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# FIRST ORDER GLOBAL ASYMPTOTICS FOR CALOGERO-SUTHERLAND GASES

DJALIL CHAFAÏ, NATHAEL GOZLAN, AND PIERRE-ANDRÉ ZITT

ABSTRACT. We study a physical system of  $N$  interacting particles in  $\mathbb{R}^d$  subject to pair repulsion and confined by an external field. We establish a large deviations principle for their empirical measure as  $N$  tends to infinity. In the case of Riesz interaction, including Coulomb interaction in arbitrary dimension  $d > 2$ , the rate function is strictly convex and admits a unique minimum, the equilibrium measure, characterized via its potential. It follows that almost surely, the empirical distribution of the particles tends to this equilibrium measure as  $N$  tends to infinity. In the more specific case of Coulomb interaction in dimension  $d > 2$ , and when the external field is a convex or increasing function of the radius, then the equilibrium measure is supported in a ring. With a quadratic external field, the equilibrium measure is uniform on a ball.

## 1. INTRODUCTION

We study in this work a system of  $N$  particles  $x_1, \dots, x_N$  lying in  $\mathbb{R}^d$ ,  $d \geq 2$ , for which the energy  $H_N(x_1, \dots, x_N)$  of the configuration  $(x_1, \dots, x_N) \in (\mathbb{R}^d)^N$  takes the form

$$H_N(x_1, \dots, x_N) := N \sum_{i=1}^N V(x_i) + \sum_{i < j} W(x_i, x_j)$$

where  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $W : \mathbb{R}^d \times \mathbb{R}^d \rightarrow (-\infty, +\infty]$  is finite outside the diagonal, and symmetric:  $W(x, y) = W(y, x)$  for all  $x, y \in \mathbb{R}^d$ . We can express the energy  $H_N(x_1, \dots, x_N)$  of the configuration in terms of the empirical distribution of the particles

$$\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}.$$

Namely, we simply have

$$H_N(x_1, \dots, x_N) = N^2 \iint_{x \neq y} \varphi(x, y) d\mu_N(x) d\mu_N(y) + N \int V(x) d\mu_N(x)$$

where the double integral is off-diagonal and where  $\varphi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  is given by

$$\varphi(x, y) := \frac{1}{2}(V(x) + V(y) + W(x, y)).$$

Our work is motivated by the following physical control problem: given the “interaction potential”  $W$ , for instance a Coulomb potential, and a target probability measure  $\mu_\star$  on  $\mathbb{R}^d$ , for instance the uniform law on the unit ball, can we tune the “external localizing potential” (or “external localizing field”)  $V$  such that  $\mu_N \rightarrow \mu_\star$  as  $N \rightarrow \infty$ ? In this direction, we provide some partial answers in Theorem 1.1, Theorem 1.2 and Example 1.3 below, and also discuss some possible extensions.

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**1.1. Main results.** From now on, and unless otherwise stated, we denote by  $|\cdot|$  the Euclidean norm of  $\mathbb{R}^d$  and we make the following additional assumptions:

**(H1)** The function  $W : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  is continuous on  $\mathbb{R}^d \times \mathbb{R}^d$ , symmetric, takes finite values on  $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x); x \in \mathbb{R}^d\}$  and satisfies the following integrability condition: for all compact set  $K \subset \mathbb{R}^d$ , the function

$$\alpha_K(u) = \sup\{W(x, y); |x - y| \geq u, x, y \in K\},$$

with the convention  $\alpha_K(u) = 0$  if  $u > \text{diam}(K)$ , is such that the function  $x \mapsto \alpha_K(|x|)$  is locally Lebesgue-integrable on  $\mathbb{R}^d$ .

**(H2)** The function  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  is continuous and such that  $\lim_{|x| \rightarrow +\infty} V(x) = +\infty$  and

$$\int_{\mathbb{R}^d} \exp(-V(x)) dx < \infty.$$

**(H3)** For some  $\varepsilon_o \in (0, 1/2)$ , the function  $(x, y) \mapsto \left(\frac{1}{2} - \varepsilon_o\right) (V(x) + V(y)) + \frac{1}{2}W(x, y)$  is bounded from below.

It is easily seen that under **(H2)**-**(H3)**, for all  $N \in \mathbb{N}^*$ ,

$$Z_N := \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \exp(-H_N(x_1, \dots, x_N)) dx_1 \cdots dx_N < \infty,$$

so that we can define the Boltzmann-Gibbs probability measure  $P_N$  on  $(\mathbb{R}^d)^N$  by

$$dP_N(x_1, \dots, x_N) := \frac{1}{Z_N} e^{-H_N(x_1, \dots, x_N)} dx_1 \dots dx_N. \quad (1.1)$$

This is the distribution of a system of  $N$  particles living on  $\mathbb{R}^d$ , subject to a confining potential  $NV$  acting on each single particle and to an interaction potential  $W$  acting on each pair of particles. Following [For10, Section 11.3.1] (see also [DX01, Section 9.6]), the model is of Calogero-Sutherland type, and the density of  $P_N$  takes the form

$$\prod_{i=1}^N f_1(x_i) \prod_{1 \leq i < j \leq N} f_2(x_i, x_j).$$

Let  $\mathcal{M}_1(\mathbb{R}^d)$  be the set of probability measures on  $\mathbb{R}^d$ . The mean-field symmetries of the model suggest to study, under the exchangeable measure  $P_N$ , the behavior as  $N \rightarrow \infty$  of the empirical measure  $\mu_N$ , which is a random variable on  $\mathcal{M}_1(\mathbb{R}^d)$ . With this asymptotic analysis in mind, we introduce the functional  $I : \mathcal{M}_1(\mathbb{R}^d) \rightarrow (-\infty, +\infty]$  given by

$$I(\mu) := \iint \varphi(x, y) d\mu(x) d\mu(y).$$

Note that **(H2)** and **(H3)** imply that  $\varphi$  is bounded from below, so that this integral makes sense (see Lemma 2.2).

Theorem 1.1 below is our first main result. It is of topological nature, inspired from the available results for logarithmic Coulomb gases in random matrix theory [BAG97, BAZ98, PH98, Har12]. We equip  $\mathcal{M}_1(\mathbb{R}^d)$  with the weak topology (defined by duality with bounded continuous functions), and for any set  $A \subset \mathcal{M}_1(\mathbb{R}^d)$  we denote by  $\text{int}(A)$ ,  $\text{clo}(A)$  the interior and closure of  $A$  with respect to this topology.

To formulate the large deviation result we need to introduce the following additional technical assumption:

**(H4)** For all  $\nu \in \mathcal{M}_1(\mathbb{R}^d)$  such that  $I(\nu) < +\infty$ , there is a sequence  $(\nu_n)_{n \in \mathbb{N}}$  of probability measures, absolutely continuous with respect to Lebesgue, such that  $\nu_n$  converges weakly to  $\nu$  and  $I(\nu_n) \rightarrow I(\nu)$ , when  $n \rightarrow \infty$ .

It turns out that assumption **(H4)** is satisfied for a large class of potentials  $V, W$ , and several examples are given in Proposition 2.8 and Theorem 1.2. We recall that the weak topology on  $\mathcal{M}_1(\mathbb{R}^d)$  can be metrized [Dud02] by the Fortet–Mourier distance defined by

$$d(\mu, \nu) := \sup_{\max(\|f\|_\infty, \|f\|_{\text{Lip}}) \leq 1} \left\{ \int f d\mu - \int f d\nu \right\}. \quad (1.2)$$

**Theorem 1.1** (Large Deviations Principle). *If **(H1)**-**(H2)**-**(H3)** are satisfied then*

- (1)  *$I$  is lower semi-continuous and has compact level sets;*
- (2) *Under  $(P_N)_N$ , the sequence  $(\mu_N)_N$  of random elements of  $\mathcal{M}_1(\mathbb{R}^d)$  equipped with the weak topology has the following asymptotic properties. For every Borel subset  $A$  of  $\mathcal{M}_1(\mathbb{R}^d)$ ,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N P_N(\mu_N \in A) \leq - \inf_{\mu \in \text{clo}(A)} I(\mu)$$

and

$$- \inf\{I(\mu); \mu \in \text{int}(A), \mu \ll \text{Lebesgue}\} \leq \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N P_N(\mu_N \in A).$$

- (3) *Under the additional assumption **(H4)**, the full Large Deviation Principle at speed  $1/N^2$  holds with the rate function*

$$I_\star := I - \inf_{\mathcal{M}_1(\mathbb{R}^d)} I.$$

More precisely, for all Borel set  $A \subset \mathcal{M}_1(\mathbb{R}^d)$ ,

$$\begin{aligned} - \inf_{\mu \in \text{int}(A)} I_\star(\mu) &\leq \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log P_N(\mu_N \in A) \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{N^2} \log P_N(\mu_N \in A) \leq - \inf_{\mu \in \text{clo}(A)} I_\star(\mu). \end{aligned}$$

- (4) *Let  $I_{\min} := \{\mu \in \mathcal{M}_1 : I_\star(\mu) = 0\} \neq \emptyset$ . If **(H4)** is satisfied and if  $(\mu_N)_{N \geq 1}$  are constructed on the same probability space, and if  $d$  stands for the Fortet–Mourier distance (1.2), then we have, almost surely,*

$$\lim_{N \rightarrow \infty} d(\mu_N, I_{\min}) = 0.$$

A careful reading of the proof of Theorem 1.2 indicates that if  $I_{\min} = \{\mu_\star\}$  is a singleton, and if **(H4)** holds for  $\nu = \mu_\star$ , then  $\mu_N \rightarrow \mu_\star$  almost surely as  $N \rightarrow \infty$ .

Our second main result, expressed in Theorem 1.2 and Example 1.3 below is of differential nature. It is based on an instance of the general Gauss problem in potential theory [Zor03a, Zor03b]. It concerns special choices of  $V$  and  $W$  for which  $I_\star$  achieves its minimum 0 for a unique and explicit  $\mu_\star \in \mathcal{M}_1(\mathbb{R}^d)$ . Recall that the *Coulomb* interactions correspond to the choice  $W(x, y) = k_\Delta(x - y)$  where  $k_\Delta$  is the *Coulomb kernel* (opposite in sign to the Newton kernel) defined on  $\mathbb{R}^d$ ,  $d \geq 1$ , by

$$k_\Delta(x) := \begin{cases} -|x| & \text{if } d = 1, \\ \log \frac{1}{|x|} & \text{if } d = 2, \\ \frac{1}{|x|^{d-2}} & \text{if } d \geq 3. \end{cases} \quad (1.3)$$

This is, up to a multiplicative constant, the fundamental solution<sup>1</sup> of the Laplace equation. In other words, denoting  $\Delta := \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$  the Laplacian, we have, in a weak sense, in

<sup>1</sup>There is no boundary conditions here, and thus the term ‘‘Green function’’ is not appropriate.

the space of Schwartz-Sobolev distributions  $\mathcal{D}'(\mathbb{R}^d)$ ,

$$-c\Delta k_\Delta = \delta_0 \quad \text{with} \quad c := \begin{cases} \frac{1}{2} & \text{if } d = 1, \\ \frac{1}{2\pi} & \text{if } d = 2, \\ \frac{1}{d(d-2)\omega_d} & \text{if } d \geq 3, \end{cases} \quad (1.4)$$

where  $\omega_d := \frac{\pi^{d/2}}{\Gamma(1+d/2)}$  is the volume of the unit ball of  $\mathbb{R}^d$ . Our notation is motivated by the fact that  $-\Delta$  is a nonnegative operator. The case of Coulomb interactions in dimension  $d = 2$  is known as “logarithmic potential with external field” and is widely studied in the literature: see [HP00, ST97, AGZ10, Har12]. To focus on novelty, we will not study the Coulomb kernel for  $d \leq 2$ . We refer to [Len63, EL62, Len61, BL75, AM81, SS13] and references therein for the Coulomb case in dimension  $d = 1$ , to [BAG97, AGZ10, Har12] to the Coulomb case in dimension  $d = 2$  with support restriction on a line, to [BAZ98, PH98, HP00, Har12, ST97, SS12] for the Coulomb case in dimension  $d = 2$ .

The asymptotic analysis of  $\mu_N$  as  $N \rightarrow \infty$  for Coulomb interactions in dimension  $d \geq 3$  motivates our next result, which is stated for the more general Riesz interactions in dimension  $d \geq 1$ . The Riesz interactions correspond to the choice  $W(x, y) = k_{\Delta_\alpha}(x - y)$  where  $k_{\Delta_\alpha}$ ,  $0 < \alpha < d$ ,  $d \geq 1$ , is the Riesz kernel defined on  $\mathbb{R}^d$ , by

$$k_{\Delta_\alpha}(x) := \frac{1}{|x|^{d-\alpha}}. \quad (1.5)$$

Up to a multiplicative constant, this is the fundamental solution of a fractional Laplace equation (which is the true Laplace equation (1.4) when  $\alpha = 2$ ), namely

$$-c\Delta_\alpha k_{\Delta_\alpha} = \mathcal{F}^{-1}(1) = \delta_0 \quad \text{with} \quad c := \frac{\pi^{\alpha-\frac{d}{2}} \Gamma(\frac{d-\alpha}{2})}{4\pi^2 \Gamma(\frac{\alpha}{2})}, \quad (1.6)$$

where the Fourier transform  $\mathcal{F}$  and the fractional Laplacian  $\Delta_\alpha$  are given by

$$\mathcal{F}(k_{\Delta_\alpha})(\xi) := \int_{\mathbb{R}^d} e^{2i\pi\xi \cdot x} k_{\Delta_\alpha}(x) dx = \frac{1}{c4\pi^2|\xi|^\alpha} \quad \text{and} \quad \Delta_\alpha f := -4\pi^2 \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(f)).$$

Note that  $\Delta_2 = \Delta$  while  $\Delta_\alpha$  is a non-local integro-differential operator when  $\alpha \neq 2$ . When  $d \geq 3$  and  $\alpha = 2$  then Riesz interactions coincide with Coulomb interactions and the constants match. Beware that our notations differ slightly from the ones of [Lan72, p. 44]. Several aspects of the Gauss problem in the Riesz case are studied in [DS07, Zor03a, Zor03b].

