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Abstract

In general there exists no relationship between the fixed point sets of the composition and of the average of a family of nonexpansive operators in Hilbert spaces. In this paper, we establish an asymptotic principle connecting the cycles generated by under-relaxed compositions of nonexpansive operators to the fixed points of the average of these operators. In the special case when the operators are projectors onto closed convex sets, we prove a conjecture by De Pierro which has so far been established only for projections onto affine subspaces.

Keywords. Cyclic projections, De Pierro's conjecture, fixed point, nonexpansive operator, projection operator, under-relaxed cycles.

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1 Introduction

Fixed points of compositions and averages of nonexpansive operators arise naturally in diverse settings; see for instance [4, 5, 10, 12] and the references therein. In general there is no simple relationship between the fixed point sets of such operators. In this paper we investigate the connection of the fixed points of the average operator with the limits of a family of under-relaxed compositions. More precisely, we consider the framework described in the following standing assumption.

Assumption 1.1 *$\mathcal{H}$ is a real Hilbert space, D is a nonempty, closed, convex subset of $\mathcal{H}$, $m \geqslant 2$ is an integer, $I = \{1, \ldots, m\}$, $(T_i)_{i\in I}$ is a family of nonexpansive operators from D to D, and $(\operatorname{Fix} T_i)_{i\in I}$ is their fixed point sets. Moreover, we set*

$$\begin{cases} T = \dfrac{1}{m}\sum_{i\in I} T_i \\ R = T_m \circ \cdots \circ T_1 \\ (\forall \varepsilon \in]0,1[)\ \ R^\varepsilon = \big(\operatorname{Id} + \varepsilon(T_m - \operatorname{Id})\big) \circ \cdots \circ \big(\operatorname{Id} + \varepsilon(T_1 - \operatorname{Id})\big). \end{cases} \tag{1.1}$$

When the operators $(T_i)_{i\in I}$ have common fixed points, $\operatorname{Fix} T = \bigcap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing$ [5, Proposition 4.32]. If, in addition, they are strictly nonexpansive in the sense that

$$(\forall i \in I)(\forall x \in D \smallsetminus \operatorname{Fix} T_i)(\forall y \in \operatorname{Fix} T_i) \quad \|T_i x - y\| < \|x - y\|, \tag{1.2}$$

it also holds that $\operatorname{Fix} R = \bigcap_{i\in I} \operatorname{Fix} T_i$ [5, Corollary 4.36], and therefore $\operatorname{Fix} R = \operatorname{Fix} T$. However, in the general case when $\bigcap_{i\in I} \operatorname{Fix} T_i = \varnothing$, the question has been long standing and remains open even for convex projection operators, e.g., [10, Section 8.3.2] and [14]. Even when $m = 2$ and T_1 and T_2 are resolvents of maximally monotone operators, there does not seem to exist a simple relationship between $\operatorname{Fix} R$ and $\operatorname{Fix} T$ [18], except for convex projection operators, in which case $\operatorname{Fix} T = (1/2)(\operatorname{Fix} R + \operatorname{Fix} R')$, with $R' = T_1 \circ T_2$ (see [3, 13] for related results, and [7] for the case of $m \geqslant 3$ resolvents).

When $(T_i)_{i\in I} = (P_i)_{i\in I}$ are projection operators onto nonempty closed convex sets $(C_i)_{i\in I}$, $\operatorname{Fix} T$ is the set of minimizers of the average square-distance function [3, 13, 15]

$$\Phi\colon \mathcal{H} \to \mathbb{R}\colon x \mapsto \frac{1}{2m}\sum_{i\in I} d_{C_i}^2(x), \tag{1.3}$$

while $\operatorname{Fix} R$ is related to the set of Nash equilibria of a cyclic projection game. Indeed, the fixed point equation $x = Rx$ can be restated as a system of equations in $(x_1, \ldots, x_m) \in \mathcal{H}^m$, namely

$$\begin{cases} x_1 &= P_1 x_m \\ x_2 &= P_2 x_1 \\ &\vdots \\ x_m &= P_m x_{m-1}, \end{cases} \tag{1.4}$$

