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FIRST ORDER GLOBAL ASYMPTOTICS FOR CALOGERO-SUTHERLAND GASES

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ABSTRACT. We study a physical system of N interacting particles in $\mathbb{R}^d$, $d \geq 1$, subject to pair repulsion and confined by an external field. We establish a large deviations principle for their empirical distribution as N tends to infinity. In the case of Riesz interaction, including Coulomb interaction in arbitrary dimension $d > 2$, the rate function is strictly convex and admits a unique minimum, the equilibrium measure, characterized via its potential. It follows that almost surely, the empirical distribution of the particles tends to this equilibrium measure as N tends to infinity. In the more specific case of Coulomb interaction in dimension $d > 2$, and when the external field is a convex or increasing function of the radius, then the equilibrium measure is supported in a ring. With a quadratic external field, the equilibrium measure is uniform on a ball.

1. INTRODUCTION

We study in this work a physical system of N particles at positions $x_1, \ldots, x_N \in \mathbb{R}^d$, $d \geq 1$, with identical "charge" $q_N := 1/N$, subject to a confining potential $V : \mathbb{R}^d \to \mathbb{R}$ coming from an external field and acting on each particle, and to an interaction potential $W : \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, +\infty]$ acting on each pair of particles. The function W is finite outside the diagonal and symmetric: for all $x, y \in \mathbb{R}^d$ with $x \neq y$, we have $W(x, y) = W(y, x) < \infty$. The energy $H_N(x_1, \ldots, x_N)$ of the configuration $(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$ takes the form

$$\begin{aligned} H_N(x_1, \ldots, x_N) &:= \sum_{i=1}^N q_N V(x_i) + \sum_{i<j} q_N^2 W(x_i, x_j) \\ &= \frac{1}{N} \sum_{i=1}^N V(x_i) + \frac{1}{N^2} \sum_{i<j} W(x_i, x_j) \\ &= \int V(x)\, d\mu_N(x) + \frac{1}{2} \iint_{\neq} W(x, y)\, d\mu_N(x)\, d\mu_N(y) \end{aligned} \tag{1.1}$$

where $\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ in the empirical measure of the particles, and where the subscript "$\neq$" indicates that the double integral is off-diagonal. The energy $H_N : (\mathbb{R}^d)^N \to \mathbb{R} \cup \{+\infty\}$ is a quadratic form functional in the variable μ_N.

From now on, and unless otherwise stated, we denote by $|\cdot|$ the Euclidean norm of $\mathbb{R}^d$ and we make the following additional assumptions:

(H1) The function $W : \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, +\infty]$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$, symmetric, takes finite values on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x); x \in \mathbb{R}^d\}$ and satisfies the following integrability condition: for all compact subset $K \subset \mathbb{R}^d$, the function

$$z \in \mathbb{R}^d \mapsto \sup\{W(x, y); |x - y| \geq |z|, x, y \in K\}$$

is locally Lebesgue-integrable on $\mathbb{R}^d$;

(H2) The function $V : \mathbb{R}^d \to \mathbb{R}$ is continuous and such that $\lim_{|x| \to +\infty} V(x) = +\infty$ and

$$\int_{\mathbb{R}^d} \exp\left(-V(x)\right) dx < \infty.$$

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(H3) There exists constants $c \in \mathbb{R}$ and $\varepsilon_o \in (0,1)$ such that for every $x, y \in \mathbb{R}^d$,

$$W(x,y) \geq c - \varepsilon_o(V(x)+V(y)).$$

Let $(\beta_N)_N$ be a sequence of positive real numbers such that $\beta_N \to +\infty$ as $N \to \infty$. Under **(H2)-(H3)**, there exists an integer N_0 depending on ε_o such that for any $N \geq N_0$, we have

$$Z_N := \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \exp\left(-\beta_N H_N(x_1,\ldots,x_N)\right) dx_1 \cdots dx_N < \infty,$$

so that we can define the Boltzmann-Gibbs probability measure P_N on $(\mathbb{R}^d)^N$ by

$$dP_N(x_1,\ldots,x_N) := \frac{\exp\left(-\beta_N H_N(x_1,\ldots,x_N)\right)}{Z_N} dx_1 \cdots dx_N. \tag{1.2}$$

Following [23, Section 11.3.1] (see also [20, Section 9.6]), the model is of Calogero-Sutherland type, and the density of P_N takes the form

$$\prod_{i=1}^N f_1(x_i) \prod_{1\leq i<j\leq N} f_2(x_i,x_j).$$

The law P_N is the equilibrium distribution of a system of N interacting Brownian particles in $\mathbb{R}^d$, at inverse temperature β_N, with equal individual "charge" $1/N$, subject to a confining potential V acting on each particle, and to an interaction potential W acting on each pair of particles, see Section 1.5.10. Note that for $\beta_N = N^2$, the quantity $\beta_N H_N$ can also be interpreted as the distribution of a system of N particles living in $\mathbb{R}^d$, with unit "charge", subject to a confining potential NV acting on each particle, and to an interaction potential W acting on each pair of particles.

Our work is motivated by the following physical control problem: given the (internal) interaction potential W, for instance a Coulomb potential, a target probability measure $\mu_\star$ on $\mathbb{R}^d$, for instance the uniform law on the unit ball, and a cooling scheme $\beta_N \to +\infty$, for instance $\beta_N = N^2$, can we tune the (external) confinement potential V (associated to an external confinement field) such that $\mu_N \to \mu_\star$ as $N \to \infty$? In this direction, we provide some partial answers in Theorem 1.1, Theorem 1.2, Corollary 1.3, and Corollary 1.4 below. We also discuss several possible extensions and related problems in Section 1.5.

Let $\mathcal{M}_1(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$. The mean-field symmetries of the model suggest to study, under the exchangeable measure P_N, the behavior as $N \to \infty$ of the empirical measure μ_N, which is a random variable on $\mathcal{M}_1(\mathbb{R}^d)$. With this asymptotic analysis in mind, we introduce the functional $I : \mathcal{M}_1(\mathbb{R}^d) \to (-\infty, +\infty]$ given by

$$I(\mu) := \frac{1}{2} \iint (V(x)+V(y)+W(x,y))\, d\mu(x) d\mu(y).$$

The assumptions **(H2)-(H3)** imply that the function under the integral is bounded from below, so that the integral defining I makes sense in $\mathbb{R} \cup \{+\infty\} = (-\infty, +\infty]$. If it is finite, then $\int V d\mu$ and $\iint W d\mu^2$ both exist (see Lemma 2.2), so that

$$I(\mu) = \int V d\mu + \frac{1}{2} \iint W d\mu^2.$$

The energy H_N defined by (1.1) is "almost" given by $I(\mu_N)$, where the infinite terms on the diagonal are forgotten.

1.1. **Large deviations principle.** Theorem 1.1 below is our first main result. It is of topological nature, inspired from the available results for logarithmic Coulomb gases in random matrix theory [4, 5, 36, 25]. We equip $\mathcal{M}_1(\mathbb{R}^d)$ with the weak topology, defined by duality with bounded continuous functions. For any set $A \subset \mathcal{M}_1(\mathbb{R}^d)$ we denote by $\mathrm{int}(A)$, $\mathrm{clo}(A)$ the interior and closure of A with respect to this topology. This topology can be metrized by the Fortet–Mourier distance defined by (see [19]):

$$d_{\mathrm{FM}}(\mu,\nu) := \sup_{\max(|f|_\infty, |f|_{\mathrm{Lip}})\leq 1} \left\{ \int f\, d\mu - \int f\, d\nu \right\}. \tag{1.3}$$

To formulate the large deviations result we need to introduce the following additional technical assumption:

(H4) For all $\nu \in \mathcal{M}_1(\mathbb{R}^d)$ such that $I(\nu) < +\infty$, there is a sequence $(\nu_n)_{n\in\mathbb{N}}$ of probability measures, absolutely continuous with respect to Lebesgue, such that ν_n converges weakly to ν and $I(\nu_n) \to I(\nu)$, when $n \to \infty$.

It turns out that assumption **(H4)** is satisfied for a large class of potentials V, W, and several examples are given in Proposition 2.8 and Theorem 1.2.

In all the paper, if $(a_N)_N$ and $(b_N)_N$ are non negative sequences, the notation $a_N \gg b_N$ means that $a_N = b_N c_N$, for some c_N that goes to $+\infty$ when $N \to \infty$.

Theorem 1.1 (Large Deviations Principle)**.** *Suppose that*

$$\beta_N \gg N \log(N).$$

*If **(H1)-(H2)-(H3)** are satisfied then*

(1) I is lower semi-continuous and has compact level sets;

(2) Under $(P_N)_N$, the sequence $(\mu_N)_N$ of random elements of $\mathcal{M}_1(\mathbb{R}^d)$ equipped with the weak topology has the following asymptotic properties. For every Borel subset A of $\mathcal{M}_1(\mathbb{R}^d)$,

$$\limsup_{N\to\infty} \frac{\log Z_N P_N(\mu_N \in A)}{\beta_N} \leq - \inf_{\mu\in \mathrm{clo}(A)} I(\mu)$$

and

$$\liminf_{N\to\infty} \frac{\log Z_N P_N(\mu_N \in A)}{\beta_N} \geq - \inf\{I(\mu); \mu \in \mathrm{int}(A), \mu \ll \text{Lebesgue}\}.$$

*(3) Under the additional assumption **(H4)**, the full Large Deviation Principle (LDP) at speed β_N holds with the rate function*

$$I_\star := I - \inf_{\mathcal{M}_1(\mathbb{R}^d)} I.$$

More precisely, for all Borel set $A \subset \mathcal{M}_1(\mathbb{R}^d)$,

$$- \inf_{\mu\in \mathrm{int}(A)} I_\star(\mu) \leq \liminf_{N\to\infty} \frac{\log P_N(\mu_N \in A)}{\beta_N}$$
$$\leq \limsup_{N\to\infty} \frac{\log P_N(\mu_N \in A)}{\beta_N} \leq - \inf_{\mu\in \mathrm{clo}(A)} I_\star(\mu).$$

*(4) Let $I_{min} := \{\mu \in \mathcal{M}_1 : I_\star(\mu) = 0\} \neq \emptyset$. If **(H4)** is satisfied and if $(\mu_N)_N$ are constructed on the same probability space, and if d stands for the Fortet–Mourier distance (1.3), then we have, almost surely,*

$$\lim_{N\to\infty} d_{\mathrm{FM}}(\mu_N, I_{min}) = 0.$$

A careful reading of the proof of Theorem 1.2 indicates that if $I_{\min} = \{\mu_\star\}$ is a singleton, and if **(H4)** holds for $\nu = \mu_\star$, then $\mu_N \to \mu_\star$ almost surely as $N \to \infty$.

1.2. **Link with Sanov theorem.** If we set $W = 0$ then the particles become i.i.d. and P_N becomes a product measure $\eta_N^{\otimes N}$ where $\eta_N \propto e^{-(\beta_N/N)V}$. When $\beta_N = N$ then $\eta_N \propto e^{-V}$ does not depend on N, and we may denote it η. To provide perspective, recall that the classical Sanov theorem [17, Theorem 6.2.10] for i.i.d. sequences means in our settings that if $W = 0$ and $\beta_N = N$ then $(\mu_N)_N$ satisfies to a large deviations principle on $\mathcal{M}_1(\mathbb{R}^d)$ at speed N and with good rate function

$$\mu \mapsto K(\mu|\eta) := \begin{cases} \int f \log(f)\, d\eta & \text{if } \mu \ll \eta, \text{ with } f := \frac{d\mu}{d\eta}; \\ +\infty & \text{otherwise} \end{cases}$$

(Kullback-Leibler relative entropy or free energy). This large deviations principle corresponds to the convergence $\lim_{N\to\infty} d_{\mathrm{FM}}(\mu_N, \eta) = 0$. Note that, if μ is absolutely continuous with respect to Lebesgue measure with density function g, then $K(\mu|\eta)$ can be decomposed in two terms

$$K(\mu|\eta) = \int V\, d\mu - H(\mu),$$

where $H(\mu)$ is the Boltzmann-Shannon "continuous" entropy $H(\mu) := -\int g(x) \log(g(x))\, dx$; therefore at the speed $\beta_N = N$, the energy factor $\int V\, d\mu$ and the Boltzmann-Shannon entropy factor $H(\mu)$ both appear in the rate function. In contrast, note that Theorem 1.1 requires a higher

inverse temperature $\beta_N \gg N\log(N)$. If we set $W = 0$ in Theorem 1.1, then P_N becomes a product measure, the particles are i.i.d. but their common law depends on N, the function $\mu \mapsto I_*(\mu) = \int V\, d\mu - \inf V$ is affine, its minimizers $I_{\min}$ over $\mathcal{M}_1(\mathbb{R}^d)$ coincide with

$$\mathcal{M}_V := \{\mu \in \mathcal{M}_1(\mathbb{R}^d) : \mathrm{supp}(\mu) \subset \arg\inf V\},$$

and Theorem 1.1 boils down to a sort of Laplace principle, which corresponds to the convergence $\lim_{N\to\infty} d_{\mathrm{FM}}(\mu_N, \mathcal{M}_V) = 0$. It is worthwhile to notice that the main difficulty in Theorem 1.1 lies in the fact that W can be infinite on the diagonal (short scale repulsion). If W is continuous and bounded on $\mathbb{R}^d \times \mathbb{R}^d$, then one may deduce the large deviations principle for $(\mu_N)_N$ from the case $W = 0$ by using the Laplace-Varadhan lemma [17, Theorem 4.3.1] (see also [4, Corollary 5.1]). To complete the picture, let us mention that if $\beta_N = N$ and if W is bounded and continuous, then the Laplace-Varadhan lemma and the Sanov theorem would yield to the conclusion that $(\mu_N)_N$ verifies a large deviations principle on $\mathcal{M}_1(\mathbb{R}^d)$ at speed N with rate function $R - \inf_{\mathcal{M}_1(\mathbb{R}^d)} R$ where the functional R is defined by

$$\begin{aligned} R(\mu) &:= K(\mu|\eta) + \frac{1}{2}\iint W(x,y)\, d\mu(x) d\mu(y) \\ &= -H(\mu) + I(\mu); \end{aligned}$$

once more, the Boltzmann-Shannon entropy factor $H(\mu)$ reappears at this rate.