In the Riesz case, to any reasonable probability measure  $\mu$ , corresponds a function  $U^\mu$  called the potential of  $\mu$  — its precise definition is recalled below, see Section 3. One can also define a notion of capacity of sets, and a property is said to hold quasi everywhere if it holds outside a set of zero capacity (once more, see Section 3 for details). The following theorem is essentially the analogue in  $\mathbb{R}^d$  of a result of Dragnev and Saff on the sphere (see [DS07]).

**Theorem 1.2** (Riesz gases). *Suppose that  $W$  is the Riesz kernel  $W(x, y) = \beta k_{\Delta_\alpha}(x - y)$ , for some  $\beta > 0$ . Then:*

- (1) *The functional  $I$  is strictly convex;*
- (2) *(H1)-(H2)-(H3)-(H4) are satisfied and Theorem 1.1 applies;*
- (3) *There exists a unique  $\mu_\star \in \mathcal{M}_1(\mathbb{R}^d)$  such that*

$$I(\mu_\star) = \inf_{\mu \in \mathcal{M}_1(\mathbb{R}^d)} I(\mu);$$

- (4) *If we define  $(\mu_N)_N$  on a unique probability space, then with probability one,*

$$\lim_{N \rightarrow \infty} \mu_N = \mu_\star.$$

If we denote by  $C_\star$  the real number

$$C_\star = \int (U^{\mu_\star} + V) d\mu_\star = J(\mu_\star) + \int V d\mu_\star,$$

then the following additional properties hold:

(5) The minimizer  $\mu_\star$  has compact support, and satisfies

$$U^{\mu_\star}(x) + V(x) \geq C_\star \quad \text{quasi everywhere,} \quad (1.7)$$

$$U^{\mu_\star}(x) + V(x) = C_\star \quad \text{for all } x \in \text{supp } \mu. \quad (1.8)$$

(6) If a compactly supported measure  $\mu$  creates a potential  $U^\mu$  such that, for some constant  $C \in \mathbb{R}$ ,

$$U^\mu(x) + V(x) = C \quad \text{on } \text{supp}(\mu), \quad (1.9)$$

$$U^\mu + V \geq C \quad \text{quasi everywhere,} \quad (1.10)$$

then  $C = C_\star$  and  $\mu = \mu_\star$ .

The constant  $C_\star$  is called the “modified Robin constant”, see e.g. [ST97], where the properties (1.7-1.8) and the characterization (1.9-1.10) are established for the logarithmic potential in dimension 2. The minimizer  $\mu_\star$  is called the *equilibrium measure*.

**Example 1.3** (Equilibrium measure of Coulomb gases with radial external fields). *Suppose that for a fixed real parameter  $\beta > 0$ , and for every  $x, y \in \mathbb{R}^d$ ,*

$$V(x) = v(|x|) \quad \text{and} \quad W(x, y) = \beta k_\Delta(x - y),$$

where  $v$  is two times differentiable. Denote by  $d\sigma_r$  the Lebesgue measure on the sphere of radius  $r$ , and let  $\sigma_d$  be the total mass of  $d\sigma_1$ . Let  $w(r) = r^{d-1}v'(r)$ , and suppose either that  $v$  is convex, or that  $w$  is increasing. Define two radii  $r_0 < R_0$  by:

$$r_0 = \inf \{r > 0; v'(r) > 0\} \quad \text{and} \quad w(R_0) = \beta(d-2).$$

Then the equilibrium measure  $\mu_\star$  is supported on the ring  $\{x; |x| \in [r_0, R_0]\}$ , and is absolutely continuous with respect to Lebesgue measure:

$$d\mu(r) = M(r)d\sigma_r dr \quad \text{where} \quad M(r) = \frac{w'(r)}{\beta(d-2)\sigma_d r^{d-1}} \mathbf{1}_{[r_0, R_0]}(r).$$

In particular, when  $v(t) = t^2$  then  $\mu_\star$  is the uniform distribution on the centered ball of radius  $(\beta(d-2)/2)^{1/d}$ .

This result on Coulomb gases with radial external fields belongs to the folklore of Potential Theory, and can be found for instance in [LG10, Proposition 2.13]. It follows quickly from the Gauss averaging principle and the characterization (1.9-1.10). For the sake of completeness, we give a (short) proof in Section 4.3.

In particular, by using Theorem 1.2 with  $\alpha = 2$  together with Example 1.3, we obtain that the empirical measure of a Coulomb gas with quadratic external field tends almost surely to the uniform distribution on a ball when  $N \rightarrow \infty$ . This phenomenon is the analogue in arbitrary dimensions of the well known result in dimension 2 for the logarithmic potential with external field (Complex Ginibre Ensemble, [BAZ98, PH98]).

**1.2. Outline of the article.** In the remainder of this introduction, we give several comments on our results, their links with different domains, and possible directions for further research. The large deviations principle stated in Theorem 1.1 is proved in Section 2. The proofs of Theorem 1.2 and Example 1.3 use several concepts and tools from Potential Theory, which we recall synthetically and discuss in Section 3 for the sake of clarity and completeness. The proofs themselves are in the final section.

**1.3. Comments, possible extensions and related topics.**

*Equilibrium measure for Riesz interaction with radial external field.* To the knowledge of the authors, the computation of the equilibrium measure for Riesz interactions with radial external field, beyond the more specific Coulomb case of Example 1.3, i.e. for  $\alpha \neq 2$ , is an open problem, due to the lack of the Gauss averaging principle when  $\alpha \neq 2$ .

*Beyond the Riesz and Coulomb interactions.* Theorem 1.2 concerns the minimization of the Riesz interaction potential with an external field  $V$ , and includes the Coulomb interaction if  $d \geq 3$ . In classical Physics, the problem of minimization of the Coulomb interaction energy with an external field is known as the Gauss variational problem [Zor03a, Zor03b]. Beyond the Riesz and Coulomb potentials, the driving structural idea behind Theorem 1.2 is that if  $W$  is of the form  $W(x, y) = k_L(x - y)$  where  $k_L$  is the fundamental solution of an equation  $\partial_t - L = 0$  where  $L$  is a local (say differential) operator, and if  $V$  is super-harmonic for  $L$ , i.e.  $LV \geq 0$ , then the density of  $\mu_*$  is roughly given by  $LV$  up to support constraints. This can be easily understood formally with Lagrange multipliers. The limiting measure  $\mu_*$  depends on  $V$  and  $W$ , and is thus non-universal in general. In the opposite direction, if  $W$  and  $\mu_*$  are prescribed, and if  $\mu \mapsto \iint W(x, y) d\mu(x) d\mu(y)$  is convex, then the proof of Theorem 1.2 suggests a natural candidate for the external field:  $V(x) = -U_W^{\mu_*}(x) := C_* - \int W(x, y) d\mu_*(y)$  for all  $x \in \text{supp}(\mu_*)$ , where  $C_*$  is a constant.

*Second order asymptotic analysis.* The asymptotic analysis of  $\mu_N - \mu_*$  as  $N \rightarrow \infty$  is a natural problem, which can be studied on various classes of test functions. It is well known that a repulsive interaction may affect dramatically the speed of convergence, and make it dependent over the regularity of the test function. In another direction, one may incorporate an inverse temperature parameter  $\beta$  in front of the energy  $H_N$ , and study the low temperature regime, i.e. asymptotics as  $\beta \rightarrow \infty$ . In the Coulomb case, this leads to Fekete points. The analysis of the second order when both  $\beta \rightarrow \infty$  and  $N \rightarrow \infty$  is conjectured to be related to an Abrikosov lattice, see [SS12, SS13], and [Bet13].

*Edge behavior.* Suppose that  $V$  is radially symmetric and that  $\mu_*$  is supported in the centered ball of radius  $r$ , like in Example 1.3. Then one may ask if the radius of the particle system  $\max_{1 \leq k \leq n} |x_k|$  converges to the edge  $r$  of the limiting support as  $N \rightarrow \infty$ . This is not provided by the weak convergence of  $\mu_N$ . The next question is the fluctuation.

*Topology.* It is known that the weak topology can be upgraded to a Wasserstein topology in the classical Sanov theorem for empirical measures of i.i.d. sequences, see [WWW10], provided that tails are strong exponentially integrable. It is then quite natural to ask about such an upgrade for Theorem 1.1.

*Connection to random matrices.* Our initial inspiration came, when writing the survey [BC12], from the role played by the logarithmic potential in the analysis of the Ginibre ensemble. When  $d = 2$  and  $V(x) = |x|^2$  and  $W(x, y) = \beta k_\Delta(x - y) = \beta \log \frac{1}{|x-y|}$  with  $\beta = 2$  then  $P_N$  is the law of the (complex) eigenvalues of the complex Ginibre ensemble:

$$dP_N(x) = Z_N^{-1} e^{-N \sum_{i=1}^N |x_i|^2} \prod_{i < j} |x_i - x_j|^2 dx.$$

(here  $\mathbb{R}^2 \equiv \mathbb{C}$  and  $P_N$  is the law of the eigenvalues of a random  $N \times N$  matrix with i.i.d. complex Gaussian entries of covariance  $\frac{1}{2N} I_2$ ). For a non-quadratic  $V$ , we may see  $P_N$  as the law of the spectrum of random normal matrices such as the ones studied in [AHM11]. On the other hand, in the case where  $d = 1$  and  $V(x) = |x|^2$  and  $W(x, y) = \beta \log \frac{1}{|x-y|}$  with  $\beta > 0$  then

$$dP_N(x) = Z_N^{-1} e^{-N \sum_{i=1}^N |x_i|^2} \prod_{i < j} |x_i - x_j|^\beta dx.$$

This is known as the  $\beta$ -Ensemble in Random Matrix Theory. For  $\beta = 1$ , we recover the law of the eigenvalues of the Gaussian Orthogonal Ensemble (GOE) of random symmetric

matrices, while for  $\beta = 2$ , we recover the law of the eigenvalues of the Gaussian Unitary Ensemble (GUE) of random Hermitian matrices. It is worthwhile to notice that  $-\log|\cdot|$  is the Coulomb potential in dimension  $d = 2$ , and not in dimension  $d = 1$ . For this reason, we may interpret the eigenvalues of GOE/GUE as being a system of charged particles in dimension  $d = 2$ , experiencing Coulomb repulsion and an external quadratic field, but constrained to stay on the real axis. We believe this type of support constraint can be incorporated in our initial model, at the price of a bit heavier notations and analysis.

*Non-compact equilibrium measures.* The assumptions made on the external field  $V$  in theorems 1.1 and 1.2 explain why the equilibrium measure  $\mu_\star$  is compactly supported. If one allows a weaker behavior of  $V$  at infinity, then one may produce equilibrium measures  $\mu_\star$  which are not compactly supported (and even heavy tailed probability measures). This requires to adapt some of the arguments, and one may use compactification as in [Har12].

*Simulation problem.* It is natural to ask about the best way to simulate the probability measure  $P_N$ . A pure rejection algorithm is too naive. Some exact algorithms are available in the determinantal case  $d = 2$  and  $W(x, y) = -2 \log|x - y|$ , see [HKPV06, Algorithm 18] and [SZT09]. One may prefer to use a non exact algorithm such as a Hastings-Metropolis algorithm. One may also use an Euler scheme to simulate a stochastic process for which  $P_N$  is invariant, or use a Metropolis adjusted Langevin approach (MALA) [RR01].

*More general energies.* One may study more general energies with many bodies interactions, of the form, for some prescribed symmetric  $W_k : (\mathbb{R}^d)^k \mapsto \mathbb{R}$ ,  $1 \leq k \leq K$ ,  $K \geq 1$ ,

$$H_N(x_1, \dots, x_N) = \sum_{k=1}^K N^{2-k} \sum_{i_1 < \dots < i_k} W_k(x_{i_1}, \dots, x_{i_k}).$$

This leads to the following candidate for the asymptotic first order global energy functional:

$$\mu \mapsto \sum_{k=1}^K 2^{-k} \int \dots \int W(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k).$$

*Stochastic processes.* Under general assumptions on  $V$  and  $W$ , see for instance [Roy07], the law  $P_N$  is the steady state solution of the Fokker-Planck evolution equation  $\partial_t - L = 0$  where  $L$  is the linear elliptic Markov diffusion operator

$$L := \Delta - \nabla H_N \cdot \nabla,$$

acting as  $Lf = \Delta f - \langle \nabla H_N, \nabla f \rangle$ . The operator  $L$  is self-adjoint in  $L^2(P_N)$ . In probabilistic terms, the law  $P_N$  is a reversible invariant probability measure for the Markov diffusion process  $(X_t)_{t \geq 0}$  solution of the system of Kolmogorov stochastic differential equations

$$dX_t = \sqrt{2} dB_t - \nabla H_N(X_t) dt$$

where  $(B_t)_{t \geq 0}$  is a standard Brownian motion on  $(\mathbb{R}^d)^N$ . In the case where  $V(x) = |x|^2$  and  $W \equiv 0$  (no interaction) then  $P_N$  is a standard Gaussian law  $\mathcal{N}(0, I_{dN})$  on  $(\mathbb{R}^d)^N$  and  $(X_t)_{t \geq 0}$  is an Ornstein-Uhlenbeck Gaussian process; while in the case where  $d = 1$  and  $V(x) = |x|^2$  and  $W(x, y) = -\beta \log|x - y|$  then  $P_N$  is the law of the spectrum of a  $\beta$ -Ensemble of random matrices and  $(X_t)_{t \geq 0}$  is a so called Dyson Brownian motion [AGZ10]. If  $\mu_{N,t}$  is the law of  $X_t$  then  $\mathbb{E}\mu_{N,t} \rightarrow \mathbb{E}\mu_N$  weakly as  $t \rightarrow \infty$ . The study of the dynamic aspects is an interesting problem connected to McKean-Vlasov models [CL97, Fon04, LLX13].