which characterize the Nash equilibria of a game in which each player $i \in I$ selects a strategy $x_i \in C_i$ to minimize the payoff $x \mapsto \|x - x_{i-1}\|$, with the convention $x_0 = x_m$. It is worth

noting that, for $m \geqslant 3$, these Nash equilibria cannot be characterized as minimizers of any function $\Psi\colon \mathcal{H}^m \to \mathbb{R}$ over $C_1 \times \cdots \times C_m$ [2], which further reinforces the lack of hope for simple connections between $\operatorname{Fix} R$ and $\operatorname{Fix} T$. It was shown in [16] that, if one of the sets is bounded, for every $y_0 \in \mathcal{H}$, the sequence $(y_{km+1}, \ldots, y_{km+m})_{k\in\mathbb{N}}$ generated by the periodic best-response dynamics

$$(\forall k \in \mathbb{N}) \quad \left\lfloor \begin{array}{ll} y_{km+1} & = P_1 y_{km} \\ y_{km+2} & = P_2 y_{km+1} \\ & \vdots \\ y_{km+m} & = P_m y_{km+m-1}, \end{array} \right. \tag{1.5}$$

converges weakly to a solution $(x_1, \ldots, x_m)$ to (1.4) (see Fig. 1). Working in a similar direction,

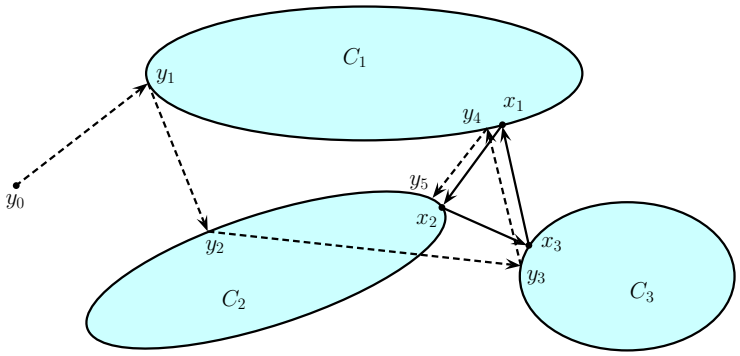

Figure 1: Cycles in the method of periodic projections.

and motivated by the work of [11] on under-relaxed projection methods for solving inconsistent systems of affine inequalities, De Pierro considered in [14] an under-relaxed version of (1.5), namely

$$(\forall k \in \mathbb{N}) \quad \left\lfloor \begin{array}{ll} y^\varepsilon_{km+1} & = \big(\operatorname{Id} + \varepsilon(P_1 - \operatorname{Id})\big) y^\varepsilon_{km} \\ y^\varepsilon_{km+2} & = \big(\operatorname{Id} + \varepsilon(P_2 - \operatorname{Id})\big) y^\varepsilon_{km+1} \\ & \vdots \\ y^\varepsilon_{km+m} & = \big(\operatorname{Id} + \varepsilon(P_m - \operatorname{Id})\big) y^\varepsilon_{km+m-1}. \end{array} \right. \tag{1.6}$$

Under mild conditions the resulting sequence $(y^\varepsilon_{km+1}, y^\varepsilon_{km+2}, \ldots, y^\varepsilon_{km+m})_{k\in\mathbb{N}}$ converges weakly to a limit cycle that satisfies the coupled equations

$$(\forall \varepsilon \in \left]0,1\right[) \quad \begin{cases} x^\varepsilon_1 & = \big(\operatorname{Id} + \varepsilon(P_1 - \operatorname{Id})\big) x^\varepsilon_m \\ x^\varepsilon_2 & = \big(\operatorname{Id} + \varepsilon(P_2 - \operatorname{Id})\big) x^\varepsilon_1 \\ & \vdots \\ x^\varepsilon_m & = \big(\operatorname{Id} + \varepsilon(P_m - \operatorname{Id})\big) x^\varepsilon_{m-1}. \end{cases} \tag{1.7}$$

In [14, Conjecture I], De Pierro conjectured that as $\varepsilon \to 0$ these limit cycles $(x^\varepsilon_1, \ldots, x^\varepsilon_m)_{\varepsilon\in\left]0,1\right[}$ shrink towards a single point which is a minimizer of Φ, i.e., a fixed point of T. In contrast with (1.4), the solutions of which do not satisfy any optimality criteria, this conjecture suggests an

asymptotic variational principle for the cycles obtained as limits of the under-relaxed version of (1.5). An important contribution was made in [6], where it was shown that De Pierro's conjecture is true for families of closed affine subspaces which satisfy a certain regularity condition.