1.3. **Equilibrium measure.** Our second main result, expressed in Theorem 1.2 and Corollary 1.3 below is of differential nature. It is based on an instance of the general Gauss problem in potential theory [45, 46]. It concerns special choices of V and W for which $I_\star$ achieves its minimum 0 for a unique and explicit $\mu_\star \in \mathcal{M}_1(\mathbb{R}^d)$. Recall that the *Coulomb* interactions correspond to the choice $W(x,y) = k_\Delta(x-y)$ where k_Δ is the *Coulomb kernel* (opposite in sign to the Newton kernel) defined on $\mathbb{R}^d$, $d \geq 1$, by

$$k_\Delta(x) := \begin{cases} -|x| & \text{if } d = 1, \\ \log \frac{1}{|x|} & \text{if } d = 2, \\ \frac{1}{|x|^{d-2}} & \text{if } d \geq 3. \end{cases} \tag{1.4}$$

This is, up to a multiplicative constant, the fundamental solution[1] of the Laplace equation. In other words, denoting $\Delta := \partial^2_{x_1} + \cdots + \partial^2_{x_d}$ the Laplacian, we have, in a weak sense, in the space of Schwartz-Sobolev distributions $\mathcal{D}'(\mathbb{R}^d)$,

$$-c\Delta k_\Delta = \delta_0 \quad \text{with} \quad c := \begin{cases} \frac{1}{2} & \text{if } d = 1, \\ \frac{1}{2\pi} & \text{if } d = 2, \\ \frac{1}{d(d-2)\omega_d} & \text{if } d \geq 3, \end{cases} \tag{1.5}$$

where $\omega_d := \frac{\pi^{d/2}}{\Gamma(1+d/2)}$ is the volume of the unit ball of $\mathbb{R}^d$. Our notation is motivated by the fact that $-\Delta$ is a nonnegative operator. The case of Coulomb interactions in dimension $d = 2$ is known as "logarithmic potential with external field" and is widely studied in the literature: see [27, 39, 3, 25]. To focus on novelty, we will not study the Coulomb kernel for $d \leq 2$. We refer to [32, 21, 31, 13, 1, 41] and references therein for the Coulomb case in dimension $d = 1$, to [4, 3, 25] to the Coulomb case in dimension $d = 2$ with support restriction on a line, to [5, 36, 27, 25, 39, 40, 44] for the Coulomb case in dimension $d = 2$. We also refer to [7] for the asymptotic analysis in terms of large deviations of Coulomb determinantal point processes on compact manifolds of arbitrary dimension.

The asymptotic analysis of μ_N as $N \to \infty$ for Coulomb interactions in dimension $d \geq 3$ motivates our next result, which is stated for the more general Riesz interactions in dimension $d \geq 1$. The *Riesz* interactions correspond to the choice $W(x,y) = k_{\Delta_\alpha}(x-y)$ where k_{Δ_α}, $0 < \alpha < d$, $d \geq 1$, is the *Riesz kernel* defined on $\mathbb{R}^d$, by

$$k_{\Delta_\alpha}(x) := \frac{1}{|x|^{d-\alpha}}. \tag{1.6}$$

[1] There is no boundary conditions here, and thus the term "Green function" is not appropriate.

Up to a multiplicative constant, this is the fundamental solution of a fractional Laplace equation (which is the true Laplace equation (1.5) when $\alpha = 2$), namely

$$-c_\alpha \Delta_\alpha k_{\Delta_\alpha} = \mathcal{F}^{-1}(1) = \delta_0 \quad \text{with} \quad c_\alpha := \frac{\pi^{\alpha-\frac{d}{2}}}{4\pi^2}\frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}, \tag{1.7}$$

where the Fourier transform $\mathcal{F}$ and the fractional Laplacian Δ_α are given by

$$\mathcal{F}(k_{\Delta_\alpha})(\xi) := \int_{\mathbb{R}^d} e^{2i\pi\xi\cdot x}\, k_{\Delta_\alpha}(x)\, dx = \frac{1}{c_\alpha 4\pi^2|\xi|^\alpha} \quad \text{and} \quad \Delta_\alpha f := -4\pi^2\mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(f)).$$

Note that $\Delta_2 = \Delta$ while Δ_α is a non-local integro-differential operator when $\alpha \neq 2$. When $d \geq 3$ and $\alpha = 2$ then Riesz interactions coincide with Coulomb interactions and the constants match. Beware that our notations differ slightly from the ones of [30, p. 44]. Several aspects of the Gauss problem in the Riesz case are studied in [18, 45, 46].

In the Riesz case, $0 < \alpha < d$, one associates to any probability measure μ on $\mathbb{R}^d$ a function $U^\mu_\alpha : \mathbb{R}^d \mapsto [0, +\infty]$ called the potential of μ as follows

$$U^\mu_\alpha(x) := \int k_{\Delta_\alpha}(x-y)\, d\mu(y), \qquad \forall x \in \mathbb{R}^d.$$

We refer to Section 3 for a review of basic definitions from potential theory. In particular, one defines there a notion of capacity of sets, and a property is said to hold quasi everywhere if it holds outside a set of zero capacity. The following theorem is essentially the analogue in $\mathbb{R}^d$ of a result of Dragnev and Saff on spheres [18].

Theorem 1.2 (Riesz gases)**.** *Suppose that W is the Riesz kernel $W(x,y) = k_{\Delta_\alpha}(x-y)$. Then:*

(1) The functional I is strictly convex;

*(2) **(H1)-(H2)-(H3)-(H4)** are satisfied and Theorem 1.1 applies;*

(3) There exists a unique $\mu_\star \in \mathcal{M}_1(\mathbb{R}^d)$ such that

$$I(\mu_\star) = \inf_{\mu \in \mathcal{M}_1(\mathbb{R}^d)} I(\mu);$$

(4) If we define $(\mu_N)_N$ on a unique probability space (for a sequence $\beta_N \gg N\log(N)$) then with probability one,

$$\lim_{N\to\infty} \mu_N = \mu_\star.$$

If we denote by $C_\star$ the real number

$$C_\star = \int (U^{\mu_\star}_\alpha + V) d\mu_\star = J(\mu_\star) + \int V d\mu_\star,$$

then the following additional properties hold:

(5) The minimizer $\mu_\star$ has compact support, and satisfies

$$U^{\mu_\star}_\alpha(x) + V(x) \geq C_\star \qquad \text{quasi everywhere}, \tag{1.8}$$

$$U^{\mu_\star}_\alpha(x) + V(x) = C_\star \qquad \text{for all } x \in \mathrm{supp}(\mu_\star). \tag{1.9}$$

(6) If a compactly supported measure μ creates a potential U^μ_α such that, for some constant $C \in \mathbb{R}$,

$$U^\mu_\alpha(x) + V(x) = C \qquad \text{on } \mathrm{supp}(\mu), \tag{1.10}$$

$$U^\mu_\alpha + V \geq C \qquad \text{quasi everywhere}, \tag{1.11}$$

then $C = C_\star$ and $\mu = \mu_\star$. The same is true under the weaker assumptions:

$$U^\mu_\alpha(x) + V(x) \leq C \qquad \text{on } \mathrm{supp}(\mu), \tag{1.12}$$

$$U^\mu_\alpha + V \geq C \qquad \text{q.e on } \mathrm{supp}(\mu_\star). \tag{1.13}$$

(7) If $\alpha \leq 2$, for any measure μ, the following "converse" to (1.12), (1.13) holds:

$$\sup_{\mathrm{supp}(\mu)} (U^\mu_\alpha + V) \geq C_\star, \tag{1.14}$$

$$\text{"}\inf_{\mathrm{supp}(\mu_\star)}\text{"}\,(U^\mu_\alpha(x) + V(x)) \leq C_\star, \tag{1.15}$$

where the "inf" means that the infimum is taken quasi-everywhere.

The constant $C_\star$ is called the "modified Robin constant", see e.g. [39], where the properties (1.8-1.9) and the characterization (1.10-1.11) are established for the logarithmic potential in dimension 2. The minimizer $\mu_\star$ is called the *equilibrium measure*.

Corollary 1.3 (Equilibrium of Coulomb gases with radial external fields in dimension ≥ 3)**.** *Suppose that for a fixed real parameter $\beta > 0$, and for every $x, y \in \mathbb{R}^d$, $d \geq 3$,*

$$V(x) = v(|x|) \quad \text{and} \quad W(x,y) = \beta k_\Delta(x-y),$$

where v is two times differentiable. Denote by $d\sigma_r$ the Lebesgue measure on the sphere of radius r, and let σ_d be the total mass of $d\sigma_1$. Let $w(r) = r^{d-1}v'(r)$, and suppose either that v is convex, or that w is increasing. Define two radii $r_0 < R_0$ by:

$$r_0 = \inf\{r > 0; v'(r) > 0\} \quad \text{and} \quad w(R_0) = \beta(d-2).$$

Then the equilibrium measure $\mu_\star$ is supported on the ring $\{x; |x| \in [r_0, R_0]\}$, and is absolutely continuous with respect to Lebesgue measure:

$$d\mu(r) = M(r)\, d\sigma_r dr \quad \text{where} \quad M(r) = \frac{w'(r)}{\beta(d-2)\sigma_d r^{d-1}} \mathbf{1}_{[r_0,R_0]}(r).$$

In particular, when $v(t) = t^2$ then $\mu_\star$ is the uniform distribution on the centered ball of radius

$$\left(\beta \frac{d-2}{2}\right)^{\frac{1}{d}}.$$

The result provided by Corollary 1.3 on Coulomb gases with radial external fields can be found for instance in [34, Proposition 2.13]. It follows quickly from the Gauss averaging principle and the characterization (1.10-1.11). For the sake of completeness, we give a (short) proof in Section 4.3. By using Theorem 1.2 with $\alpha = 2$ together with Corollary 1.3, we obtain that the empirical measure of a Coulomb gas with quadratic external field in dimension $d \geq 3$ tends almost surely to the uniform distribution on a ball when $N \to \infty$. This phenomenon is the analogue in arbitrary dimension $d \geq 3$ of the well known result in dimension $d = 2$ for the logarithmic potential with quadratic radial external field (where the uniform law on the disc or "circular law" appears as a limit for the Complex Ginibre Ensemble, see for instance [5, 36]). The study of the equilibrium measure for Coulomb interaction with non radially symmetric external fields was initiated recently in dimension $d = 2$ by Bleher and Kuijlaars in a beautiful work [9] by using orthogonal polynomials.

The following proposition shows that in the Riesz case, it is possible to construct a good confinement potential V so that the equilibrium measure is prescribed in advance.

Corollary 1.4 (Riesz gases: external field for prescribed equilibrium measure)**.** *Let $0 < \alpha < d$, $d \geq 1$, and $W(x,y) := k_{\Delta_\alpha}$. Let $\mu_\star$ be a probability measure with a compactly supported density $f_\star \in \mathbf{L}^p(\mathbb{R}^d)$ for some $p > d/\alpha$. Define the confinement potential*

$$V(x) := -U_\alpha^{\mu_\star}(x) + [|x|^2 - R]_+, \qquad x \in \mathbb{R}^d,$$

*where $U_\alpha^{\mu_\star}$ is the Riesz potential created by $\mu_\star$ and $R > 0$ is such that $\mathrm{supp}(\mu_\star) \subset B(0,R)$. Then the couple of functions (V,W) satisfy **(H1)-(H2)-(H3)-(H4)** and the functional*

$$\mu \in \mathcal{M}_1(\mathbb{R}^d) \mapsto I(\mu) := \int V\, d\mu + \frac{1}{2}\iint k_{\Delta_\alpha}(x-y)\, d\mu(x) d\mu(y) \in \mathbb{R} \cup \{+\infty\}$$

admits $\mu_\star$ as unique minimizer. In particular, the probability $\mu_\star$ is the almost sure limit of the sequence $(\mu_N)_N$ (constructed on the same probability space), as soon as $\beta_N \gg N \log(N)$.

1.4. **Outline of the article.** In the remainder of this introduction (Section 1.5), we give several comments on our results, their links with different domains, and possible directions for further research. Section 2 provides the proof of Theorem 1.1 (large deviations principle). Section 4 provides the proof of Theorem 1.2, Corollary 1.3, and Corollary 1.4. These proofs rely on several concepts and tools from Potential Theory, which we recall synthetically and discuss in Section 3 for the sake of clarity and completeness.

1.5. **Comments, possible extensions and related topics.**

1.5.1. *Non-compactly supported equilibrium measures.* The assumptions made on the external field V in theorems 1.1 and 1.2 explain why the equilibrium measure $\mu_\star$ is compactly supported. If one allows a weaker behavior of V at infinity, then one may produce equilibrium measures $\mu_\star$ which are not compactly supported (and may even be heavy tailed). This requires to adapt some of the arguments, and one may use compactification as in [25]. This might allow to extend Corollary 1.4 beyond the compactly supported case.

1.5.2. *Equilibrium measure for Riesz interaction with radial external field.* To the knowledge of the authors, the computation of the equilibrium measure for Riesz interactions with radial external field, beyond the more specific Coulomb case of Corollary 1.3, is an open problem, due to the lack of the Gauss averaging principle when $\alpha \neq 2$.

1.5.3. *Beyond the Riesz and Coulomb interactions.* Theorem 1.2 concerns the minimization of the Riesz interaction potential with an external field V, and includes the Coulomb interaction if $d \geq 3$. In classical Physics, the problem of minimization of the Coulomb interaction energy with an external field is known as the Gauss variational problem [45, 46]. Beyond the Riesz and Coulomb potentials, the driving structural idea behind Theorem 1.2 is that if W is of the form $W(x,y) = k_D(x-y)$ where k_D is the fundamental solution of an equation $\partial_t - D = 0$ where D is a local differential operator, and if V is super-harmonic for D, i.e. $DV \geq 0$, then the density of $\mu_\star$ is roughly given by DV up to support constraints. This can be easily understood formally with Lagrange multipliers. The limiting measure $\mu_\star$ depends on V and W, and is thus non-universal in general.

1.5.4. *Second order asymptotic analysis.* The asymptotic analysis of $\mu_N - \mu_\star$ as $N \to \infty$ is a natural problem, which can be studied on various classes of tests functions. It is well known that a repulsive interaction may affect dramatically the speed of convergence, and make it dependent over the regularity of the test function. In another direction, one may take $\beta_N = \beta N^2$ and study the low temperature regime $\beta \to \infty$ at fixed N. In the Coulomb case, this leads to Fekete points. We refer to [40, 12, 41, 8] for the analysis of the second order when both $\beta \to \infty$ and $N \to \infty$. In the one-dimensional case, another type of local universality inside the limiting support is available in [24].

1.5.5. *Edge behavior.* Suppose that V is radially symmetric and that $\mu_\star$ is supported in the centered ball of radius r, like in Corollary 1.3. Then one may ask if the radius of the particle system $\max_{1\leq k\leq n} |x_k|$ converges to the edge r of the limiting support as $N \to \infty$. This is not provided by the weak convergence of μ_N. The next question is the fluctuation. In the two-dimensional Coulomb case, a universality result is available for a class of external fields in [15].

1.5.6. *Topology.* It is known that the weak topology can be upgraded to a Wasserstein topology in the classical Sanov theorem for empirical measures of i.i.d. sequences, see [43], provided that tails are strong exponentially integrable. It is then quite natural to ask about such an upgrade for Theorem 1.1.

