



HAL
open science

Detection and Location of Moving Point Sources in Contaminant Transport Models. Uniqueness and Minimal Observations

Faker Ben Belgacem, Henda El Fekih, Khiari Souad

► **To cite this version:**

Faker Ben Belgacem, Henda El Fekih, Khiari Souad. Detection and Location of Moving Point Sources in Contaminant Transport Models. Uniqueness and Minimal Observations. 2013. hal-00818370

HAL Id: hal-00818370

<https://hal.science/hal-00818370>

Submitted on 29 Apr 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Detection and Location of Moving Point Sources in Contaminant Transport Models. Uniqueness and Minimal Observations

F. Ben Belgacem* H. El Fekih † S. Khiari ‡

April 26, 2013

Abstract

We are interested in an inverse problem of recovering the position of a pollutant or contaminant source in a stream water. Advection, dispersive transport and the reaction of the solute is commonly modeled by a linear or non-linear parabolic equation. In former works, it is established that a point-wise source is fully identifiable from measurements recorded by a couple of sensors placed, one up-stream and the other down-stream of the pollution source. The observability question we try to solve here is related to the redundancy of sensors when additional information is available on the point-wise source. It may occur, in hydrological engineering, that the intensity of the pollutant is known in advance. In this case, we pursue an identifiability result of a moving source location using a single observation. The chief mathematical tools to prove identifiability are the unique continuation theorem together with an appropriate maximum principle for the parabolic equation under investigation.

KEYWORDS: Reaction-dispersion equation, point-wise sources, maximum principle, unique continuation theorem, identifiability.

1 Introduction

Mathematical models are increasingly used in monitoring rivers and channels and for predicting the effects on the environmental media of contaminant transport. Advective-dispersive-reactive equations are popular in theoretical and numerical ecological engineering. Many of computing programs, developed for water quality analysis to assess the impact of the pollutants on stream waters, are based on one dimensional modeling. We refer to QUAL2E (the last release is QUAL2K) the most known of the water quality softwares, developed in the US Environmental Protection

*LMAC, EA 2222, Génie Informatique, Université de Technologie de Compiègne, BP 20529, 60205 COMPIEGNE Cedex, FRANCE.

†LAMSIN, Ecole Nationale d'Ingénieurs de Tunis.

‡LMAC, Université de Technologie de Compiègne and LAMSIN, Ecole Nationale d'Ingénieurs de Tunis.

Agency (see [5]). The contaminant concentration $c(\cdot, \cdot)$ is governed by the following transport equation

$$\partial_t c - (Dc' + Vc)' + Rc = f(t)\delta_{x-s(t)}, \quad \text{in } I \times (0, T). \quad (1)$$

I is the interval $[0, L]$, $x \in I$ is the curvilinear abscissa, $t \in [0, T]$ the time and the symbol $'$ denotes the space derivative ∂_x . The physical parameters D, V and R are the dispersion, advection and the reaction coefficients. The right hand side describes, most often, the load of an accidental contaminant or polluting spill, where $f(\cdot)$ is the intensity of the source and $s(\cdot)$ is its time-dependent position. The symbol δ_{x-s} stands for the Dirac distribution with mass unity and supported at the point s . Only for seek of simplicity, we select the initial condition where there is no contamination nor pollution at the time origin. At the extremities of the channel, we will consider the most standard boundary conditions. Let us fix them for a while to Robin conditions. We have then

$$\begin{aligned} c(\cdot, 0) &= 0 & \text{in } I, \\ -\alpha_0 Dc'(0, \cdot) + c(0, \cdot) &= \alpha_L Dc'(L, \cdot) + c(L, \cdot) = 0 & \text{in } (0, T). \end{aligned}$$

α_0, α_L are positive real-numbers. There is a large literature handling different aspects of the contaminant transport. We refer to [25] for a wide description of the mathematical modeling.

The inverse problem of source detection mostly dealt with in the specialized literature consists in the determination of $F = (f(\cdot), s(\cdot))$ from some given observations on $c(\cdot, \cdot)$. we refer to [6, 7, 14, 8, 12, 11, 16, 24, 13] without being exhaustive. To make diagnostic statements about a possible contamination, assume two sensors placed at the points $\zeta, \eta \in [0, L]$, *framing the source* which means that $\zeta < s(\cdot) < \eta$. The observation operator is then defined by

$$B_{\zeta, \eta}[F](\cdot) = (c(\zeta, \cdot), c(\eta, \cdot)).$$

The question now is the following. Consider that the measurement functions $(h_\zeta(\cdot), h_\eta(\cdot))$ are known. Are they significant and sufficient to discriminate the source F . This is actually related to the uniqueness for the inverse problem: *find F satisfying*

$$B_{\zeta, \eta}[F](\cdot) = (h_\zeta(\cdot), h_\eta(\cdot)).$$

This problem is mathematically ill-posed. No existence of F is ensured and, if so, it suffers from severe instability with respect to the outputs, that is a small perturbation on the data $(h_\zeta(\cdot), h_\eta(\cdot))$ produce erratic solutions. The main feature dealt with has to do with the uniqueness, it is the identifiability. We believe it worthwhile to emphasize on the fact that this identifiability issue arise harder work and requires then sharper analysis in one than in higher dimension. Mathematical results has been stated progressively and took several steps. The first identifiability we know of dates back to [8, 2005], we refer also to [12, 11] for further developments. The proofs made there

are based on some assumptions that are drastically restrictive. The physical parameters (D, V, R) should be constant, the source should become inactive at some time anterior to T and at least one sensor should be placed at a ‘strategic’ point. To make only one comment about these hypothesis, we say that the last condition on the sensor placement⁽¹⁾ is hard and even impossible to meet in the practice. Few improvements have been realized before the problem has been fully solved in [2, 2011] in the general context. Parameters are space-varying, no limitation are needed on the source activity and no stringent hypothesis are made on the location of the sensors.

The purpose, in this contribution, is to focus on the detection of the location $s(\cdot)$ from only one observations. The question to ask therefore is related to the observability of the system and can be reformulated as follows

Assume the source intensity $f(\cdot)$ known, can one identify its moving location $s(\cdot)$ by means of a single observation?

In case of an affirmative answer (it is in fact the case), this allows the economy of placing redundant sensors and spare useless expenses. Indeed, collecting observations for some contaminant species generates substantial costs. The issue of detecting the location of point-wise sources may be encountered quite often in the real-life. It may occur for instance that an accidentally broken pipe spills hazardous matter in rivers or channels, with a well-known (positive) debit. Engineers should therefore cope with finding its location before implementing and activating suitable mechanisms for reparation.

The paper is organized as follows. In Section 2, we recall some useful regularity results for the solution of the reaction-diffusion problem. They are required by the application more than once of the unique continuation theorem. Section 3 is devoted to the inverse problem of detecting the location of a point-wise source in (1) from a single observation. The central result we state is the identifiability for a moving source. Then, we briefly discuss the ill-posedness issue. Section 4 is a focus on the same detection problem of the location of point sources in a coupled parabolic system. This model is obtained after adding the Taylor dispersion term to the well known linear Streeter-Phelps reacting equations, currently used in the water quality analysis. One of the sources to identify describes a polluting spill in a river and the other represents an oxygen pump. We also state a uniqueness result for both sources. Two extensions are proposed in Section 5. We address the identifiability for multi point sources with a minimum observations. Then, we consider the case of a semi-linear equation where the reaction kinetics are of the logistic and Michaelis-Menten type.

Notation. For a given $p \in [1, \infty[$, the Lebesgue space of functions p -integrable over I is denoted by

¹To provide a clue of this difficulty, we may compare it to the problem of placing the sensor at non-rational location in the interval $(0, L)$.