In this paper we investigate the asymptotic behavior of the under-relaxed cycles

$$\begin{cases} x_1^\varepsilon &= \big(\mathrm{Id} + \varepsilon(T_1 - \mathrm{Id})\big)x_m^\varepsilon \\ x_2^\varepsilon &= \big(\mathrm{Id} + \varepsilon(T_2 - \mathrm{Id})\big)x_1^\varepsilon \\ &\vdots \\ x_m^\varepsilon &= \big(\mathrm{Id} + \varepsilon(T_m - \mathrm{Id})\big)x_{m-1}^\varepsilon \end{cases} \tag{1.8}$$

as $\varepsilon \to 0$ in the general setting of Assumption 1.1. In Section 2 we present a first general convergence result, which establishes conditions under which the limits as $\varepsilon \to 0$ of the m curves $(x_i^\varepsilon)_{\varepsilon\in]0,1[}$ $(i \in I)$ exist and all coincide with a fixed point of T. This result not only gives conditions under which De Pierro's conjecture is true, but also extends its scope from projection operators to arbitrary nonexpansive operators. In Section 3 we revisit the problem from a constructive angle. Given an initial point $y_0 \in D$ and $\varepsilon \in]0,1[$, it is known [5, Theorem 5.22] that the cycles in (1.8) can be constructed iteratively as the weak limit of the periodic process

$$(\forall k \in \mathbb{N}) \quad \begin{cases} y_{km+1}^\varepsilon &= \big(\mathrm{Id} + \varepsilon(T_1 - \mathrm{Id})\big)y_{km}^\varepsilon \\ y_{km+2}^\varepsilon &= \big(\mathrm{Id} + \varepsilon(T_2 - \mathrm{Id})\big)y_{km+1}^\varepsilon \\ &\vdots \\ y_{km+m}^\varepsilon &= \big(\mathrm{Id} + \varepsilon(T_m - \mathrm{Id})\big)y_{km+m-1}^\varepsilon. \end{cases} \tag{1.9}$$

We analyze the connection between this iterative process and the trajectories of the evolution equation

$$\begin{cases} x'(t) + x(t) = Tx(t) \ \text{ on } \ ]0,+\infty[\\ x(0) = y_0, \end{cases} \tag{1.10}$$

and then establish extended versions of De Pierro's conjecture under various assumptions.

Notation. The scalar product of $\mathcal{H}$ is denoted by $\langle \cdot \mid \cdot \rangle$ and the associated norm by $\|\cdot\|$. The symbols $\rightharpoonup$ and $\to$ denote, respectively, weak and strong convergence, and Id denotes the identity operator. The closed ball of center $x \in \mathcal{H}$ and radius $\rho \in]0,+\infty[$ is denoted by $B(x;\rho)$. Given a nonempty closed convex subset $C \subset \mathcal{H}$, the distance function to C and the projection operator onto C are respectively denoted by d_C and P_C.

2 Convergence of general families of under-relaxed cycles

We investigate the asymptotic behavior of the cycles $(x_i^\varepsilon)_{i\in I}$ defined by (1.8) when $\varepsilon \to 0$. Let us remark that such a cycle $(x_i^\varepsilon)_{i\in I}$ is in bijection with the fixed points of the composition R^ε of (1.1). Indeed, $z^\varepsilon = x_m^\varepsilon$ is a fixed point of R^ε; conversely, each $z^\varepsilon \in \operatorname{Fix} R^\varepsilon$ generates a cycle by setting, for every $i \in I$, $x_i^\varepsilon = (\mathrm{Id} + \varepsilon(T_i - \mathrm{Id}))x_{i-1}^\varepsilon$, where $x_0^\varepsilon = z^\varepsilon$. This motivates our second standing assumption.