1.5.7. *Connection to random matrices.* Our initial inspiration came, when writing the survey [11], from the role played by the logarithmic potential in the analysis of the Ginibre ensemble. When $d = 2$, $\beta_N = N^2$, $V(x) = |x|^2$ and $W(x,y) = \beta k_\Delta(x-y) = \beta \log \frac{1}{|x-y|}$ with $\beta = 2$ then P_N is the law of the (complex) eigenvalues of the complex Ginibre ensemble:

$$dP_N(x) = Z_N^{-1} e^{-N\sum_{i=1}^N |x_i|^2} \prod_{i<j} |x_i - x_j|^2 dx.$$

(here $\mathbb{R}^2 \equiv \mathbb{C}$ and P_N is the law of the eigenvalues of a random $N \times N$ matrix with i.i.d. complex Gaussian entries of covariance $\frac{1}{2N} I_2$). For a non-quadratic V, we may see P_N as the law of the spectrum of random normal matrices such as the ones studied in [2]. On the other hand, in the case where $d = 1$ and $V(x) = |x|^2$ and $W(x,y) = \beta \log \frac{1}{|x-y|}$ with $\beta > 0$ then

$$dP_N(x) = Z_N^{-1} e^{-N\sum_{i=1}^N |x_i|^2} \prod_{i<j} |x_i - x_j|^\beta \, dx.$$

This is known as the β-Ensemble in Random Matrix Theory. For $\beta = 1$, we recover the law of the eigenvalues of the Gaussian Orthogonal Ensemble (GOE) of random symmetric matrices, while for

$\beta = 2$, we recover the law of the eigenvalues of the Gaussian Unitary Ensemble (GUE) of random Hermitian matrices. It is worthwhile to notice that $-\log|\cdot|$ is the Coulomb potential in dimension $d = 2$, and not in dimension $d = 1$. For this reason, we may interpret the eigenvalues of GOE/GUE as being a system of charged particles in dimension $d = 2$, experiencing Coulomb repulsion and an external quadratic field, but constrained to stay on the real axis. We believe this type of support constraint can be incorporated in our initial model, at the price of a bit heavier notations and analysis.

1.5.8. *Simulation problem and numerical approximation of the equilibrium measure.* It is natural to ask about the best way to simulate the probability measure P_N. A pure rejection algorithm is too naive. Some exact algorithms are available in the determinantal case $d = 2$ and $W(x, y) = -2\log|x - y|$, see [28, Algorithm 18] and [42]. One may prefer to use a non exact algorithm such as a Hastings-Metropolis algorithm. One may also use an Euler scheme to simulate a stochastic process for which P_N is invariant, or use a Metropolis adjusted Langevin approach (MALA) [37]. In this context, a very natural way to approximate numerically the equilibrium measure $\mu_\star$ is to use a simulated annealing stochastic algorithm.

1.5.9. *More general energies.* One may study more general energies with many bodies interactions, of the form, for some prescribed symmetric $W_k : (\mathbb{R}^d)^k \mapsto \mathbb{R}$, $1 \le k \le K$, $K \ge 1$,

$$H_N(x_1, \dots, x_N) = \sum_{k=1}^{K} \sum_{i_1 < \cdots < i_k} N^{-k} W_k(x_{i_1}, \dots, x_{i_k}).$$

This leads to the following candidate for the asymptotic first order global energy functional:

$$\mu \mapsto \sum_{k=1}^{K} 2^{-k} \int \cdots \int W(x_1, \dots, x_k)\, d\mu(x_1) \cdots d\mu(x_k).$$

1.5.10. *Stochastic processes.* Under general assumptions on V and W, see for instance [38], the law P_N is the invariant probability measure of a well defined (the absence of explosion comes from the assumptions on V and W) reversible Markov diffusion process $(X_t)_{t\in\mathbb{R}_+}$ with state space

$$\left\{x \in (\mathbb{R}^d)^N : H_N(x) < \infty\right\} = \left\{ x \in (\mathbb{R}^d)^N : \sum_{i<j} W(x_i, x_j) < \infty \right\},$$

solution of the system of Kolmogorov stochastic differential equations

$$dX_t = \sqrt{2\frac{\alpha_N}{\beta_N}}\, dB_t - \alpha_N \nabla H_N(X_t)\, dt$$

where $(B_t)_{t\ge 0}$ is a standard Brownian motion on $(\mathbb{R}^d)^N$, and where $\alpha_N > 0$ is an arbitrary scale parameter (natural choices being $\alpha_N = 1$ and $\alpha_N = \beta_N$). The law P_N is the equilibrium distribution of a system of N interacting Brownian particles $(X_{1,t})_{t\ge 0}, \dots, (X_{N,t})_{t\ge 0}$ in $\mathbb{R}^d$ at inverse temperature β_N, with equal individual "charge" $q_N := 1/N$, subject to a confining potential $\alpha_N V$ acting on each particle and to an interaction potential $\alpha_N W$ acting on each pair of particles, and one can rewrite the stochastic differential equation above as the system of coupled stochastic differential equations ($1 \le i \le N$)

$$dX_{i,t} = \sqrt{2\frac{\alpha_N}{\beta_N}}\, dB_{i,t} - q_N \alpha_N \nabla V(X_{i,t}) - \sum_{j\ne i} q_N^2 \alpha_N \nabla_1 W(X_{i,t}, X_{j,t})\, dt$$

where $(B_t^{(1)})_{t\ge 0}, \dots, (B_t^{(N)})_{t\ge 0}$ are i.i.d. standard Brownian motions on $\mathbb{R}^d$. From a partial differential equations point of view, the probability measure P_N is the steady state solution of the Fokker-Planck evolution equation $\partial_t - L = 0$ where L is the elliptic Markov diffusion operator (second order linear differential operator without constant term)

$$L := \frac{\alpha_N}{\beta_N}(\Delta - \beta_N \nabla H_N \cdot \nabla),$$

acting as $Lf = \frac{\alpha_N}{\beta_N}(\Delta f - \langle \beta_N \nabla H_N, \nabla f\rangle)$. This self-adjoint operator in $\mathrm{L}^2(P_N)$ is the infinitesimal generator of the Markov semigroup $(P_t)_{t\ge 0}$, $P_t(f)(x) := \mathbb{E}(f(X_t)|X_0 = x)$. Let us take $\alpha_N = \beta_N$

for convenience. In the case where $V(x) = |x|^2$ and $W \equiv 0$ (no interaction) then P_N is a standard Gaussian law $\mathcal{N}(0, I_{dN})$ on $(\mathbb{R}^d)^N$ and $(X_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck Gaussian process; while in the case where $d = 1$ and $V(x) = |x|^2$ and $W(x, y) = -\beta \log|x - y|$ of some fixed parameter $\beta > 0$ then P_N is the law of the spectrum of a β-Ensemble of random matrices and $(X_t)_{t\geq 0}$ is a so called Dyson Brownian motion [3]. If $\mu_{N,t}$ is the law of X_t then $\mathbb{E}\mu_{N,t} \to \mathbb{E}\mu_N$ weakly as $t \to \infty$. The study of the dynamic aspects is an interesting problem connected to McKean-Vlasov models [14, 22, 33].

1.5.11. *Calogero-(Moser-)Sutherland Schrödinger operators.* Let us keep the notations used above. We define $U_N := \beta_N H_N$ and we take $\beta_N = N^2$ for simplicity. Let us consider the isometry $\Theta : \mathrm{L}^2(P_N) \to \mathrm{L}^2(dx)$ defined by

$$\Theta(f)(x) := f(x)\sqrt{\frac{dP_N(x)}{dx}} = f(x)e^{-\frac{1}{2}(U_N(x)+\log(Z_N))}.$$

The differential operator $S := -\Theta L \Theta^{-1}$ is a Schrödinger operator:

$$S := -\Theta L \Theta^{-1} = -\Delta + Q, \quad Q := \frac{1}{4}|\nabla U_N|^2 - \frac{1}{2}\Delta U_N$$

which acts as $Sf = -\Delta f + Qf$. The operator S is self-adjoint in $\mathrm{L}^2(dx)$. Being isometrically conjugated, the operators $-L$ and S have the same spectrum, and their eigenspaces are isometric. In the case where $V(x) = |x|^2$ and $W \equiv 0$ (no interactions), we find that and $Q = \frac{1}{2}(1 - V)$, and S is a harmonic oscillator. On the other hand, following [23, Proposition 11.3.1], in the case $d = 1$ and $W(x, y) = -\log|x - y|$ (Coulomb interaction), then S is a Calogero-(Moser-)Sutherland Schrödinger operator:

$$S = -\Delta - E_0 + \frac{1}{4}\sum_{i=1}^N x_i^2 - \frac{1}{2}\sum_{1\leq i<j\leq N} \frac{1}{(x_i - x_j)^2}, \quad E_0 := \frac{N}{2} + \frac{N(N-1)}{2}.$$

More examples are given in [23, Proposition 11.3.2], related to classical ensembles of random matrices. The study of the spectrum and eigenfunctions of such operators is a wide subject, connected to Dunkl operators. These models attracted some attention due to the fact that for several natural choices of the potentials V, W, they are exactly solvable (or integrable). We refer to [23, Section 11.3.1], [20, Section 9.6], [16, Section 2.7] and references therein.

2. Proof of the large deviations principle — Theorem 1.1

The proof of Theorem 1.1 is split is several steps.

2.1. **A standard reduction.** To prove Theorem 1.1, we will use the following standard reduction (see for instance [17, Chapter 4]).

Proposition 2.1 (Standard reduction)**.** *Let $(Q_N)_{N\in\mathbb{N}^*}$ be a sequence of probability measures on some Polish space $(\mathcal{X}, d)$, $(Z_N)_{N\in\mathbb{N}^*}$ and $(\varepsilon_N)_{N\in\mathbb{N}^*}$ two sequences of positive numbers with $\varepsilon_N \to 0$ and $\mathcal{I} : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ be a function bounded from below.*

(1) Suppose that the sequence $(Q_N)_{N\in\mathbb{N}^*}$ satisfies the following conditions

(a) The sequence $(Z_N Q_N)_{N\in\mathbb{N}^*}$ is exponentially tight: for all $L \geq 0$ there exists a compact set $K_L \subset \mathcal{X}$ such that

$$\limsup_{N\to\infty} \varepsilon_N \log Z_N Q_N(\mathcal{X} \setminus K_L) \leq -L.$$

(b) For all $x \in \mathcal{X}$,

$$\lim_{r\to 0} \limsup_{N\to\infty} \varepsilon_N \log Z_N Q_N(B(x, r)) \leq -\mathcal{I}(x).$$

Then the sequence $(Z_N Q_N)_{N\in\mathbb{N}^*}$ satisfies the following large deviation upper bound: for all Borel set $A \subset \mathcal{X}$, it holds

$$\limsup_{N\to\infty} \varepsilon_N \log Z_N Q_N(A) \leq -\inf\{\mathcal{I}(\mu; \mu \in \mathrm{clo}(A))\}. \tag{2.1}$$

(2) If in addition $(Z_N Q_N)_{N\in\mathbb{N}^*}$ satisfies the following large deviation lower bound: for all Borel set $A \subset \mathcal{X}$,

$$-\inf\{\mathcal{I}(x); x \in \mathrm{int}(A)\} \leq \liminf_{N\to\infty} \varepsilon_N \log Z_N Q_N(A), \tag{2.2}$$

then $(Q_N)_{N\in\mathbb{N}^*}$ satisfies the full Large Deviation Principle with speed ε_N and rate function $\mathcal{I}_\star = \mathcal{I} - \inf_{x\in\mathcal{X}} \mathcal{I}(x)$, namely for all Borel set $A \subset \mathcal{X}$,

$$\begin{aligned}-\inf\{\mathcal{I}_\star(x); x \in \mathrm{int}(A)\} &\leq \liminf_{N\to\infty} \varepsilon_N \log Q_N(A)\\ &\leq \limsup_{N\to\infty} \varepsilon_N \log Q_N(A) \leq -\inf\{\mathcal{I}_\star(x); x \in \mathrm{clo}(A)\}.\end{aligned}$$

Proof. Let us begin by (1). Let $\delta > 0$; by assumption, for any $x \in \mathcal{X}$, there is $\eta_x > 0$ such that

$$\limsup_{N\to\infty} \varepsilon_N \log Z_N Q_N(B(x,\eta_x)) \leq -\mathcal{I}(x) + \delta.$$

If $F \subset \mathcal{X}$ is compact, there is a finite family $(x_i)_{1\leq i\leq m}$ of points of F such that $F \subset \cup_{i=1}^m B(x_i,\eta_{x_i})$. Therefore,

$$\begin{aligned}\limsup_{N\to\infty} \varepsilon_N \log Z_N Q_N(F) &\leq \limsup_{N\to\infty} \varepsilon_N \log\left(\sum_{i=1}^N Z_N Q_N(B(x_i,\eta_{x_i}))\right)\\ &= \max_{1\leq i\leq m} \limsup_{N\to\infty} \varepsilon_N \log\left(Z_N Q_N(B(x_i,\eta_{x_i}))\right)\\ &\leq \max_{1\leq i\leq m} -\mathcal{I}(x_i) + \delta \leq -\inf_F \mathcal{I} + \delta.\end{aligned}$$

Letting $\delta \to 0$ yields to (2.1) for $A = F$ compact.

Now if F is an arbitrary closed set, then for all $L > 0$, since $F \cap \mathcal{K}_L$ is compact, it holds

$$\begin{aligned}&\limsup_{N\to\infty} \varepsilon_N \log Z_N Q_N(F)\\ &\leq \max\left(\limsup_{N\to\infty} \varepsilon_N \log Z_N Q_N(F\cap K_L), \limsup_{N\to\infty} \varepsilon_N \log Z_N Q_N(K_L^c)\right)\\ &\leq \max(-\inf_{F\cap K_L} \mathcal{I}; -L).\end{aligned}$$

Letting $L \to \infty$ shows that (2.1) is true for arbitrary closed sets F. Since $A \subset \mathrm{clo}(A)$, the upper bound (2.1) holds for arbitrary Borel sets A.

To prove (2), take $A = \mathcal{X}$ in (2.2) and (2.1) to get:

$$\lim_{N\to\infty} \varepsilon_N \log(Z_N) = -\inf \mathcal{I} \in \mathbb{R}.$$

Subtracting this to (2.2) and (2.1) gives the large deviations principle with rate function $\mathcal{I}_\star$. □

In our context, $\mathcal{X} = \mathcal{M}_1(\mathbb{R}^d)$ is equipped with the Fortet–Mourier distance (1.3).

2.2. **Properties of the rate function.** In the following lemma, we prove different properties of the rate function $I_\star$ including those announced in Theorem 1.1, point (1).

Lemma 2.2 (Properties of the rate function)**.** *Under Assumptions **(H1)-(H2)-(H3)**,*

(1) $I_\star$ is well defined;
(2) $I_\star(\mu) < \infty$ implies $\int |V|\, d\mu < \infty$ and $\iint |W|\, d\mu^2 < \infty$;
(3) $I_\star(\mu) < \infty$ for any compactly supported probability μ with a bounded density with respect to Lebesgue;
(4) $I_\star$ is a good rate function (i.e. the levels sets $\{I_\star \leq k\}$ are compact).

Proof. Let us define $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, +\infty]$ by $\varphi(x,y) := \frac{1}{2}(V(x) + V(y) + W(x,y))$.

(1) Since V is continuous and $V(x) \to \infty$ as $|x| \to \infty$ thanks to **(H2)**, the function V is bounded from below. Using **(H3)** it follows that φ is bounded from below. The functional $I_\star$ is thus well defined with values in $[0,\infty]$.

(2) Assume that $I(\mu) = \iint \varphi d\mu^2 < \infty$. Since V is bounded from below, $[V]_- \in \mathbf{L}^1(\mu)$. From **(H3)** and the definition of φ,

$$2\varphi(x,y) = V(x) + V(y) + W(x,y) \geq c + (1-\varepsilon_0)(V(x) + V(y)).$$

Therefore

$$(1-\varepsilon_0)([V]_+(x)+[V]_+(y)) \le 2\varphi(x,y) - c + (1-\varepsilon_0)([V]_-(x)+[V]_-(y)),$$

so $[V]_+ \in \mathbf{L}^1(\mu)$, and $\int |V|\,d\mu < \infty$. Since

$$c - \varepsilon_0 V(x) - \varepsilon_0 V(y) \le W(x,y) \le 2\varphi(x,y) - V(x) - V(y), \tag{2.3}$$

this implies that $W \in \mathbf{L}^1(\mu^2)$.

