,k_0)))>0, \tag{57}$$

is also a sufficient condition for directional metric regularity at $\bar{y}$, uniformly in p of $F(x,p) := f(x,p) - K$ at (x_0,p_0). Here, $D^*_{\lim} g_{p_0}(x_0)$ denotes the sequential limiting subdifferential of $D^* g_p(x)$:

$$D^*_{\lim} g_{p_0}(x_0)(y_1^*, y_2^*) := \liminf_{\substack{(x,p)\to(x_0,p_0)\\ (z_1^*,z_2^*)\overset{w^*}{\to}(y_1^*,y_2^*)}} D^* g_p(x)(z_1^*, z_2^*),\ \ y_1^*, y_2^* \in Y^*.$$

COROLLARY 5. *With the assumptions of Corollary 4, suppose further that f is Fréchet differential with respect to x near (x_0,p_0), and its derivative with respect to x is continuous at (x_0,p_0). Then, the mapping $F(x,p) := f(x,p) - K$, $(x,p) \in X \times P$ is directionally metrically regular in direction $\bar{y}$ uniformly in p if and only if*

$$\liminf_{\substack{(k_1,k_2)\to k_0\\ \delta\downarrow 0^+}} d_*(0, g_x'^*(x_0,p_0)(T(\bar{y},\delta) \cap (N(K,k_1) \times N(K,k_2))) > m > 0. \tag{58}$$

Here, $f_x'^*(x,p)$ stands for the adjoint operator of $f_x'(x,p)$ Moreover, if K is normally sequentially compact, then (58) is equivalent to

$$d_*(0, g_x'^*(x_0,p_0)(T(\bar{y},0) \cap (N(K,k_0) \times N(K,k_0))) > 0. \tag{59}$$

Proof. For the sufficiently part, suppose that

$$\liminf_{\substack{(k_1,k_2)\to(k_0,k_0)\\ \delta\downarrow 0^+}} d_*(0, g_x'^*(x_0,p_0)(T(\bar{y},\delta)) \cap N(K,k_1) \times N(K,k_2)) > m > 0.$$

Since f_x' is continuous at (x_0,p_0), for any $\varepsilon > 0$, there exist $\delta > 0$ and a neighborhood W of p_0 such that

$$\|g_x'(x,p) - g_x'(x_0,p_0)\| < \varepsilon \quad \text{for all } (x,p) \in B(x_0,\varepsilon) \times W.$$

Therefore, for all $\delta > 0$,

$$\|g_x'(x,p)(y_1^*,y_2^*) - g_x'(x_0,p_0)(y_1^*,y_2)\| < \varepsilon,$$
$$\text{for all } (x,p) \in B(x_0,\varepsilon) \times W,\ k_1,k_2 \in B(k_0,\varepsilon),\ (y_1^*,y_2^*) \in T(\bar{y},\delta) \cap (N(K,k_1) \times N(K,k_2)).$$

Consequently,

$$\begin{aligned}&\liminf_{\substack{(x,p,k_1,k_2)\to(x_0,p_0,k_0,k_0)\\ \delta\downarrow 0^+}} d_*(0, g_x'^*(x,p)(T(\bar{y},\delta) \cap (N(K,k_1) \times N(K,k_2)))\\ &= \liminf_{\substack{k\to k_0\\ \delta\downarrow 0^+}} d_*(0, g_x'^*(x_0,p_0)(T(\bar{y},\delta) \cap (N(K,k_1) \times N(K,k_2))) > m > 0\end{aligned}.$$

The conclusion follows from Corollary 4. The proof of the necessary part is analogous to the one of Proposition 2. The equivalence between (58) and (59) follows from Remark 2. □

Corollary 5 subsumes the following result, established by Arutyunov, Avakov and Izmailov in [1].

COROLLARY 6. *([1], Theorem 2.3) With the assumptions of Corollary 5, if*

$$\operatorname{cone}\{\bar{y}\} \cap \operatorname{Int}(f(x_0,p_0) + \operatorname{Im} f'(x_0,p_0) - K) \neq \emptyset, \tag{60}$$

then the mapping $F(x,p) := f(x,p) - K$, $(x,p) \in X \times P$, is directionally metrically regular in direction $\bar{y}$ uniformly in p at (x_0,p_0).