Assumption 2.1 For every $\varepsilon \in]0,1[$, R^ε is given by (1.1) and

$$(\exists\, \eta \in]0,1])(\exists\, \beta \in]0,+\infty[)(\forall \varepsilon \in]0,\eta[)(\exists\, z^\varepsilon \in \operatorname{Fix} R^\varepsilon) \quad \|z^\varepsilon\| \leqslant \beta. \tag{2.1}$$

For later reference, we record the fact that under this assumption the cycles in (1.8) can be obtained as weak limits of the iterative process (1.9).

Proposition 2.2 *Suppose that Assumptions 1.1 and 2.1 are satisfied. Let $y_0 \in D$ and $\varepsilon \in]0,\eta[$. Then the sequence $(y^\varepsilon_{km+1},\ldots,y^\varepsilon_{km+m})_{k\in\mathbb{N}}$ produced by (1.9) converges weakly to an m-tuple $(x^\varepsilon_1,\ldots,x^\varepsilon_m)$ which satisfies (1.8).*

Proof. This follows from [5, Theorem 5.22]. □

The following result provides sufficient conditions for Assumption 2.1 to hold.

Proposition 2.3 *Suppose that Assumption 1.1 holds, together with one of the following.*

(i) *For some $j \in I$, T_j has bounded range.*

(ii) *D is bounded.*

Then Assumption 2.1 is satisfied.

Proof. It is clear that (ii) is a special case of (i). Suppose that (i) holds. Fix $\varepsilon \in]0,1]$ and $y \in D$, and take $\rho \in \big[\max_{i\in I\smallsetminus\{j\}} \|T_i y - y\|, +\infty\big[$ such that $T_j(D) \subset B(y;\rho)$. Furthermore, let $x \in D$, set $x_0 = x$, and define recursively $x_i = (1-\varepsilon)x_{i-1} + \varepsilon T_i x_{i-1}$, so that $x_m = R^\varepsilon x$. Then

$$\begin{aligned}(\forall i \in I \smallsetminus \{j\}) \quad \|x_i - y\| &= \|(1-\varepsilon)(x_{i-1}-y) + \varepsilon(T_i x_{i-1} - y)\| \\ &\leqslant (1-\varepsilon)\|x_{i-1}-y\| + \varepsilon\|T_i x_{i-1} - T_i y\| + \varepsilon\|T_i y - y\| \\ &\leqslant \|x_{i-1}-y\| + \varepsilon\rho \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\|x_j - y\| &\leqslant (1-\varepsilon)\|x_{j-1}-y\| + \varepsilon\|T_j x_{j-1} - y\| \\ &\leqslant (1-\varepsilon)\|x_{j-1}-y\| + \varepsilon\rho. \end{aligned} \tag{2.3}$$

By applying inductively (2.2) and (2.3) to majorize $\|x_m - y\|$, we obtain

$$\|R^\varepsilon x - y\| = \|x_m - y\| \leqslant (1-\varepsilon)\|x-y\| + \varepsilon m\rho. \tag{2.4}$$

This implies that R^ε maps $D \cap B(y; m\rho)$ to itself. Hence, the Browder–Göhde–Kirk theorem (see [5, Theorem 4.19]) asserts that R^ε has a fixed point in $B(y;m\rho)$. Moreover, if x is a fixed point of R^ε, (2.4) gives $\|x-y\| \leqslant m\rho$, which shows that (2.1) holds with $\eta = 1$ and $\beta = \|y\| + m\rho$. □

To illustrate Assumption 2.1, it is instructive to consider the following examples.