*Calogero-(Moser-)Sutherland Schrödinger operators.* Let us keep the notations used above, and let us consider the isometry  $\Theta : L^2(P_N) \rightarrow L^2(dx)$  defined by

$$\Theta(f)(x) := f(x) \sqrt{\frac{dP_N(x)}{dx}} = f(x) e^{-\frac{1}{2}(H_N(x) + \log(Z_N))}.$$

The differential operator  $S := -\Theta L \Theta^{-1}$  is a Schrödinger operator:

$$S := -\Theta L \Theta^{-1} = -\Delta + U, \quad U := \frac{1}{4} |\nabla H_N|^2 - \frac{1}{2} \Delta H_N$$

which acts as  $Sf = -\Delta f + Uf$ . The operator  $S$  is self-adjoint in  $L^2(dx)$ . Being isometrically conjugated, the operators  $-L$  and  $S$  have the same spectrum, and their eigenspaces are isometric. In the case where  $V(x) = |x|^2$  and  $W \equiv 0$  (no interactions), we find that  $U = \frac{1}{2}(1 - V)$ , and  $S$  is a harmonic oscillator. On the other hand, following [For10, Proposition 11.3.1], in the case  $d = 1$  and  $W(x, y) = -\log|x - y|$  (Coulomb interaction), then  $S$  is a Calogero-(Moser-)Sutherland Schrödinger operator:

$$S = -\Delta - E_0 + \frac{1}{4} \sum_{i=1}^N x_i^2 - \frac{1}{2} \sum_{1 \leq i < j \leq N} \frac{1}{(x_i - x_j)^2}, \quad E_0 := \frac{N}{2} + \frac{N(N-1)}{2}.$$

More examples are given in [For10, Proposition 11.3.2], related to classical ensembles of random matrices. The study of the spectrum and eigenfunctions of such operators is a wide subject, connected to Dunkl operators. These models attracted some attention due to the fact that for several natural choices of the potentials  $V, W$ , they are exactly solvable (or integrable). We refer to [For10, Section 11.3.1], [DX01, Section 9.6], [CDR<sup>+</sup>08, Section 2.7] and references therein.

## 2. PROOF OF THE LARGE DEVIATIONS PRINCIPLE — THEOREM 1.1

The proof of Theorem 1.1 is split in several steps.

**2.1. A standard reduction.** To prove Theorem 1.1, we will use the following standard reduction explained in the following proposition (see for instance [DZ98, Chapter 4]).

**Proposition 2.1.** *Let  $(Q_N)_{N \in \mathbb{N}^*}$  be a sequence of probability measures on some Polish space  $(\mathcal{X}, d)$ ,  $(Z_N)_{N \in \mathbb{N}^*}$  and  $(\varepsilon_N)_{N \in \mathbb{N}^*}$  two sequences of positive numbers with  $\varepsilon_N \rightarrow 0$  and  $\mathcal{I} : \mathcal{X} \rightarrow \mathbb{R}$  be a function bounded from below.*

- (1) *Suppose that the sequence  $(Q_N)_{N \in \mathbb{N}^*}$  satisfies the following conditions*
- (a) *The sequence  $(Z_N Q_N)_{N \in \mathbb{N}^*}$  is exponentially tight: for all  $L \geq 0$  there exists a compact set  $K_L \subset \mathcal{X}$  such that*

$$\limsup_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N(\mathcal{X} \setminus K_L) \leq -L.$$

- (b) *For all  $x \in \mathcal{X}$ ,*

$$\lim_{r \rightarrow 0} \limsup_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N(B(x, r)) \leq -\mathcal{I}(x).$$

*Then the sequence  $(Z_N Q_N)_{N \in \mathbb{N}^*}$  satisfies the following large deviation upper bound: for all Borel set  $A \subset \mathcal{X}$ , it holds*

$$\limsup_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N(A) \leq -\inf \{ \mathcal{I}(\mu); \mu \in \text{clo}(A) \}. \quad (2.1)$$

- (2) *If in addition  $(Z_N Q_N)_{N \in \mathbb{N}^*}$  satisfies the following large deviation lower bound: for all Borel set  $A \subset \mathcal{X}$ ,*

$$-\inf \{ \mathcal{I}(x); x \in \text{int}(A) \} \leq \liminf_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N(A), \quad (2.2)$$

then  $(Q_N)_{N \in \mathbb{N}^*}$  satisfies the full Large Deviation Principle with speed  $\varepsilon_N$  and rate function  $\mathcal{I}_\star = \mathcal{I} - \inf_{x \in \mathcal{X}} \mathcal{I}(x)$ , namely for all Borel set  $A \subset \mathcal{X}$ ,

$$\begin{aligned} -\inf\{\mathcal{I}_\star(x); x \in \text{int}(A)\} &\leq \liminf_{N \rightarrow \infty} \varepsilon_N \log Q_N(A) \\ &\leq \limsup_{N \rightarrow \infty} \varepsilon_N \log Q_N(A) \leq -\inf\{\mathcal{I}_\star(x); x \in \text{clo}(A)\}. \end{aligned}$$

*Proof.* Let us begin by (1). Let  $\delta > 0$ ; by assumption, for any  $x \in \mathcal{X}$ , there is  $\eta_x > 0$  such that

$$\limsup_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N(B(x, \eta_x)) \leq -\mathcal{I}(x) + \delta.$$

If  $F \subset \mathcal{X}$  is compact, there is a finite family  $(x_i)_{1 \leq i \leq m}$  of points of  $F$  such that  $F \subset \cup_{i=1}^m B(x_i, \eta_{x_i})$ . Therefore,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N(F) &\leq \limsup_{N \rightarrow \infty} \varepsilon_N \log \left( \sum_{i=1}^m Z_N Q_N(B(x_i, \eta_{x_i})) \right) \\ &= \max_{1 \leq i \leq m} \limsup_{N \rightarrow \infty} \varepsilon_N \log (Z_N Q_N(B(x_i, \eta_{x_i}))) \\ &\leq \max_{1 \leq i \leq m} -\mathcal{I}(x_i) + \delta \leq -\inf_F \mathcal{I} + \delta. \end{aligned}$$

Letting  $\delta \rightarrow 0$  yields to (2.1) for  $A = F$  compact.

Now if  $F$  is an arbitrary closed set, then for all  $L > 0$ , since  $F \cap K_L$  is compact, it holds

$$\begin{aligned} \limsup_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N(F) &\leq \max \left( \limsup_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N(F \cap K_L), \limsup_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N(K_L^c) \right) \\ &\leq \max(-\inf_{F \cap K_L} \mathcal{I}; -L). \end{aligned}$$

Letting  $L \rightarrow \infty$  shows that (2.1) is true for arbitrary closed sets  $F$ . Since  $A \subset \text{clo}(A)$ , the upper bound (2.1) holds for arbitrary Borel sets  $A$ .

To prove (2), take  $A = \mathcal{X}$  in (2.2) and (2.1) to get:

$$\lim_{N \rightarrow \infty} \varepsilon_N \log(Z_N) = -\inf \mathcal{I} \in \mathbb{R}.$$

Subtracting this to (2.2) and (2.1) gives the LDP with rate function  $\mathcal{I}_\star$ .  $\square$

In our context,  $\mathcal{X} = \mathcal{M}_1(\mathbb{R}^d)$  is equipped with the Fortet–Mourier distance (1.2).

**2.2. Properties of the rate function.** Let us prove the properties of the rate function  $I_\star$  announced in Theorem 1.1, point (1).

**Lemma 2.2** ( $I_\star$  is a good rate function). *Under Assumptions (H2)-(H3), the functional  $I_\star$  is well defined and is a good rate function (the levels sets  $\{I \leq k\}$  are compact).*

*Proof.* According to Assumption (H3), there exists  $\varepsilon_o \in (0, 1/2)$  such that

$$\varphi(x, y) \geq \varepsilon_o(V(x) + V(y)) + c, \quad (2.3)$$

with  $c = \inf_{x,y} (1/2 - \varepsilon_o)(V(x) + V(y)) + (1/2)W(x, y) > -\infty$ . It follows from Assumption (H2) that  $\varphi$  is bounded from below. The functional  $I_\star$  is thus well defined with values in  $[0, \infty]$ . Furthermore, according to the monotone convergence theorem,  $I = \sup_{n \in \mathbb{N}} I_n$ , where  $I_n(\mu) = \iint \min(\varphi(x, y); n) d\mu(x)d\mu(y)$ . The functions  $\min(\varphi, n)$  being bounded and continuous, it follows that the functionals  $I_n$  are continuous for the weak topology (see for

instance [DZ98, Lemma 7.3.12]). Being a supremum of continuous functions,  $I$  is lower semi-continuous. Set  $b_\star = \inf \varphi$ ; we have, for every  $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ ,  $L > 0$ ,

$$\begin{aligned} I(\mu) - b_\star &= \iint (\varphi(x, y) - b_\star) d\mu(x) d\mu(y) \\ &\geq \iint \mathbf{1}_{|x|>L, |y|>L} (\varphi(x, y) - b_\star) d\mu(x) d\mu(y) \\ &\geq (b_L - b_\star) \mu(|x| > L)^2, \end{aligned}$$

where  $b_L := \inf_{|x|>L, |y|>L} \varphi(x, y)$ . According to (2.3) and Assumption (H2), we see that  $b_L \rightarrow +\infty$  as  $L \rightarrow +\infty$ . Therefore, there exists  $L_\star > 0$  such that  $b_L > b_\star$  for every  $L > L_\star$ . We get then for every real number  $r \geq b_\star$ ,

$$\{\mu \in \mathcal{M}_1(\mathbb{R}^d) : I(\mu) \leq r\} \subset \left\{ \mu \in \mathcal{M}_1(\mathbb{R}^d) : \mu(|x| > L) \leq \sqrt{\frac{r - b_\star}{b_L - b_\star}}, L > L_\star \right\}.$$

Since  $b_L \rightarrow +\infty$  as  $L \rightarrow +\infty$ , the subset of  $\mathcal{M}_1(\mathbb{R}^d)$  in the right hand side is tight, and the Prohorov theorem implies then that it is relatively compact for the topology of  $\mathcal{M}_1(\mathbb{R}^d)$ . Since  $I$  is lower semi-continuous, the set  $\{I \leq r\}$  is also closed, which completes the proof.  $\square$

**2.3. Proof of the upper bound.** For all  $N \in \mathbb{N}^*$ , one denotes by  $Q_N$  the law of  $\mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$  under the probability  $P_N$  defined by (1.1):  $Q_N$  is an element of  $\mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^d))$ .

**Lemma 2.3** (Exponential tightness). *Under Assumptions (H2)-(H3), the sequence of measures  $(Z_N Q_N)_{N \in \mathbb{N}^*}$  is exponentially tight: for all  $L \geq 0$  there exists a compact set  $K_L \subset \mathcal{M}_1(\mathbb{R}^d)$  such that*

$$\limsup_{N \rightarrow \infty} \varepsilon_N \log Z_N Q_N (\mathcal{M}_1(\mathbb{R}^d) \setminus K_L) \leq -L.$$

*Proof.* Let us consider the probability  $\nu_V \in \mathcal{M}_1(\mathbb{R}^d)$  by  $\nu_V(dx) = \frac{1}{C} e^{-V(x)} dx$ , with  $C = \int e^{-V(x)} dx$ . For all  $L \geq 0$ ,

$$\begin{aligned} &Z_N Q_N \left( \int V d\mu_N > L \right) \\ &= C^N \int \mathbf{1}_{\{\int V d\mu_N > L\}} \exp \left( -N^2 \iint_{x \neq y} \varphi d\mu_N^2 \right) d\nu_V^N \\ &\leq C^N \int \mathbf{1}_{\{\int V d\mu_N > L\}} \exp \left( -N^2 \iint_{x \neq y} \varepsilon_o (V(x) + V(y)) + b d\mu_N(x) d\mu_N(y) \right) d\nu_V^N \\ &= C^N e^{-bN(N-1)} \int \mathbf{1}_{\{\int V d\mu_N > L\}} \exp \left( -2N^2(1 - N^{-1}) \varepsilon_o \int V d\mu_N \right) d\nu_V^N \\ &\leq C^N e^{-bN(N-1)} \exp \left( -2N^2(1 - N^{-1}) \varepsilon_o L \right) \end{aligned}$$

where the first inequality follows from (2.3), with  $\varepsilon_o \in (0, 1/2)$  and  $b \in \mathbb{R}$  independent of  $N$ . This implies that

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N Q_N \left( \int V d\mu_N > L \right) \leq b - 2\varepsilon_o L.$$

Since  $V(x) \rightarrow \infty$  when  $|x| \rightarrow +\infty$  and  $V$  is continuous, it follows from Prohorov theorem that the sets  $K_L := \{\mu \in \mathcal{M}_1(\mathbb{R}^d); \int V d\mu \leq L\}$ ,  $L \geq 0$  are compact. This completes the proof.  $\square$

**Proposition 2.4** (Upper bound). *Under Assumptions (H2)-(H3), for all  $r \geq 0$ , for all  $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ ,*

$$\lim_{r \rightarrow 0} \limsup_{N \rightarrow +\infty} \frac{1}{N^2} \log Z_N Q_N (B(\mu, r)) \leq -I(\mu),$$

where the ball  $B(\mu, r)$  is defined for the Fortet–Mourier distance (1.2).

*Proof.* Recall that  $\nu_V(dx) = C^{-1}e^{-V(x)} dx$  and let  $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ ,  $r \geq 0$  and  $n \in \mathbb{N}^*$ ; it holds

$$\begin{aligned} & Z_N Q_N(B(\mu, r)) \\ &= C^N \int \mathbf{1}_{B(\mu, r)}(\mu_N) \exp\left(-N^2 \iint_{x \neq y} \varphi(x, y) d\mu_N(x) d\mu_N(y)\right) d\nu_V^N \\ &\leq C^N \int \mathbf{1}_{B(\mu, r)}(\mu_N) \exp\left(-N^2 \iint_{x \neq y} \min(\varphi(x, y); n) d\mu_N(x) d\mu_N(y)\right) d\nu_V^N \\ &\leq C^N \int \mathbf{1}_{B(\mu, r)}(\mu_N) \exp\left(-N^2 \iint \min(\varphi(x, y); n) d\mu_N(x) d\mu_N(y)\right) \\ &\quad \times \exp\left(N \int \min(\varphi(x, x); n) d\mu_N(x)\right) d\nu_V^N \\ &\leq C^N e^{nN} \exp\left(-N^2 \inf_{\nu \in B(\mu, r)} \iint \min(\varphi(x, y); n) d\nu(x) d\nu(y)\right). \end{aligned}$$

Therefore,

$$\limsup_{N \rightarrow +\infty} \frac{1}{N^2} \log Z_N Q_N(B(\mu, r)) \leq - \inf_{\nu \in B(\mu, r)} I_n(\nu),$$

where  $I_n(\nu) := \iint \min(\varphi(x, y); n) d\nu(x) d\nu(y)$ . Arguing as in the proof of Lemma 2.2, we see that  $I_n$  is bounded continuous. Therefore,  $\lim_{r \rightarrow 0} \inf_{\nu \in B(\mu, r)} I_n(\nu) = I_n(\mu)$ . Finally, the monotone convergence theorem implies that  $\sup_{n \in \mathbb{N}^*} I_n(\mu) = I(\mu)$ , which ends the proof.  $\square$

Using this Proposition, Lemma 2.3 and the first point of Proposition 2.1, we get the upper bound of Theorem 1.1, point (2).