Proof. It suffices to show that (60) implies (58). Indeed, assume (60) holds, and assume to contrary that (58) fails to be hold. Then, there exist sequences $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n \downarrow 0$; $(k_n^1)_{n\in\mathbb{N}}, (k_n^2)_{n\in\mathbb{N}} \subseteq K$, $k_n^i \to k_0 = f(x_0,p_0)$ $(i=1,2)$, $(y_n^{1*})_{n\in\mathbb{N}}, (y_n^{2*})_{n\in\mathbb{N}}$ with $(y_n^{1*}, y_n^{2*}) \in T(\bar{y},\delta_n) \cap [N(K,k_n^1) \times N(K,k_n^2)]$ and $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ such that

$$x_n^* = (y_n^{1*} + y_n^{2*}) \circ f'_x(x_0,p_0);\ \ \|x_n^*\| \to 0.$$

By (60), there exist $\lambda \geq 0$, such that

$$0 \in \operatorname{Int}(f(x_0,p_0) + \operatorname{Im} f'(x_0,p_0) - K) - \lambda\bar{y}. \tag{61}$$

Set $C_{nm} := nf'(x_0,p_0)(B(0,1)) + m(f(x_0,p_0) - K - \lambda\bar{y})$, $n \in \mathbb{N}$. Then, $\bigcup_{n,m=1}^{\infty} C_{nm} = Y$. According to the Baire theorem, at least one of the $\operatorname{cl} C'_{nm}s$ has a nonempty interior. Therefore, consider $y \in Y$, $\alpha > 0$ and $\varepsilon > 0$ such that

$$B(y,\varepsilon) \subseteq \operatorname{cl}(f'(x_0,p_0)(B(0,\alpha)) + f(x_0,p_0) - K - \lambda\bar{y}).$$

On the other hand, from (61), there are $t, r > 0$ such that

$$-ty \in f'(x_0,p_0)(B(0,\alpha)) + r(f(x_0,p_0) - K - \lambda\bar{y}).$$

Hence,

$$B(0,t\varepsilon) \subseteq -ty + B(ty,t\varepsilon) \subseteq \operatorname{cl}((1+t)f'(x_0,p_0)(B(0,\alpha)) + (t+r)(f(x_0,p_0) - K - \lambda\bar{y}).$$

Equivalently, for $\gamma := t\varepsilon/(t+r)$, $\beta := (1+t)\alpha/(t+r)$,

$$B(\lambda\bar{y},\gamma) \subseteq \operatorname{cl}(f'(x_0,p_0)(B(0,\beta)) + f(x_0,p_0) - K). \tag{62}$$

For each n, let $u_n \in B_X$ be chosen such that $\langle y_n^{1*} + y_n^{2*}, u_n\rangle < -1/2$. Since $(y_n^{1*}, y_n^{2*}) \in T(\bar{y},\delta_n)$, then

$$\limsup_{n\to\infty} \langle y_n^{1*} + y_n^{2*}, \lambda\bar{y} + \gamma u_n\rangle \leq -\gamma/2.$$

On the other hand, by (62), for each n, we can find $x_n \in B(0,\beta)$, $z_n \in K$ such that

$$\|\lambda\bar{y} + \gamma u_n - (k_0 + f'_x(x_0,p_0)(x_n) - z_n)\| < 1/n.$$

Since $y_n^{i*} \in N(K,k_n^i)$ $(i=1,2)$; $\|x_n^*\| \to 0$; (x_n) is bounded, and $k_n^i \to k_0$, one has

$$\limsup_{n\to\infty} \langle y_n^{1*} + y_n^{2*}, \lambda\bar{y} + \gamma u_n\rangle = \limsup_{n\to\infty} (\langle y_n^{1*} + y_n^{2*}, k_0 - z_n\rangle + \langle x_n^*, x_n\rangle) \geq 0,$$

a contradiction. □

From Theorem 5, we can derive directly the following result due to Ioffe ([38]) on directional metric regularity of a closed convex multifunction for the case in which the convex multifunction under consideration is assumed to be pseudo-Lipschitz.

Corollary 7. *([38], Proposition 15) Let X, Y be Banach spaces and $F : X \rightrightarrows Y$ be a closed convex multifunction and let $(x_0,y_0) \in \operatorname{gph} F$ and $\bar{y} \in Y$. Suppose that $x_0 \in \operatorname{Int} F^{-1}(Y)$. Then F is directionally metrically regular in direction $\bar{y}$ at (x_0,y_0) if and only if*

$$\operatorname{cone}\{\bar{y}\} \cap \operatorname{Int}(F(X) - y_0) \neq \emptyset. \tag{63}$$

Proof. For the sufficiency part, under the assumption $x_0 \in \operatorname{Int} F^{-1}(Y)$, according to the Robinson-Ursescu Theorem ([57], [60]), then F is pseudo-Lipschitz near x_0. By virtue of Theorem 5, we only need to show that

$$\liminf_{\substack{(x,y_1,y_2)\overset{G}{\to}(x_0,y_0,y_0)\\ \delta\downarrow 0^+}} d_*(0, D^*G(x,y_1,y_2)(T(\bar{y},\delta))) > 0. \tag{64}$$