Example 2.4 The following variant of the example discussed in [14, Section 3] shows that (2.1) is a non trivial assumption: $\mathcal{H}$ is the Euclidean plane, $m = 3$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\gamma \in]0, +\infty[$, $\varepsilon \in]0, 1[$, and $(T_i)_{1\leqslant i\leqslant 3}$ are, respectively, the projection operators onto the sets

$$C_1 = \mathbb{R} \times \{\alpha\}, \quad C_2 = \mathbb{R} \times \{\beta\}, \quad \text{and} \quad C_3 = \big\{(\xi_1, \xi_2) \in]0, +\infty[^2 \;\big|\; \xi_1\xi_2 \geqslant \gamma\big\}. \tag{2.5}$$

Then we have

$$\begin{cases} \operatorname{Fix} T = \big\{(\xi_1, \xi_2) \in C_3 \;\big|\; \xi_2 = (\alpha + \beta)/2\big\} \\ \operatorname{Fix} R = \big\{(\xi_1, \xi_2) \in C_3 \;\big|\; \xi_2 = \beta\big\} \\ \operatorname{Fix} R^\varepsilon = \big\{(\xi_1, \xi_2) \in C_3 \;\big|\; \xi_2 = \big((1-\varepsilon)\alpha + \beta\big)/(2-\varepsilon)\big\}. \end{cases} \tag{2.6}$$

Thus, depending on the values of α and β, we can have $\operatorname{Fix} T = \operatorname{Fix} R \neq \varnothing$, $\operatorname{Fix} T = \operatorname{Fix} R = \varnothing$, $\operatorname{Fix} T \neq \operatorname{Fix} R = \varnothing$, $\operatorname{Fix} R \neq \operatorname{Fix} T = \varnothing$, or $\varnothing \neq \operatorname{Fix} R \neq \operatorname{Fix} T \neq \varnothing$. Now set $\eta = 1 + \beta/\alpha$. Then, under the assumption that $\alpha + \beta < 0 < \beta$, we have $\eta \in]0, 1[$ and $\operatorname{Fix} R^\varepsilon = \varnothing$ if $\varepsilon \leqslant \eta$, while $\operatorname{Fix} R^\varepsilon \neq \varnothing$ if $\varepsilon > \eta$. On the other hand, under the assumption that $\beta < 0 < \alpha + \beta$, $\eta \in]0, 1[$ and $\operatorname{Fix} R^\varepsilon \neq \varnothing$ if $\varepsilon < \eta$, while $\operatorname{Fix} R^\varepsilon = \varnothing$ if $\varepsilon \geqslant \eta$. Moreover, setting

$$(\forall \varepsilon \in]0, \eta[) \quad \begin{cases} y^\varepsilon = \left(\dfrac{2\gamma}{(1-\varepsilon)\alpha + \beta} + \dfrac{1}{\varepsilon}, \dfrac{(1-\varepsilon)\alpha + \beta}{2-\varepsilon}\right) \in \operatorname{Fix} R^\varepsilon \\ z^\varepsilon = \left(\dfrac{(2-\varepsilon)\gamma}{(1-\varepsilon)\alpha + \beta}, \dfrac{(1-\varepsilon)\alpha + \beta}{2-\varepsilon}\right) \in \operatorname{Fix} R^\varepsilon. \end{cases} \tag{2.7}$$

we see that $(y^\varepsilon)_{\varepsilon\in]0,\eta[}$ is an unbounded curve, while $(z^\varepsilon)_{\varepsilon\in]0,\eta[}$ is bounded.

Example 2.5 In Example 2.4 the sets $(\operatorname{Fix} T_i)_{1\leqslant i\leqslant 3}$ are nonempty, and one may ask whether this plays a role in the nonemptiness of $\operatorname{Fix} R$, $\operatorname{Fix} T$, or $\operatorname{Fix} R^\varepsilon$. To see that such is not the case, define T_3 as in Example 2.4, and consider the modified operators $T_1 \colon (\xi_1, \xi_2) \mapsto (\xi_1 + \mu, \alpha)$ and $T_2 \colon (\xi_1, \xi_2) \mapsto (\xi_1 - \mu, \beta)$, where $\mu > 0$. Although now the nonexpansive operators T_1 and T_2 have no fixed points, the operators T, R, and R^ε remain unchanged.