(3) It is clearly enough to prove that W is locally Lebesgue integrable on $\mathbb{R}^d \times \mathbb{R}^d$. Let K be a compact of $\mathbb{R}^d$; according to **(H2)** and **(H3)** the function W is bounded from below on $K \times K$. On the other hand, letting

$$\alpha_K(z) = \sup\{W(x,y); |z-y| \ge |z|, x, y \in K\},$$

we have $W(x,y) \le \alpha_K(x-y)$, for all $x, y \in K$. Assumption **(H1)** then easily implies that $(x,y) \mapsto \alpha_K(x-y)$ is integrable on $K \times K$.

(4) According to the monotone convergence theorem,

$$I = \sup_{n\in\mathbb{N}} I_n, \quad I_n(\mu) := \iint \min(\varphi(x,y); n)\, d\mu(x) d\mu(y).$$

The functions $\min(\varphi, n)$ being bounded and continuous, it follows that the functionals I_n are continuous for the weak topology (see for instance [17, Lemma 7.3.12]). Being a supremum of continuous functions, I is lower semi-continuous. Set $b_\star = \inf \varphi$; we have, for every $\mu \in \mathcal{M}_1(\mathbb{R}^d)$, $L > 0$,

$$\begin{aligned} I(\mu) - b_\star &= \iint (\varphi(x,y) - b_\star)\, d\mu(x) d\mu(y) \\ &\ge \iint \mathbf{1}_{|x|>L, |y|>L} (\varphi(x,y) - b_\star)\, d\mu(x) d\mu(y) \\ &\ge (b_L - b_\star)\mu(|x| > L)^2, \end{aligned}$$

where $b_L := \inf_{|x|>L, |y|>L} \varphi(x,y)$. According to **(H2)** and **(H3)**, we see that $b_L \to +\infty$ as $L \to +\infty$. Therefore, there exists $L_\star > 0$ such that $b_L > b_\star$ for every $L > L_\star$. We get then for every real number $r \ge b_\star$,

$$\{\mu \in \mathcal{M}_1(\mathbb{R}^d) : I(\mu) \le r\} \subset \left\{ \mu \in \mathcal{M}_1(\mathbb{R}^d) : \mu(|x| > L) \le \sqrt{\frac{r - b_\star}{b_L - b_\star}}, L > L_\star \right\}.$$

Since $b_L \to +\infty$ as $L \to +\infty$, the subset of $\mathcal{M}_1(\mathbb{R}^d)$ in the right hand side is tight, and the Prohorov theorem implies then that it is relatively compact for the topology of $\mathcal{M}_1(\mathbb{R}^d)$. Since I is lower semi-continuous, the set $\{I \le r\}$ is also closed, which completes the proof.

□

2.3. **Proof of the upper bound.** For all $N \ge 1$, one denotes by Q_N the law of $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ under the probability P_N defined by (1.2): Q_N is an element of $\mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^d))$.

Lemma 2.3 (Exponential tightness)**.** *If $\beta_N \gg N$ then, under Assumptions **(H2)-(H3)**, the sequence of measures $(Z_N Q_N)_N$ is exponentially tight: for all $L \ge 0$ there exists a compact set $K_L \subset \mathcal{M}_1(\mathbb{R}^d)$ such that*

$$\limsup_{N\to\infty} \frac{\log Z_N Q_N\big(\mathcal{M}_1(\mathbb{R}^d) \setminus K_L\big)}{\beta_N} \le -L. \tag{2.4}$$

Proof. For any $L \ge 0$, let $L' := \frac{L - c/2}{1-\varepsilon_o}$ and set $K_L := \{\mu \in \mathcal{M}_1(\mathbb{R}^d); \int V\,d\mu \le L'\}$. Since **(H2)** holds, $V(x) \to \infty$ when $|x| \to +\infty$ and V is continuous. By Prohorov's theorem on tightness this implies that K_L is compact in $\mathcal{M}_1(\mathbb{R}^d)$.

It remains to check (2.4). Let us consider the law $\nu_V \in \mathcal{M}_1(\mathbb{R}^d)$ defined by

$$d\nu_V(x) := \frac{e^{-V(x)}}{C_V}\,dx, \qquad C_V := \int e^{-V(x)}\,dx > 0. \tag{2.5}$$

Using (2.3) to bound W from below, we get

$$\begin{aligned}
&Z_N Q_N\Big(\int V\,d\mu_N > L'\Big)\\
&\quad= \int_{(\mathbb{R}^d)^N} \mathbf{1}_{\{\int V\,d\mu_N > L'\}} \exp\left(-\frac{\beta_N}{2}\iint_{\neq} W\,d\mu_N^2 - \beta_N\int V\,d\mu_N\right)dx\\
&\quad\le \int_{(\mathbb{R}^d)^N} \mathbf{1}_{\{\int V\,d\mu_N > L'\}} \exp\left(-\frac{\beta_N}{2}\iint_{\neq} (c-\varepsilon_o(V(x)+V(y)))\,d\mu_N^2 - \beta_N\int V\,d\mu_N\right)dx\\
&\quad= \int_{(\mathbb{R}^d)^N} \mathbf{1}_{\{\int V\,d\mu_N > L'\}} \exp\left(-\frac{\beta_N}{2}c\frac{N-1}{N} - \beta_N\Big(1-\varepsilon_o\frac{N-1}{N}\Big)\int V\,d\mu_N\right)dx\\
&\quad= C_V^N \int_{(\mathbb{R}^d)^N} \mathbf{1}_{\{\int V\,d\mu_N > L'\}} \exp\left(-\frac{\beta_N}{2}c\frac{N-1}{N} - \left(\beta_N\Big(1-\varepsilon_o\frac{N-1}{N}\Big)-N\right)\int V\,d\mu_N\right)d\nu_V^{\otimes N}(x).
\end{aligned}$$

Now, if N is large enough, then $\beta_N\big(1-\varepsilon_o\frac{N-1}{N}\big)\ge N$, so that

$$Z_N Q_N\Big(\int V\,d\mu_N > L'\Big) \le C_V^N \exp\left(-\frac{\beta_N}{2}c\frac{N-1}{N}\right)\exp\left(-\left(\beta_N\Big(1-\varepsilon_o\frac{N-1}{N}\Big)-N\right)L'\right).$$

Therefore, when N is large enough, using the fact that $\beta_N \gg N$,

$$\begin{aligned}
\frac{\log Z_N Q_N\big(\int V\,d\mu_N > L'\big)}{\beta_N} &\le \frac{N\log C_V}{\beta_N} - \frac{1}{2}c\frac{N-1}{N} - \left(\Big(1-\varepsilon_o\frac{N-1}{N}\Big)-\frac{N}{\beta_N}\right)L'\\
&= -\frac{1}{2}c - (1-\varepsilon_o)L' + o_{N\to\infty}(1)\\
&= -L + o_{N\to\infty}(1).
\end{aligned}$$

This implies (2.4) and concludes the proof. □

Proposition 2.4 (Upper bound)**.** *If $\beta_N \gg N$ then, under Assumptions **(H2)-(H3)**, for all $r\ge 0$, for all $\mu\in\mathcal{M}_1(\mathbb{R}^d)$,*

$$\lim_{r\to 0}\limsup_{N\to+\infty}\frac{\log Z_N Q_N(B(\mu,r))}{\beta_N} \le -I(\mu),$$

where the ball $B(\mu,r)$ is defined for the Fortet–Mourier distance (1.3).

Proof. In contrast with the proof of Lemma 2.3, our objective now is to keep enough empirical terms inside the exponential in order to get $I(\mu)$ at the limit. Introduce $\varphi(x,y)=\frac{1}{2}(W(x,y)+V(x)+V(y))$, $x,y\in\mathbb{R}^d$. According to **(H3)**, it holds

$$\varphi(x,y)\ge \frac{c}{2}+\frac{1-\varepsilon_o}{2}(V(x)+V(y)),\qquad \forall x,y\in\mathbb{R}^d, \tag{2.6}$$

for some $c\in\mathbb{R}$ and $\varepsilon_o\in(0,1)$. Define $\lambda_N=\frac{N^2}{(1-\varepsilon_o)(N-1)}$ and let us bound the function H_N from below using (2.6) at the third line: for all $n\in\mathbb{N}$, it holds

$$\begin{aligned}
\beta_N H_N(x) &= \beta_N\left(\frac{1}{2}\iint_{\neq} W\,d\mu_N^2 + \int V\,d\mu_N\right) = \beta_N\left(\iint_{\neq}\varphi\,d\mu_N^2 + \frac{1}{N}\int V\,d\mu_N\right)\\
&\ge (\beta_N-\lambda_N)\iint_{\neq}\varphi\,d\mu_N^2 + \lambda_N\iint_{\neq}\varphi\,d\mu_N^2 + \frac{\beta_N}{N}\min V\\
&\ge (\beta_N-\lambda_N)\iint_{\neq}\varphi\,d\mu_N^2 + \lambda_N\frac{(N-1)c}{2N} + N\int V\,d\mu_N + \frac{\beta_N}{N}\min V\\
&\ge (\beta_N-\lambda_N)\iint \varphi\wedge n\,d\mu_N^2 - (\beta_N-\lambda_N)\frac{n}{N} + \lambda_N\frac{(N-1)c}{2N} + N\int V\,d\mu_N + \frac{\beta_N}{N}\min V\\
&= (\beta_N-\lambda_N)\iint \varphi\wedge n\,d\mu_N^2 + N\int V\,d\mu_N + o(\beta_N),
\end{aligned}$$

since $\beta_N\gg N$ and $\lambda_N = O(N)$.

Denoting by $I_n(\nu) = \iint \varphi \wedge n \, d\nu^2$, $\nu \in \mathcal{M}_1(\mathbb{R}^d)$, and using the preceding lower bound, we see that for every $\mu \in \mathcal{M}_1(\mathbb{R}^d)$, $r \geq 0$ and $N \gg 1$, we have

$$\begin{aligned} Z_N Q_N(B(\mu,r)) &= \int_{(\mathbb{R}^d)^N} \mathbf{1}_{B(\mu,r)}(\mu_N) \exp\left(-\beta_N H_N(x)\right) dx \\ &\leq e^{o(\beta_N)} \int_{(\mathbb{R}^d)^N} \mathbf{1}_{B(\mu,r)}(\mu_N) \exp\left(-(\beta_N - \lambda_N) I_n(\mu_N)\right) \prod_{i=1}^N e^{-V(x_i)} dx \\ &= C_V^N e^{o(\beta_N)} \int_{(\mathbb{R}^d)^N} \mathbf{1}_{B(\mu,r)}(\mu_N) \exp\left(-(\beta_N - \lambda_N) I_n(\mu_N)\right) d\nu_V^N \\ &\leq C_V^N e^{o(\beta_N)} e^{-(\beta_N - \lambda_N) \inf_{\nu \in B(\mu,r)} I_n(\nu)}, \end{aligned}$$

where the definition of ν_V is given by (2.5).

Therefore, since $\beta_N \gg N$ and $\lambda_N = O(N)$,

$$\limsup_{N \to +\infty} \frac{\log Z_N Q_N(B(\mu,r))}{\beta_N} \leq - \inf_{\nu \in B(\mu,r)} I_n(\nu).$$

Since $\varphi \wedge n$ is bounded continuous, the functional I_n is continuous for the weak topology. As a result, it holds

$$\lim_{r \to 0} \inf_{\nu \in B(\mu,r)} I_n(\nu) = I_n(\mu).$$

Finally, the monotone convergence theorem implies that $\sup_{n \geq 1} I_n(\mu) = I(\mu)$, which ends the proof. □

Using this Proposition, Lemma 2.3 and the first point of Proposition 2.1, we get the upper bound of Theorem 1.1, point (2).

2.4. **The lower bound and the full LDP.** In what follows, we denote by $|A|$ the Lebesgue measure of a Borel set $A \subset \mathbb{R}^n$.

Proposition 2.5 (Lower bound for regular probabilities)**.** *Under the assumptions **(H1)-(H2)-(H3)**, if $\beta_N \gg N \log(N)$, then for all probability measure μ on $\mathbb{R}^d$ supported in a box $B = \prod_{i=1}^d [a_i, b_i]$, $a_i, b_i \in \mathbb{R}$, with a density h with respect to the Lebesgue measure such that, for some $\delta > 0$, $\delta \leq h \leq \delta^{-1}$ on B, it holds*

$$\liminf_{N \to \infty} \frac{\log Z_N Q_N(B(\mu,r))}{\beta_N} \geq -I(\mu), \qquad \forall r \geq 0,$$

where $B(\mu,r)$ is the open ball of radius r centered at μ for the Fortet–Mourier distance (1.3).

If B is the box $\prod [a_k, b_k]$ in $\mathbb{R}^d$, let $l(B)$ and $L(B)$ be the minimum (resp. maximum) edge length:

$$l(B) = \min_k (b_k - a_k), \quad L(B) = \max_k (b_k - a_k)$$

We admit for a moment the following result:

Lemma 2.6 (Existence of nice partitions)**.** *For all d and all $\delta > 0$ there exists a constant $C(d,\delta)$ such that the following holds. For any box B, any integer n, and any measure μ with a density h w.r.t. Lebesgue measure, if $\delta \leq h \leq \delta^{-1}$, then there exists a partition $(B_1, B_2 \dots B_n)$ of B in n sub-boxes, such that:*

(1) B is split in equal parts: for all i, $\mu(B_i) = \frac{1}{n}\mu(B)$;
(2) The edge lengths of the B_i are controlled:

$$\frac{1}{C(d,\delta) n^{1/d}} l(B) \leq l(B_i) \leq L(B_i) \leq \frac{C(d,\delta)}{n^{1/d}} L(B).$$

Proof of Proposition 2.5. For each N we apply Lemma 2.6 to obtain a partition of B in N boxes $B_1^N, \dots B_N^N$. Let d_N be the maximum diameter of the boxes: by the lemma, since $\mu(B) = 1$,

$$\frac{c_1}{N^{1/d}} \leq l(B_i^N) \quad \text{and} \quad d_N \leq \frac{c_2}{N^{1/d}},$$

where c_1 and c_2 only depend on B, d and δ.