By (63), there is some $\lambda \geq 0$ such that $0 \in \operatorname{Int}(F(X) - y_0 - \lambda\bar{y})$. Then, by the convexity of the multifunction F,

$$\bigcup_{n,m=1}^{\infty} (F(B(x_0,n)) - m(y_0 - \lambda\bar{y})) = Y.$$

Thanks to the Baire Category Theorem, similarly to the proof of Corollary 6, we can find $\gamma > 0$, $\beta > 0$ such that

$$B(\lambda\bar{y},\gamma) \subseteq \operatorname{cl}(F(B(x_0,\beta)) - y_0). \tag{65}$$

Let $(x_n, y_n^1, y_n^2) \in \operatorname{gph} G$; $(x_n^*)_{n\in\mathbb{N}}$, $(y_n^{1*})_{n\in\mathbb{N}}$, $(y_n^{2*})_{n\in\mathbb{N}}$, $(\delta_n)_{n\in\mathbb{N}}$ such that

$$(x_n, y_n^1, y_n^2) \to (x_0, y_0, y_0);\ \delta_n \downarrow 0^+;\ x_n^* \in D^*F(x_n, y_n^1, y_n^2)(y_n^{1*}, y_n^{2*});\ (y_n^{1*}, y_n^{2*}) \in T(\bar{y}, \delta_n).$$

For each n, take $u_n \in B_Y$ such that $\langle y_n^{1*} + y_n^{2*}, u_n\rangle < -1 + \delta_n$. Then,

$$\langle y_n^{1*} + y_n^{2*}, \lambda\bar{y} + \gamma u_n\rangle < \delta_n - (1-\delta_n)\gamma.$$

On the other hand, by (65), we can select $z_n \in B(x_0,\beta)$; $v_n \in F(z_n)$ such that

$$\|\lambda\bar{y} + \gamma u_n - (v_n - y_0)\| < \delta_n.$$

Therefore,

$$\langle y_n^{1*} + y_n^{2*}, v_n - y_0\rangle < \langle y_n^{1*} + y_n^{2*}, \lambda\bar{y} + \gamma u_n\rangle + \delta_n < 2\delta_n - (1-\delta_n)\gamma.$$

Since $(x_n^*, -y_n^{1*}, -y_n^{2*}) \in N(\operatorname{gph} G, (x_n, y_n^1, y_n^2))$, then

$$\liminf_{n\to\infty}(\langle x_n^*, z_n - x_n\rangle - \langle y_n^{1*} + y_n^{2*}, v_n - y_0\rangle) \leq 0.$$

Consequently,

$$\liminf_{n\to\infty} \|x_n^*\|\|z_n - x_n\| \geq \liminf_{n\to\infty} \langle y_n^{1*} + y_n^{2*}, v_n - y_0\rangle \geq (1-\delta_n)\gamma - 2\delta_n.$$

By letting $n \to \infty$, one obtains $\liminf_{n\to\infty} \|x_n^*\| \geq \gamma/\alpha$, which shows (64).

Suppose now that there exist $\tau > 0$, $\delta > 0$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \text{ for all } (x,y) \in B(x_0,\delta) \times B(y_0,\delta),\ y \in F(x) + \operatorname{cone} B(\bar{y},\delta).$$

In particular, one has

$$F^{-1}(y) \neq \emptyset \quad \text{for all } y \in B(y_0,\varepsilon) \cap (y_0 + \operatorname{cone} B(\bar{y},\delta)).$$

Let $\varepsilon > 0$ $\alpha > 0$ be sufficiently small such that $\varepsilon < \delta\lambda$ and $\|\lambda\bar{y}\| + \varepsilon < \delta$. Then, for all $u \in B_Y$

$$z := y_0 + \lambda\bar{y} + \varepsilon u \in B(y_0,\delta) \cap (y_0 + \operatorname{cone} B(\bar{y},\delta)).$$

Hence, $F^{-1}(z) \neq \emptyset$. It follows that $B(\lambda\bar{y},\varepsilon) \subseteq F(X) - y_0$, and the proof is complete. □

4. Robustness of directional metric regularity The characterizations of directional metric regularity established in Theorem 4, enable us to derive the following result on the stability of directional metric regularity under perturbation. This result has been first obtained in [1] under the inner semicontinuity assumption. Then, when the image space Y is a Banach space, Ioffe in [38] has extended this stability result (without the inner semicontinuityassumption) with estimates sharper than the one in [1]. Here, based on the mentioned characterizations, we prove this result for which, the completeness of Y is not necessary.