Example 2.6 By considering products of sets of the form (2.5) one can build an example in which $\operatorname{Fix} T$ is nonempty but the sets $(\operatorname{Fix} R^\varepsilon)_{\varepsilon\in]0,1[}$ are empty. More precisely, let $\mathcal{H} = \ell^2(\mathbb{N})$, and let $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, and $(\gamma_n)_{n\in\mathbb{N}}$ be sequences in $\ell^2(\mathbb{N})$ such that $(\gamma_n/(\alpha_n+\beta_n))_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and $(\forall n \in \mathbb{N})$ $\beta_n < 0 < \alpha_n + \beta_n$ and $\gamma_n > 0$. Set

$$\begin{cases} C_1 = \big\{(\xi_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \;\big|\; (\forall n \in \mathbb{N})\;\; \xi_{2n} = \alpha_n\big\} \\ C_2 = \big\{(\xi_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \;\big|\; (\forall n \in \mathbb{N})\;\; \xi_{2n} = \beta_n\big\} \\ C_3 = \big\{(\xi_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \;\big|\; (\forall n \in \mathbb{N})\;\; \xi_n > 0 \;\text{ and }\; \xi_{2n-1}\xi_{2n} \geqslant \gamma_n\big\}. \end{cases} \tag{2.8}$$

Then $\operatorname{Fix} T \neq \varnothing$ but, for $\varepsilon \in]0, 1[$, we have $\operatorname{Fix} R^\varepsilon \neq \varnothing$ if and only if $(\forall n \in \mathbb{N})$ $\varepsilon < 1 + \beta_{2n+1}/\alpha_{2n+1}$. In particular if we take, for every $n \in \mathbb{N} \smallsetminus \{0\}$, $\alpha_n = (n+1)/n^2$, $\beta_n = -1/n$, and $\gamma_n = 1/n^3$, then $\operatorname{Fix} R^\varepsilon = \varnothing$ for every $\varepsilon \in]0, 1[$.

Example 2.7 ([6, Example 4.1]) Let $m = 2$, and let T_1 and T_2 be the projection operators onto closed affine subspaces $C_1 \subset \mathcal{H}$ and $C_2 \subset \mathcal{H}$, respectively. If $\mathcal{H}$ is finite-dimensional, the sets $\operatorname{Fix} R$, $(\operatorname{Fix} R^\varepsilon)_{\varepsilon\in]0,1[}$, and $\operatorname{Fix} T$ are nonempty; if $\mathcal{H}$ is infinite-dimensional, there exist C_1 and C_2 such that these sets are all empty. However, if the vector subspace $(C_1 - C_1) + (C_2 - C_2)$ is closed, then $\operatorname{Fix} T \neq \varnothing$ and $(\forall \varepsilon \in]0, 1[)$ $\operatorname{Fix} R^\varepsilon \neq \varnothing$.

The next result establishes conditions for the convergence of the cycles of (1.8) when the relaxation parameter ε vanishes.

Theorem 2.8 *Suppose that Assumptions 1.1 and 2.1 are satisfied. Then* $\operatorname{Fix} T \neq \varnothing$. *Now let* $(x_m^\varepsilon)_{\varepsilon\in]0,\eta[} = (z^\varepsilon)_{\varepsilon\in]0,\eta[}$ *be the bounded curve provided by* (2.1) *and denote by* $(x_1^\varepsilon, \ldots, x_m^\varepsilon)_{\varepsilon\in]0,\eta[}$ *the associated family of cycles arising from* (1.8). *Then* $(x_1^\varepsilon, \ldots, x_m^\varepsilon)_{\varepsilon\in]0,\eta[}$ *is bounded and each of its weak sequential cluster points is of the form* $(x, \ldots, x)$, *where* $x \in \operatorname{Fix} T$. *Moreover,*

$$(\forall i \in I) \quad \lim_{\varepsilon\to 0} \|x_i^\varepsilon - x_{i-1}^\varepsilon\| = 0, \quad \text{where} \quad (\forall \varepsilon \in]0,\eta[) \quad x_0^\varepsilon = x_m^\varepsilon. \tag{2.9}$$

In addition, suppose that one of the following holds.

(i) $(\forall x \in \operatorname{Fix} T)(\forall y \in \operatorname{Fix} T)\ \langle x_m^\varepsilon \mid x - y\rangle$ *converges as* $\varepsilon \to 0$.

(ii) $(\forall x \in \operatorname{Fix} T)\ \|x_m^\varepsilon - x\|$ *converges as* $\varepsilon \to 0$.