Note that, for all 1-Lipschitz function f with $\|f\|_\infty \leq 1$, if $x_i \in B_i^N$ for all $i \leq N$, since $\mu(B_i) = 1/N$ we have:

$$\left|\frac{1}{N}\sum_{i=1}^N f(x_i) - \int f\,d\mu\right| \leq \sum_{i=1}^N \int_{B_i^N} |f(x) - f(x_i)|\,d\mu(x) \leq d_N.$$

If N is large enough, $d_N \leq r$, which implies that

$$\left\{(x_1, \dots x_n) \in B_1^N \times \cdots \times B_N^N\right\} \subset \{\mu_N \in B(\mu, r)\}.$$

Let us denote by $C_i^N \subset B_i^N$ the box obtained from B_i^N by an homothetic transformation of center the center of B_i^N and ratio (say) $1/2$. It holds,

$$Z_N Q_N(B(\mu, r)) \geq \exp\left(-\frac{\beta_N}{N}\sum_{i=1}^N \max_{C_i^N} V - \frac{\beta_N}{N^2}\sum_{i<j} \max_{C_i^N \times C_j^N} W\right) \prod_{i=1}^N |C_i^N|.$$

Since $\left|C_i^N\right| \geq (l(B_N^i)/2)^d \geq c_3/N$ for some absolute constant c_3, we have

$$\frac{\log \prod_{i=1}^N |C_i^N|}{\beta_N} \geq \frac{N\log(c_3)}{\beta_N} - \frac{N\log(N)}{\beta_N} \underset{N\to\infty}{\longrightarrow} 0$$

thus we conclude that

$$\liminf_{N\to+\infty} \frac{\log\left(Z_N Q_N(B(\mu, r))\right)}{\beta_N} \geq -\limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \max_{C_i^N} V - \limsup_{N\to\infty} \frac{1}{N^2}\sum_{i<j} \max_{C_i^N \times C_j^N} W.$$

For all N, consider the locally constants functions $V_N : B \to \mathbb{R}$ and $W_N : B \times B \to \mathbb{R}$ defined by

$$\forall x \in B_i^N,\ V_N(x) := \max_{C_i^N} V \quad \text{and} \quad \forall (x, y) \in B_i^N \times B_j^N,\ W_N(x, y) := \max_{C_i^N \times C_j^N} W.$$

Since $\mu(B_i^N) = 1/N$, it holds

$$\frac{1}{N}\sum_{i=1}^N \max_{C_i^N} V = \int_B V_N(x)\,d\mu(x) \quad \text{and} \quad \frac{1}{N^2}\sum_{i<j} \max_{C_i^N \times C_j^N} W = \frac{1}{2}\int_{x\neq y} W_N(x, y)\,d\mu(x)d\mu(y).$$

The uniform continuity of V on B immediately implies that V_N converges uniformly to V, and so

$$\int V_N\,d\mu \to \int V\,d\mu.$$

For the same reason W_N converges uniformly to W on

$$(B \times B) \cap \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |x - y| \geq u\},$$

for all $u > 0$. According to **(H2)** and **(H3)**, the function W is bounded from below on $B \times B$. It follows that the functions W_N are bounded from below by some constant independent on N. To apply the dominated convergence theorem it remains to bound W_N from above by some integrable function. Let

$$\alpha_B(u) := \sup_{|x-y|\geq u} W(x, y),$$

so that $W(x, y) \leq \alpha_B(|x - y|)$. Obviously

$$\max_{(x,y)\in B_i^N \times B_j^N} |x - y| \leq 2d_N + \min_{(x,y)\in C_i^N \times C_j^N} |x - y|.$$

By construction, since $i \neq j$, we have

$$\min_{(x,y)\in C_i^N \times C_j^N} |x - y| \geq \frac{1}{4}(l(B_i^N) + l(B_j^N)) \geq \frac{c_1}{4}N^{-1/d} \geq \frac{c_1}{4c_2}d_N.$$

Therefore, there is an absolute constant c_4 such that

$$\min_{(x,y)\in C_i^N \times C_j^N} |x - y| \geq c_4 \max_{(x,y)\in B_i^N \times B_j^N} |x - y|.$$

Since the function α_B is non-increasing, it holds

$$\max_{C_i^N \times C_j^N} \alpha_B(|x - y|) \leq \min_{(x,y)\in B_i^N \times B_j^N} \alpha_B(c_4|x - y|).$$

We conclude from this that $W_N(x,y) \leq \alpha_B(c_4|x-y|)$, $x \neq y$. It follows from Assumption **(H1)** that the function $\alpha_B(c_4|x-y|)$ is integrable on $B \times B$ with respect to Lebesgue. Since the density of μ with respect to Lebesgue is bounded from above this function is integrable on $B \times B$ with respect to μ^2. Applying the dominated convergence theorem, we conclude that

$$\begin{aligned}\liminf_{N\to\infty} \frac{\log\left(Z_N Q_N(B(\mu,r))\right)}{\beta_N} &\geq -\int V(x)\,d\mu(x) - \frac{1}{2}\iint W(x,y)\,d\mu(x)d\mu(y)\\ &= -I(\mu).\end{aligned}$$

□

Let us now prove that "nice" partitions exist.

Proof of Lemma 2.6. The proof is an induction on the dimension d.

Base case. Let $d = 1$, and suppose that $B = [a_0, b_0]$. Since μ has a density, there exists "quantiles" $a_0 = q_0 < q_1 \ldots < q_n = b_0$ such that

$$\forall 1 \leq i \leq n, \quad \mu([q_{i-1}, q_i]) = \frac{1}{n}\mu(B).$$

In this simple case $l(B) = L(B) = b_0 - a_0$, and $l(B_i) = L(B_i) = (q_i - q_{i-1})$. The boundedness assumption on h implies that

$$\begin{aligned}\delta(q_i - q_{i-1}) \leq \mu([q_{i-1}, q_i]) \leq \frac{1}{\delta}(q_i - q_{i-1})\\ \delta(b_0 - a_0) \leq \mu(B) \leq \frac{1}{\delta}(b_0 - a_0),\end{aligned}$$

and the claim holds for $d = 1$ with $C(1,\delta) = 1/\delta^2$.

Induction step. Suppose that the statement holds for a dimension $d-1$. Let $B = [a_0, b_0] \times B'$ be a box in dimension d (where B' is a $(d-1)$-dimensional box). Let μ_0 be the first marginal of μ (this is a measure on $[a_0, b_0] \subset \mathbb{R}$).

Let $b = \lfloor n^{1/d} \rfloor$ be the integer part of $n^{1/d}$, and let $b_0 = 1/\left(2^{1/d} - 1\right)$. If $b \leq b_0$, we reason as in the base case, on the one-dimensional measure μ_0, to find a partition of B in n slices of mass $\mu(B)/n$. Since the number of slices is less than the constant $(b_0+1)^d$, the edge length is controlled as needed.

If $b > b_0$, we look for a decomposition of n as a sum of b integers n_i, each as close to $n^{(d-1)/d}$ as possible: the idea is to cut B along the first dimension in b slices, and to apply the induction hypothesis to cut the slice i in n_i parts.

To this end, decompose the integer n in base b:

$$\exists \alpha_0, \alpha_1, \ldots \alpha_d \in \{0, \ldots b-1\}^{d+1}, \quad n = \sum_{k=0}^{d} \alpha_k b^k.$$

The condition $b > b_0$ guarantees that $b + 1 < 2^{1/d} b$, which implies that $\alpha_d = 1$. Therefore:

$$\exists \alpha_0, \alpha_1, \ldots \alpha_{d-1} \in \{0, \ldots b-1\}^{d}, \quad n = b^d + \sum_{k=0}^{d-1} \alpha_k b^k.$$

Writing $\alpha_k = \sum_{i=1}^{b} \mathbf{1}_{\{i \leq \alpha_k\}}$ we get

$$n = \sum_{i=1}^{b} \left(b^{d-1} + \sum_{k=0}^{d-1} \mathbf{1}_{\{i \leq \alpha_k\}} b^k \right) = \sum_{i=1}^{b} n_i,$$

where $n_i = b^{d-1} + \sum_{k=0}^{d-1} \mathbf{1}_{\{i \leq \alpha_k\}} b^k$. From this expression, we get the bound $b^{d-1} \leq n_i \leq (b^d - 1)/(b-1)$. Since $k - 1 \geq k/2$ whenever $k \geq 2$, using the inequalities $b \leq n^{1/d}$ and $b \geq n^{1/d} - 1 \geq \frac{1}{2} n^{1/d}$, we get

$$\frac{1}{2^{d-1}} n^{(d-1)/d} \leq n_i \leq 2 n^{(d-1)/d}.$$

Now let us cut B along its first dimension. Recall that μ_0 is the first marginal of μ. By continuity there exist quantiles $a_0 = q_0 < q_1 < \ldots q_b = b_0$ such that

$$\forall 1 \leq i \leq b, \quad \mu_1([q_{i-1}, q_i]) = \mu([q_{i-1}, q_i] \times B') = \frac{n_i}{n}\mu(B).$$

We apply the induction hypothesis separately for each $1 \leq i \leq b$, to the $(d-1)$-dimensional box B', with the measure

$$\mu_i(\cdot) = \mu([q_{i-1}, q_i] \times \cdot\,)$$

and the integer n_i to obtain a decomposition $B' = \bigcup_{j=1}^{n_i} B'_{i,j}$ such that:

(1) the edge lengths $B'_{i,j}$ are controlled,
(2) $\mu_i(B'_{i,j}) = \frac{1}{n_i}\mu_i(B')$.

Finally, for all $1 \leq i \leq b$ and all $1 \leq j \leq n_i$, let

$$B_{i,j} = [q_{i-1}, q_i] \times B'_{i,j}.$$

Let us check that the partition $B = \bigcup_i \bigcup_j B_{i,j}$ satisfies the requirements. By definition,

$$\mu(B_{i,j}) = \mu_i(B'_{i,j}) = \frac{1}{n_i}\mu_i(B') = \frac{1}{n}\mu(B),$$

so the first requirement is met. To control the edge lengths, first remark that

$$l(B) = \min(b_0 - a_0, l(B')) \qquad L(B) = \max(b_0 - a_0, L(B'))$$
$$l(B_{i,j}) = \min(q_i - q_{i-1}, l(B'_{i,j})) \qquad L(B_{i,j}) = \max(q_i - q_{i-1}, L(B'_{i,j})).$$

By the induction hypothesis, the bounds on n_i and the fact that $L(B') \leq L(B)$ we get:

$$\begin{aligned} L(B'_{i,j}) &\leq \frac{C(d-1,\delta)}{n_i^{1/(d-1)}} L(B') \\ &\leq \frac{2C(d-1,\delta)}{n^{1/d}} L(B). \end{aligned}$$

On the other hand, reasoning as in the proof of the base case,

$$\begin{aligned} (q_i - q_{i-1})|B'| &\leq \frac{1}{\delta}\frac{n_i}{n}\mu(B) \\ \mu(B) &\leq \frac{1}{\delta}(b_0 - a_0)|B'| \end{aligned}$$

so

$$(q_i - q_{i-1}) \leq (b_0 - a_0)\delta^{-2}\frac{n_i}{n} \leq L(B)\frac{2\delta^{-2}}{n^{1/d}}.$$

Therefore $L(B_{i,j}) \leq C(d,\delta)n^{-1/d}L(B)$. The proof of the lower bound on $l(B_{i,j})$ follows the same lines and is omitted. This concludes the induction step, and the lemma is proved. □

Corollary 2.7 (Lower bound)**.** *Under the assumptions* ***(H1)-(H2)-(H3)****, if* $\beta_N \gg N\log(N)$*, then for all* $A \subset \mathcal{M}_1(\mathbb{R}^d)$*, it holds*

$$\liminf_{N\to\infty} \frac{\log Z_N Q_N(A)}{\beta_N} \geq -\inf\{I(\eta); \eta \in \mathrm{int}(A), \eta \ll \text{Lebesgue}\}.$$

Proof. Let $A \subset \mathcal{M}_1(\mathbb{R}^d)$ be a Borel set and let $\eta \in \mathrm{int}(A)$ be absolutely continuous with respect to Lebesgue with density h and such that $I(\eta) < +\infty$. For some sequence $(\varepsilon_n)_{n\geq 1}$ converging to 0, let us define, for all $n \geq 1$,

$$\eta_n := (1-\varepsilon_n)\nu_n + \varepsilon_n\lambda_n,$$

where $d\nu_n(x) = \frac{1}{C_n}\min(h(x); n)\mathbf{1}_{[-n;n]^d}(x)\,dx$ and $d\lambda_n(x) = \frac{1}{(2n)^d}\mathbf{1}_{[-n;n]^d}(x)\,dx$, where the normalizing constant $C_n \to 1$, when $n \to +\infty$.

According to point (3) of Lemma 2.2, we see that

$$I(\nu_n) < \infty, \quad I(\lambda_n) < \infty, \quad \iint \varphi(x,y)\,d\nu_n(x)d\lambda_n(y) < \infty$$

where $\varphi(x,y) := \frac{1}{2}(V(x) + V(y) + W(x,y))$ (this function takes its values in $(-\infty, +\infty]$ and is bounded from below thanks to **(H3)**, see the proof of Lemma 2.2). It holds

$$I(\eta_n) = (1-\varepsilon_n)^2 I(\nu_n) + 2\varepsilon_n(1-\varepsilon_n)\iint \varphi(x,y)\,d\nu_n(x)d\lambda_n(y) + \varepsilon_n^2 I(\lambda_n).$$

Choose ε_n converging to 0 sufficiently fast so that the last two terms above converge to 0 when $n \to \infty$. According to point (1) of Lemma 2.2, $V \in \mathbf{L}^1(\mu)$ and $W \in \mathbf{L}^1(\mu^2)$; it follows then easily

from the dominated convergence theorem that $I(\nu_n) \to I(\eta)$ when $n \to \infty$ and that η_n converges to η for the weak topology.

Let $r > 0$ be such that $B(\eta, 2r) \subset A$; for all n large enough, $B(\eta_n, r) \subset B(\eta, 2r) \subset A$. Since η_n satisfies the assumptions of Proposition 2.5, we conclude that for n large enough

$$\liminf_{N\to\infty} \frac{\log Z_N Q_N(A)}{\beta_N} \geq \liminf_{N\to\infty} \frac{\log Z_N Q_N(B(\eta_n, r))}{\beta_N} \geq -I(\eta_n).$$

Letting $n \to \infty$ and optimizing over $\{\eta \in A, \eta \ll \text{Lebesgue}\}$ gives the conclusion. □

End of the proof of Theorem 1.1. The properties of $I_\star$ and the upper bound in point (2) are already known. The lower bound of point (2) is given by Corollary 2.7.

To prove point (3), let $A \subset \mathcal{M}_1(\mathbb{R}^d)$ be some Borel set and take $\mu \in \text{int}(A)$. According to Assumption **(H4)**, there exists a sequence of absolutely continuous probability measures ν_n converging weakly to μ and such that $I(\nu_n) \to I(\mu)$, when $n \to \infty$. For all n large enough, $\nu_n \in A$ so applying Corollary 2.7, we conclude that

$$\liminf_{N\to\infty} \frac{\log Z_N Q_N(A)}{\beta_N} \geq -I(\nu_n).$$

Letting $n \to \infty$ and then optimizing over $\mu \in \text{int}(A)$ we arrive at

$$\liminf_{N\to\infty} \frac{\log Z_N Q_N(A)}{\beta_N} \geq -\inf\{I(\mu); \mu \in \text{int}(A)\}.$$

According to point (2) of Proposition 2.1, we conclude that Q_N obeys the full LDP. □

2.5. **Proof of the almost-sure convergence.** Let us establish the last part of Theorem 1.1. First note that since $I_\star$ has compact sublevel sets and is bounded from below, $I_\star$ attains is infimum, so $I_{\min}$ is not empty. For an arbitrary fixed real $\varepsilon > 0$, consider the complement of the ε-neighborhood of $I_{\min}$ for the Fortet–Mourier distance:

$$A_\varepsilon := (I_{\min})^c_\varepsilon := \{\mu \in \mathcal{M}_1 : d_{\text{FM}}(\mu, I_{\min}) > \varepsilon\}.$$

Since I is lower semi-continuous, $c_\varepsilon := \inf_{\mu\in A_\varepsilon} I(\mu) > 0$, thus $\mathbb{P}(\mu_N \in A_\varepsilon) \leq \exp(-\beta_N c_\varepsilon)$, by the upper bound of the full large deviation principle. By the first Borel–Cantelli lemma, it follows that almost surely, $\lim_{N\to\infty} d_{\text{FM}}(\mu_N, I_{\min}) = 0$.