$L^p(I)$. The scale of Sobolev spaces $H^\sigma(I)$, with $\sigma \in \mathbb{R}$, are defined as in [1]. We have in particular $H^0(I) = L^2(I)$. For a given $T > 0$, we use also the anisotropic Sobolev spaces $L^2(0, T; H^\sigma(I))$ and $H^\sigma(0, T; L^2(I))$ whose definitions are given in [18]. Then, we indicate by $\mathcal{C}(I)$ the space of continuous functions in I and $\mathcal{C}^m(I)$ is the space of functions whose derivative to the order m are continuous. Finally, we need also the Hölder space $\mathcal{C}^{0,1}(I)$ containing the Lipschitzian functions. We refer to [1] for general Hölder spaces.

2 Regularity

The mathematical complications of the inverse problem we are concerned with are essentially originated from the dispersion term in the transport equation. The advection has doubtless an important physical effect. It does not however bring further insuperable difficulties to the study. Therefore, and as already explained in [2], operating a suitable transform to the equation (1) restricts the scope to the reaction-dispersion equation instead of the full problem. Regarding these remarks, the advection velocity of the river can be put to $V = 0$. The only incidence is the changing of the boundary conditions to prescribe to the channel extremities. Well then, as we intend to address all the classical boundary conditions, such an assumption ($V = 0$) has no real limitation. The problem under investigation reads thus as follows

$$\partial_t c - (Dc')' + Rc = f(t)\delta_{x-s(t)}, \quad \text{in } Q_T, \quad (2)$$

$$c(\cdot, 0) = 0 \quad \text{in } I, \quad (3)$$

$$-\alpha_0 Dc'(0, \cdot) + c(0, \cdot) = \alpha_L Dc'(L, \cdot) + c(L, \cdot) = 0 \quad \text{in } (0, T). \quad (4)$$

The symbol Q_T is the space-time domain $I \times]0, T[$.

Remark 2.1 *In the geometry of the space-time domain plotted in Fig. 1, the trajectory $(s(t), t)_{0 \leq t \leq T}$ splits Q_T into two sub-domains Q_- and Q_+ . The reaction-dispersion equation (2) may be then reformulated differently. The source can be put to zero outside the trajectory of the source support, when the equation is restricted to Q_- and Q_+ . Across that trajectory, a transmission condition holds*

$$[c](s(t), t) = 0, \quad [Dc'](s(t), t) = f(t), \quad \forall t \in (0, T).$$

The symbol $[c]$ stands for the jump ($c_+ - c_-$).

Further regularity of $c(\cdot, \cdot)$ is necessary to the analysis to undertake. We need to introduce some assumptions on the physical parameters. We assume that the dispersion coefficient $D = D(\cdot)$ is differentiable and the reaction parameter $R(\cdot)$ is continuous. Extension to piecewise continuously differentiable $D(\cdot)$ and piecewise continuous $R(\cdot)$ does not bring arguments other than those related to the technical mathematics to use. We suppose also that $D(\cdot)$ has both sided bounds that is

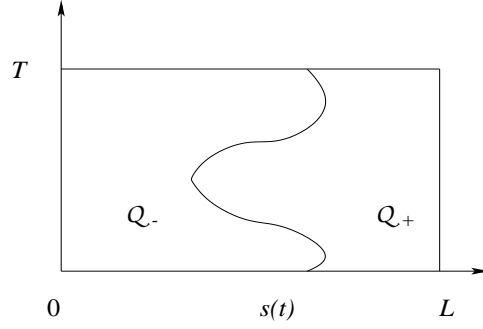


Figure 1: The trajectory of the source divides \mathcal{Q}_T into two disjoint parts denoted by \mathcal{Q}_- and \mathcal{Q}_+ .

$D_* \leq D(x) \leq D^*$ for all $x \in I$, where D_* and D^* are positive constants. The reaction $R(\cdot)$ is positive and continuous. In addition, we consider that the source intensity $f(\cdot)$ is in $L^2(0, T)$ and the location function $s(\cdot) \in \mathcal{C}^{0,1}([0, T])$. The following result holds.

Lemma 2.1 *Problem (2)-(4) has a unique solution $c(\cdot, \cdot)$ in $\mathcal{C}([0, T], L^2(I)) \cap L^2(0, T, H^1(I))$. We have also the stability*

$$\|c(t)\|_{L^2(I)} + \|c\|_{L^2(0, T, H^1(I))} \leq \gamma \|f\|_{L^2(0, T)}, \quad \forall t \geq 0.$$

γ is a positive real-number.

The solution $c(\cdot, \cdot)$ enjoys more regularity. Results stated in [18] allows to write that

$$c \in L^2(0, T, H^{(3/2)-}(I)) \cap H^{(3/4)-}(0, T, L^2(I)). \quad (5)$$

Additional and useful regularity may be obtained on $c(\cdot, \cdot)$, away from the source trajectory $(s(t), t)_{0 \leq t \leq T}$. The proof of the following lemma may be achieved following the lines of [4, proposition 3.4].

Lemma 2.2 *Assume that $f \in L^2(0, T)$. The solution $c(\cdot, \cdot)$ of problem (2)-(4) satisfies the following regularity*

$$\begin{aligned} (Dc')' &\in L^2_{loc}(\mathcal{Q}_-) \cup L^2_{loc}(\mathcal{Q}_+), \\ Dc' &\in \mathcal{C}(\mathcal{Q}_-) \cup \mathcal{C}(\mathcal{Q}_+). \end{aligned}$$

Remark 2.2 *Parabolic regularity yields that the smoothness of $c(\cdot, \cdot)$, away from the source trajectory, is limited only by the regularity of (D, R) . For instance, in case these last parameters are in $\mathcal{C}^\infty(\bar{I})$, then $c(\cdot, \cdot) \in \mathcal{C}^\infty(\mathcal{Q}_-) \cup \mathcal{C}^\infty(\mathcal{Q}_+)$. This is the hypo-ellipticity of the heat operator (see [10]).*

3 Identifiability

We investigate, first, the uniqueness of the point source location $s(\cdot)$ where the intensity function is known a-priori. Assume first that the sign of $f(\cdot)$ is constant, let us say $f(\cdot) \geq 0$. As we are interested only in active sources we make the natural hypothesis that $f(\cdot) \not\equiv 0$. Extension to a broader class of intensity functions will be discussed later on. The aim pursued here is that the identifiability is ensured by only one observation.

Consider that the position of the sensor is fixed at ζ . Let the observation operator B_ζ be defined as

$$B_\zeta[s] := c(\zeta, \cdot) \quad t \in (0, T).$$

Then, given the observed function $h_\zeta(\cdot)$, we are involved in solving the problem: *find s such that*

$$B_\zeta[s] = h_\zeta(\cdot) \quad t \in (0, T). \quad (6)$$

Obviously, this problem is non-linear. The question we are addressing is the identifiability. Let $s_j(\cdot), j = 1, 2$, stand for the locations of two point-wise sources that are both solutions of (6). Do they coincide? A successful answer needs an supplementary (minor) information about the position of the sensor with respect to the source location. We shall know whether the sensor is placed up-stream or down-stream of the point source. From now on, we assume it is located down-stream that is $s(\cdot) \in]0, \zeta[$. Then, we have

Theorem 3.1 *Let $f(\cdot)$ be in $L^2(0, T)$ with $f(\cdot) \geq 0$ and not identically zero. The following identifiability result holds. If $B_\zeta[s_1] = B_\zeta[s_2]$, then the trajectories of both sources coincide, then*

$$s_1(\cdot) = s_2(\cdot), \quad \text{in } (0, T).$$

Once the identifiability result stated, we are left with solving the identification problem (6). As a by-product there holds the following

Corollary 3.2 *Assume that $f \in L^2(0, T)$ with $f(\cdot) \geq 0$. Let the observation function $h_\zeta(\cdot)$ be given, The identification problem (6) has at most one solution $s(\cdot) < \zeta$.*

To establish the equality of $s_1(\cdot)$ and $s_2(\cdot)$, we choose to supply the guidance for the identifiability process in the case of fixed sources that is when s is not time dependent. Necessary modifications enabling the extension of the result to moving sources are exposed in an independent subsection.