**2.4. The lower bound and the full LDP.** In what follows, we denote by  $|A|$  the Lebesgue measure of a Borel set  $A \subset \mathbb{R}^n$ .

**Proposition 2.5** (Lower bound for regular probabilities). *Under the assumptions (H1)-(H2)-(H3), for all probability measure  $\mu$  on  $\mathbb{R}^d$  supported in a box  $B = \prod_{i=1}^d [a_i, b_i]$ ,  $a_i, b_i \in \mathbb{R}$ , with a density  $h$  with respect to the Lebesgue measure such that, for some  $\delta > 0$ ,  $\delta \leq h \leq \delta^{-1}$  on  $B$  it holds*

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N Q_N(B(\mu, r)) \geq -I(\mu), \quad \forall r \geq 0,$$

where  $B(\mu, r)$  denotes the open ball of center  $\mu$  and radius  $r$  for the Fortet–Mourier distance (1.2).

If  $B$  is the box  $\prod [a_k, b_k]$  in  $\mathbb{R}^d$ , let  $l(B)$  and  $L(B)$  be the minimum (resp. maximum) edge length:

$$l(B) = \min_k (b_k - a_k), \quad L(B) = \max_k (b_k - a_k)$$

We admit for a moment the following result:

**Lemma 2.6** (Existence of nice partitions). *For all  $d$  and all  $\delta > 0$  there exists a constant  $C(d, \delta)$  such that the following holds. For any box  $B$ , any integer  $n$ , and any measure  $\mu$  with a density  $h$  w.r.t. Lebesgue measure, if  $\delta \leq h \leq \delta^{-1}$ , then there exists a partition  $(B_1, B_2 \dots B_n)$  of  $B$  in  $n$  sub-boxes, such that:*

- (1)  $B$  is split in equal parts: for all  $i$ ,  $\mu(B_i) = \frac{1}{n} \mu(B)$  ;
- (2) The edge lengths of the  $B_i$  are controlled:

$$\frac{1}{C(d, \delta) n^{1/d}} l(B) \leq l(B_i) \leq L(B_i) \leq \frac{C(d, \delta)}{n^{1/d}} L(B).$$

*Proof of Proposition 2.5.* For each  $N$  we apply Lemma 2.6 to obtain a partition of  $B$  in  $N$  boxes  $B_1^N, \dots, B_N^N$ . We let  $d_N$  be the maximum diameter of the boxes: by the lemma, since  $\mu(B) = 1$ ,

$$\frac{c_1}{N^{1/d}} \leq l(B_i^N), \quad d_N \leq \frac{c_2}{N^{1/d}},$$

where  $c_1$  and  $c_2$  only depend on  $B$ ,  $d$  and  $\delta$ .

Note that, for all 1-Lipschitz function  $f$  with  $\|f\|_\infty \leq 1$ , if  $x_i \in B_i^N$  for all  $i \leq N$ , since  $\mu(B_i) = 1/N$  we have:

$$\left| \frac{1}{N} \sum_{i=1}^N f(x_i) - \int f d\mu \right| \leq \sum_{i=1}^N \int_{B_i^N} |f(x) - f(x_i)| d\mu(x) \leq d_N.$$

Therefore, if  $N$  is large enough,

$$B_1^N \times \dots \times B_N^N \subset \{\mu_N \in B(\mu, r)\}.$$

Let us denote by  $C_i^N \subset B_i^N$  the box obtained from  $B_i^N$  by an homothetic transformation of center the center of  $B_i^N$  and ratio (say)  $1/2$ . It holds,

$$Z_N Q_N(B(\mu, r)) \geq \exp \left( -N \sum_{i=1}^N \max_{C_i^N} V - \sum_{i < j} \max_{C_i^N \times C_j^N} W \right) \prod_{i=1}^N |C_i^N|.$$

Since  $|C_i^N| \geq (l(B_i^N)/2)^d \geq c_3/N$  for some absolute constant  $c_3$ , we conclude that

$$\liminf_{N \rightarrow +\infty} \frac{1}{N^2} \log (Z_N Q_N(B(\mu, r))) \geq - \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \max_{C_i^N} V - \limsup_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i < j} \max_{C_i^N \times C_j^N} W.$$

For all  $N$ , consider the locally constants functions  $V_N : B \rightarrow \mathbb{R}$  and  $W_N : B \times B \rightarrow \mathbb{R}$  defined by  $V_N(x) = \max_{C_i^N} V$ , if  $x \in B_i^N$  and  $W_N(x, y) = \max_{C_i^N \times C_j^N} W$ , if  $x \in B_i^N$  and  $y \in B_j^N$ . Since  $\mu(B_i^N) = 1/N$ , it holds

$$\frac{1}{N} \sum_{i=1}^N \max_{C_i^N} V = \int_B V_N(x) d\mu(x) \quad \text{and} \quad \frac{1}{N^2} \sum_{i < j} \max_{C_i^N \times C_j^N} W = \frac{1}{2} \int_{x \neq y} W_N(x, y) d\mu(x) d\mu(y).$$

The uniform continuity of  $V$  on  $B$  immediately implies that  $V_N$  converges uniformly to  $V$ , and so  $\int V_N d\mu \rightarrow \int V d\mu$ . For the same reason  $W_N$  converges uniformly to  $W$  on  $(B \times B) \cap \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |x - y| \geq u\}$ , for all  $u > 0$ . To apply the dominated convergence theorem it remains to bound  $W_N$  from above by some integrable function. Recall that  $\alpha_B(u) = \sup_{|x-y| \geq u} W(x, y)$ , so that  $W(x, y) \leq \alpha_B(|x - y|)$ . Obviously

$$\max_{(x,y) \in B_i^N \times B_j^N} |x - y| \leq 2d_N + \min_{(x,y) \in C_i^N \times C_j^N} |x - y|.$$

By construction, since  $i \neq j$ ,  $\min_{(x,y) \in C_i^N \times C_j^N} |x - y| \geq \frac{1}{4}(l(B_i^N) + l(B_j^N)) \geq \frac{c_1}{4} N^{-1/d} \geq \frac{c_1}{4c_2} d_N$ . Therefore, the right hand side is bounded by  $(8c_2/c_1 + 1) \min_{C_i^N \times C_j^N} |x - y|$ , so there is an absolute constant  $c_4$  such that

$$\min_{(x,y) \in C_i^N \times C_j^N} |x - y| \geq c_4 \max_{(x,y) \in B_i^N \times B_j^N} |x - y|.$$

Since the function  $\alpha_B$  is non-increasing, it holds

$$\max_{C_i^N \times C_j^N} \alpha_B(|x - y|) \leq \min_{(x,y) \in B_i^N \times B_j^N} \alpha_B(c_4 |x - y|).$$

We conclude from this that  $W_N(x, y) \leq c_5 \alpha_B(c_4 |x - y|)$ ,  $x \neq y$ , for some absolute constant  $c_5$ , and using Assumption **(H1)**, it is not difficult to see that the function in the right

hand side is integrable on  $B$  with respect to  $\mu \times \mu$ . Applying the dominated convergence theorem, we conclude that

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log(Z_N Q_N(B(\mu, r))) &\geq - \int V(x) d\mu(x) - \frac{1}{2} \iint W(x, y) d\mu(x) d\mu(y) \\ &= -I(\mu). \end{aligned} \quad \square$$

Let us now prove that “nice” partitions exist.

*Proof of Lemma 2.6.* The proof is an induction on the dimension  $d$ .

**Base case.** Let  $d = 1$ , and suppose that  $B = [a_0, b_0]$ . Since  $\mu$  has a density, there exists “quantiles”  $a_0 = q_0 < q_1 \dots < q_n = b_0$  such that

$$\forall 1 \leq i \leq n, \quad \mu([q_{i-1}, q_i]) = \frac{1}{n} \mu(B).$$

In this simple case  $l(B) = L(B) = b_0 - a_0$ , and  $l(B_i) = L(B_i) = (q_i - q_{i-1})$ . The boundedness assumption on  $h$  implies that

$$\begin{aligned} \delta(q_i - q_{i-1}) &\leq \mu([q_{i-1}, q_i]) \leq \frac{1}{\delta} (q_i - q_{i-1}) \\ \delta(b_0 - a_0) &\leq \mu(B) \leq \frac{1}{\delta} (b_0 - a_0), \end{aligned}$$

and the claim holds for  $d = 1$  with  $C(1, \delta) = \frac{1}{\delta^2}$ .

**Induction step.** Suppose that the statement holds for a dimension  $d - 1$ . Let  $B = [a_0, b_0] \times B'$  be a box in dimension  $d$  (where  $B'$  is a  $(d - 1)$ -dimensional box). Let  $\mu_0$  be the first marginal of  $\mu$  (this is a measure on  $[a_0, b_0] \subset \mathbb{R}$ ).

Let  $b = \lfloor n^{1/d} \rfloor$  be the integer part of  $n^{1/d}$ . If  $b = 1$ ,  $n \leq 2^d$ , and we can reason as in the base case, on the one-dimensional measure  $\mu_0$ , to find a partition of  $B$  in  $n$  slices of mass  $\mu(B)/n$ . Since the number of slices is less than the constant  $2^d$ , the edge length is controlled as needed.

If  $b \geq 2$ , we look for a decomposition of  $n$  as a sum of  $b$  integers  $n_i$ , each as close to  $n^{(d-1)/d}$  as possible: the idea is to cut  $B$  along the first dimension in  $b$  slices, and to apply the induction hypothesis to cut the slice  $i$  in  $n_i$  parts.

To this end, decompose the integer  $n$  in base  $b$ :

$$\exists \alpha_0, \alpha_1, \dots, \alpha_{d-1} \in \{0, \dots, b-1\}^d, \quad n = b^d + \sum_{k=0}^{d-1} \alpha_k b^k.$$

Writing  $\alpha_k = \sum_{i=1}^b \mathbf{1}_{\{i \leq \alpha_k\}}$  we get

$$n = \sum_{i=1}^b \left( b^{d-1} + \sum_{k=0}^{d-1} \mathbf{1}_{\{i \leq \alpha_k\}} b^k \right) = \sum_{i=1}^b n_i,$$

where  $n_i = b^{d-1} + \sum_{k=0}^{d-1} \mathbf{1}_{\{i \leq \alpha_k\}} b^k$ . From this expression, we get the bound  $b^{d-1} \leq n_i \leq (b^d - 1)/(b - 1)$ . Since  $k - 1 \geq k/2$  whenever  $k \geq 2$ , using the inequalities  $b \leq n^{1/d}$  and  $b \geq n^{1/d} - 1 \geq \frac{1}{2} n^{1/d}$ , we get

$$\frac{1}{2^{d-1}} n^{(d-1)/d} \leq n_i \leq 2n^{(d-1)/d}.$$

Now let us cut  $B$  along its first dimension. Recall that  $\mu_0$  is the first marginal of  $\mu$ . By continuity there exist quantiles  $a_0 = q_0 < q_1 < \dots < q_b = b_0$  such that

$$\forall 1 \leq i \leq b, \quad \mu_1([q_{i-1}, q_i]) = \mu([q_{i-1}, q_i] \times B') = \frac{n_i}{n} \mu(B).$$

We apply the induction hypothesis separately for each  $1 \leq i \leq b$ , to the  $(d-1)$ -dimensional box  $B'$ , with the measure

$$\mu_i(\cdot) = \mu([q_{i-1}, q_i] \times \cdot)$$

and the integer  $n_i$  to obtain a decomposition  $B' = \bigcup_{j=1}^{n_i} B'_{i,j}$  such that:

- (1) the edge lengths  $B'_{i,j}$  are controlled,
- (2)  $\mu_i(B'_{i,j}) = \frac{1}{n_i} \mu_i(B')$ .