(iii) $\operatorname{Fix} T$ *is a singleton.*

Then there exists $\overline{x} \in \operatorname{Fix} T$ *such that, for every* $i \in I$, $x_i^\varepsilon \rightharpoonup \overline{x}$ *as* $\varepsilon \to 0$. *Finally, suppose that* $\operatorname{Id} - T$ *is demiregular on* $\operatorname{Fix} T$, *i.e.,*

$$\big(\forall (y_k)_{k\in\mathbb{N}} \in D^{\mathbb{N}}\big)\big(\forall y \in \operatorname{Fix} T\big) \quad \begin{cases} y_k \rightharpoonup y \\ y_k - Ty_k \to 0 \end{cases} \quad \Rightarrow \quad y_k \to y. \tag{2.10}$$

Then, for every $i \in I$, $x_i^\varepsilon \to \overline{x}$ *as* $\varepsilon \to 0$.

Proof. Fix $z \in D$. By nonexpansiveness of the operators $(T_i)_{i\in I}$, we have

$$\begin{aligned}(\forall i \in I) \quad \|T_i x_{i-1}^\varepsilon - x_{i-1}^\varepsilon\| &\leqslant \|T_i x_{i-1}^\varepsilon - T_i z\| + \|T_i z - z\| + \|z - x_{i-1}^\varepsilon\| \\ &\leqslant 2\|x_{i-1}^\varepsilon - z\| + \|T_i z - z\|. \end{aligned} \tag{2.11}$$

In particular, for $i = 1$, it follows from the boundedness of $(x_m^\varepsilon)_{\varepsilon\in]0,\eta[}$ that $(T_1 x_m^\varepsilon - x_m^\varepsilon)_{\varepsilon\in]0,\eta[}$ is bounded. In turn, we deduce from (1.8) that $(x_1^\varepsilon)_{\varepsilon\in]0,\eta[}$ is bounded. Continuing this process, we obtain the boundedness of $(x_1^\varepsilon, \ldots, x_m^\varepsilon)_{\varepsilon\in]0,\eta[}$ and the fact that

$$(\forall i \in I) \quad (T_i x_{i-1}^\varepsilon - x_{i-1}^\varepsilon)_{\varepsilon\in]0,\eta[} \quad \text{is bounded.} \tag{2.12}$$

On the other hand, adding all the equalities in (1.8), we get

$$(\forall \varepsilon \in]0,\eta[) \quad \sum_{i\in I} T_i x_{i-1}^\varepsilon = \sum_{i\in I} x_i^\varepsilon, \tag{2.13}$$

from which it follows that

$$\begin{aligned}(\forall \varepsilon \in]0,\eta[) \quad Tx_m^\varepsilon - x_m^\varepsilon &= \frac{1}{m}\sum_{i=1}^m T_i x_m^\varepsilon - x_m^\varepsilon \\ &= \frac{1}{m}\sum_{i=1}^m T_i x_{i-1}^\varepsilon + \frac{1}{m}\sum_{i=2}^m \big(T_i x_m^\varepsilon - T_i x_{i-1}^\varepsilon\big) - x_m^\varepsilon \\ &= \frac{1}{m}\sum_{i=1}^m x_i^\varepsilon + \frac{1}{m}\sum_{i=2}^m \big(T_i x_m^\varepsilon - T_i x_{i-1}^\varepsilon\big) - x_m^\varepsilon \\ &= \frac{1}{m}\sum_{i=1}^{m-1}(x_i^\varepsilon - x_m^\varepsilon) + \frac{1}{m}\sum_{i=1}^{m-1}\big(T_{i+1}x_m^\varepsilon - T_{i+1}x_i^\varepsilon\big). \end{aligned} \tag{2.14}$$

Hence, using the nonexpansiveness of the operators $(T_i)_{i\in I}$, we obtain

$$(\forall \varepsilon \in]0,\eta[) \quad \|Tx_m^\varepsilon - x_m^\varepsilon\| \leqslant \frac{2}{m}\sum_{i=1}^{m-1}\big\|x_m^\varepsilon - x_i^\varepsilon\big\|. \tag{2.15}$$