2.6. **Sufficient conditions for (H4).** The following proposition gives several sufficient conditions under which assumption **(H4)** holds true. Even if some of these conditions are quite general, it is an open problem to find an even more general and natural condition. One may possibly find some inspiration in [10].

Proposition 2.8 (Sufficient conditions for **(H4)**). *Let $V : \mathbb{R}^d \to \mathbb{R}$ and $W : \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, +\infty]$ be symmetric, finite on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x); x \in \mathbb{R}^d\}$ and such that **(H2)** and **(H3)** hold true. Assumption **(H4)** holds in each of the following cases:*

(1) W is finite and continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

(2) For all $x \in \mathbb{R}^d$, the function $y \mapsto W(x, y)$ is super harmonic, i.e. W satisfies

$$W(x, y) \geq \frac{1}{|B(y, r)|} \int_{B(y,r)} W(x, z)\, dz, \qquad \forall r > 0,$$

where $|B(y, r)|$ denotes the Lebesgue measure of the ball of center y and radius r.

(3) The function W is such that $W(x+a, y+a) = W(x, y)$ for all $x, y, a \in \mathbb{R}^d$ and the function J defined by

$$J(\mu) = \iint W(x, y)\, d\mu(x) d\mu(y) \tag{2.7}$$

is convex on the set of compactly supported probability measures.

Proof. Let $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ be such that $I(\mu) < \infty$. Recall that, according to point (1) of Lemma 2.2, under the assumptions **(H2) - (H3)**, the condition $I(\mu) < \infty$ implies that

$$\int |V|\, d\mu < +\infty \quad \text{and} \quad \iint |W|\, d\mu^2 < +\infty.$$

Moreover, it follows follows from **(H2)** and **(H3)** that W is bounded from below on every compact, and so the definition (2.7) of $J(\mu)$ makes sense if μ is compactly supported.

For all $R > 0$, let us define μ_R as the normalized restriction of μ to $[-R;R]^d$. Using the dominated convergence theorem and point (1) of Lemma 2.2, it is not hard to see that μ_R converges weakly to μ and that $I(\mu_R) \to I(\mu)$ when $R \to +\infty$. To regularize μ_R, we consider $\mu_{R,\varepsilon} = \mathrm{Law}(X_R + \varepsilon U)$, $\varepsilon \leq 1$, where X_R is distributed according to μ_R and U is uniformly distributed on the Euclidean unit ball B_1 of $\mathbb{R}^d$. It is clear that $\mu_{R,\varepsilon}$ has a density with respect to Lebesgue. Moreover, $\mu_{R,\varepsilon} \to \mu_R$, when $\varepsilon \to 0$. Indeed, if $f : \mathbb{R}^d \to \mathbb{R}$ is continuous, it is bounded on $[-R;R]^d + B_1$, and it follows that

$$\int f \, d\mu_{R,\varepsilon} = \mathbb{E}[f(X_R + \varepsilon U)] \underset{\varepsilon \to 0}{\to} \mathbb{E}[f(X_R)],$$

This applies in particular to $f = V$. Now let us show in each cases that $J(\mu_{R,\varepsilon})$ converges to $J(\mu_R)$, when ε goes to 0. Let us write

$$J(\mu_{R,\varepsilon}) = \mathbb{E}[W(X_R + \varepsilon U, Y_R + \varepsilon V)],$$

where X_R, Y_R, U, V are independent and such that $Y_R \overset{(d)}{=} X_R$ and $V \overset{(d)}{=} U$.

(1) If W is finite and continuous on $\mathbb{R}^d \times \mathbb{R}^d$, then using the boundedness of W on $([-R;R]^d + B_1) \times ([-R;R]^d + B_1)$, it follows that $J(\mu_{R,\varepsilon}) \to J(\mu_R)$ when $\varepsilon \to 0$.

(2) If W is super harmonic, then $W_\varepsilon(x,y) := \mathbb{E}_{U,V}[W(x + \varepsilon U, y + \varepsilon V)] \leq W(x,y)$ for all x, y. Moreover, it follows from the continuity of W outside the diagonal that, for all $x \neq y$, $W_\varepsilon(x,y) \to W(x,y)$ when $\varepsilon \to 0$. Since $I(\mu) < +\infty$, μ does not have atoms and so the diagonal is of measure 0 for μ^2. It follows from the dominated convergence theorem that $J(\mu_{R,\varepsilon}) \to J(\mu_R)$ as $\varepsilon \to 0$.

(3) Denoting by μ_R^x the law of $X_R + x$, we see that $\mu_{R,\varepsilon} = \mathbb{E}_U[\mu_R^{\varepsilon U}]$. Therefore, the convexity of J yields to

$$J(\mu_{R,\varepsilon}) \leq \mathbb{E}_U[J(\mu_R^{\varepsilon U})] = \mathbb{E}_U\left[\iint W(x + \varepsilon U, y + \varepsilon U)\, d\mu_R(x) d\mu_R(y)\right] = J(\mu_R),$$

where the last equality comes from the property $W(x + a, y + a) = W(x,y)$. On the other hand, Fatou's lemma implies that $\liminf_{\varepsilon \to 0} J(\mu_{R,\varepsilon}) \geq J(\mu_R)$. Therefore $J(\mu_{R,\varepsilon}) \to J(\mu_R)$, when ε goes to 0.

We conclude from the above discussion that for any $\delta > 0$, it is possible to choose R sufficiently large and then ε sufficiently small so that $d_{\mathrm{FM}}(\mu_{R,\varepsilon}, \mu) \leq \delta$ and $|I(\mu_{R,\varepsilon}) - I(\mu)| \leq \delta$. This completes the proof. □

3. Tools from Potential Theory

In this section, we recall results from Potential Theory that will prove useful when we discuss the proof of Theorem 1.2 and Corollary 1.3. There are many textbooks on Potential Theory, with different point of views; our main source is [30], where the Riesz case is well-developed.

In this section, and unless otherwise stated, we set $k_\alpha := k_{\Delta_\alpha}$ and we take $W(x,y) := k_\alpha(x-y)$, $0 < \alpha < d$, $d \geq 1$. We denote respectively by $\mathcal{M}_1 \subset \mathcal{M}_\infty \subset \mathcal{M}_+ \subset \mathcal{M}_\pm$ the sets of probability measures, of positive measures integrating $k_\alpha(\cdot)\mathbf{1}_{|\cdot|>1}$, of positive measures, and of signed measures on $\mathbb{R}^d$.

3.1. Potentials and interaction energy.

We benefit from the constant sign of the Riesz kernel: $k_\alpha \geq 0$, contrary to the Coulomb kernel in dimension $d = 2$ and its logarithm. Following [30, p. 58], the *potential* of $\mu \in \mathcal{M}_+$ is the function $U_\alpha^\mu : \mathbb{R}^d \to [0, \infty]$ defined for every $x \in \mathbb{R}^d$ by

$$U_\alpha^\mu(x) := \int W(x,y)\, d\mu(y) = \int k_\alpha(x-y)\, d\mu(y). \tag{3.1}$$

Note that $U_\alpha^\mu(x) = \infty$ if μ has a Dirac mass at point x. By using the Fubini theorem, for every $\mu \in \mathcal{M}_+$, we have $U_\alpha^\mu < \infty$ Lebesgue almost everywhere iff $\mu \in \mathcal{M}_\infty$. This explains actually the condition $0 < \alpha < d$ taken in the Riesz potential, which is related to polar coordinates ($dx = r^{d-1}\, dr d\sigma_d$). In fact if $\mu \in \mathcal{M}_\infty$ then U_α^μ is a locally Lebesgue integrable function. Moreover, as Schwartz distributions, we have $U_\alpha^\mu = k_\alpha * \mu$ and, with the notations of (1.7),

$$-c_\alpha \Delta_\alpha U_\alpha^\mu = (-c_\alpha \Delta_\alpha k_\alpha) * \mu = \mu.$$

The interaction energy is the quadratic functional $J_\alpha : \mathcal{M}_+ \mapsto [0,\infty]$ defined by

$$J_\alpha(\mu) := \iint W(x,y)\, d\mu(x) d\mu(y) = \int U_\alpha^\mu \, d\mu.$$

Note that $J_\alpha(\mu) = \infty$ if μ has a Dirac mass, and in particular $J_\alpha(\mu_N) = \infty$.

In the Coulomb case where $\alpha = 2$, we have $c_2 J_2(\mu) = -\int U_2^\mu \Delta U_2^\mu \, dx = \int |\nabla U_2^\mu|^2 \, dx$. The quantity ∇U_2^μ is the (electric) field generated by the (Coulomb) potential U_2^μ and this explains the term "carré-du-champ" ("square of the field" in French) used for $J_2(\mu)$.

Lemma 3.1 (Positivity and convexity on $\mathcal{M}_+$)**.**

- *For every $\mu \in \mathcal{M}_+$ we have $J_\alpha(\mu) \geq 0$ with equality iff $\mu = 0$;*
- *$J_\alpha : \mathcal{M}_+ \mapsto [0,\infty]$ is strictly convex: for every $\mu, \nu \in \mathcal{E}_+$ with $\mu \neq \nu$, we have*

$$\forall t \in (0,1), \quad J_\alpha(t\mu + (1-t)\nu) < t J_\alpha(\mu) + (1-t) J_\alpha(\nu);$$

- *$\mathcal{E}_{\alpha,+} := \{\mu \in \mathcal{M}_+ : J_\alpha(\mu) < \infty\}$ is a convex cone.*

We recall that in classical Harmonic Analysis, a function $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a *positive definite kernel* when $\sum_{i=1}^n x_i K(x_i, x_j) \bar{x}_j \geq 0$ for every $n \geq 1$ and every $x \in \mathbb{C}^n$. If this holds only when $x_1 + \cdots + x_n = 0$, the kernel is said to be *weakly positive definite*. The famous Bochner theorem states that a kernel is positive definite if and only if it is the Fourier transform of a *finite* Borel measure. The famous Schoenberg theorem states for every $f : \mathbb{R}_+ \to \mathbb{R}_+$, the kernel $(x,y) \mapsto f(|x-y|^2)$ is positive definite on $\mathbb{R}^d$ for every $d \geq 1$ if and only if f is the Laplace transform of a *finite* Borel measure on $\mathbb{R}_+$. The famous Bernstein theorem states that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\mathcal{C}^\infty((0,\infty))$ then f is the Laplace transform of a finite Borel measure on $\mathbb{R}_+$ if and only if f is *completely monotone*: $(-1)^n f^{(n)} \geq 0$ for every $n \geq 0$. For all these notions, we refer to [6, 29].

The proof of Lemma 3.1 is short and self-contained. It relies on the fact that the convexity of the functional is equivalent to the fact that W is a weakly positive definite kernel, which is typically the case when W is a mixture of shifted Gaussian kernels, which are the most useful weakly positive definite kernels. For example this works if for some arbitrary measurable $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ and Borel measure η, and every $x, y \in \mathbb{R}^d$,

$$W(x,y) = w(|x-y|) = \int_0^\infty \left(e^{-\alpha^2(t)|x-y|^2} + \beta(t) \right) d\eta(t).$$

The shift β can be < 0, which allows non positive definite kernels such as the logarithmic kernel (note that the Riesz kernel is positive definite). The method is used for the logarithmic kernel in [4, Proof of Property 2.1 (4)] with the following mixture:

$$\log \frac{1}{|x-y|} = \int_0^\infty \frac{1}{2t} \left(e^{-\frac{|x-y|^2}{2t}} - e^{-\frac{1}{2t}} \right) dt.$$

This kernel has a sign change and a double singularity near zero and infinity, which can be circumvented by using a cutoff. Alternatively, one may proceed by regularization and use the Bernstein theorem with the completely monotone function $f(t) = (\varepsilon + t)^{-\beta}$, $\beta, \varepsilon > 0$, and then the Schoenberg theorem, see e.g. [35]. For instance, for the logarithmic kernel, the following representation is used in [27, Chapter 5]:

$$\log \frac{1}{\varepsilon + |x-y|} = \int_0^\infty \left(\frac{1}{\varepsilon + 1 + |x-y|} - \frac{1}{1+t} \right) dt.$$

Finally, let us mention that for the Riesz kernel, yet another short proof of Lemma 3.1, based on the formula $k_\alpha = c k_{\alpha/2} * k_{\alpha/2}$, can be found in [30, Theorem 1.15 p. 79].

Proof of Lemma 3.1. Set $\beta := d - \alpha$. We start from the identity,

$$\Gamma(1+\alpha) = c^{1+\alpha} \int_0^\infty t^\alpha e^{-ct} \, dt, \quad c > 0, \ \alpha > -1.$$

Taking $c = |x-y|^2$ and $1 + \alpha = \beta/2$, we get, for every $x, y \in \mathbb{R}^d$,

$$k_\alpha(x-y) = \int_0^\infty f(t)\, e^{-t|x-y|^2} \, dt, \quad \text{where} \quad f(t) := \frac{t^{\beta/2-1}}{\Gamma(\beta/2)}.$$

Now for every $\mu \in \mathcal{M}_+$ such that $J_\alpha(\mu) < \infty$,

$$J_\alpha(\mu) = \int_0^\infty f(t) \left(\iint e^{-t|x-y|^2} \, d\mu(x) d\mu(y) \right) dt.$$

Expressing the Gaussian kernel as the Fourier transform of a Gaussian kernel, we get, by writing $e^{i\langle x-y,w\rangle} = e^{i\langle x,w\rangle} e^{-i\langle y,w\rangle}$ and using the Fubini theorem,

$$\begin{aligned} \iint e^{-t|x-y|^2} \, d\mu(x) d\mu(y) &= (4\pi t)^{-d/2} \iint \left(\int_{\mathbb{R}^d} e^{i\langle x-y,w\rangle} e^{\frac{1}{4t}|w|^2} \, dw \right) d\mu(x) d\mu(y) \\ &= (4\pi t)^{-d/2} \int_{\mathbb{R}^d} \underbrace{\left| \int e^{i\langle x,w\rangle} \, d\mu(x) \right|^2}_{K_w(\mu)} e^{-\frac{1}{4t}|w|^2} \, dw. \end{aligned}$$

Now K_w is clearly convex since for every $\mu, \nu \in \mathcal{M}_1(\mathbb{R}^d)$ and every $t \in (0,1)$,

$$\frac{tK_w(\mu) + (1-t)K_w(\nu) - K_w(t\mu + (1-t)\nu)}{t(1-t)} = K_w(\mu - \nu) = \left| \int e^{i\langle x,w\rangle} \, d(\mu-\nu)(x) \right|^2 \geq 0.$$

It follows then that J_α is also convex as a conic combination of convex function. Let us establish now the strict convexity of J_α. Let us suppose that $\mu, \nu \in \mathcal{M}_1(\mathbb{R}^d)$ with $J_\alpha(\mu) < \infty$ and $J_\alpha(\nu) < \infty$ and $tJ_\alpha(\mu) + (1-t)J_\alpha(\nu) = J_\alpha(t\mu + (1-t)\nu)$ for some $t \in (0,1)$. Then

$$J_\alpha(\mu - \nu) = \frac{tJ_\alpha(\mu) + (1-t)J_\alpha(\nu) - J_\alpha(t\mu + (1-t)\nu)}{t(1-t)} = 0.$$

Arguing as before, we find

$$0 = J_\alpha(\mu - \nu) = \int_0^\infty f(t) \left[(4\pi t)^{-d/2} \int_{\mathbb{R}^d} K_w(\mu - \nu) \, e^{-\frac{1}{4t}|w|^2} \, dw \right] dt.$$

Hence, for every $t > 0$ (a single $t > 0$ suffices in what follows)

$$\int_{\mathbb{R}^d} K_w(\mu - \nu) \, e^{-\frac{1}{4t}|w|^2} \, dw = 0.$$

Thus, the Fourier transform of $\mu - \nu$ vanishes almost everywhere, and therefore $\mu = \nu$.