Finally, for all  $1 \leq i \leq b$  and all  $1 \leq j \leq n_i$ , let

$$B_{i,j} = [q_{i-1}, q_i] \times B'_{i,j}.$$

Let us check that the partition  $B = \bigcup_i \bigcup_j B_{i,j}$  satisfies the requirements. By definition,

$$\mu(B_{i,j}) = \mu_i(B'_{i,j}) = \frac{1}{n_i} \mu_i(B') = \frac{1}{n} \mu(B),$$

so the first requirement is met. To control the edge lengths, first remark that

$$\begin{aligned} l(B) &= \min(b_0 - a_0, l(B')) & L(B) &= \max(b_0 - a_0, L(B')) \\ l(B_{i,j}) &= \min(q_i - q_{i-1}, l(B'_{i,j})) & L(B_{i,j}) &= \max(q_i - q_{i-1}, L(B'_{i,j})). \end{aligned}$$

By the induction hypothesis, the bounds on  $n_i$  and the fact that  $L(B') \leq L(B)$  we get:

$$\begin{aligned} L(B'_{i,j}) &\leq \frac{C(d-1, \delta)}{n_i^{1/(d-1)}} L(B') \\ &\leq \frac{2C(d-1, \delta)}{n^{1/d}} L(B). \end{aligned}$$

On the other hand, reasoning as in the proof of the base case,

$$\begin{aligned} (q_i - q_{i-1})|B'| &\leq \frac{1}{\delta} \frac{n_i}{n} \mu(B) \\ \mu(B) &\leq \frac{1}{\delta} (b_0 - a_0) |B'| \end{aligned}$$

so

$$(q_i - q_{i-1}) \leq (b_0 - a_0) \delta^{-2} \frac{n_i}{n} \leq L(B) \frac{2\delta^{-2}}{n^{1/d}}.$$

Therefore  $L(B_{i,j}) \leq C(d, \delta) n^{-1/d} L(B)$ . The proof of the lower bound on  $l(B_{i,j})$  follows the same lines and is omitted. This concludes the induction step, and the lemma is proved.  $\square$

**Corollary 2.7.** *Under the assumptions (H1)-(H2)-(H3), for all  $A \subset \mathcal{M}_1(\mathbb{R}^d)$ , it holds*

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N Q_N(A) \geq -\inf\{I(\eta); \eta \in \text{int}(A), \eta \ll \text{Lebesgue}\}.$$

*Proof.* Let  $A \subset \mathcal{M}_1(\mathbb{R}^d)$  be a Borel set and let  $\eta \in \text{int}(A)$  be absolutely continuous with respect to Lebesgue with density  $h$  and such that  $I(\eta) < +\infty$ . For all  $n \in \mathbb{N}^*$ , define, for some sequence  $\varepsilon_n$  converging to 0,

$$\eta_n = (1 - \varepsilon_n) \nu_n + \varepsilon_n \lambda_n,$$

where  $d\nu_n(x) = \frac{1}{C_n} \min(h(x); n) \mathbf{1}_{[-n; n]^d}(x) dx$  and  $d\lambda_n(x) = \frac{1}{(2n)^d} \mathbf{1}_{[-n; n]^d}(x) dx$ , where the normalizing constant  $C_n \rightarrow 1$ , when  $n \rightarrow +\infty$ . It holds

$$I(\eta_n) = (1 - \varepsilon_n)^2 I(\nu_n) + 2\varepsilon_n(1 - \varepsilon_n) \iint \varphi(x, y) d\nu_n(x) d\lambda_n(y) + \varepsilon_n^2 I(\lambda_n).$$

Choose  $\varepsilon_n$  converging to 0 sufficiently fast so that the last two terms above converge to 0 when  $n \rightarrow \infty$ . Using the monotone convergence theorem it is not difficult to check that  $I(\nu_n) \rightarrow I(\eta)$  when  $n \rightarrow \infty$  and that  $\eta_n$  converges to  $\eta$  for the weak topology.

Let  $r > 0$  be such that  $B(\eta, 2r) \subset A$ ; for all  $n$  large enough,  $B(\eta_n, r) \subset B(\eta, 2r) \subset A$ . Since  $\eta_n$  satisfies the assumptions of Proposition 2.5, we conclude that for  $n$  large enough

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N Q_N(A) \geq \liminf_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N Q_N(B(\eta_n, r)) \geq -I(\eta_n).$$

Letting  $n \rightarrow \infty$  and optimizing over  $\{\eta \in A, \eta \ll \text{Lebesgue}\}$  gives the conclusion.  $\square$

*End of the proof of Theorem 1.1.* The properties of  $I_*$  and the upper bound in point (2) are already known. The lower bound of point (2) is given by Corollary 2.7.

To prove point (3), let  $A \subset \mathcal{M}_1(\mathbb{R}^d)$  be some Borel set and take  $\mu \in \text{int}(A)$ . According to Assumption (H4), there exists a sequence of absolutely continuous probability measures  $\nu_n$  converging weakly to  $\mu$  and such that  $I(\nu_n) \rightarrow I(\mu)$ , when  $n \rightarrow \infty$ . For all  $n$  large enough,  $\nu_n \in A$  so applying Corollary 2.7, we conclude that  $\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N Q_N(A) \geq -I(\nu_n)$ . Letting  $n \rightarrow \infty$  and then optimizing over  $\mu \in \text{int}(A)$  we arrive at

$$\liminf_{N \rightarrow \infty} \frac{1}{N^2} \log Z_N Q_N(A) \geq -\inf\{I(\mu); \mu \in \text{int}(A)\}.$$

According to point (2) of Proposition 2.1, we conclude that  $Q_N$  obeys the full LDP.  $\square$

**2.5. Proof of the almost-sure convergence.** Let us establish the last part of Theorem 1.1. First note that since  $I_*$  has compact sublevel sets and is bounded from below,  $I_*$  attains its infimum, so  $I_{\min}$  is not empty. For an arbitrary fixed real  $\varepsilon > 0$ , consider the complement of the  $\varepsilon$ -neighborhood of  $I_{\min}$  for the Fortet–Mourier distance:

$$A_\varepsilon := (I_{\min})_\varepsilon^c := \{\mu \in \mathcal{M}_1 : d(\mu, I_{\min}) > \varepsilon\}.$$

Since  $I$  is lower semi-continuous,  $c_\varepsilon := \inf_{\mu \in A_\varepsilon} I(\mu) > 0$ , thus  $\mathbb{P}(\mu_N \in A_\varepsilon) \leq \exp(-N^2 c_\varepsilon)$ , by the upper bound of the full large deviation principle. By the first Borel–Cantelli lemma, it follows that almost surely,  $\lim_{N \rightarrow \infty} d(\mu_N, I_{\min}) = 0$ .

**2.6. Sufficient conditions for (H4).** The following proposition gives several sufficient conditions under which assumption (H4) holds true. Even if some of these conditions are quite general. It is an open problem to find an even more general and natural condition. One may possibly find some inspiration in [Blo11].

**Proposition 2.8** (Sufficient conditions for (H4)). *Let  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $W : \mathbb{R}^d \times \mathbb{R}^d \rightarrow (-\infty, +\infty]$  be symmetric, finite on  $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x); x \in \mathbb{R}^d\}$  and such that (H2) and (H3) hold true. Assumption (H4) holds in each of the following cases:*

- (1)  $W$  is finite and continuous on  $\mathbb{R}^d \times \mathbb{R}^d$ .
- (2) For all  $x \in \mathbb{R}^d$ , the function  $y \mapsto W(x, y)$  is super harmonic, i.e.  $W$  satisfies

$$W(x, y) \geq \frac{1}{|B(y, r)|} \int_{B(y, r)} W(x, z) dz, \quad \forall r > 0,$$

where  $|B(y, r)|$  denotes the Lebesgue measure of the ball of center  $y$  and radius  $r$ .

- (3) The function  $W$  is such that  $W(x + a, y + a) = W(x, y)$  for all  $x, y, a \in \mathbb{R}^d$  and the function  $J$  defined by

$$J(\mu) = \iint W(x, y) d\mu(x) d\mu(y) \tag{2.4}$$

is convex on the set of compactly supported probability measures.

*Proof.* Let  $\mu \in \mathcal{M}_1(\mathbb{R}^d)$  be such that  $I(\mu) < \infty$ . First note that, under the assumptions (H2) - (H3), the condition  $I(\mu) < \infty$  easily implies that  $\int |V| d\mu < +\infty$  and  $\iint |W| d\mu^2 < +\infty$ . Moreover, it follows from (H3) that  $W(x, y) \geq (\varepsilon_o - 1/2)(V(x) + V(y)) + c$ , for some finite constant  $c$ . Therefore  $W$  is bounded from below on every compact, and so the definition (2.4) of  $J(\mu)$  makes sense if  $\mu$  is compactly supported.



For all  $R > 0$ , let us define  $\mu_R$  as the normalized restriction of  $\mu$  to  $[-R; R]^d$ . Using the dominated convergence theorem, it is not hard to see that  $\mu_R$  converges weakly to  $\mu$  and that  $I(\mu_R) \rightarrow I(\mu)$  when  $R \rightarrow +\infty$ . To regularize  $\mu_R$ , we consider  $\mu_{R,\varepsilon} = \text{Law}(X_R + \varepsilon U)$ ,  $\varepsilon \leq 1$ , where  $X_R$  is distributed according to  $\mu_R$  and  $U$  is uniformly distributed on the Euclidean unit ball  $B_1$  of  $\mathbb{R}^d$ . It is clear that  $\mu_{R,\varepsilon}$  has a density with respect to Lebesgue. Moreover,  $\mu_{R,\varepsilon} \rightarrow \mu_R$ , when  $\varepsilon \rightarrow 0$ . Indeed, if  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is continuous, it is bounded on  $[-R; R]^d + B_1$ , and it follows that  $\int f d\mu_{R,\varepsilon} = \mathbb{E}[f(X_R + \varepsilon U)] \rightarrow \mathbb{E}[f(X_R)]$ , when  $\varepsilon \rightarrow 0$ . This applies in particular to  $f = V$ . Now let us show in each cases that  $J(\mu_{R,\varepsilon})$  converges to  $J(\mu_R)$ , when  $\varepsilon$  goes to 0. Let us write

$$J(\mu_{R,\varepsilon}) = \mathbb{E}[W(X_R + \varepsilon U, Y_R + \varepsilon V)],$$

where  $X_R, Y_R, U, V$  are independent and such that  $Y_R \stackrel{(d)}{=} X_R$  and  $V \stackrel{(d)}{=} U$ .

- (1) If  $W$  is finite and continuous on  $\mathbb{R}^d \times \mathbb{R}^d$ , then using the boundedness of  $W$  on  $([-R; R]^d + B_1) \times ([-R; R]^d + B_1)$ , it follows that  $J(\mu_{R,\varepsilon}) \rightarrow J(\mu_R)$  when  $\varepsilon \rightarrow 0$ .
- (2) If  $W$  is super harmonic, then  $W_\varepsilon(x, y) := \mathbb{E}_{U,V}[W(x + \varepsilon U, y + \varepsilon V)] \leq W(x, y)$  for all  $x, y$ . Moreover, it follows from the continuity of  $W$  outside the diagonal that, for all  $x \neq y$ ,  $W_\varepsilon(x, y) \rightarrow W(x, y)$  when  $\varepsilon \rightarrow 0$ . Since  $I(\mu) < +\infty$ ,  $\mu$  does not have atoms and so the diagonal is of measure 0 for  $\mu^2$ . It follows from the dominated convergence theorem that  $J(\mu_{R,\varepsilon}) \rightarrow J(\mu_R)$  as  $\varepsilon \rightarrow 0$ .
- (3) Denoting by  $\mu_R^x$  the law of  $X_R + x$ , we see that  $\mu_{R,\varepsilon} = \mathbb{E}_U[\mu_R^{\varepsilon U}]$ . Therefore, the convexity of  $J$  yields to

$$J(\mu_{R,\varepsilon}) \leq \mathbb{E}_U[J(\mu_R^{\varepsilon U})] = \mathbb{E}_U \left[ \iint W(x + \varepsilon U, y + \varepsilon U) d\mu_R(x) d\mu_R(y) \right] = J(\mu_R),$$

where the last equality comes from the property  $W(x + a, y + a) = W(x, y)$ . On the other hand, Fatou's lemma implies that  $\liminf_{\varepsilon \rightarrow 0} J(\mu_{R,\varepsilon}) \geq J(\mu_R)$ . Therefore  $J(\mu_{R,\varepsilon}) \rightarrow J(\mu_R)$ , when  $\varepsilon$  goes to 0.

We conclude from the above discussion that for any  $\delta > 0$ , it is possible to choose  $R$  sufficiently large and then  $\varepsilon$  sufficiently small so that  $d(\mu_{R,\varepsilon}, \mu) \leq \delta$  and  $|I(\mu_{R,\varepsilon}) - I(\mu)| \leq \delta$ . This completes the proof.  $\square$

### 3. TOOLS FROM POTENTIAL THEORY

In this section we recall results from Potential Theory, that will prove useful when we discuss the proof of Theorem 1.2 and Example 1.3. There are many textbooks on Potential Theory, with different point of views; our main source is [Lan72], where the Riesz case is well-developed.

In this section, and unless otherwise stated, we set  $k_\alpha := k_{\Delta_\alpha}$  and we take  $W(x, y) := k_\alpha(x - y)$ ,  $0 < \alpha < d$ ,  $d \geq 1$ . We denote respectively by  $\mathcal{M}_1 \subset \mathcal{M}_\infty \subset \mathcal{M}_+ \subset \mathcal{M}_\pm$  the sets of probability measures, of positive measures integrating  $k_\alpha(\cdot)\mathbf{1}_{|\cdot|>1}$ , of positive measures, and of signed measures on  $\mathbb{R}^d$ .

**3.1. Potentials and interaction energy.** We benefit from the constant sign of the Riesz kernel:  $k_\alpha \geq 0$ , contrary to the Coulomb kernel in dimension  $d = 2$  and its logarithm. Following [Lan72, p. 58], the *potential* of  $\mu \in \mathcal{M}_+$  is the function  $U_\alpha^\mu : \mathbb{R}^d \rightarrow [0, \infty]$  defined for every  $x \in \mathbb{R}^d$  by

$$U_\alpha^\mu(x) := \int W(x, y) d\mu(y) = \int k_\alpha(x - y) d\mu(y). \quad (3.1)$$

Note that  $U_\alpha^\mu(x) = \infty$  if  $\mu$  has a Dirac mass at point  $x$ . By using the Fubini theorem, for every  $\mu \in \mathcal{M}_+$ , we have  $U_\alpha^\mu < \infty$  Lebesgue almost everywhere iff  $\mu \in \mathcal{M}_\infty$ . This explains actually the condition  $0 < \alpha < d$  taken in the Riesz potential, which is related to polar coordinates ( $dx = r^{d-1} dr d\sigma_d$ ). In fact if  $\mu \in \mathcal{M}_\infty$  then  $U_\alpha^\mu$  is a locally Lebesgue

integrable function. Moreover, as Schwartz distributions, we have  $U_\alpha^\mu = k_\alpha * \mu$  and, with the notations of (1.6),

$$-c\Delta_\alpha U_\alpha^\mu = (-c\Delta_\alpha k_\alpha) * \mu = \mu.$$

The interaction energy is the quadratic functional  $J_\alpha : \mathcal{M}_+ \mapsto [0, \infty]$  defined by

$$J_\alpha(\mu) := \iint W(x, y) d\mu(x)d\mu(y) = \int U_\alpha^\mu d\mu.$$

Note that  $J_\alpha(\mu) = \infty$  if  $\mu$  has a Dirac mass, and in particular  $J_\alpha(\mu_N) = \infty$ .