THEOREM 8. *Let X be a complete metric space and Y be a normed space. Let $F : X \rightrightarrows Y$ be a closed multifunction and $(x_0, y_0) \in \operatorname{gph} F$. Suppose that F is metrically regular with a modulus $\tau > 0$ in the direction $\bar{y} \in Y$, i.e., there exist $\varepsilon > 0$, $\delta > 0$ such that*

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \text{ for all } (x, y) \in B((x_0, y_0), \varepsilon) \cap V(\bar{y}, \delta) \text{ with } d(y, F(x)) < \varepsilon. \tag{66}$$

Let a mapping $g : X \to Y$ be locally Lipschitz around x_0 with a Lipschitz constant $L > 0$. Then $F + g$ is metrically regular in the direction $\bar{y}$ at $(x_0, y_0 + g(x_0))$ with

$$\operatorname{reg}_{\bar{y}}(F + g)(x_0, y_0 + g(x_0)) \leq \left(\frac{1 - \gamma}{\tau(1 + \gamma)} - L \right)^{-1},$$

provided

$$\alpha \in (0, 1),\ \gamma := \frac{\alpha \|\bar{y}\|}{\|\bar{y}\| + \delta(1 - \alpha)};\ L < \frac{\delta(1 - \alpha)\alpha}{\tau((1 + \alpha)\|\bar{y}\| + \delta(1 - \alpha))}.$$

Proof. Let $\varepsilon, \delta, \gamma, \alpha, L$ as in Theorem 8. Let $g : X \to Y$ be Lipschitz with constant L on $B(x_0, \varepsilon)$. To simplify the notations, denote by

$$V_F(\delta) := \{(x, y) :\ y \in F(x) + \operatorname{cone} B(\bar{y}, \delta)\};\ V_{(F+g)}(\delta) := \{(x, y) :\ y \in F(x) + g(x) + \operatorname{cone} B(\bar{y}, \delta)\};$$

$$V_{F,y}(\delta) := \{x \in X :\ y \in F(x) + \operatorname{cone} B(\bar{y}, \delta)\};\ V_{F+g,y}(\delta) := \{x \in X :\ y \in F(x) + g(x) + \operatorname{cone} B(\bar{y}, \delta)\},$$

and $\varphi_{V_F}(x, y)$, (resp. $\varphi_{V_{F+g}}(x, y)$) the lower semicontinuous envelope relative to V_F (resp. V_{F+g}) of F (resp. $F + g$). Obviously,

$$\varphi_{V_{F+g}}(x, y) = \varphi_{V_F}(x, y - g(x)),\ \text{for all } (x, y) \in X \times Y.$$

According to Theorem 4 (ii), it suffices to prove that

$$|\Gamma \varphi_{V_{F+g}}(\cdot, y)|(x) \geq \left(\frac{1 - \gamma}{\tau(1 + \gamma)} - L \right), \tag{67}$$

whenever

$$(x, y) \in B((x_0, y_0 + g(x_0)), \eta) \text{ satisfies } x \in \operatorname{cl} V_{F+g,y}(\rho);\ d(y, F(x) + g(x)) < \eta, \tag{68}$$

where $\rho := \delta(1 - \alpha)$ and $\eta = \min\{\varepsilon/(L + 2), \varepsilon/(8\tau)\}$.

Let x, y be as in (68). Then select sequences $(\lambda_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}, (x_n))_{n\in\mathbb{N}}$ satisfying $\lambda_n > 0, z_n \in B_X, (x_n) \to x$ and such that

$$y - g(x_n) \in F(x_n) + \lambda_n(\bar{y} + \rho z_n), \quad \lim_{n\to\infty} d(y, F(x_n) + g(x_n)) = \varphi_{V_{F+g}(\rho)}(x, y). \tag{69}$$

Note that since (x_n) tends to x and $x \in B(x_0, \eta)$, then for n large we have

$$d(y, F(x_n) + g(x_n)) < \eta. \tag{70}$$

Setting

$$t_n := \alpha\varphi_{V_{F+g}(\rho)}(x_n, y)/(\|\bar{y}\| + \rho), \tag{71}$$

we observe that

$$t_n\|\bar{y}\| < \varphi_{V_{F+g}(\rho)}(x_n, y) < \eta, \tag{72}$$

and

$$d(y, F(x_n) + g(x_n)) \le \lambda_n\|\bar{y} + \rho z_n\| \le \lambda_n(\|\bar{y}\| + \rho) \text{ for some } z_n \text{ with } |z_n\| = 1.$$