3.1 Fixed Source

Let us order the sources so that $s_1 \leq s_2$. Recall that they are both $< \zeta$. Then, we consider the difference function $\epsilon = (c_2 - c_1)$. It is solution of

$$\partial_t \epsilon - (D\epsilon)' + R\epsilon = f(t)(\delta_{x-s_2} - \delta_{x-s_1}) \quad \text{in } Q_T, \quad (7)$$

$$\epsilon(\cdot, 0) = 0 \quad \text{in } I, \quad (8)$$

$$-\alpha_0 D\epsilon'(0, \cdot) + \epsilon(0, \cdot) = \alpha_L D\epsilon'(L, \cdot) + \epsilon(L, \cdot) = 0 \quad \text{in } (0, T). \quad (9)$$

We proceed by a contradiction argument. We state that $s_1 \neq s_2$ results necessarily in $f(\cdot) = 0$. Both sources F_1 and F_2 are thus switched off and no contamination is affecting the channel. This result can be obtained if we show that $\epsilon(\cdot, \cdot)$ is fully zero in Q_T . This is gradually established following the scheme of Fig. 2.

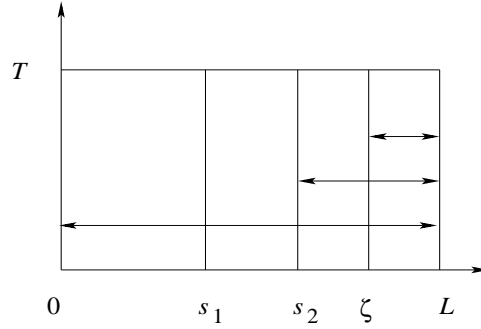


Figure 2: The space-time domain Q_T and illustration of some notations.

Lemma 3.3 *Assume that $f(\cdot) \in L^2(0, T)$. Suppose that $B_\zeta[s_1] = B_\zeta[s_2]$, then*

$$\epsilon(\cdot, \cdot) = 0, \quad \text{in } [\zeta, L] \times [0, T].$$

Proof: Restricting the problem to $(0, T) \times (\zeta, L)$ produces the sub-problem

$$\begin{aligned} \partial_t \epsilon - (D\epsilon)' + R\epsilon &= 0 & \text{in } (\zeta, L) \times (0, T), \\ \epsilon(\cdot, 0) &= 0 & \text{in } (0, \zeta), \\ \epsilon(\zeta, \cdot), \quad \alpha_L D\epsilon'(L, \cdot) + \epsilon(L, \cdot) &= 0 & \text{in } (0, T). \end{aligned}$$

This is because the source in equation (7) lies at the left of the sensor position ζ . Dirichlet's condition at point ζ comes from the observations (6) satisfied by F_1 and F_2 . The unique solution of this homogeneous parabolic problem is $\epsilon(\cdot, \cdot) = 0$. The proof is complete. \blacksquare

The result can be extended to cover the larger strip $(0, T) \times (s_2, L)$. The following result holds.

Lemma 3.4 Assume that $f \in L^2(0, T)$. Suppose that $B_\zeta[s_1] = B_\zeta[s_2]$, then

$$\epsilon(\cdot, \cdot) = 0, \quad \text{in } (0, T) \times (s_2, L).$$

Proof: The support of the source term in (7) does not intersect with $(0, T) \times]s_2, L)$, we derive hence

$$\begin{aligned} \partial_t \epsilon - (D\epsilon')' + R\epsilon &= 0 & \text{in }]s_2, L) \times (0, T), \\ \epsilon(\cdot, 0) &= 0 & \text{in } (s_2, L), \\ \alpha_L D\epsilon'(L, \cdot) + \epsilon(L, \cdot) &= 0 & \text{in } (0, T). \end{aligned}$$

The smoothness on $c_j(\cdot, \cdot)$, $j = 1, 2$, stated in Lemma 2.2 together with Lemma 3.3 showing that $\epsilon(\cdot, \cdot)$ vanishes in $(\zeta, L) \times (0, T)$ suggest to apply the unique continuation theorem (see [20, Theorem 1.1]). This yields that $\epsilon(\cdot, \cdot) = 0$ in $(s_2, L) \times (0, T)$. The proof is complete. \blacksquare

We are now left with the reaction-dispersion problem (7)-(9) set in the sub-domain $(0, s_2) \times (0, T)$,

$$\begin{aligned} \partial_t \epsilon - (D\epsilon')' + R\epsilon &= -f(t)\delta_{x-s_1} & \text{in } (0, s_2) \times (0, T), \\ \epsilon(\cdot, 0) &= 0 & \text{in } I, \\ -\alpha_0 D\epsilon'(0, \cdot) + \epsilon(0, \cdot) &= 0, \quad \epsilon(s_2, \cdot) = 0 & \text{in } (0, T). \end{aligned}$$

The Dirichlet condition at s_2 is ensued from Lemma 3.4. Next, calling for a weak form of the maximum principle (see [10]) we come up with the following. Given that $f(\cdot) \geq 0$, then we have

$$\epsilon(\cdot, \cdot) \leq 0 \quad \text{in } (0, s_2) \times (0, T). \quad (10)$$

This result contributes to prove the

Lemma 3.5 Assume that $f \in L^2(0, T)$ with $f(\cdot) \geq 0$. Suppose that $B_\zeta[s_1] = B_\zeta[s_2]$, then

$$\epsilon(\cdot, \cdot) \equiv 0, \quad \text{in } (0, s_2) \times (0, T).$$

Proof: For convenience, let us first set

$$m(t) = \int_{(0, s_2)} \epsilon(x, t) dx.$$

Integrate equation (7) with respect to the space variable x . We obtain

$$\partial_t m(t) - \int_{(0, s_2)} (D\epsilon')'(x, t) dx + \int_{(0, s_2)} R\epsilon(x, t) dx = -f(t).$$

This formula is due to $\langle \delta_{x-s_1}, 1 \rangle = 1$. Then, easy computations yield that

$$\partial_t m(t) - D\epsilon'(s_2, t) + D\epsilon'(0, t) + \int_{(0, s_2)} R\epsilon(x, t) dx = -f(t).$$

A straightforward consequence of Lemma 3.4 is that $(D\epsilon')_+(s_2, \cdot) = 0$. Thus, thanks to Remark 2.1, used with $\epsilon(\cdot, \cdot)$, we deduce that

$$(D\epsilon')_-(s_2, t) = f(t), \quad \forall t \in (0, T).$$

This allows to derive that

$$\partial_t m(t) = -D\epsilon'(0, t) - \int_{(0, s_2)} R\epsilon(x, t) dx.$$

Owing to (10), the integral term in the right hand side is non-positive. Remains to investigate the sign of $D\epsilon'(0, t)$. This is dependent on the boundary condition, enforced there, and then on the real number $\alpha_0 \geq 0$. We check each of the possible boundary conditions.

- *Neumann condition* ($\alpha_0 = +\infty$) — We have naturally that $D\epsilon'(0, t) = 0$.
- *Robin condition* ($0 < \alpha_0 < +\infty$) — According to (10), we have in particular that $\epsilon(0, t) \leq 0$ and since $\alpha_0 D\epsilon'(0, t) = \epsilon(0, t)$, this results in $D\epsilon'(0, t) \leq 0$.
- *Dirichlet condition* ($\alpha_0 = 0$) — Given that $\epsilon(0, t) = 0$ and $\epsilon(\cdot, t) \leq 0$ at the vicinity of $x = 0$. Then, $\epsilon(\cdot, t)$ decreases to the right of zero. As a result we have necessarily that $D\epsilon'(0, t) \leq 0$.