**Lemma 3.1** (Positivity and convexity on  $\mathcal{M}_+$ ).

- For every  $\mu \in \mathcal{M}_+$  we have  $J_\alpha(\mu) \geq 0$  with equality iff  $\mu = 0$ ;
- $J_\alpha : \mathcal{M}_+ \mapsto [0, \infty]$  is strictly convex: for every  $\mu, \nu \in \mathcal{E}_+$  with  $\mu \neq \nu$ , we have

$$\forall t \in (0, 1), \quad J_\alpha(t\mu + (1-t)\nu) < tJ_\alpha(\mu) + (1-t)J_\alpha(\nu);$$

- $\mathcal{E}_{\alpha,+} := \{\mu \in \mathcal{M}_+ : J_\alpha(\mu) < \infty\}$  is a convex cone.

We recall that in classical Harmonic Analysis, a function  $K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is called a *positive definite kernel* when  $\sum_{i=1}^n x_i K(x_i, x_j) \bar{x}_j \geq 0$  for every  $n \geq 1$  and every  $x \in \mathbb{C}^n$ . If this holds only when  $x_1 + \dots + x_n = 0$ , the kernel is said to be *weakly positive definite*. The famous Bochner theorem states that a kernel is positive definite if and only if it is the Fourier transform of a *finite* Borel measure. The famous Schoenberg theorem states for every  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , the kernel  $(x, y) \mapsto f(|x - y|^2)$  is positive definite on  $\mathbb{R}^d$  for every  $d \geq 1$  if and only if  $f$  is the Laplace transform of a *finite* Borel measure on  $\mathbb{R}_+$ . The famous Bernstein theorem states that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $C^\infty((0, \infty))$  then  $f$  is the Laplace transform of a finite Borel measure on  $\mathbb{R}_+$  if and only if  $f$  is *completely monotone*:  $(-1)^n f^{(n)} \geq 0$  for every  $n \geq 0$ . For all these notions, we refer to [BCR84, Kol05].

The proof of Lemma 3.1 is short and self-contained. It relies on the fact that the convexity of the functional is equivalent to the fact that  $W$  is a weakly positive definite kernel, which is typically the case when  $W$  is a mixture of shifted Gaussian kernels, which are the most useful weakly positive definite kernels. For example this works if for some arbitrary measurable  $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}$  and Borel measure  $\eta$ , and every  $x, y \in \mathbb{R}^d$ ,

$$W(x, y) = w(|x - y|) = \int_0^\infty \left( e^{-\alpha^2(t)|x-y|^2} + \beta(t) \right) d\eta(t).$$

The shift  $\beta$  can be  $< 0$ , which allows non positive definite kernels such as the logarithmic kernel (note that the Riesz kernel is positive definite). The method is used for the logarithmic kernel in [BAG97, Proof of Property 2.1 (4)] with the following mixture:

$$\log \frac{1}{|x - y|} = \int_0^\infty \frac{1}{2t} \left( e^{-\frac{|x-y|^2}{2t}} - e^{-\frac{1}{2t}} \right) dt.$$

This kernel has a sign change and a double singularity near zero and infinity, which can be circumvented by using a cutoff. Alternatively, one may proceed by regularization and use the Bernstein theorem with the completely monotone function  $f(t) = (\varepsilon + t)^{-\beta}$ ,  $\beta, \varepsilon > 0$ , and then the Schoenberg theorem, see e.g. [Mat97]. For instance, for the logarithmic kernel, the following representation is used in [HP00, Chapter 5]:

$$\log \frac{1}{\varepsilon + |x - y|} = \int_0^\infty \left( \frac{1}{\varepsilon + 1 + |x - y|} - \frac{1}{1 + t} \right) dt.$$

Finally, let us mention that for the Riesz kernel, yet another short proof of Lemma 3.1, based on the formula  $k_\alpha = ck_{\alpha/2} * k_{\alpha/2}$ , can be found in [Lan72, Theorem 1.15 p. 79].

*Proof of Lemma 3.1.* Set  $\beta := d - \alpha$ . We start from the identity,

$$\Gamma(1 + \alpha) = c^{1+\alpha} \int_0^\infty t^\alpha e^{-ct} dt, \quad c > 0, \alpha > -1.$$

Taking  $c = |x - y|^2$  and  $1 + \alpha = \beta/2$ , we get, for every  $x, y \in \mathbb{R}^d$ ,

$$k_\alpha(x - y) = \int_0^\infty f(t) e^{-t|x-y|^2} dt, \quad \text{where } f(t) := \frac{t^{\beta/2-1}}{\Gamma(\beta/2)}.$$

Now for every  $\mu \in \mathcal{M}_+$  such that  $J_\alpha(\mu) < \infty$ ,

$$J_\alpha(\mu) = \int_0^\infty f(t) \left( \iint e^{-t|x-y|^2} d\mu(x) d\mu(y) \right) dt.$$

Expressing the Gaussian kernel as the Fourier transform of a Gaussian kernel, we get, by writing  $e^{i\langle x-y, w \rangle} = e^{i\langle x, w \rangle} e^{-i\langle y, w \rangle}$  and using the Fubini theorem,

$$\begin{aligned} \iint e^{-t|x-y|^2} d\mu(x) d\mu(y) &= (4\pi t)^{-d/2} \iint \left( \int_{\mathbb{R}^d} e^{i\langle x-y, w \rangle} e^{-\frac{1}{4t}|w|^2} dw \right) d\mu(x) d\mu(y) \\ &= (4\pi t)^{-d/2} \int_{\mathbb{R}^d} \underbrace{\left| \int_{\mathbb{R}^d} e^{i\langle x, w \rangle} d\mu(x) \right|^2}_{K_w(\mu)} e^{-\frac{1}{4t}|w|^2} dw. \end{aligned}$$

Now  $K_w$  is clearly convex since for every  $\mu, \nu \in \mathcal{M}_1(\mathbb{R}^d)$  and every  $t \in (0, 1)$ ,

$$\frac{tK_w(\mu) + (1-t)K_w(\nu) - K_w(t\mu + (1-t)\nu)}{t(1-t)} = K_w(\mu - \nu) = \left| \int e^{i\langle x, w \rangle} d(\mu - \nu)(x) \right|^2 \geq 0.$$

It follows then that  $J_\alpha$  is also convex as a conic combination of convex function. Let us establish now the strict convexity of  $J_\alpha$ . Let us suppose that  $\mu, \nu \in \mathcal{M}_1(\mathbb{R}^d)$  with  $J_\alpha(\mu) < \infty$  and  $J_\alpha(\nu) < \infty$  and  $tJ_\alpha(\mu) + (1-t)J_\alpha(\nu) = J_\alpha(t\mu + (1-t)\nu)$  for some  $t \in (0, 1)$ . Then

$$J_\alpha(\mu - \nu) = \frac{tJ_\alpha(\mu) + (1-t)J_\alpha(\nu) - J_\alpha(t\mu + (1-t)\nu)}{t(1-t)} = 0.$$

Arguing as before, we find

$$0 = J_\alpha(\mu - \nu) = \int_0^\infty f(t) \left[ (4\pi t)^{-d/2} \int_{\mathbb{R}^d} K_w(\mu - \nu) e^{-\frac{1}{4t}|w|^2} dw \right] dt.$$

Hence, for every  $t > 0$  (a single  $t > 0$  suffices in what follows)

$$\int_{\mathbb{R}^d} K_w(\mu - \nu) e^{-\frac{1}{4t}|w|^2} dw = 0.$$

Thus, the Fourier transform of  $\mu - \nu$  vanishes almost everywhere, and therefore  $\mu = \nu$ .

Finally,  $\mathcal{E}_{\alpha,+}$  is clearly a cone, and its convexity comes from the convexity of  $J_\alpha$ .  $\square$

Following [Lan72, p. 62], for any  $\mu = \mu_+ - \mu_- \in \mathcal{M}_\pm$  such that  $\mu_\pm \in \mathcal{M}_\infty$ , we have  $U_\alpha^{\mu_\pm} < \infty$  Lebesgue almost everywhere, and we may define for Lebesgue almost every  $x$

$$U_\alpha^\mu(x) := U_\alpha^{\mu_+}(x) - U_\alpha^{\mu_-}(x) \in (-\infty, +\infty).$$

Following [Lan72, p. 77], for every  $\mu = \mu_+ - \mu_- \in \mathcal{M}_\pm$  such that  $\mu_\pm \in \mathcal{M}_\infty$  and

$$\int U_\alpha^{\mu_+} d\mu_- < \infty \quad \text{and} \quad \int U_\alpha^{\mu_-} d\mu_+ < \infty,$$

we may define  $J_\alpha(\mu) \in (-\infty, +\infty]$  as (thanks to the Fubini theorem)

$$J_\alpha(\mu) := \int U_\alpha^\mu d\mu = \int U_\alpha^{\mu_+} d\mu_+ + \int U_\alpha^{\mu_-} d\mu_- - \int U_\alpha^{\mu_+} d\mu_- - \int U_\alpha^{\mu_-} d\mu_+.$$

More generally, for every  $\mu_1, \mu_2 \in \mathcal{M}_\pm$  such that  $\mu_{1\pm}, \mu_{2\pm} \in \mathcal{M}_\infty$  and

$$\int U_\alpha^{\mu_{1\pm}} d\mu_{2\mp} < \infty$$

we may define  $J_\alpha(\mu_1, \mu_2) \in (-\infty, +\infty]$  by

$$J_\alpha(\mu_1, \mu_2) := \int U_\alpha^{\mu_1} d\mu_2 = \int U_\alpha^{\mu_1+} d\mu_{2+} + \int U_\alpha^{\mu_1-} d\mu_{2-} - \int U_\alpha^{\mu_1+} d\mu_{2-} - \int U_\alpha^{\mu_1-} d\mu_{2+}.$$

Following [Lan72, p. 78], since  $k_\alpha$  is symmetric, then the *reciprocity law* holds:

$$J_\alpha(\mu_1, \mu_2) = J_\alpha(\mu_2, \mu_1) \quad \text{i.e.} \quad \int U_\alpha^{\mu_1} d\mu_2 = \int U_\alpha^{\mu_2} d\mu_1.$$

Let  $\mathcal{E}_\alpha$  be the set of elements of  $\mathcal{M}_\pm$  for which  $J_\alpha$  makes sense and is finite. As pointed out by N.S. Landkof [Lan72] in his preface, a very nice idea going back to H. Cartan consists in seeing  $J_\alpha$  as a Hilbert structure on  $\mathcal{E}_\alpha$ . This idea is simply captured by the following lemma, which is the analogue of Lemma 3.1 for signed measures of finite energy.

**Lemma 3.2** (Properties of  $(\mathcal{E}_\alpha, J_\alpha)$ ).

- $J_\alpha$  is lower semi-continuous on  $\mathcal{E}_\alpha$  for the vague topology (i.e. the weak topology with respect to continuous functions vanishing at infinity);
- $\mathcal{E}_\alpha$  is a vector space and  $(\mu_1, \mu_2) \mapsto J_\alpha(\mu_1, \mu_2)$  defines a scalar product on  $\mathcal{E}_\alpha$ .

In particular for every  $\mu \in \mathcal{E}_\alpha$ , we have  $J_\alpha(\mu) = J_\alpha(\mu, \mu) \geq 0$  with equality iff  $\mu = 0$ ; and moreover,  $J_\alpha : \mathcal{E}_\alpha \mapsto (-\infty, \infty)$  is strictly convex: for every  $\mu, \nu \in \mathcal{E}_\alpha$  with  $\mu \neq \nu$ ,

$$\forall t \in (0, 1), \quad \frac{tJ_\alpha(\mu) + (1-t)J_\alpha(\nu) - J_\alpha(t\mu + (1-t)\nu)}{t(1-t)} = J_\alpha(\mu - \nu) > 0.$$

*Proof.* The lower semi-continuity for the vague convergence follows from the fact that  $k_\alpha \geq 0$ , see [Lan72, p. 78]. The vector space nature of  $\mathcal{E}_\alpha$  is immediate from its definition. The bilinearity of  $(\mu_1, \mu_2) \mapsto J_\alpha(\mu_1, \mu_2)$  is immediate. By reasoning as in the proof of Lemma 3.1, we get  $J_\alpha(\mu, \mu) \geq 0$  for every  $\mu \in \mathcal{E}_\alpha$ , with equality iff  $\mu = 0$ .  $\square$

Following [Lan72, Theorem 1.18 and 1.19 p. 90], for this pre-Hilbertian topology, it can be shown that  $\mathcal{E}_{\alpha,+}$  is complete while  $\mathcal{E}_\alpha$  is not complete if  $\alpha > 1$ , and that  $J_\alpha$  is not continuous for the vague topology.

**3.2. Capacity and “approximately/quasi everywhere”.** The notion of capacity is central in Potential Theory. We just need basic facts on zero-capacity sets. Once more we follow the presentation of Landkof ([Lan72, Chapter II.1]), to which we refer for additional details, references and proofs.

For any compact set  $K$  consider the minimization problem

$$W_\alpha(K) = \inf\{J_\alpha(\nu); \nu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha, \text{supp}(\nu) \subset K\}.$$

The boundedness of  $K$  implies that  $W_\alpha(K) \in (0, \infty]$ . Its inverse  $C_\alpha(K)$  is called the *capacity* of the compact set  $K$ . The capacity of  $K$  is zero if and only if there is no measure of finite energy supported in  $K$ .

On general sets one can define an “inner capacity” and an “outer capacity” by

$$\begin{aligned} \underline{C}_\alpha(A) &= \sup\{C_\alpha(K), K \subset A, K \text{ compact}\}, \\ \overline{C}_\alpha(A) &= \inf\{\underline{C}_\alpha(O), A \subset O, O \text{ open}\}. \end{aligned}$$

It can be shown (see [Lan72, Theorem 2.8]) that if  $A$  is a Borel set, these two quantities coincide —  $A$  is said to be “capacitable” and the common value is called the capacity of  $A$ .

A property  $P(x)$  is said to hold “approximately everywhere” if the set  $A = \{x, P(x) \text{ is false}\}$  has zero inner capacity, and “quasi-everywhere” if it has zero outer capacity. For many “reasonable”  $P(x)$ , the set  $A$  is Borel and the two notions coincide. The following result ([Lan72, Theorem 2.1 & 2.2]) shows that, for such “reasonable” properties, “quasi-everywhere” means “ $\nu$ -almost surely, for all measure  $\nu$  of finite energy”.