This yields,

$$t_n(\|\bar{y}\| + \rho)/\alpha \le \varphi_{V_{F+g}(\rho)}(x_n, y) \le d(y, F(x_n) + g(x_n)) \le \lambda_n(\|\bar{y}\| + \rho).$$

Consequently,

$$t_n/\lambda_n \le \alpha. \tag{73}$$

Observe also that

$$\begin{aligned}\lambda_n(\bar{y} + \rho z_n) - t_n\bar{y} &\\ &= (\lambda_n - t_n)\bar{y} + \lambda_n\rho z_n\\ &= (\lambda_n - t_n)(\bar{y} + \tfrac{\lambda_n\rho}{\lambda_n - t_n} z_n).\end{aligned}$$

According to (73)

$$\frac{\lambda_n - t_n}{\lambda_n} = 1 - \frac{t_n}{\lambda_n} \ge 1 - \alpha$$

and therefore

$$\frac{\lambda_n}{\lambda_n - t_n}\rho \le \frac{\rho}{1-\alpha} = \delta.$$

Hence, $\lambda_n(\bar{y} + \rho z_n) - t_n\bar{y} \in \operatorname{cone} B(\bar{y}, \varepsilon)$ and thanks to (69), this yields

$$y - g(x_n) - t_n\bar{y} \in F(x_n) + \operatorname{cone} B(\bar{y}, \delta). \tag{74}$$

Moreover,

$$\|y - g(x_n) - t_n\bar{y} - y_0\| \le \|y - g(x_0) - y_0\| + \|g(x_n) - g(x_0)\| + t_n\|\bar{y}\| < (2 + L)\eta = \varepsilon; \tag{75}$$

and combining 70) and (72) we also have

$$d(y - g(x_n) - t_n\bar{y}, F(x_n)) \le d(y - g(x_n), F(x_n)) + t_n\|\bar{y}\| < 2\eta < \frac{2\varepsilon}{L+2} < \varepsilon. \tag{76}$$

From (75) and (76) we deduce that

- $y - g(x_n) - t_n\bar{y} \in B(y_0, \varepsilon)$;
- $d(y - g(x_n) - t_n\bar{y}, F(x_n)) < \varepsilon$;
- $(x_n, y - g(x_n) - t_n\bar{y}) \in V_F(\delta)$.

Hence according to Proposition 4 (ii) we have

$$\begin{aligned}d(x_n, F^{-1}(y - g(x_n) - t_n\bar{y}))&\\ &< \tau\varphi_{V_{F(\rho)}}(x_n, y - g(x_n) - t_n\bar{y})\\ &\le \tau(\varphi_{V_{F+g}(\rho)}(x_n, y) + t_n\|\bar{y}\|)\\ &= \tau t_n\frac{\Big((1+\alpha)\|\bar{y}\| + \rho\Big)}{\alpha} \qquad \text{thanks to (71).}\end{aligned} \tag{77}$$

Using the fact that $t_n\|\bar{y}\| < \eta$ and $\varphi_{V_{F+g}(\rho)}(x_n, y) \le d(y - g(x_n), F(x_n) < \eta$, we obtain

$$d(x_n, F^{-1}(y - g(x_n) - t_n\bar{y}) < 2\tau\eta.$$

By the choice of η, we derive $d(x_n, F^{-1}(y-g(x_n)-t_n\bar{y}) < \varepsilon/2$, and therefore for any $r \in (0,1)$, the existence of some $u_n \in F^{-1}(y-g(x_n)-t_n)$ such that

$$d(x_n,u_n) < \tau(1+r)t_n((1+\alpha)\|\bar{y}\|+\rho)/\alpha < \varepsilon/2.$$

Since $(x_n) \to x \in B(x_0,\eta)$, for n sufficiently large we have $d(x_n,x_0) \le d(x_n,x)+d(x,x_0) < \varepsilon/2+\eta < \varepsilon$, so that $u_n \in B(x_0,\varepsilon)$. Since $u_n \in F^{-1}(y-g(x_n)-t_n\bar{y}) \cap B(x_0,\varepsilon)$ and by the Lipschitz property of g on $B(x_0,\varepsilon)$:

$$\|g(u_n)-g(x_n)\| \le Ld(u_n,x_n),$$

then

$$y \in F(u_n)+g(x_n)+t_n\bar{y} \subseteq F(u_n)+g(u_n)+t_n\Big(\bar{y}+L\frac{d(u_n,x_n)}{t_n}B_Y\Big).$$