Finally, $\mathcal{E}_{\alpha,+}$ is clearly a cone, and its convexity comes from the convexity of J_α. □

Following [30, p. 62], for any $\mu = \mu_+ - \mu_- \in \mathcal{M}_\pm$ such that $\mu_\pm \in \mathcal{M}_\infty$, we have $U_\alpha^{\mu_\pm} < \infty$ Lebesgue almost everywhere, and we may define for Lebesgue almost every x

$$U_\alpha^\mu(x) := U_\alpha^{\mu_+}(x) - U_\alpha^{\mu_-}(x) \in (-\infty, +\infty).$$

Following [30, p. 77], for every $\mu = \mu_+ - \mu_- \in \mathcal{M}_\pm$ such that $\mu_\pm \in \mathcal{M}_\infty$ and

$$\int U_\alpha^{\mu_+} \, d\mu_- < \infty \quad \text{and} \quad \int U_\alpha^{\mu_-} \, d\mu_+ < \infty,$$

we may define $J_\alpha(\mu) \in (-\infty, +\infty]$ as (thanks to the Fubini theorem)

$$J_\alpha(\mu) := \int U_\alpha^\mu \, d\mu = \int U_\alpha^{\mu_+} \, d\mu_+ + \int U_\alpha^{\mu_-} \, d\mu_- - \int U_\alpha^{\mu_+} \, d\mu_- - \int U_\alpha^{\mu_-} \, d\mu_+.$$

More generally, for every $\mu_1, \mu_2 \in \mathcal{M}_\pm$ such that $\mu_{1\pm}, \mu_{2\pm} \in \mathcal{M}_\infty$ and

$$\int U_\alpha^{\mu_{1\pm}} d\mu_{2\mp} < \infty$$

we may define $J_\alpha(\mu_1, \mu_2) \in (-\infty, +\infty]$ by

$$J_\alpha(\mu_1, \mu_2) := \int U_\alpha^{\mu_1} \, d\mu_2 = \int U_\alpha^{\mu_{1+}} \, d\mu_{2+} + \int U_\alpha^{\mu_{1-}} \, d\mu_{2-} - \int U_\alpha^{\mu_{1+}} \, d\mu_{2-} - \int U_\alpha^{\mu_{1-}} \, d\mu_{2+}.$$

Following [30, p. 78], since k_α is symmetric, then the *reciprocity law* holds:

$$J_\alpha(\mu_1, \mu_2) = J_\alpha(\mu_2, \mu_1) \quad \text{i.e.} \quad \int U_\alpha^{\mu_1} \, d\mu_2 = \int U_\alpha^{\mu_2} \, d\mu_1.$$

Let $\mathcal{E}_\alpha$ be the set of elements of $\mathcal{M}_\pm$ for which J_α makes sense and is finite. As pointed out by N.S. Landkof [30] in his preface, a very nice idea going back to H. Cartan consists in seeing J_α

as a Hilbert structure on $\mathcal{E}_\alpha$. This idea is simply captured by the following lemma, which is the analogue of Lemma 3.1 for signed measures of finite energy.

Lemma 3.2 (Properties of $(\mathcal{E}_\alpha, J_\alpha)$)**.**

- *J_α is lower semi-continuous on $\mathcal{E}_\alpha$ for the vague topology (i.e. with respect to continuous functions with compact support);*
- *$\mathcal{E}_\alpha$ is a vector space and $(\mu_1, \mu_2) \mapsto J_\alpha(\mu_1, \mu_2)$ defines a scalar product on $\mathcal{E}_\alpha$.*

In particular for every $\mu \in \mathcal{E}_\alpha$, we have $J_\alpha(\mu) = J_\alpha(\mu, \mu) \geq 0$ with equality iff $\mu = 0$; and moreover, $J_\alpha : \mathcal{E}_\alpha \mapsto (-\infty, \infty)$ is strictly convex: for every $\mu, \nu \in \mathcal{E}_\alpha$ with $\mu \neq \nu$,

$$\forall t \in (0,1), \quad \frac{tJ_\alpha(\mu) + (1-t)J_\alpha(\nu) - J_\alpha(t\mu + (1-t)\nu)}{t(1-t)} = J_\alpha(\mu - \nu) > 0.$$

Proof. The lower semi-continuity for the vague convergence follows from the fact that $k_\alpha \geq 0$, see [30, p. 78]. The vector space nature of $\mathcal{E}_\alpha$ is immediate from its definition. The bilinearity of $(\mu_1, \mu_2) \mapsto J_\alpha(\mu_1, \mu_2)$ is immediate. By reasoning as in the proof of Lemma 3.1, we get $J_\alpha(\mu, \mu) \geq 0$ for every $\mu \in \mathcal{E}_\alpha$, with equality iff $\mu = 0$. □

Following [30, Theorem 1.18 and 1.19 p. 90], for this pre-Hilbertian topology, it can be shown that $\mathcal{E}_{\alpha,+}$ is complete while $\mathcal{E}_\alpha$ is not complete if $\alpha > 1$, and that J_α is not continuous for the vague topology.

3.2. **Capacity and "approximately/quasi everywhere".** The notion of capacity is central in Potential Theory. We just need basic facts on zero-capacity sets. Once more we follow the presentation of Landkof ([30, Chapter II.1]), to which we refer for additional details, references and proofs.

For any compact set K consider the minimization problem

$$W_\alpha(K) = \inf\{J_\alpha(\nu); \nu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha, \mathrm{supp}(\nu) \subset K\}.$$

The boundedness of K implies that $W_\alpha(K) \in (0, \infty]$. Its inverse $C_\alpha(K)$ is called the *capacity* of the compact set K. The capacity of K is zero if and only if there is no measure of finite energy supported in K.

On general sets on can define an "inner capacity" and an "outer capacity" by

$$\underline{C}_\alpha(A) = \sup\{C_\alpha(K), K \subset A, K \text{ compact}\},$$
$$\overline{C}_\alpha(A) = \inf\{\underline{C}_\alpha(O), A \subset O, O \text{ open}\}.$$

It can be shown (see [30, Theorem 2.8]) that if A is a Borel set, these two quantities coincide — A is said to be "capacitable" and the common value is called the capacity of A.

A property $P(x)$ is said to hold "approximately everywhere" if the set A of x such that $P(x)$ is false, has zero inner capacity, and "quasi-everywhere" if it has zero outer capacity. For many "reasonable" $P(x)$, the set A is Borel and the two notions coincide. The following result ([30, Theorem 2.1 & 2.2]) shows that, for such "reasonable" properties, "quasi-everywhere" means "ν-almost surely, for all measure ν of finite energy".

Theorem 3.3 (Zero capacity Borel sets)**.** *A Borel set A has zero capacity if and only if, for any measure ν of finite energy, $\nu(A) = 0$. In particular, if $C_\alpha(A) > 0$, A has a positive inner capacity, and there exists a compact $K \subset A$ and a probability measure ν of finite energy such that* $\mathrm{supp}(\nu) \subset K$.

3.3. **The Gauss averaging principle.** In the classical Coulombian case ($\alpha = 2$), we will need the following result, known as Gauss' averaging principle. In $\mathbb{R}^d$, for all $r > 0$, let σ_r be the surface measure on the sphere $\partial B(0, r)$; its total mass is $\sigma_d r^{d-1}$ where σ_d is the surface of the unit sphere.

Theorem 3.4 (Gauss' Averaging Principle)**.** *In $\mathbb{R}^d$,*

$$\frac{1}{r^{d-1}\sigma_d} \int_{\partial B(0,r)} \frac{1}{|x-y|^{d-2}}\, d\sigma_r(y) = \begin{cases} \frac{1}{r^{d-2}} & \text{if } |x| < r \\ \frac{1}{|x|^{d-2}} & \text{if } |x| > r. \end{cases}$$

This result can be found in [26], Lemma 1.6.1 p. 21.

4. Proof of the properties of the minimizing measure

The proof of Theorem 1.2 is decomposed in two steps. We begin by proving the existence, uniqueness and the support properties of $\mu_\star$ in Section 4.1. The characterization of $\mu_\star$ is proved in Section 4.2.

Recall that, for a probability measure μ, we have defined

$$I(\mu) = \frac{1}{2} J_\alpha(\mu) + \int V \, d\mu.$$

In this section we consider the following minimization problem:

$$\mathcal{P} : \inf \{ I(\mu), \mu \in \mathcal{M}_1 \}. \tag{4.1}$$

4.1. Existence, uniqueness and compactness of the support.

The existence of a minimizer for $\mathcal{P}$ is clear since we have already seen that I has compact level sets.

Since $I(\mu) < \infty$ implies that $\mu \in \mathcal{E}_\alpha$ and $\int V \, d\mu < \infty$, the problem $\mathcal{P}$ is equivalent to

$$\mathcal{P}_\alpha : \inf \left\{ I(\mu), \mu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha \text{ such that } V \in L^1(\mu) \right\} \tag{4.2}$$

in that they have the same values and the same minimizers. Let us call p the common value.

Suppose μ and ν are two measures in $\mathcal{M}_1 \cap \mathcal{E}_\alpha$ such that $V \in \mathrm{L}^1(\mu) \cap \mathrm{L}^1(\nu)$. Let $\psi : t \in [0,1] \mapsto [0, \infty)$ by

$$\begin{aligned} \psi(t) :&= I((1-t)\mu + t\nu) \\ &= \frac{1}{2} J_\alpha((1-t)\mu + t\nu) + (1-t) \int V \, d\mu + t \int V \, d\nu. \end{aligned} \tag{4.3}$$

By Lemma 3.2, ψ is strictly convex if $\mu \neq \nu$. If μ and ν minimize I, then they are in $\mathcal{M}_1 \cap \mathcal{E}_\alpha$, so ψ is well-defined, and since $\psi(0) = I(\mu) = I(\nu) = \psi(1)$, μ must be equal to ν. Therefore the minimizer $\mu_\star$ is unique.

Let us now prove that $\mu_\star$ has compact support. This result also holds in dimension 2 with the logarithmic potential, see [39, Theorem 1.3, p. 27]. To this end, let us define, for any compact K, a new minimization problem:

$$\mathcal{P}_K : \inf \{ I(\mu), \mu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha, \mathrm{supp}(\mu) \subset K \},$$

and let p_K be the value of $\mathcal{P}_K$.

Lemma 4.1 (Reduction to restricted optimization problem)**.** *Let K be a compact set, and suppose that $V(x) \geq 2p+3$ when $x \notin K$, where p is the common value of $\mathcal{P}$, $\mathcal{P}_\alpha$ (defined by (4.1) and (4.2)). Then the problems $\mathcal{P}$ and $\mathcal{P}_K$ are equivalent: their values p and p_K are equal, the minimizer exists and is the same. In particular, the minimizer $\mu_\star$ of the original problem $\mathcal{P}$ satisfies* $\mathrm{supp}(\mu_\star) \subset K$.

Proof. Suppose μ is such that $I(\mu) \leq p+1$. We will prove that, if $\mu(K) < 1$, we can find a μ_K, supported in K such that $I(\mu_K) < I(\mu)$. This clearly implies that the two values p_K and p coincide. Since we know that the minimizer $\mu_\star$ of the original problem exists, this also proves that it must be supported in K.

Let us now construct μ_K as the renormalized restriction of μ to K. First, remark that $\mu(K)$ cannot be zero, since

$$p + 1 \geq I(\mu) \geq (1 - \mu(K))(2p+3).$$

Therefore we can define

$$\mu_K(A) = \frac{1}{\mu(K)} \mu(K \cap A).$$

Since by assumption $\mu(K) < 1$, we may similarly define μ_{K^c}. The measure μ is the convex combination

$$\mu = \mu(K)\mu_K + (1 - \mu(K))\mu_{K^c}.$$

The positivity of V, W, and the choice of K imply that

$$\begin{aligned} I(\mu) &= \frac{1}{2} J_\alpha(\mu) + \mu(K) \int V \, d\mu_K + (1-\mu(K)) \int V \, d\mu_{K^c} \\ &\geq \frac{1}{2} \mu(K)^2 J_\alpha(\mu_K) + \mu(K)^2 \int V \, d\mu_K + (1-\mu(K))(2p+3), \end{aligned}$$

since $J_\alpha(\mu_{K^c})$ and the interaction energy $J_\alpha(\mu_K, \mu_{K^c})$ are both non negative. Therefore

$$I(\mu) \geq \mu(K)^2 I(\mu_K) + (1-\mu(K))(2p+3).$$

Assume that $I(\mu_K) \geq I(\mu)$. Then

$$I(\mu)(1-\mu(K)^2) \geq (1-\mu(K))(2p+3).$$

Using the fact that $I(\mu) \leq p+1$, and dividing by $1-\mu(K)$, we get

$$2(p+1) \geq (p+1)(1+\mu(K)) \geq 2p+3,$$

a contradiction. Therefore $I(\mu_K) < I(\mu)$, and the proof is complete. □

4.2. **A criterion of optimality.** In this section we prove the items (5), (6) and (7) of Theorem 1.2. The corresponding result in dimension 2 for the logarithmic potential can be found in [39, Theorem 3.3, p. 44]. We adapt it, using fully the pre-Hilbertian structure rather than the principle of domination when it is possible.