**Theorem 3.3.** *A Borel set  $A$  has zero capacity if and only if, for any measure  $\nu$  of finite energy,  $\nu(A) = 0$ .*

*In particular, if  $C_\alpha(A) > 0$ ,  $A$  has a positive inner capacity, and there exists a compact  $K \subset A$  and a probability measure  $\nu$  of finite energy such that  $\text{supp}(\nu) \subset K$ .*

**3.3. The Gauss averaging principle.** In the classical Coulombian case ( $\alpha = 2$ ), we will need the following result, known as Gauss' averaging principle. In  $\mathbb{R}^d$ , for all  $r > 0$ , let  $\sigma_r$  be the surface measure on the sphere  $\partial B(0, r)$ ; its total mass is  $\sigma_d r^{d-1}$  where  $\sigma_d$  is the surface of the unit sphere.

**Theorem 3.4** (Gauss' Averaging Principle). *In  $\mathbb{R}^d$ ,*

$$\frac{1}{r^{d-1}\sigma_d} \int_{\partial B(0,r)} \frac{1}{|x-y|^{d-2}} d\sigma_r(y) = \begin{cases} \frac{1}{r^{d-2}} & \text{if } |x| < r \\ \frac{1}{|x|^{d-2}} & \text{if } |x| > r. \end{cases}$$

This result can be found in [Hel09], Lemma 1.6.1 p. 21.

#### 4. PROOF OF THE PROPERTIES OF THE MINIMIZING MEASURE

The proof of Theorem 1.2 is decomposed in two steps. We begin by proving the existence, uniqueness and the support properties of  $\mu_\star$  in Section 4.1. The characterization of  $\mu_\star$  is proved in Section 4.2

Recall that, for a probability measure  $\mu$ , we have defined

$$I(\mu) = \frac{1}{2}J_\alpha(\mu) + \int V d\mu.$$

In this section we consider the following minimization problem:

$$\mathcal{P} : \inf \{I(\mu), \mu \in \mathcal{M}_1\}. \quad (4.1)$$

**4.1. Existence, uniqueness and compactness of the support.** The existence of a minimizer for  $\mathcal{P}$  is clear since we have already seen that  $I$  has compact level sets.

Since  $I(\mu) < \infty$  implies that  $\mu \in \mathcal{E}_\alpha$  and  $\int V d\mu < \infty$ , the problem  $\mathcal{P}$  is equivalent to

$$\mathcal{P}_\alpha : \inf \left\{ I(\mu), \mu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha \text{ such that } V \in L^1(\mu) \right\} \quad (4.2)$$

in that they have the same values and the same minimizers. Let us call  $p$  the common value.

Suppose  $\mu$  and  $\nu$  are two measures in  $\mathcal{M}_1 \cap \mathcal{E}_\alpha$  such that  $V \in L^1(\mu) \cap L^1(\nu)$ . Let  $\psi : t \in [0, 1] \mapsto [0, \infty)$  by

$$\begin{aligned} \psi(t) &:= I((1-t)\mu + t\nu) \\ &= \frac{1}{2}J_\alpha((1-t)\mu + t\nu) + (1-t) \int V d\mu + t \int V d\nu. \end{aligned} \quad (4.3)$$

By Lemma 3.2,  $\psi$  is strictly convex if  $\mu \neq \nu$ . If  $\mu$  and  $\nu$  minimize  $I$ , then they are in  $\mathcal{M}_1 \cap \mathcal{E}_\alpha$ , so  $\psi$  is well-defined, and since  $\psi(0) = I(\mu) = I(\nu) = \psi(1)$ ,  $\mu$  must be equal to  $\nu$ . Therefore the minimizer  $\mu_\star$  is unique.

Let us now prove that  $\mu_\star$  has compact support. This result also holds in dimension 2 with the logarithmic potential, see [ST97, Theorem 1.3, p. 27]. To this end, let us define, for any compact  $K$ , a new minimization problem:

$$\mathcal{P}_K : \inf \{I(\mu), \mu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha, \text{supp}(\mu) \subset K\},$$

and let  $p_K$  be the value of  $\mathcal{P}_K$ .

**Lemma 4.1.** *Let  $K$  be a compact set, and suppose that  $V(x) \geq 2p+3$  when  $x \notin K$ , where  $p$  is the common value of  $\mathcal{P}$ ,  $\mathcal{P}_\alpha$  (defined by (4.1) and (4.2)). Then the problems  $\mathcal{P}$  and  $\mathcal{P}_K$  are equivalent: their values  $p$  and  $p_K$  are equal, the minimizer exists and is the same. In particular, the minimizer  $\mu_\star$  of the original problem  $\mathcal{P}$  satisfies  $\text{supp}(\mu_\star) \subset K$ .*

*Proof.* Suppose  $\mu$  is such that  $I(\mu) \leq p + 1$ . We will prove that, if  $\mu(K) < 1$ , we can find a  $\mu_K$ , supported in  $K$  such that  $I(\mu_K) < I(\mu)$ . This clearly implies that the two values  $p_K$  and  $p$  coincide. Since we know that the minimizer  $\mu_\star$  of the original problem exists, this also proves that it must be supported in  $K$ .

Let us now construct  $\mu_K$  as the renormalized restriction of  $\mu$  to  $K$ . First, remark that  $\mu(K)$  cannot be zero, since

$$p + 1 \geq I(\mu) \geq (1 - \mu(K))(2p + 3).$$

Therefore we can define

$$\mu_K(A) = \frac{1}{\mu(K)}\mu(K \cap A).$$

Since by assumption  $\mu(K) < 1$ , we may similarly define  $\mu_{K^c}$ . The measure  $\mu$  is the convex combination

$$\mu = \mu(K)\mu_K + (1 - \mu(K))\mu_{K^c}.$$

The positivity of  $V$ ,  $W$ , and the choice of  $K$  imply that

$$\begin{aligned} I(\mu) &= \frac{1}{2}J_\alpha(\mu) + \mu(K) \int V d\mu_K + (1 - \mu(K)) \int V d\mu_{K^c} \\ &\geq \frac{1}{2}\mu(K)^2 J_\alpha(\mu_K) + \mu(K)^2 \int V d\mu_K + (1 - \mu(K))(2p + 3), \end{aligned}$$

since  $J_\alpha(\mu_{K^c})$  and the interaction energy  $J_\alpha(\mu_K, \mu_{K^c})$  are both non negative. Therefore

$$I(\mu) \geq \mu(K)^2 I(\mu_K) + (1 - \mu(K))(2p + 3).$$

Assume that  $I(\mu_K) \geq I(\mu)$ . Then

$$I(\mu)(1 - \mu(K)^2) \geq (1 - \mu(K))(2p + 3).$$

Using the fact that  $I(\mu) \leq p + 1$ , and dividing by  $1 - \mu(K)$ , we get

$$2(p + 1) \geq (p + 1)(1 + \mu(K)) \geq 2p + 3,$$

a contradiction. Therefore  $I(\mu_K) < I(\mu)$ , and the proof is complete.  $\square$

**4.2. A criterion of optimality.** In this section we prove the items (5) and (6) of Theorem 1.2. The corresponding result in dimension 2 for the logarithmic potential can be found in [ST97, Theorem 3.3, p. 44], and its proof carries over quite smoothly.

*Proof of item (5) of Theorem 1.2.* We already know that  $\mu_\star$  has compact support. The first step is to show that  $\mu_\star$  satisfies (1.7) and (1.8). Let  $\mu = \mu_\star$ , and let  $\nu$  be in  $\nu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha$  such that  $V \in L^1(\nu)$ . Recall the function  $\psi$  from (4.3):

$$\psi(t) = I((1 - t)\mu_\star + t\nu).$$

Since  $\mu_\star$  minimizes  $I$ ,  $\psi'(0^+)$  must be non-negative. Since  $J_\alpha$  is quadratic we get

$$\begin{aligned} \psi(t) &= \int V d\mu_\star + t \int V d(\nu - \mu_\star) + \frac{1}{2}J_\alpha(\mu_\star + t(\nu - \mu_\star)) \\ &= \int V d\mu_\star + t \int V d(\nu - \mu_\star) + \frac{1}{2}(J_\alpha(\mu_\star) + t^2 J_\alpha(\nu - \mu_\star) + 2tJ_\alpha(\mu_\star, \nu - \mu_\star)) \end{aligned}$$

Therefore

$$\begin{aligned} \psi'(0^+) &= \int V d(\nu - \mu_\star) + J_\alpha(\mu_\star, \nu - \mu_\star) \\ &= \int V d\nu + J_\alpha(\mu_\star, \nu) - \left( \int V d\mu_\star + J_\alpha(\mu_\star) \right) \\ &= \int (V + U_\alpha^{\mu_\star}) d\nu - C_\star \\ &= \int (V + U_\alpha^{\mu_\star} - C_\star) d\nu. \end{aligned}$$

This must be non negative, so

$$\forall \nu \in \mathcal{M}_1 \cap \mathcal{E}_\alpha, \quad \int (V + U_\alpha^{\mu_\star} - C_\star) d\nu \geq 0. \quad (4.4)$$

Since this holds for all  $\nu$ ,  $V + U_\alpha^{\mu_\star}$  is greater than  $C_\star$  quasi everywhere. Indeed, let  $A = \{x, V(x) + U_\alpha^{\mu_\star}(x) < C_\star\}$ . Since  $V + U_\alpha^{\mu_\star}$  is measurable this is a Borel set. Suppose by contradiction that its capacity is strictly positive. By Proposition 3.3, there exist a compact set  $K \subset A$  and a measure  $\nu$  with finite energy supported in  $K$ . For this measure  $\int V + U_\alpha^{\mu_\star} d\nu < C_\star$ , which contradicts (4.4). This proves (1.7).

Let us prove (1.8). Suppose  $V(x) + U_\alpha^{\mu_\star}(x) > C_\star$  for some  $x \in \text{supp}(\mu_\star)$ . Since  $V + U_\alpha^{\mu_\star}$  is lower semi-continuous, we can find a neighborhood  $\mathcal{U}$  of  $x$ , and an  $\eta > 0$  such that

$$\forall x \in \mathcal{U}, \quad V(x) + U_\alpha^{\mu_\star}(x) \geq C_\star + \eta.$$

Therefore

$$\int V + U_\alpha^{\mu_\star} d\mu_\star \geq (C_\star + \eta)\mu_\star(\mathcal{U}) + \int_{\mathbb{R}^d \setminus \mathcal{U}} (V + U_\alpha^{\mu_\star}) d\mu_\star.$$

Since  $V + U_\alpha^{\mu_\star} \geq C_\star$  quasi everywhere, and  $\mu_\star$  has finite energy, this holds  $\mu_\star$  almost surely, so

$$C_\star = \int V + U_\alpha^{\mu_\star} d\mu_\star \geq C_\star + \eta\mu_\star(\mathcal{U}).$$

This is impossible since  $\mu_\star(\mathcal{U}) > 0$ , by definition of the support. Therefore (1.8) holds.  $\square$

*Proof of item (6) of Theorem 1.2.* Let  $\nu \in \mathcal{E}_\alpha \cap \mathcal{M}_1(\mathbb{R}^d)$  such that  $V \in L^1(\nu)$  and consider again the function  $\psi(t) = I((1-t)\mu + t\nu)$ ,  $t \in [0, 1]$ . According to Lemma 3.1, this function is convex. Therefore,

$$I(\nu) = \psi(1) \geq \psi(0) + \psi'(0^+) = I(\mu) + \psi'(0^+).$$

To prove that  $\mu$  minimizes  $I_\alpha$ , it is thus enough to prove that  $\psi'(0^+) \geq 0$ , for all choice of  $\nu$ . According to a calculation made above,

$$\psi'(0^+) = \int V + U_\alpha^\mu d\nu - \int V + U_\alpha^\mu d\mu$$

According to (1.10),  $V + U_\alpha^\mu \geq C$  quasi everywhere. The set where  $V + U_\alpha^\mu < C$  is Borel, so according to Theorem 3.3, it has  $\nu$  measure 0. Therefore,

$$\int V + U_\alpha^\mu d\nu \geq C.$$

On the other hand, (1.9) implies that

$$\int V + U_\alpha^\mu d\mu = \int_{\text{supp } \mu} V + U_\alpha^\mu d\mu = C.$$

This proves that  $\psi'(0^+) \geq 0$  and completes the proof.  $\square$

Let us sketch an alternative argument for item (6) of Theorem 1.2, based on the so called Principle of Domination. This result holds only for  $\alpha \leq 2$ .

**Theorem 4.2** (Principle of domination). *Suppose  $\alpha \leq 2$ . Let  $\mu$  and  $\nu$  be two probability measures in  $\mathcal{E}_\alpha$ , and  $c$  a real constant. If the inequality*

$$U^\mu(x) \leq U^\nu(x) + c$$

*holds  $\mu$ -almost surely, then it holds for all  $x \in \mathbb{R}^d$ .*

*Proof.* In the Coulomb case  $\alpha = 2$ , [Lan72, Theorem 1.27, p. 110] applies, since  $U^\nu$  is super-harmonic. If  $\alpha < 2$ , the potential  $U^\nu$  is  $\alpha$ -superharmonic, so we can apply [Lan72, Theorem 1.29, p. 115] and get the result.  $\square$

*Alternative proof of item (6) of Theorem 1.2 in the case  $\alpha \leq 2$ .* Suppose that  $\mu$  satisfies the hypotheses (1.9), (1.10) of the theorem. Let us prove first that  $C = C_*$ . By (1.9),  $U^\mu + V = C$   $\mu$ -almost surely, and by (1.7),  $U^{\mu_*} + V \geq C_*$   $\mu$ -almost surely. Therefore  $U^\mu \leq U^{\mu_*} + C - C_*$ ,  $\mu$ -almost surely. According to Theorem 4.2, this inequality holds everywhere:

$$\forall x \in \mathbb{R}^d, \quad U^\mu(x) \leq U^{\mu_*}(x) + C - C_*. \quad (4.5)$$

In particular it holds at infinity where the potentials go to zero (this follows from the compactness of the support of  $\mu_*$  and  $\mu$ ). Therefore  $C \geq C_*$ . Reversing the roles of  $\mu$  and  $\mu_*$ , we find that  $C$  must be equal to the “modified Robin constant”  $C_*$ .