By the definition of L, for r sufficiently small, one obtains

$$y \in F(u_n)+g(u_n)+\operatorname{cone} B(\bar{y},\rho).$$

Therefore,

$$\varphi_{V_{F+g}(\rho)}(u_n,y) \le d(y-g(u_n),F(u_n)) \le t_n\|\bar{y}\|+Ld(x_n,u_n). \tag{78}$$

As $t_n\|\bar{y}\| \le \alpha\varphi_{V_{F+g}(\rho)}(x_n,y)$ with $\alpha \in (0,1)$, it follows that $\liminf_{n\to\infty} d(x_n,u_n) > 0$. Therefore, one has

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\varphi_{V_{F+g}(\rho)}(x,y)\varphi_{V_{F+g}(\rho)}(u_n,y)}{d(x,u_n)}\\
&= \liminf_{n\to\infty} \frac{\varphi_{V_{F+g}(\rho)}(x_n,y)-\varphi_{V_{F+g}(\rho)}(u_n,y)}{d(x_n,u_n)}\\
&\ge \liminf_{n\to\infty} \frac{t_n(\|\bar{y}\|+\rho)/\alpha - t_n\|\bar{y}\|}{t_n((1+r)(\|\bar{y}\|+\rho)/\alpha + t_n\|\bar{y}\|} - L\\
&= \frac{(\|\bar{y}\|+\rho)/\alpha - \|\bar{y}\|}{((1+r)(\|\bar{y}\|+\rho)/\alpha + \|\bar{y}\|} - L.
\end{aligned}$$

As $r>0$ is arbitrary small, one obtains

$$|\Gamma\varphi_{V_{F+g}(\rho)}(\cdot,y)|(x) \ge \frac{1-\gamma}{\tau(1+\gamma)} - L,$$

which completes the proof. □

Acknowledgments. Research of Huynh Van Ngai was supported by VIASM (Vietnam Institute of Avanced Study on Mathematics).

Research of Michel Théra was partially supported by by the Australian Research Council under grant DP-110102011 and by LIA "FormathVietnam".

References

[1] Arutyunov, A.V., E.R. Avakov, A.F. Izmailov. 2007. Directional regularity and metric regularity. *SIAM J. Optim.* **18**(3) 810–833. doi:10.1137/060651616.

[2] Azé., D. 2003. A survey on error bounds for lower semicontinuous functions. *Proceedings of 2003 MODE-SMAI Conference*, *ESAIM Proc.*, vol. 13. EDP Sci., Les Ulis, 1–17.

[3] Azé, D. 2006. A unified theory for metric regularity of multifunctions. *J. Convex Anal.* **13**(2) 225–252.

[4] Azé, D., J.-N. Corvellec. 2004. Characterizations of error bounds for lower semicontinuous functions on metric spaces. *ESAIM Control Optim. Calc. Var.* **10**(3) 409–425. doi:10.1051/cocv:2004013.

[5] Bonnans, J. F., A. Shapiro. 2000. *Perturbation analysis of optimization problems*. Springer Series in Operations Research, Springer-Verlag, New York.

[6] Borwein, J. M., Q.J. Zhu. 2005. *Techniques of Variational Analysis*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 20, Springer-Verlag, New York.

[7] Borwein, J. M., D. M. Zhuang. 1988. Verifiable necessary and sufficient conditions for openness and regularity for set-valued and single-valued mapps. *J. Math. Anal. Appl.* **134** 441–459.

[8] Borwein, J.M., A.L. Dontchev. 2003. On the Bartle-Graves theorem. *Proc. Amer. Math. Soc.* **131**(8) 2553–2560. doi:10.1090/S0002-9939-03-07229-0.

[9] Borwein, J.M., Q.J. Zhu. 1996. Viscosity solutions and viscosity subderivatives in smooth Banach spaces with applications to metric regularity. *SIAM J. Contr. Optim.* **34** 1568–1591.

[10] Bosch, P., A. Jourani, R. Henrion. 2004. Sufficient conditions for error bounds and applications. *Appl. Math. Optim.* **50**(2) 161–181.

[11] Clarke, F.H. 1983. *Optimization and Nonsmooth Analysis*. Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons Inc., New York. A Wiley-Interscience Publication.

[12] Cominetti, R. 1990. Metric regularity, tangent sets, and second-order optimality conditions. *Appl. Math. Optim.* **21**(3) 265–287. doi:10.1007/BF01445166.

[13] Conway, J.B. 1990. *A course in functional analysis*, *Graduate Texts in Mathematics*, vol. 96. 2nd ed. Springer-Verlag, New York.