Hence, the fact that $D\epsilon'(0, t) \leq 0$ holds true whatever the boundary condition at the origin is. We directly deduce that

$$\partial_t m(t) \geq 0, \quad \forall t \in (0, T).$$

The function $m(\cdot)$ is therefore non-decreasing, is non-positive with $m(0) = 0$. This implies $m(t) = 0$ for all $t \in (0, T)$. In view of its non-positivity of $\epsilon(\cdot, \cdot)$ we conclude that $\epsilon(\cdot, \cdot)$ vanishes in $(0, s_2) \times (0, T)$. The proof is complete. ■

These lemmas are helpful toward the proof of the identifiability theorem of the source location using a single observation.

Proof of Theorem 3.1 (Part 1): — We start by the case of fixed sources and assume that $s_1 \neq s_2$. Using Lemmas 3.3, 3.4 and 3.5, the equality $B_\zeta[s_1] = B_\zeta[s_2]$ yields that $c_1(\cdot, \cdot) = c_2(\cdot, \cdot)$ in Q_T . A direct consequence is that $f(\cdot) \equiv 0$. This is a contradiction. ■

Remark 3.1 *The assumption concerning the position of the sensor with respect to the source location is important for the uniqueness. Otherwise, the identifiability fails. The following counter example is liable to sweep away any doubt. Let $I = (-1, 1)$ and $s \in]0, 1[$. Then, we place the sensor at $\zeta = 0$. Consider c_β with $\beta \in \{-1, +1\}$, the solution of*

$$\begin{aligned} \partial_t c_\beta - c_\beta'' &= f(t)\delta_{x-\beta s}, & \text{in } Q_T, \\ c_\beta(\cdot, 0) &= 0 & \text{in } I, \\ c_\beta'(0, \cdot) = c_\beta'(L, \cdot) &= 0 & \text{in } (0, T). \end{aligned}$$

It is easily checked that $c_{-1}(t, x) = c_{+1}(t, -x)$ in Q_T . We have then the same observation $c_{-1}(t, 0) = c_{+1}(t, 0)$ for both sources supports $-s$ and $+s$ located at different sides of ζ .

3.2 Moving Source

We proceed now with the proofs for a moving source. The canvas of the proof is fundamentally unchanged. Lemma 3.3 is the same. For Lemma 3.4, some adaptations are needed. The unique continuation theorem yields therefore that $\epsilon(\cdot, \cdot)$ vanishes in the non-cylindrical sub-domain

$$\mathcal{Q}^+ = \left\{ (x, t); \quad \max(s_1(t), s_2(t)) \leq x \leq L, \quad t \in (0, T) \right\}.$$

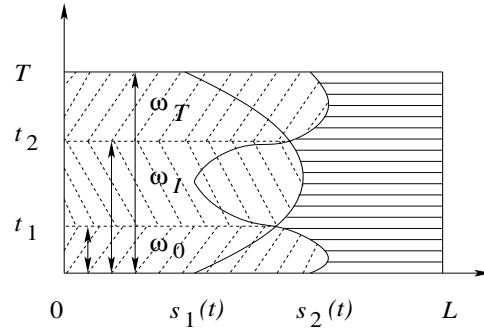


Figure 3: The function $\epsilon(\cdot, \cdot)$ is zero in \mathcal{Q}^+ , the horizontally hashed sub-domain.

We revisit the maximum principle fulfilled by $\epsilon(\cdot, \cdot)$. In the moving sources context, it can not be globally in \mathcal{Q}^- (the complementary region of \mathcal{Q}^+). Actually, we need to proceed by sliced sub-domains, according to the strips where $\max(s_1(t), s_2(t)) = s_1(t)$ or $s_2(t)$.

Lemma 3.6 *Assume that $f \in L^2(0, T)$ with $f(\cdot) \geq 0$. Suppose that $B_\zeta[s_1] = B_\zeta[s_2]$, then*

$$\epsilon(\cdot, \cdot) = 0, \quad \text{in } Q_T.$$

Proof: To fix the ideas, we consider the example of Figure 3. The trajectories $(s_1(t), t)$, $(s_2(t), t)$ split \mathcal{Q}^- into three sub-regions ω_0, ω_I and ω_T related to the time intervals $(0, T_0)$, (T_0, T_I) and (T_I, T) . The fact that remains to check is that $\epsilon(\cdot, \cdot) = 0$ in $\mathcal{Q}^- = \cup_{i \in \{0, I, T\}} \omega_i$. It takes three steps. We start by the first sub-domain

$$\omega_0 = \left\{ (x, t); \quad 0 \leq x \leq s_2(t), \quad t \in (0, T_0) \right\}.$$

The sub-problem set in ω_0 reads as

$$\begin{aligned} \partial_t \epsilon - (D\epsilon)' + R\epsilon &= -f(t)\delta_{x-s_1(t)} && \text{in } \omega_0, \\ \epsilon(\cdot, 0) &= 0, && \text{in } (0, s_2(0)), \\ \epsilon(0, \cdot) = \epsilon(s_2(\cdot), \cdot) &= 0, && \text{in } (0, T_0). \end{aligned}$$

Using standard arguments, we may show that the weak maximum principle is valid

$$\epsilon(\cdot, \cdot) \leq 0, \quad \text{in } \omega_0. \quad (11)$$

The function $\epsilon(\cdot, \cdot)$ vanishes in $Q_{T_0} \setminus \omega_0$ and is non-positive in ω_0 . Next, we follow the proof developed for fixed sources. The point here is that because of the non-cylindrical shape of ω_0 , we are led to integrate $\epsilon(\cdot, \cdot)$ in x on the whole interval I . We obtain then

$$\partial_t \left(\int_I \epsilon(x, t) dx \right) - \int_I (D\epsilon')'(x, t) dx + \int_I R\epsilon(x, t) dx = 0, \quad \forall t \in (0, T_0).$$

Accounting for $\epsilon(x, t) = 0$ for all $x \geq s_2(t)$, we derive that

$$\partial_t \left(\int_{(0, s_2(t))} \epsilon(x, t) dx \right) = -D\epsilon'(0, t) - \int_{(0, s_2(t))} R\epsilon(x, t) dx, \quad \forall t \in (0, T_0).$$

The proof is then achieved following the same process as in the one for Lemma 3.5 using the maximum principle (11). The result that $\epsilon(\cdot, \cdot) = 0$ in Q_{T_0} is thus demonstrated.

The second step is to extend that result to Q_{T_I} and we are then involved in proving it in $Q_{T_I} \setminus Q_{T_0}$. We have already established it outside ω_I . Restricting the problem to

$$\omega_I = \left\{ (x, t); \quad 0 \leq x \leq s_1(t), \quad t \in (T_0, T_I) \right\},$$

we get that

$$\begin{aligned} \partial_t \epsilon - (D\epsilon')' + R\epsilon &= f(t)\delta_{x-s_2(t)} && \text{in } \omega_I, \\ \epsilon(x, T_0) &= 0, && \text{in } (0, s_1(T_0)), \\ \epsilon(0, t) = \epsilon(s_1(t), t) &= 0, && \text{in } (T_0, T_I). \end{aligned}$$

The initial condition is inherited from the previous step, hence the necessity to proceed sequentially. Applying the maximum principle yields that

$$\epsilon(\cdot, \cdot) \geq 0, \quad \text{in } \omega_I, \quad (12)$$

so that $\epsilon(\cdot, \cdot)$ is non-negative in ω_I and vanishes outside (of ω_I). Performing the same proof as above yields that

$$\partial_t \left(\int_{(0, s_1(t))} \epsilon(x, t) dx \right) = -D\epsilon'(0, t) - \int_{(0, s_1(t))} R\epsilon(x, t) dx, \quad \text{in } (T_0, T_I).$$

In view of the maximum principle (12), it can be stated that the right hand sides is non-positive.

The function

$$m(t) = \int_{(0, s_1(t))} \epsilon(x, t) dx,$$

is thus non-increasing and non-negative with $m(T_0) = 0$. This implies necessarily that $m(t) = 0$, for all $t \in (T_0, T_I)$. Calling once again for the non-negativity provides that $\epsilon(\cdot, \cdot) = 0$ in Q_{T_I} .