Coming back to (4.5), we see that  $U^\mu \leq U^{\mu_*}$  everywhere, and similarly  $U^{\mu_*} \leq U^\mu$ . Therefore  $J(\mu, \mu) = J(\mu, \mu_*) = J(\mu_*)$ , so

$$\begin{aligned} J(\mu - \mu_*, \mu - \mu_*) &= J(\mu, \mu) - J(\mu, \mu_*) + J(\mu_*, \mu_*) - J(\mu, \mu_*) \\ &= 0, \end{aligned}$$

and  $\mu = \mu_*$ . □

**4.3. Radial external fields in the Coulomb case — Example 1.3.** For the sake of completeness, let us finally give a proof of the result mentioned in Example 1.3. Recall that  $V$  is supposed to be radially symmetric: there exists  $v : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that

$$V(x) = v(|x|).$$

In the following we suppose that  $v$  is  $\mathcal{C}^2$ . Standard computations show that

$$\Delta V(x) = \frac{d-1}{|x|} v'(|x|) + v''(|x|) = \frac{1}{r^{d-1}} w'(r),$$

where  $w(r) = r^{d-1} v'(r)$ . Suppose additionally that  $\Delta V \geq 0$ , i.e. that  $w$  is increasing.

In this case it is natural to look for a radially symmetric equilibrium measure. Guided by the results of [ST97], we also suppose that  $\mu$  has a density with respect to Lebesgue measure. Let  $M : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be compactly supported and let  $d\mu = M(r) d\sigma_r dr$ . Suppose that the total mass

$$\sigma_d \int_0^\infty M(r) r^{d-1} dr$$

equals 1. Let us compute the potential of  $\mu$ . Using the Gauss’ averaging principle (Theorem 3.4), it holds

$$\begin{aligned} U^\mu(x) &= \iint M(r) W(x, y) d\sigma_r(y) dr \\ &= \int M(r) \int_{\partial B(0, r)} \frac{\beta}{|x - y|^{d-2}} d\sigma_r(y) dr \\ &= \sigma_d \beta \int M(r) r^{d-1} \left( \frac{\mathbf{1}_{\{|x| > r\}}}{|x|^{d-2}} + \frac{\mathbf{1}_{\{|x| \leq r\}}}{r^{d-2}} \right) dr \\ &= \frac{\beta \sigma_d}{|x|^{d-2}} \int_0^{|x|} M(r) r^{d-1} dr + \beta \sigma_d \int_{|x|}^\infty M(r) r dr. \end{aligned} \quad (4.6)$$

To continue, we make the additional assumption that  $\mu$  satisfies  $\Delta V = \mu$  on  $\text{supp } \mu$ . In other words, we fix a radius  $R_0$  and define

$$\begin{aligned} M(r) &= C(R_0) \left( \frac{d-1}{r} v'(r) + v''(r) \right) \mathbf{1}_{\{|x| \leq R_0\}} \\ &= C(R_0) \frac{w'(r)}{r^{d-1}} \mathbf{1}_{\{|x| \leq R_0\}}. \end{aligned}$$



The constant  $C(R_0)$  must be chosen so that  $\mu$  is a probability measure, so

$$C(R_0)^{-1} = \sigma_d w(R_0). \quad (4.7)$$

Since  $M(r) = 0$  for  $r \geq R_0$ , we have two cases. If  $|x| > R_0$ , the second integral in (4.6) disappears, and

$$U^\mu(x) = \beta \sigma_d C(R_0) \frac{1}{|x|^{d-2}} \int_0^{R_0} M(r) r^{d-1} dr = \frac{\beta \sigma_d C(R_0)}{|x|^{d-2}} w(R_0).$$

If  $|x| \leq R_0$ , remarking that the derivative of  $rv'(r) + (d-2)v(r)$  is  $rM(r)$  (for  $r < R_0$ ), we get

$$\begin{aligned} U^\mu(x) &= \frac{\beta \sigma_d}{|x|^{d-2}} \int_0^{|x|} M(r) r^{d-1} dr + \beta \sigma_d \int_{|x|}^{R_0} M(r) r dr \\ &= \frac{C(R_0) \beta \sigma_d}{|x|^{d-2}} w(|x|) + C(R_0) \beta \sigma_d [rv'(r) + (d-2)v(r)]_{|x|}^{R_0} \\ &= C(R_0) \beta \sigma_d (|x|v'(|x|) + R_0 v'(R_0) + (d-2)v(R_0) - |x|v'(|x|) - (d-2)v(|x|)) \\ &= C(R_0) \beta \sigma_d (R_0 v'(R_0) + (d-2)v(R_0) - (d-2)v(|x|)) \end{aligned}$$

To satisfy the requirement of the characterization theorem, the term that depends on  $|x|$  in the last equation has to cancel exactly with the exterior potential  $V$ . This occurs if  $C(R_0) \beta \sigma_d (d-2) = 1$ , which can be rewritten using Equation (4.7) as:

$$w(R_0) = \beta(d-2).$$

If this holds, the total potential  $U^\mu + V$  is given by:

$$U^\mu(x) + V(x) = \begin{cases} \frac{1}{d-2} \frac{w(R_0)}{|x|^{d-2}} + v(|x|) & \text{if } |x| > R_0, \\ \frac{1}{d-2} (R_0 v'(R_0) + (d-2)v(R_0)) & \text{if } |x| \leq R_0. \end{cases}$$

The potential of  $\mu$  is such that the hypotheses (1.9) and (1.10) of Theorem 1.2 (item 6) hold: we have found the minimizing measure.

If the external field  $V$  is quadratic, i.e. if  $v(r) = r^2$ ,  $w(r) = 2r^d$ , so  $r_0 = 0$ ,  $M(r) = 2dC(R_0) \mathbf{1}_{\{|x| \leq R_0\}}$ , where  $R_0$  satisfies  $w(R_0) = 2R_0^d = \beta(d-2)$ , so  $R_0 = (\beta(d-2))^{1/d}$ . In other words the equilibrium measure is the uniform measure on the ball centered in 0 of radius  $(\beta(d-2))^{1/d}$ .

## REFERENCES

- [AGZ10] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics, vol. 118, Cambridge University Press, Cambridge, 2010. [MR-2760897](#)
- [AHM11] Yacin Ameur, Håkan Hedenmalm, and Nikolai Makarov, *Fluctuations of eigenvalues of random normal matrices*, Duke Math. J. **159** (2011), no. 1, 31–81. [MR-2817648](#)
- [AM81] Michael Aizenman and Philippe A. Martin, *Structure of Gibbs states of one-dimensional Coulomb systems*, Comm. Math. Phys. **78** (1980/81), no. 1, 99–116. [MR-597033](#)
- [BAG97] Gérard Ben Arous and Alice Guionnet, *Large deviations for Wigner’s law and Voiculescu’s non-commutative entropy*, Probab. Theory Related Fields **108** (1997), no. 4, 517–542. [MR-1465640](#)
- [BAZ98] Gérard Ben Arous and Ofer Zeitouni, *Large deviations from the circular law*, ESAIM Probab. Statist. **2** (1998), 123–134 (electronic). [MR-1660943](#)
- [BC12] Charles Bordenave and Djalil Chafaï, *Around the circular law*, Probab. Surv. **9** (2012), 1–89. [MR-2908617](#)
- [BCR84] Christian Berg, Jens Peter Reus Christensen, and Paul Ressel, *Harmonic analysis on semi-groups*, Graduate Texts in Mathematics, vol. 100, Springer-Verlag, New York, 1984, Theory of positive definite and related functions. [MR-747302](#)
- [Bet13] Laurent Betermin, *PhD thesis*, under preparation, 2013.

- [BL75] H.J. Brascamp and E.H. Lieb, *Some inequalities for Gaussian measures and the long-range order of the one-dimensional plasma*, Functional integration and its applications. Proceedings of the international conference held in London in April 1974 (A.M. Arthur, ed.), Clarendon Press, 1975, pp. 1–14.
- [Blo11] Thomas Bloom, *Voiculescu’s entropy and potential theory*, Ann. Fac. Sci. Toulouse Math. (6) **20** (2011), no. Fascicule Special, 57–69. [MR-2858167](#)
- [CDR<sup>+</sup>08] Oleksandr Chybiryakov, Nizar Demni, Margit Rösler, Mickael Voit, and Marc Yor, *Harmonic & stochastic analysis of dunkl processes*, Hermann, 2008.
- [CL97] Emmanuel Cépa and Dominique Lépingle, *Diffusing particles with electrostatic repulsion*, Probab. Theory Related Fields **107** (1997), no. 4, 429–449. [MR-1440140](#)
- [DS07] P. D. Draznev and E. B. Saff, *Riesz spherical potentials with external fields and minimal energy points separation*, Potential Anal. **26** (2007), no. 2, 139–162. [MR-2276529](#)
- [Dud02] R. M. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, Cambridge, 2002, Revised reprint of the 1989 original. [MR-1932358](#)
- [DX01] Charles F. Dunkl and Yuan Xu, *Orthogonal polynomials of several variables*, Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge University Press, Cambridge, 2001. [MR-1827871](#)
- [DZ98] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, second ed., Applications of Mathematics (New York), vol. 38, Springer-Verlag, New York, 1998. [MR-MR1619036](#)
- [EL62] S. F. Edwards and A. Lenard, *Exact statistical mechanics of a one-dimensional system with Coulomb forces. II. The method of functional integration*, J. Mathematical Phys. **3** (1962), 778–792. [MR-0147214](#)
- [Fon04] J. Fontbona, *Uniqueness for a weak nonlinear evolution equation and large deviations for diffusing particles with electrostatic repulsion*, Stochastic Process. Appl. **112** (2004), no. 1, 119–144. [MR-2062570](#)
- [For10] P. J. Forrester, *Log-gases and random matrices*, London Mathematical Society Monographs Series, vol. 34, Princeton University Press, Princeton, NJ, 2010. [MR-2641363](#)
- [Har12] Adrien Hardy, *A note on large deviations for 2D Coulomb gas with weakly confining potential*, Electron. Commun. Probab. **17** (2012), no. 19, 12. [MR-2926763](#)
- [Hel09] L.L.V. Helms, *Potential theory*, Universitext, Springer London, 2009.
- [HKPV06] J. Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág, *Determinantal processes and independence*, Probab. Surv. **3** (2006), 206–229. [MR-2216966](#)
- [HP00] Fumio Hiai and Dénes Petz, *The semicircle law, free random variables and entropy*, Mathematical Surveys and Monographs, vol. 77, American Mathematical Society, Providence, RI, 2000. [MR-1746976](#)
- [Kol05] Alexander Koldobsky, *Fourier analysis in convex geometry*, Mathematical Surveys and Monographs, vol. 116, American Mathematical Society, Providence, RI, 2005. [MR-2132704](#)
- [Lan72] N. S. Landkof, *Foundations of modern potential theory*, Springer-Verlag, New York, 1972, Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180. [MR-0350027](#)
- [Len61] A. Lenard, *Exact statistical mechanics of a one-dimensional system with Coulomb forces*, J. Mathematical Phys. **2** (1961), 682–693. [MR-0129874](#)
- [Len63] Andrew Lenard, *Exact statistical mechanics of a one-dimensional system with Coulomb forces. III. Statistics of the electric field*, J. Mathematical Phys. **4** (1963), 533–543. [MR-0148411](#)
- [LG10] A. López García, *Greedy energy points with external fields*, Recent trends in orthogonal polynomials and approximation theory, Contemp. Math., vol. 507, Amer. Math. Soc., Providence, RI, 2010, pp. 189–207. [MR-2647570](#)
- [LLX13] Songzi Li, Xiang-Dong Li, and Yong-Xiao Xie, *Generalized Dyson Brownian motion, McKean-Vlasov equation and eigenvalues of random matrices*, preprint [arXiv:1303.1240](#), 2013.
- [Mat97] L. Mattner, *Strict definiteness of integrals via complete monotonicity of derivatives*, Trans. Amer. Math. Soc. **349** (1997), no. 8, 3321–3342. [MR-1422615](#)
- [PH98] Dénes Petz and Fumio Hiai, *Logarithmic energy as an entropy functional*, Advances in differential equations and mathematical physics (Atlanta, GA, 1997), Contemp. Math., vol. 217, Amer. Math. Soc., Providence, RI, 1998, pp. 205–221. [MR-1606719](#)
- [Roy07] Gilles Royer, *An initiation to logarithmic Sobolev inequalities*, SMF/AMS Texts and Monographs, vol. 14, American Mathematical Society, Providence, RI, 2007, Translated from the 1999 French original by Donald Babbitt. [MR-2352327](#)
- [RR01] Gareth O. Roberts and Jeffrey S. Rosenthal, *Optimal scaling for various Metropolis-Hastings algorithms*, Statist. Sci. **16** (2001), no. 4, 351–367. [MR-1888450](#)

- [SS12] E. Sandier and S. Serfaty, *2D Coulomb Gases and the Renormalized Energy*, ArXiv e-prints (2012).
- [SS13] Etienne Sandier and Sylvia Serfaty, *1D Log Gases and the Renormalized Energy: Crystallization at Vanishing Temperature*, preprint arXiv:1303.2968, 2013.
- [ST97] Edward B. Saff and Vilmos Totik, *Logarithmic potentials with external fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 316, Springer-Verlag, Berlin, 1997, Appendix B by Thomas Bloom. [MR-1485778](#)
- [SZT09] Antonello Scardicchio, Chase Zachary, and Salvatore Torquato, *Statistical properties of determinantal point processes in high-dimensional Euclidean spaces*, Phys. Rev. E **79** (2009), no. 4, Article 041108.
- [WWW10] Ran Wang, Xinyi Wang, and Liming Wu, *Sanov's theorem in the Wasserstein distance: a necessary and sufficient condition*, Statist. Probab. Lett. **80** (2010), no. 5-6, 505–512. [MR-2593592](#)
- [Zor03a] N. V. Zorii, *Equilibrium potentials with external fields*, Ukrain. Mat. Zh. **55** (2003), no. 9, 1178–1195. [MR-2075132](#)
- [Zor03b] ———, *Equilibrium problems for potentials with external fields*, Ukrain. Mat. Zh. **55** (2003), no. 10, 1315–1339. [MR-2073873](#)

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