[14] DeGiorgi, E., A. Marino, M. Tosques. 1980. Problems of evolution in metric spaces and maximal decreasing curve. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **68**(3) 180–187.

[15] Dmitruk, A. V., A.A. Milyutin, N.P. Osmolovsky. 1980. Lyusternik's theorem and the theory of extrema. *Uspekhy Mat. Nauk* **35** 11–46. In Russian.

[16] Dmitruk, A.V., A. Y. Kruger. 2008. Metric regularity and systems of generalized equations. *J. Math. Anal. Appl.* **342**(2) 864–873. doi:10.1016/j.jmaa.2007.12.057.

[17] Dmitruk, A.V., A. Y. Kruger. 2009. Extensions of metric regularity. *Optimization* **58**(5) 561–584. doi:10.1080/02331930902928674.

[18] Dontchev, A.L. 1996. The Graves theorem revisited. *J. Convex Anal.* **3**(1) 45–53.

[19] Dontchev., A.L., A.S. Lewis, R.T. Rockafellar. 2003. The radius of metric regularity. *Trans. Amer. Math. Soc.* **355**(2) 493–517. doi:10.1090/S0002-9947-02-03088-X.

[20] Dontchev, A.L., R.T. Rockafellar. 2009. *Implicit Functions and Solution Mappings. A View from Variational Analysis*. Springer Monographs in Mathematics, Springer, Dordrecht.

[21] Ekeland, I. 1974. On the variational principle. *J. Math. Anal. Appl.* **47** 324–353.

[22] Fabian, M. J., R. Henrion, A.Y. Kruger, J.V. Outrata. 2010. Error bounds: necessary and sufficient conditions. *Set-Valued Var. Anal.* **18**(2) 121–149. doi:10.1007/s11228-010-0133-0.

[23] Fabian, M.J., R. Henrion, A.Y. Kruger, J. V. Outrata. 2012. About error bounds in metric spaces. Diethard Klatte, Hans-Jacob Lüthi, Karl Schmedders, eds., *Operations Research Proceedings 2011. Selected papers of the Int. Conf. Operations Research (OR 2011), August 30 – September 2, 2011, Zurich, Switzerland*. Springer-Verlag, Berlin, 33–38.

[24] Giannessi, F. 2005. *Constrained Optimization and Image Space Analysis. Vol. 1: Separation of Sets and Optimality Conditions*, *Mathematical Concepts and Methods in Science and Engineering*, vol. 49. Springer, New York.

[25] Graves, L.M. 1950. Some mapping theorems. *Duke Math. J.* **17** 111–114.

[26] Hoffman, A.J. 1952. On approximate solutions of systems of linear inequalities. *J. Research Nat. Bur. Standards* **49** 263–265.

[27] Huynh, V.N., A. Y. Kruger, M. Théra. 2010. Stability of error bounds for semi-infinite convex constraint systems. *SIAM J. Optim.* **20**(4) 2080–2096. doi:10.1137/090767819.

[28] Huynh, V.N., A.Y. Kruger, M. Théra. 2012. Slopes of multifunctions and extensions of metric regularity. *Vietnam J. Math.* **40**(2-3) 355–369.

[29] Huynh, V.N., H.T. Nguyen, M. Théra. 2013. Implicit multifunctions theorems in complete metric spaces. *Math. Program.* doi:10.1007/s10107-013-0673-9.

[30] Huynh, V.N., M. Théra. 2001. Metric inequality, subdifferential calculus and applications. *Set-Valued Anal.* **9**(1-2) 187–216. doi:10.1023/A:1011291608129. Wellposedness in optimization and related topics (Gargnano, 1999).

[31] Huynh, V.N., M. Théra. 2004. Error bounds and implicit multifunction theorem in smooth Banach spaces and applications to optimization. *Set-Valued Anal.* **12**(1-2) 195–223. doi:10.1023/B:SVAN.0000023396.58424.98.

[32] Huynh, V.N., M. Théra. 2005. Error bounds for convex differentiable inequality systems in Banach spaces. *Math. Program., Ser. B* **104**(2-3) 465–482.

[33] Huynh, V.N., M. Théra. 2008. Error bounds in metric spaces and application to the perturbation stability of metric regularity. *SIAM J. Optim.* **19**(1) 1–20.

[34] Huynh, V.N., M. Théra. 2009. Error bounds for systems of lower semicontinuous functions in Asplund spaces. *Math. Program., Ser. B* **116**(1-2) 397–427.

[35] Ioffe, A.D. 1979. Regular points of Lipschitz functions. *Trans. Amer. Math. Soc.* **251** 61–69. doi:10.2307/1998683.