The last step is to switch to ω_T which is made by reproducing the proof in ω_0 with some slight modifications. The final result is that $\epsilon(\cdot, \cdot)$ vanishes in the whole Q_T . The proof is complete. ■

Proof of Theorem 3.1 (Part 2): — To conclude with the identifiability for a moving source, we argue in the same way as for a fixed source, where Lemma 3.5 is replaced by Lemma 3.6. ■

3.3 Identifiability for a broader class of intensity functions

The identifiability result stated in Theorem 3.1 are so far limited to intensity functions $f(\cdot) \in L^2(0, T)$ possessing a constant sign. However, they may be extended to a broader class of intensities. Assume, for instance, that for a given sequence $t_0 = 0 < t_1 < \dots < t_{K-1} < t_K = T$, the sign of the intensity function $f(\cdot)$ remains constant in $([t_k, t_{k+1}])_{(0 \leq k \leq K-1)}$, the sign, whatever it is, does not matter. Then the identifiability may be processed sequentially step by step as in the proof of Theorem 3.1 elaborated for a moving source location. First, we prove the result in $I \times [0, t_1]$, then on $I \times [t_1, t_2]$ and so on until the last $I \times [t_{K-1}, T]$. This class includes the continuous functions $f(\cdot)$, for which the zeroes, in $[0, T]$, where $f(\cdot)$ changes sign are isolated. An example for which the approach fails is provided by the intensity function

$$f(t) = t \sin\left(\frac{1}{t}\right), \quad t \in (0, T).$$

Although it is continuous at $t = 0$, this function changes sign permanently at the vicinity of zero. The step-by-step arguing hardly works in this case, we do not even see how to start. Nevertheless, our feeling is that the identifiability still holds true. Conceiving a different methodology to prove this result is thus an open question.

3.4 Ill-posedness

The issue of ill-posedness degree is probably hard to investigate in details. This is beyond our scope. The modest purpose we are assigned is to briefly illustrate the severe ill-posedness of problem (6). The way to reach this objective consists in proving that for all $s(\cdot) \in \mathcal{C}^{0,1}([0, T])$, the operator $B'_\zeta[s]$, mapping $\mathcal{C}^{0,1}([0, T])$ into $L^2(0, T)$, is compact. B'_ζ is the Fréchet derivative of B_ζ . Then, we supply an explanation of why the compactness degree of $B'_\zeta[s]$ is infinite which is a clue of severe ill-posedness (see [23]).

We investigate the particular case where $D(\cdot)$ and $R(\cdot)$ are indefinitely smooth. Hence, they both lie in $\mathcal{C}^\infty([0, T])$. In addition, to alleviate the presentation, we fix the amplitude function to $f(\cdot) \equiv 1$ and we consider the case where conditions (3) are of (Dirichlet, Neumann) type. Let now $(ds)(\cdot) \in \mathcal{C}^{0,1}([0, T])$ be a given small perturbation of the source $s(\cdot)$, we define $\psi \in L^2(Q_T)$ as the solution of the boundary value problem

$$\begin{aligned} \partial_t \psi - (D\psi)' + R\psi &= -(ds)(t)\delta'_{x-s(t)}, & \text{in } Q_T, \\ \psi(\cdot, 0) &= 0 & \text{in } (0, T), \\ \psi(0, \cdot) = D\psi'(L, \cdot) &= 0 & \text{in } I. \end{aligned}$$

It may be checked without pain that

$$\langle B'_\zeta[s], (ds) \rangle = \psi(\zeta, \cdot), \quad \text{in } (0, T).$$

The well posedness of the problem in $\psi(\cdot, \cdot)$ may be stated by the duality method (see [17]). Actually, the source term expresses that in the sub-domains \mathcal{Q}_- and \mathcal{Q}_+ represented in Fig. 1, the source in the reaction-diffusion vanishes. Then, across the interface $(s(t), t)$, one has the following transmission conditions

$$[D\psi'](s(t), t) = 0, \quad [\psi](s(t), t) = (ds)(t), \quad \forall t \in (0, T).$$

The following result holds

Lemma 3.7 *For all $s(\cdot) \in \mathcal{C}^{0,1}([0, T])$, the linear operator $B'_\zeta[s]$ mapping $\mathcal{C}^{0,1}([0, T])$ into $L^2(0, T)$ is compact.*

Proof: The hypo-ellipticity of the heat operator yields that $\psi(\cdot, \cdot)$ is indefinitely smooth outside the support of the source term. In consequence, we infer that $\psi(\zeta, \cdot)$ belongs to $\mathcal{C}^\infty([0, T])$. Furthermore, for arbitrary $m \geq 0$ we have the estimate

$$\|\langle B'_\zeta[s], (ds) \rangle\|_{\mathcal{C}^m([0, T])} = \|\psi(\zeta, \cdot)\|_{\mathcal{C}^m([0, T])} \leq C(m) \|(ds)\|_{\mathcal{C}^{0,1}([0, T])}.$$

Then, invoking the compactness of the embedding from $\mathcal{C}^\infty([0, T])$ into $L^2(0, T)$, we conclude to the compactness of $B'_\zeta[s]$ when considered as an operator ranging from $\mathcal{C}^{0,1}([0, T])$ into $L^2(0, T)$. The proof is complete. ■

Remark 3.2 *The compactness of $B'_\zeta[s]$, for any $s(\cdot)$, suffices to assert the ill-posedness of the inverse problem (6). However, one may be interested in further information about the ill-posedness degree. The severe ill-posedness is in general related to the smoothing effects of $B'_\zeta[s]$ which are unlimited here. As mentioned in the proof above, $\langle B'_\zeta[s], (ds) \rangle$ is indefinitely smooth for all $(ds)(\cdot) \in \mathcal{C}^{0,1}([0, T])$. This suggests that the the problem is in reality severely ill posed.*

4 A deoxygenation reaeration model

In the analysis of water quality in rivers, engineers are most often interested in two tracers. One is the *biochemical oxygen demand* concentration $b(\cdot, \cdot)$ measures the amount of oxygen consumed by bacteria in the oxidation or biodegradation process of polluting organic matter. The other indicator is the *dissolved oxygen* density generally denoted $c(\cdot, \cdot)$ describing the amount of oxygen available in the water and absorbed by the river from the atmosphere (see [22, 21]). In the subsequent, we use the acronyms (bod) and (do) for convenience. The model currently used is derived from the Streeter-Phelps coupled reaction system to which the Taylor dispersion is added. It is a linear model in which the vertical and transversal transport are instantaneously completed processes (see

[5]). In this respect, the dispersive (bod-do) system reads then as follows

$$\begin{aligned}
\partial_t b - (Db')' + Rb &= f(t)\delta_{x-s(t)} && \text{in } Q_T \\
\partial_t c - (Dc')' + Rc + Rb &= R\chi + g(t)\delta_{x-r(t)} && \text{in } Q_T \\
Db'(L, \cdot) = 0, \quad b(0, \cdot) &= 0 && \text{in } (0, T) \\
Dc'(L, \cdot) = 0, \quad c(0, \cdot) &= \chi && \text{in } (0, T) \\
b(\cdot, 0) &= 0 && \text{in } I \\
c(\cdot, 0) &= \chi && \text{in } I.
\end{aligned}$$

The constant χ is the saturation level of the oxygen in the water. Before the activation of the point-wise sources we consider that the equilibrium of the river is settled at the oxygen saturation. This explains the initial condition on $c(\cdot, \cdot)$. The Dirichlet condition says that, upstream, sufficiently away from the sources, the river is at the equilibrium state and no pollution occurs there. The term Rb in the second equation indicates that the increase in (bod) is interpreted as the oxygen uptake in the water. Together with the source $g(t)\delta_{x-r(t)}$ (if $g(\cdot) < 0$), they contribute to the de-oxygenation of the river. Oppositely, the $R\chi$ competes to the re-aeration of the river. Notice that here again, we do not account for the advection despite its physical important meaning. We mainly focus on the effects of the dispersion which is responsible of the most intrincating mathematical difficulties.