Proof of item (5) of Theorem 1.2. We already know that $\mu_\star$ has compact support. The first step is to show that $\mu_\star$ satisfies (1.8) and (1.9). Let $\mu = \mu_\star$, and let ν be in $\nu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha$ such that $V \in \mathrm{L}^1(\nu)$. Recall the function ψ from (4.3):

$$\psi(t) = I((1-t)\mu_\star + t\nu).$$

Since J_α is quadratic we get

$$\begin{aligned}\psi(t) &= \int V\, d\mu_\star + t\int V\, d(\nu-\mu_\star) + \frac{1}{2}J_\alpha(\mu_\star + t(\nu-\mu_\star)) \\ &= \int V\, d\mu_\star + t\int V\, d(\nu-\mu_\star) + \frac{1}{2}\big(J_\alpha(\mu_\star) + t^2 J_\alpha(\nu-\mu_\star) + 2tJ_\alpha(\mu_\star, \nu-\mu_\star)\big)\end{aligned}$$

Therefore

$$\psi'(t) = \int V\, d(\nu-\mu_\star) + tJ_\alpha(\nu-\mu_\star) + J_\alpha(\mu_\star, \nu-\mu_\star). \tag{4.4}$$

Since $\mu_\star$ minimizes I, $\psi'(0^+)$ must be non-negative.

$$\begin{aligned}0 &\leq \int V\, d(\nu-\mu_\star) + J_\alpha(\mu_\star, \nu-\mu_\star) \\ &\leq \int V\, d\nu + J_\alpha(\mu_\star, \nu) - \left(\int V\, d\mu_\star + J_\alpha(\mu_\star)\right) \\ &\leq \int (V + U_\alpha^{\mu_\star})\, d\nu - C_\star.\end{aligned}$$

Therefore:

$$\forall \nu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha, \quad \int (V + U_\alpha^{\mu_\star} - C_\star)\, d\nu \geq 0. \tag{4.5}$$

Since this holds for all ν, $V + U_\alpha^{\mu_\star}$ is greater than $C_\star$ quasi everywhere. Indeed, let $A = \{x, V(x) + U_\alpha^{\mu_\star}(x) < C_\star\}$. Since $V + U_\alpha^{\mu_\star}$ is measurable this is a Borel set. Suppose by contradiction that its capacity is strictly positive. By Proposition 3.3, there exist a compact set $K \subset A$ and a measure ν with finite energy supported in K. For this measure $\int V + U_\alpha^{\mu_\star}\, d\nu < C_\star$, which contradicts (4.5). This proves (1.8).

Let us prove (1.9). Suppose $V(x) + U_\alpha^{\mu_\star}(x) > C_\star$ for some $x \in \mathrm{supp}(\mu_\star)$. Since $V + U_\alpha^{\mu_\star}$ is lower semi-continuous, we can find a neighborhood $\mathcal{U}$ of x, and an $\eta > 0$ such that

$$\forall x \in \mathcal{U}, \quad V(x) + U_\alpha^{\mu_\star}(x) \geq C_\star + \eta.$$

Therefore

$$\int (V + U_\alpha^{\mu_\star})\, d\mu_\star \geq (C_\star + \eta)\mu_\star(\mathcal{U}) + \int_{\mathbb{R}^d \setminus \mathcal{U}} (V + U_\alpha^{\mu_\star})\, d\mu_\star.$$

Since $V + U_\alpha^{\mu_\star} \geq C_\star$ quasi everywhere, and $\mu_\star$ has finite energy, this holds $\mu_\star$ almost surely, so

$$C_\star = \int V + U_\alpha^{\mu_\star} d\mu_\star \geq C_\star + \eta\mu_\star(\mathcal{U}).$$

This is impossible since $\mu_\star(\mathcal{U}) > 0$, by definition of the support. Therefore (1.9) holds. □

Proof of item (6) of Theorem 1.2. Let $\mu \in \mathcal{E}_\alpha \cap \mathcal{M}_1(\mathbb{R}^d)$ be such that $V \in \mathrm{L}^1(\nu)$. It is enough to show that, if (1.12) and (1.13) hold, then $\mu = \mu_\star$. We argue by contradiction and suppose $\mu \neq \mu_\star$. Consider again the function ψ (with $\nu = \mu$): $\psi(t) = I((1-t)\mu_\star + t\mu)$, $t \in [0,1]$. According to Lemma 3.1, this function is strictly convex, therefore $\psi'(1) > \psi'(0) \geq 0$. The explicit expression of ψ' (Equation (4.4)) gives:

$$\begin{aligned} 0 < \psi'(1) &= \int V d(\mu - \mu_\star) + J_\alpha(\mu - \mu_\star) + J_\alpha(\mu_\star, \mu - \mu_\star) \\ &= \int V \, d\mu - \int V \, d\mu_\star + J_\alpha(\mu) - J_\alpha(\mu, \mu_\star). \end{aligned}$$

Therefore:

$$\int (U_\alpha^\mu + V) \, d\mu_\star < \int (U_\alpha^\mu + V) \, d\mu. \tag{4.6}$$

On the other hand, integrating (1.12) with respect to μ and (1.13) with respect to $\mu_\star$ yields:

$$\int (U_\alpha^\mu + V) \, d\mu \leq C \leq \int (U_\alpha^\mu + V) \, d\mu_\star,$$

which contradicts (4.6) and concludes the proof. □

To prove the last result of Theorem 1.2 we recall the following classical result.

Theorem 4.2 (Principle of domination)**.** *Suppose $\alpha \leq 2$. Let μ and ν be two positive measures in $\mathcal{E}_\alpha$, and c a non negative constant. If the inequality*

$$U_\alpha^\mu(x) \leq U_\alpha^\nu(x) + c$$

holds μ-almost surely, then it holds for all $x \in \mathbb{R}^d$.

Proof. In the Coulomb case $\alpha = 2$, [30, Theorem 1.27, p. 110] applies, since U_α^ν is positive and super-harmonic. If $\alpha < 2$, the potential U_α^ν is α-superharmonic, so we can apply [30, Theorem 1.29, p. 115] and get the result. □

Proof of item (7) of Theorem 1.2. We follow the proof of Theorem 1.3 in [18]. Arguing by contradiction, let us suppose that, for some measure μ, and some $\epsilon > 0$,

$$\sup_{\mathrm{supp}(\mu)} (U_\alpha^\mu + V) \leq C_\star - \epsilon.$$

By (1.9), this implies that

$$U_\alpha^\mu(x) + \epsilon \leq U_\alpha^{\mu_\star}(x),$$

for all $x \in \mathrm{supp}(\mu)$. Let η be the equilibrium (probability) measure of $\mathrm{supp}(\mu)$: $U_\alpha^\eta(x) = C_\eta$ on $\mathrm{supp}(\mu)$, therefore

$$U_\alpha^{\mu + (\epsilon/C_\eta)\eta} \leq U_\alpha^{\mu_\star}$$

for all x in $\mathrm{supp}(\mu)$. By the principle of domination this holds at infinity. Since for any compactly supported μ, $U_\alpha^\mu(x) \sim \frac{\mu(\mathbb{R}^d)}{|x|^{d-\alpha}}$ at infinity, we get a contradiction:

$$(1 + \epsilon/C_\eta) \leq 1.$$

Similarly, if

$$\text{“} \inf_{\mathrm{supp}(\mu_\star)} \text{”} \, (U_\alpha^\mu(x) + V(x)) > C_\star,$$

then $U_\alpha^\mu + V \geq C_\star + \epsilon$ q.e. on $\mathrm{supp}(\mu_\star)$, so

$$U_\alpha^\mu(x) \geq U_\alpha^{\mu_\star}(x) + \epsilon, \quad \mu_\star - \text{a.s.}$$

The same proof as before applies to get a contradiction. □

4.3. **Radial external fields in the Coulomb case — Corollary 1.3.** For the sake of completeness, let us finally give a proof of the result mentioned in Corollary 1.3.

Changing V into βV, we can assume without loss of generality that $\beta = 1$.

Recall that V is supposed to be radially symmetric and of class $\mathcal{C}^2$: there exists $v : \mathbb{R}_+ \to \mathbb{R}$ such that $V(x) = v(|x|)$.

In this case it is thus natural to look for a radially symmetric equilibrium probability measure. Guided by the results of [39], let us consider an absolutely continuous probability measure μ, such that $\mathrm{supp}(\mu) = \{x \in \mathbb{R}^d; r_0 \le |x| \le R_0\}$ for some $0 \le r_0 < R_0$ and such that $d\mu = M(r)\, d\sigma_r dr$, where $M : [r_0, R_0] \to \mathbb{R}_+$ is assumed to be continuous.

First let us calculate the potential of μ. Using the Gauss' averaging principle (Theorem 3.4), it holds

$$\begin{aligned} U_2^\mu(x) &= \iint M(r) W(x,y)\, d\sigma_r(y) dr \\ &= \int M(r) \int_{\partial B(0,r)} \frac{1}{|x-y|^{d-2}}\, d\sigma_r(y) dr \\ &= \sigma_d \int M(r) r^{d-1} \left(\frac{\mathbf{1}_{\{|x|>r\}}}{|x|^{d-2}} + \frac{\mathbf{1}_{\{|x|\le r\}}}{r^{d-2}} \right) dr \\ &= \frac{\sigma_d}{|x|^{d-2}} \int_0^{|x|} M(r) r^{d-1}\, dr + \sigma_d \int_{|x|}^\infty M(r) r\, dr. \end{aligned} \tag{4.7}$$

Thus $U_2^\mu(x) = u(|x|)$, for some function u of class $\mathcal{C}^1$.

Now, let us consider Condition (1.10). It holds if and only if there exists some C such that $u(r) = C - v(r)$ for all $r \in [r_0, R_0]$. This is obviously equivalent to the conditions $u'(r) = -v'(r)$ for all $r \in [r_0, R_0]$ and $u(R_0) = 1/R_0^{d-2} = C - v(R_0)$ (here we use that $\sigma_d \int_0^{R_0} M(r) r^{d-1}\, dr = 1$). Observing that

$$u'(r) = -\frac{\sigma_d (d-2)}{r^{d-1}} \int_0^r M(t) t^{d-1}\, dt,$$

we see that $u' = -v'$ on $[r_0, R_0]$ if and only if $u'(R_0) = -v'(R_0)$ which amounts to $w(R_0) = d-2$ and $M(t) = \frac{1}{\sigma_d (d-2)} \frac{\omega'(t)}{t^{d-1}}$, for all $t \in [r_0, R_0]$, where we recall that $w(t) = t^{d-1} v'(t)$, $t \ge 0$. The condition $\sigma_d \int_0^{R_0} M(r) r^{d-1}\, dr = 1$ implies that $\frac{1}{d-2}(w(R_0) - w(r_0)) = 1$ and so $w(r_0) = 0$. In the case where w is increasing this determines uniquely $r_0 = 0$ and $R_0 = w^{-1}(d-2)$. In the case where v is supposed to be convex, we see that w is increasing on $[a_0, \infty[$ with $a_0 = \inf\{t > 0; v'(t) > 0\}$ and $w \le 0$ on $[0, a_0]$. Therefore R_0 is uniquely defined and reasoning on the support of μ easily yields to the conclusion that $r_0 = a_0$. In all cases, the probability μ is uniquely determined and $C = 1/R_0^{d-2} + v(R_0)$.

It remains to check that this probability μ satisfies also Condition (1.11). If $r = |x| \ge R_0$, then $U_2^\mu(x) + V(x) = \frac{1}{r^{d-2}} + v(r) \ge \frac{1}{R_0^{d-2}} + v(R_0) = C$, since it is easy to check that $r \mapsto \frac{1}{r^{d-2}} + v(r)$ is increasing on $[R_0, \infty)$. In the case where v is convex and $r \le r_0$, an integration by parts yields to

$$\begin{aligned} U_2^\mu(x) &= \frac{1}{(d-2)} \int_{r_0}^{R_0} \frac{w'(t)}{t^{d-2}}\, dt = \frac{1}{d-2} \int_{r_0}^{R_0} (d-1) v'(t) + t v''(t)\, dt \\ &= v(R_0) - v(r_0) + \frac{1}{R_0^{d-2}} = C - v(r_0) \ge C - v(r), \end{aligned}$$

since v is non-increasing on $[0, r_0]$. Therefore, in all cases $U_2^\mu(x) + V(x) \ge C$ for every $x \in \mathbb{R}^d$, which completes the proof of the characterization of the equilibrium measure.

Finally, if the external field V is quadratic, i.e. if $v(r) = r^2$, then $w(r) = 2r^d$ and so $r_0 = 0$, $R_0 = ((d-2)/2)^{1/d}$ and $M(r) = \frac{2d}{\sigma_d (d-2)} \mathbf{1}_{\{|x| \le R_0\}}$. In other words, the equilibrium probability measure is uniform on the ball centered in 0 and of radius $((d-2)/2)^{1/d}$.

4.4. **Prescribed equilibrium measure.** In this section we prove Corollary 1.4. We will need the following elementary lemma.

Lemma 4.3 (Regularity of Riesz potential)**.** *Let $0 < \alpha < d$, $d \geq 1$, and let μ be a probability measure with a density $f \in \mathbf{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p > d/\alpha$. Then U^μ_α is continuous and finite everywhere on $\mathbb{R}^d$.*

Proof. For all $n \geq 1$, define

$$R_n(x) := \int f(y) \min(n; k_{\Delta_\alpha}(x-y))\, dy$$

and

$$S_n(x) := U^\mu_\alpha(x) - R_n(x) = \int f(y)[k_{\Delta_\alpha}(x-y) - n]_+\, dy.$$

It follows from the dominated convergence theorem that R_n is continuous on $\mathbb{R}^d$. Let us show that S_n converges to 0 uniformly on compact sets, which will prove the claim. Let $q := p/(p-1)$ be the conjugate exponent of p; applying Hölder inequality yields to

$$\begin{aligned} 0 \leq S_n(x) &\leq \int f(y) \frac{1}{|x-y|^{d-\alpha}} \mathbf{1}_{B(x,n^{-1/(d-\alpha)})}(y)\, dy \\ &\leq \|f\|_{p,B(x,1)} \left(\int \frac{1}{|x-y|^{q(d-\alpha)}} \mathbf{1}_{B(x,n^{-1/(d-\alpha)})}(y)\, dy \right)^{1/q} \\ &= \|f\|_{p,B(x,1)} \varepsilon_n, \end{aligned}$$

where $\varepsilon_n := \sigma_d^{1/q} \left(\int_0^{n^{-1/(d-\alpha)}} \frac{1}{u^{q(d-\alpha)-d+1}}\, du \right)^{1/q}$ and where σ_d is the surface of the unit Euclidean ball. The condition $p > d/\alpha$ is equivalent to $q(d-\alpha)-d+1 < 1$ and so ε_n is finite for all n and $\varepsilon_n \to 0$ as $n \to \infty$. We conclude from this that if K is a compact set of $\mathbb{R}^d$ and $K_1 = \{x \in \mathbb{R}^d; d(x,K) \leq 1\}$, it holds

$$\sup_{x \in K} |S_n|(x) \leq \|f\|_{p,K_1} \varepsilon_n,$$

which completes the proof. □

Proof of Corollary 1.4. Lemma 4.3 above shows that $U^{\mu_\star}_\alpha$ is continuous and everywhere finite on $\mathbb{R}^d$. Since $\mu_\star$ is compactly supported, $U^{\mu_\star}_\alpha(x) \to 0$ as $|x| \to \infty$. Therefore $V(x) \to \infty$, when $|x| \to \infty$. This proves **(H2)**. The other assumptions are straightforward. By the very definition of V, it holds

$$U^{\mu_\star}_\alpha(x) + V(x) \geq 0, \qquad \forall x \in \mathbb{R}^d,$$

with equality on $B(0,R) \supseteq \mathrm{supp}(\mu)$. According to point (6) of Theorem 1.2, this proves that μ_* is the (unique) minimizer of I. The last assertion follows from point (4) of Theorem 1.2. □

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