[36] Ioffe, A.D. 2000. Metric regularity and subdifferential calculus. *Uspekhi Mat. Nauk* **55**(3(333)) 103–162. In Russian.

[37] Ioffe, A.D. 2001. Towards metric theory of metric regularity. *Approximation, optimization and mathematical economics (Pointe-à-Pitre, 1999)*. Physica, Heidelberg, 165–176.

[38] Ioffe, A.D. 2010. On regularity concepts in variational analysis. *J. Fixed Point Theory Appl.* **8**(2) 339–363. doi:10.1007/s11784-010-0021-0.

[39] Ioffe, A.D. 2013. Convexity and variational analysis. *Math. Program.* **To appear**.

[40] Ioffe, A.D., J.V. Outrata. 2008. On metric and calmness qualification conditions in subdifferential calculus. *Set-Valued Anal.* **16**(2-3) 199–227.

[41] Jourani, A. 2000. Hoffman's error bound, local controllability, and sensitivity analysis. *SIAM J. Control Optim.* **38**(3) 947–970. doi:10.1137/S0363012998339216.

[42] Jourani, A., L. Thibault. 1995. Metric regularity and subdifferential calculus in Banach spaces. *Set-Valued Anal.* **3**(1) 87–100. doi:10.1007/BF01033643.

[43] Jourani, A., L. Thibault. 1999. Coderivatives of multivalued mappings, locally compact cones and metric regularity. *Nonlinear Anal.* **35**(7, Ser. A: Theory Methods) 925–945. doi:10.1016/S0362-546X(98)00031-5.

[44] Klatte, D., B. Kummer. 2002. *Nonsmooth equations in optimization*, *Nonconvex Optimization and its Applications*, vol. 60. Kluwer Academic Publishers, Dordrecht.

[45] Klatte, D., B. Kummer. 2002. *Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications*, *Nonconvex Optimization and its Applications*, vol. 60. Kluwer Academic Publishers, Dordrecht.

[46] Kruger, A. Y., V.N. Huynh, M. Théra. 2010. Stability of error bounds for convex constraint systems in Banach spaces. *SIAM J. Optim.* **20**(6) 3280–3296. doi:10.1137/100782206.

[47] Kummer, B. 1999. Metric regularity: Characterizations, nonsmooth variations and successive approximation. *Optimization* **46** 247–281.

[48] Lyusternik, L.A. 1934. On conditional extrema of functionals. *Math. Sbornik* **41** 390–401. In Russian.

[49] Mordukhovich, B. S. 2006. *Variational analysis and generalized differentiation. I*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 330. Springer-Verlag, Berlin. Basic theory.

[50] Mordukhovich, B. S. 2006. *Variational analysis and generalized differentiation. II*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 331. Springer-Verlag, Berlin. Applications.

[51] Mordukhovich, B. S., J.V. Outrata. 2007. Coderivative analysis of quasi-variational inequalities with applications to stability and optimization. *SIAM J. Optim.* **18**(2) 389–412. doi:10.1137/060665609.

[52] Mordukhovich, B. S., Y. Shao. 1997. Stability of set-valued mappings in infinite dimensions: point criteria and applications. *SIAM J. Control Optim.* **35**(1) 285–314. doi:10.1137/S0363012994278171.

[53] Mordukhovich, B. S., B. Wang. 2004. Restrictive metric regularity and generalized differential calculus in Banach spaces. *Int. J. Math. Math. Sci.* (49-52) 2653–2680. doi:10.1155/S0161171204405183.

[54] Outrata, J., M. Kočvara, J. Zowe. 1998. *Nonsmooth approach to optimization problems with equilibrium constraints*, *Nonconvex Optimization and its Applications*, vol. 28. Kluwer Academic Publishers, Dordrecht. Theory, applications and numerical results.

[55] Penot, J.-P. 1989. Metric regularity, openness and Lipschitz behavior of multifunctions. *Nonlinear Anal.* **13** 629–643.

[56] Penot, J.-P. 2013. *Calculus Without Derivatives*. Graduate Texts in Mathematics, Springer Verlag, Berlin.

[57] Robinson, S.M. 1973. An inverse-function theorem for a class of multivalued functions. *Proc. Amer. Math. Soc.* **41** 211–218.

[58] Robinson, S.M. 1980. Strongly regular generalized equations. *Math. Oper. Res.* **5**(1) 43–62.

[59] Rockafellar, R.T., R.J.-B. Wets. 1998. *Variational analysis*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 317. Springer-Verlag, Berlin.

[60] Ursescu, C. 1975. Multifunctions with convex closed graph. *Czechoslovak Math. J.* **25(100)**(3) 438–441.