The inverse problem we consider is the reconstruction of the sources $f(\cdot)\delta_{x-s(\cdot)}$ and $g(\cdot)\delta_{x-r(\cdot)}$ by observing the depletion of the (do) concentration $c(\cdot, \cdot)$ caused by the elevation of the (bod) density. No direct measurements are available for the density $b(\cdot, \cdot)$. The reason is that observations on (do) are much easier to conduct. Recording observations on (bod) follows a strict chemical protocol that may last five days, too long a period when treating accidental spills is pursued. The complication of the identifiability of the sources is currently augmented by the lack of direct records on $b(\cdot, \cdot)$. To correctly solve this issue, we will extract from [3] the results that will help us to be in the context described in the previous sections.

We will be in particular interested in the non-linear identifiability of the sources location. Only $(s(\cdot), r(\cdot))$ are to be detected. The intensities $f(\cdot)$ and $g(\cdot)$ are available beforehand and are fixed. We suppose that they are signed, let us say for instance that $f(\cdot) \geq 0$ and $g(\cdot) \leq 0$. The assumption on $f(\cdot)$ is natural and expresses that $f(\cdot)\delta_{x-s(\cdot)}$ is a polluting source, while the one on $g(\cdot)$ tells that $g(\cdot)\delta_{x-r(\cdot)}$ describes the action of an oxygen pump. Define the following observations operator

$$D_\zeta [s, r] = \left\{ (c(\zeta, \cdot), D(\zeta)c'(\zeta, \cdot)), \quad 0 \leq t \leq T \right\}.$$

Given two observation functions $h_\zeta(\cdot), k_\zeta(\cdot)$ we pursue the reconstruction of $(s(\cdot), r(\cdot))$ satisfying

$$D_\zeta [s, r] = (h_\zeta(\cdot), k_\zeta(\cdot)). \tag{13}$$

The identifiability result concludes that the trajectories of the pollution sources may be recovered if the variations in (do) concentration are recorded at an observation station. The only further information required is the knowledge about the position of the sources with respect to station location ζ . The case we are able to study is when the positions $s(\cdot)$ and $r(\cdot)$ of both sources are at the same side of ζ and we need to know at which side. Henceforth, we consider that they are permanently contained in $(0, \zeta)$. Then, we have $s(t), r(t) < \zeta$ for all $t \in (0, T)$. The identifiability of the source for the coupled (bod-do) system is given in

Theorem 4.1 *Assume that $f(\cdot) \geq 0$ and $g(\cdot) \leq 0$. Then, for any $h_\zeta(\cdot)$ and $k_\zeta(\cdot)$ in $L^2(0, T)$, problem (13) has at most one solution $(s(\cdot), r(\cdot))$ with $s(\cdot), r(\cdot) < \zeta$.*

To proceed with the proof, we follow the lines of [3]. Consider $(s_1(\cdot), r_1(\cdot))$ and $(s_2(\cdot), r_2(\cdot))$ two solutions to the inverse problem (13). Without loss of generality, we suppose that $s_1(\cdot) < s_2(\cdot)$ and $r_1(\cdot) < r_2(\cdot)$. Indeed, if the trajectories cross each other, we split the interval into slices where either $s_1(\cdot) \leq s_2(\cdot)$ or $s_1(\cdot) \geq s_2(\cdot)$ following the methodology elaborated in Subsection 3.2. The ultimate objective is to show that

$$(D_\zeta [s_1, r_1] = D_\zeta [s_2, r_2]) \text{ implies that } ((b_1, c_1) = (b_2, c_2)).$$

This assertion produces the uniqueness result, that is $s_1(\cdot) = s_2(\cdot)$ and $r_1(\cdot) = r_2(\cdot)$, which is the ultimate purpose. We begin by some preparatory lemmas.

Lemma 4.2 *Suppose that $D_\zeta [s_1, r_1] = D_\zeta [s_2, r_2]$, then*

$$(b_1, c_1) = (b_2, c_2) \quad \text{in } (\zeta, L) \times (0, T).$$

Proof: Introduce the notations $\epsilon = (b_2 - b_1)$ and $\kappa = (c_2 - c_1)$. The following system holds

$$\begin{aligned} \partial_t \epsilon - (D\epsilon)' + R\epsilon &= 0 & \text{in } (\zeta, L) \times (0, T) \\ \partial_t \kappa - (D\kappa)' + R\kappa + R\epsilon &= 0 & \text{in } (\zeta, L) \times (0, T) \\ D\epsilon'(L, t) = D\kappa'(L, t) &= 0 & \text{in } (0, T) \\ D\kappa'(\zeta, t) = 0, \quad \kappa(\zeta, t) &= 0 & \text{in } (0, T) \\ \epsilon(\cdot, 0) = 0, \quad \kappa(\cdot, 0) &= 0 & \text{in } (\zeta, L), \end{aligned}$$

The particularity consists in the boundary conditions at ζ . Neumann and Dirichlet conditions are enforced on $\kappa(\cdot, \cdot)$ whereas $\epsilon(\cdot, \cdot)$ is free of any condition. It is checked in [3] that this system is ill-posed. Nevertheless, a uniqueness result has been proven in [3, Theorem 2.7]. It is easily seen that this homogeneous system has the trial solution $(\epsilon(\cdot, \cdot), \kappa(\cdot, \cdot)) = (0, 0)$ which therefore unique. The proof is complete. ■

We are in position to achieve the first step, the identifiability for the (bod) source $s(\cdot)$. Indeed, there holds that

Lemma 4.3 Assume that $f(\cdot) \geq 0$. If $D_\zeta [s_1, r_1] = D_\zeta [s_2, r_2]$, then

$$s_1(\cdot) = s_2(\cdot), \quad \text{in } (0, T).$$

Proof: Setting once again $\epsilon = (b_2 - b_1)$. We derive that

$$\begin{aligned} \partial_t \epsilon - (D\epsilon)' + R\epsilon &= f(t)\delta_{x-s_2(t)} - f(t)\delta_{x-s_1(t)} && \text{in } Q_T \\ D\epsilon'(L, t) = 0 \quad \epsilon(0, t) &= 0 && \text{in } (0, T) \\ \epsilon(\cdot, 0) &= 0 && \text{in } I. \end{aligned}$$

In addition, we know now that $\epsilon(\cdot, \cdot) = 0$ in $(\zeta, L) \times (0, T)$. Using that $f(\cdot) \geq 0$, we deduce that $\epsilon(\cdot, \cdot) \leq 0$ in Q_T . Then, to end the proof we follow the argument of the proof of Lemma 3.6. This results in $\epsilon(\cdot, \cdot) \equiv 0$ in Q_T . Identifiability for $s(\cdot)$ is a directly ensued. The proof is complete. ■

We are one step away from concluding to the global identifiability result for both sources $s(\cdot)$ and $r(\cdot)$. Theorem 4.1 is proved if we show the following

Proposition 4.4 Assume that $f(\cdot) \geq 0$ and $g(\cdot) \leq 0$. If $D_\zeta [s_1, r_1] = D_\zeta [s_2, r_2]$, then

$$(s_1(\cdot), r_1(\cdot)) = (s_2(\cdot), r_2(\cdot)) \quad \text{in } (0, T).$$

Proof: It remains only to check the identifiability for $r(\cdot)$. Accounting for the fact that $\epsilon(\cdot, \cdot) \equiv 0$ obtained in Lemma 4.3, the following reaction-dispersion equation on $\kappa(\cdot, \cdot)$ holds,

$$\begin{aligned} \partial_t \kappa - (D\kappa)' + R\kappa &= g(t)\delta_{x-r_2(t)} - g(t)\delta_{x-r_1(t)} && \text{in } Q_T \\ D\kappa'(L, t) = 0, \quad \kappa(0, t) &= 0 && \text{in } (0, T), \\ \kappa(\cdot, 0) &= 0 && \text{in } I. \end{aligned}$$

Calling once again for Lemma 4.2 yields that the $\kappa(\cdot, \cdot) = 0$ in the strip $(\zeta, L) \times (0, T)$. In view of the sign constancy, $g(\cdot) \leq 0$, we are able to show that $\kappa(\cdot, \cdot) \geq 0$ in Q_T . We are exactly in the same context as in the proof of Lemma 2.1. The only difference is the sign of $\kappa(\cdot, \cdot)$. Using analogous arguments concludes to the same result $\kappa(\cdot, \cdot) \equiv 0$ in Q_T . This yields that $r_1(\cdot) = r_2(\cdot)$ in $(0, T)$. The proof is complete. ■

5 Extensions

We present and comment two extensions. We generalize the forgoing identifiability to the case of multi point-wise sources. Then, we check the reasons why the same results still hold true for Fisher's equation where the reaction term is non-linear.

5.1 Multiple point-wise sources.

Engineers may be facing the detection of more than one pollution spill. The source term in the reaction-diffusion equation (2)-(4) corresponds to the superimposition of multiple point-wise sources,

$$\partial_t c - (Dc')' + Rc = \mathbf{f}(t)\delta_{x-\mathbf{s}(t)}, \quad \text{in } Q_T, \quad (14)$$

$$c(\cdot, 0) = 0 \quad \text{in } I, \quad (15)$$

$$c(0, \cdot) = Dc'(L, \cdot) = 0 \quad \text{in } (0, T). \quad (16)$$

The source is the given by

$$\mathbf{f}(t)\delta_{x-\mathbf{s}(t)} = \sum_{1 \leq k \leq k_*} f_k(t)\delta_{x-s_k(t)}, \quad \forall t \in (0, T).$$

The amplitudes $\mathbf{f}(\cdot) = (f_k(\cdot))_{1 \leq k \leq k_*}$ are known and the detection will be exclusively concerned with the sources location $\mathbf{s}(\cdot) = (s_k(\cdot))_{1 \leq k \leq k_*}$.

We study here a particular category of sources. Their locations $\mathbf{s}(\cdot)$ should satisfy the following ordering

$$s_1(\cdot) < s_2(\cdot) < \cdots < s_{k_*-1}(\cdot) < s_{k_*}(\cdot), \quad \text{in } (0, T).$$

Recording k_* observations at k_* distinct points $\zeta = (\zeta_k)_{1 \leq k \leq k_*}$ turns out to be sufficient for the statement of the identifiability for k_* point-wise sources. They should be distributed in a particular way. Between each pair of neighboring sources one should place an observation station,

$$\zeta_1 < s_1(\cdot) < \zeta_2 < s_2(\cdot) < \zeta_3 < \cdots < s_{k_*-1}(\cdot) < \zeta_{k_*} < s_{k_*}(\cdot), \quad \text{in } (0, T). \quad (17)$$

Remark 5.1 *Assumption (17) tells that to each source $s_k(\cdot)$ may be attached an observation point ζ_k located at its the left side (of $s_k(\cdot)$). This demands of course some a-priori knowledge on the location of the polluting sources. The symmetric situation where ζ_k is positioned at the right side of $s_k(\cdot)$ is also admissible.*

Basically, the mathematical process toward the identifiability consists in using repeatedly the arguments elaborated above in a given order that depends on the position of the observations and with respect to the sources. The aim is to analyze the observation operator and to figure out its properties liable to help us reach the identifiability objective. This operator is thus defined by

$$B_\zeta[\mathbf{s}] = \left\{ (c(\zeta_1, \cdot), c(\zeta_2, \cdot), \cdots, c(\zeta_{k_*}, \cdot)), \quad 0 \leq t \leq T \right\}.$$

Now, given k_* observation functions $\mathbf{h}_\zeta(\cdot) = (h_{\zeta_k}(\cdot))_{1 \leq k \leq k_*}$ we consider the inverse problem: *find $\mathbf{s}(\cdot)$ satisfying*

$$B_\zeta[\mathbf{s}] = \mathbf{h}_\zeta(\cdot). \quad (18)$$

The following identifiability holds true

Proposition 5.1 *Assume that $\mathbf{f}(\cdot) \geq 0$ and that assumption (17) is fulfilled. If $B_\zeta[\mathbf{s}_1] = B_\zeta[\mathbf{s}_2]$, then*

$$\mathbf{s}_1(\cdot) = \mathbf{s}_2(\cdot) \quad \text{in } (0, T).$$

Proof: Only for seek of simplicity, we sketch the proof in the case of two sources ($k_* = 2$). The ideas exposed here work as well for an arbitrary number of sources provided that assumption (17) is valid. The notation $\mathbf{s}_k(\cdot) = (s_{k,1}(\cdot), s_{k,2}(\cdot))$ is suitable and will be adopted during the proof. Now, assume that $B_\zeta[\mathbf{s}_1] = B_\zeta[\mathbf{s}_2]$ and $\epsilon = (c_2 - c_1)$. The following holds

$$\partial_t \epsilon - (D\epsilon)' + R\epsilon = f_1(t)(\delta_{x-s_{2,1}} - \delta_{x-s_{1,1}}) + f_2(t)(\delta_{x-s_{2,2}} - \delta_{x-s_{1,2}}) \quad \text{in } Q_T. \quad (19)$$

The initial and boundary conditions are still homogeneous. Owing to the observations we obtain that

$$\epsilon(\zeta_1, \cdot) = \epsilon(\zeta_2, \cdot) = 0, \quad \text{in } (0, T).$$

Let us now start the identifiability proof by looking at equation (19), restricted to the strip $(0, \zeta_2) \times (0, T)$. The assumption (17) re-transcribed yields that $\zeta_1 < s_{k,1} \leq \zeta_2 < s_{k,2}$, for $k = 1, 2$. In the reaction-dispersion equation set $(0, \zeta_2) \times (0, T)$ only the sources $s_{1,1}$ and $s_{2,1}$ are kept there. The new situation turns out to be similar to the forgoing one studied in Subsection 3.2. Arguing alike, we are able to reach the fact that $\epsilon(\cdot, \cdot) \equiv 0$ in $(0, \zeta_2) \times (0, T)$. This results in $s_{1,1}(\cdot) \equiv s_{2,1}(\cdot)$. Back to equation (19), the first part of the source, the one involving $f_1(\cdot)$ is canceled. Only remain the sources located at the right of the sensor ζ_2 whose amplitude $f_2(\cdot)$. This problem can be handled as previously done and ends to the coincidence of the source locations $s_{1,2}(\cdot)$ and $s_{2,2}(\cdot)$. The proof is complete. \blacksquare

Remark 5.2 *The non-negative sign of $\mathbf{f} = (f_k(\cdot))_{1 \leq k \leq k_*}$ is of course only indicative. Some of the intensities $f_k(\cdot)$ may be non-negative and others may be non-positive. The only mandatory feature is that each $f_k(\cdot)$ should be signed.*

Remark 5.3 *The proof is unfolded into two steps (k_* steps in the general case), it was necessary to proceed from the left to the right. Checking the identifiability $s_{1,1}(\cdot) = s_{2,1}(\cdot)$ first and then $s_{1,2}(\cdot) = s_{2,2}(\cdot)$. Otherwise if the observation points and the sources are distributed differently, that is $s_{k,1} \leq \zeta_1 < s_{k,2} \leq \zeta_2$, one could and should proceed from the right to the left. The identifiability $s_{1,2}(\cdot) = s_{2,2}(\cdot)$ should be established first.*

Corollary 5.2 *Assume that $\mathbf{f}(\cdot)$ is signed. Problem (18) has at most one solution fulfilling assumption (17).*

5.2 Fisher's equation

In many real-situations, the reaction term turns out to be non-linear. We are then interested in the detection of a point-wise source for a semi-linear diffusion-reaction equation of Fisher's type.

The equation (2) is therefore transformed into the Fisher equation

$$\partial_t c - (Dc')' + R(c)c = f(t)\delta_{x-s(t)}, \quad \text{in } Q_T$$

The reaction term $R(c)c$ is non-linear. This means that $R(\cdot)$ is dependent on c . This is not a rare event and may occur for instance for the biochemical oxygen demand (bod) in a river that is heavily polluted by an organic matter. If the intensity of the polluting spill gets high, then reasonably the (bod) concentration cannot increase without restriction. The fact is that the amount of oxygen dissolved in the river is limited. A relevant modeling introduces a non-linear reaction in order to slow down an hypothetical strong growth of the (bod) density $b(\cdot, \cdot)$. The term $R(b) = R(1 + \varrho|b|)$, known as the logistic non-linearity. The semi-linear partial differential equation to deal with henceforth is

$$\partial_t b - (Db')' + R(1 + \varrho|b|)b = f(t)\delta_{x-s(t)} \quad \text{in } Q_T. \quad (20)$$

The logistic parameter ϱ is a positive real-number and so is R . Another example of non-linearity is the Michaelis-Menten reaction $R(n) = \frac{\varrho}{K+|n|}$. The constant K is the Michaelis constant and ϱ is the maximum rate reaction. The Michaelis-Menten Fisher's equation reads therefore as

$$\partial_t n - (Dn')' + \frac{\varrho n}{K + |n|} = f(t)\delta_{x-s(t)}, \quad \text{in } Q_T. \quad (21)$$

The fate of ammonia nitrogen may be modeled by this equation (see [9]). The point-wise source describes a nitrogen contamination. Another example picked-up in chemical physiology is that the oxygen uptake in a cell is governed by equation (21) (see [19]).

Now, considering that the function $f(\cdot)$ is known and signed, the inverse problem of identifying the source location $s(\cdot)$ from data $h_\zeta(\cdot)$, recorded by only one sensor located at the point ζ , is expressed exactly as in (6). Establishing the identifiability of point-wise sources in Fisher's equation may be achieved after combining the methodology followed here and the technical approach presented in [4]. The first result suffer from some limitations when the source is mobile. Indeed, the methodology fails in the general case. In particular, we need to work with an intensity function $f(\cdot)$ that belongs to $L^\infty(0, T)$. This does not seem to be a strong assumption. The stringent one however is involved with the diffusion coefficient which should be constant. Following [4], the constancy of D is mandatory. Under these restrictions we have the

Proposition 5.3 *Assume the diffusion parameter D is a positive constant and let $f(\cdot)$ be in $L^\infty(0, T)$ with $f(\cdot) \geq 0$. Then problem (6), for the logistic Fisher equation (20) or the Michaelis-Menten Fisher equation (21), has at most one solution.*

6 Conclusion

Results proven here represent substantial enhancement in the identifiability chapter in the detection of point-wise sources location for parabolic equations. The forthcoming complementary part

consists in tackling the identification issue to assess numerically the predictions supplied here, in particular the minimality and the position of the observations necessary to locate a finite number of point sources. The simulation and validation of the detection process of point-wise sources are part of the Ph. D. thesis of S. Khiari (see [15]). The scalar contaminant transport equation and the dispersive deoxygenation-reaeration model will be both investigated. Hopefully, the numerical counterpart of the current paper may be achieved sometime during the thesis.

References

- [1] ADAMS, R. A., AND FOURNIER, J. J. F. *Sobolev Spaces*. Academic Press, 2003.
- [2] ANDRLE, M., BEN BELGACEM, F., AND EL BADIA, A. Identification of Moving Pointwise Sources in an Advection-Dispersion-Reaction Equation. *Inverse Problems*, 27 (2011), 025007.
- [3] BEN BELGACEM, F. Uniqueness for an ill-posed reaction-dispersion model. Application to organic pollution in stream-waters. *Inverse Problems and Imaging* 6, 2 (2012), 163 – 181.
- [4] BEN BELGACEM, F. Identifiability for the pointwise source detection in fisher’s reaction-diffusion equation. *Inverse Problems*, 28 (2012), 065015.
- [5] BROWN, L. C., AND BARNWELL, T. O. *The Enhanced Stream Water Quality Models QUAL2E and QUAL2E-UNCAS: Documentation and User Manual*. Environmental Protection Agency, 1987.
- [6] CANNON, J. R. Determination of an Unknown Heat Source from Overspecified Boundary Data. *SIAM Journal of Numerical Analysis* 5, 2 (1968), 275–286.
- [7] CANNON, J. R., AND DUCHATEAU, P. Structural identification of an unknown source term in a heat equation. *Inverse Problems* 14 (1998), 535–551.
- [8] EL BADIA, A., HA-DUONG, T., AND HAMDI, A. Identification of a point source in a linear advection-dispersion-reaction equation: application to a pollution source problem. *Inverse Problems*, 21 (2005), 1121–1136.
- [9] FRIED, J. J. Geographically based models, surface and estuarine waters, including river/aquifer interface considerations. *SGOMSEC 5, Mexico* (1991).
- [10] FRIEDMAN, A. *Partial Differential Equations of Parabolic Type*. Prentice Hall Englewood, NJ, 1964.
- [11] HAMDI, A. *Identification de sources de pollution dans les eaux de surface*. Ph. D. These Université de Technologie de Compiègne, 2005.
- [12] HAMDI, A. The recovery of a time-dependent point source in a linear transport equation: application to surface water pollution. *Inverse Problems* 25, 7 (2009), 075006.
- [13] HAMDI, A., AND MAHFOUDHI, I. Inverse source problem in a one-dimensional evolution linear transport equation with spatially varying coefficients: Application to surface water pollution. *Inverse Problems in Science and Engineering, to appear* (2013).
- [14] HETTLICH, F., AND RUNDELL, W. Identification of a discontinuous source in the heat equation. *Inverse Problems* 17 (2001), 1465–1482.

- [15] KHIARI, S. *Observabilité et Identifiabilité de points sources dans les modèles paraboliques de transport de contaminants*. Ph. D. Thesis, Université de Technologie de Compiègne., 2015, expected.
- [16] KUSIAK, S., AND WEATHERWAX, J. Identification and Characterization of a Mobile Source in a General Parabolic Differential Equation with Constant Coefficients. *SIAM Journal of Applied Math* 68, 3 (2008), 784–805.
- [17] LATTÈS, R., AND LIONS, J.-L. . Lecture Notes in Mathematics 1341. Springer-Verlag., 1988.
- [18] LIONS, J. L., AND MAGENES, E. *Problèmes aux Limites Non Homogènes et Applications*, vol. 2. Dunod, 1968.
- [19] NICHOLSON, C. Diffusion and related transport mechanisms in brain tissue. *Rep. Prog. Phys.*, 60 (2001), 815.
- [20] SAUT, J.-C., AND SCHEURER, B. Unique Continuation for Some Evolution Equations. *Journal of Differential Equations*, 66 (1987), 118–139.
- [21] SAWYER, C. N., MCCARTY, P. L., AND PARKIN, G. F. *Chemistry for Environmental Engineering and Science*. McGraw-Hill, 2003.
- [22] STREETER, H. W., AND PHELPS, E. B. A study of the pollution and natural purification of the ohio river. *US Public Health Bull* 146 (1925).
- [23] WAHBA, G. *Ill posed problems: Numerical and statistical methods for mildly, moderately and severely ill posed problems with noisy data*. University of Wisconsin, Madison. Unpublished proceedings of the Delaware Conference on Ill Posed Inverse Problems, TR 595, 1980.
- [24] WANG, Z., AND LIU, J. Identification of the pollution source from one-dimensional parabolic equation models. *Applied Mathematics and Computation* 219, 8 (2012), 3403–3413.
- [25] ZHENG, C., AND BENNETT, G. D. *Applied Contaminant Transport Modeling*. John Wiley & Sons, New York, 2nd Edition,, 2002.