Consequently, since (1.8) and (2.12) also imply that

$$(\forall i \in I) \quad \|x_i^\varepsilon - x_{i-1}^\varepsilon\| = \varepsilon\|T_i x_{i-1}^\varepsilon - x_{i-1}^\varepsilon\| \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{2.16}$$

thus proving (2.9), the triangle inequality gives $\|x_m^\varepsilon - x_i^\varepsilon\| \to 0$, which, combined with (2.15), yields

$$Tx_m^\varepsilon - x_m^\varepsilon \to 0. \tag{2.17}$$

Hence, we can invoke the demiclosed principle [5, Corollary 4.18] to deduce that every weak sequential cluster point of the bounded curve $(x_m^\varepsilon)_{\varepsilon\in]0,\eta[}$ belongs to $\operatorname{Fix} T$, which is therefore nonempty. In view of (2.16), we therefore deduce that every weak sequential cluster point of $(x_1^\varepsilon,\ldots,x_m^\varepsilon)_{\varepsilon\in]0,\eta[}$ is of the form $(x,\ldots,x)$, where $x \in \operatorname{Fix} T$. It remains to show that under any of the conditions (i), (ii), or (iii), the curve $(x_m^\varepsilon)_{\varepsilon\in]0,1[}$ is weakly convergent. Clearly (iii) implies (i), and the same holds for (ii) since

$$(\forall (x,y) \in \mathcal{H}^2)(\forall \varepsilon \in]0,\eta[) \quad \langle x_m^\varepsilon \mid x-y\rangle = \frac{1}{2}\big(\|x_m^\varepsilon - y\|^2 - \|x_m^\varepsilon - x\|^2 + \|x\|^2 - \|y\|^2\big). \tag{2.18}$$

Thus, it suffices to show that under (i) the curve $(x_m^\varepsilon)_{\varepsilon\in]0,\eta[}$ has a unique weak sequential cluster point. Let x and y be two weak sequential cluster points and choose sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n')_{n\in\mathbb{N}}$ in $]0,\eta[$ converging to 0 such that $x_m^{\varepsilon_n} \rightharpoonup x$ and $x_m^{\varepsilon_n'} \rightharpoonup y$ as $n \to +\infty$. As shown above, we have x and y lie in $\operatorname{Fix} T$ and, therefore, it follows from (i) that $\langle x \mid x-y\rangle = \lim_{n\to+\infty}\langle x_m^{\varepsilon_n} \mid x-y\rangle = \lim_{n\to+\infty}\Big\langle x_m^{\varepsilon_n'} \mid x-y\Big\rangle = \langle y \mid x-y\rangle$. This yields $\|x-y\|^2 = 0$ proving our claim.

Finally, let us establish the strong convergence assertion. To this end, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $]0,\eta[$ converging to 0. Then, as just proved, $x_m^{\varepsilon_n} \rightharpoonup \overline{x} \in \operatorname{Fix} T$ as $n \to +\infty$. On the other hand, (2.17) yields $x_m^{\varepsilon_n} - Tx_m^{\varepsilon_n} \to 0$ as $n \to +\infty$. Hence, we derive from (2.10) that $x_m^{\varepsilon_n} \to \overline{x}$ as $n \to +\infty$. This shows that $x_m^\varepsilon \to \overline{x}$ as $\varepsilon \to 0$. In view of (2.16), the proof is complete. □

Remark 2.9 The demiregularity condition (2.10) is a specialization of a notion introduced in [1, Definition 2.3] for set-valued operators (see also [19, Definition 27.1]). It follows from [1, Proposition 2.4] that (2.10) is satisfied in each of the following cases.

(i) $\operatorname{Id} - T$ is uniformly monotone at every $y \in \operatorname{Fix} T$.

(ii) $\operatorname{Id} - T$ is strongly monotone at every $y \in \operatorname{Fix} T$.

(iii) $T = \operatorname{Id} - \nabla f$, where $f \in \Gamma_0(\mathcal{H})$ is uniformly convex at every $y \in \operatorname{Fix} T$.

(iv) D is boundedly compact: its intersection with every closed ball is compact.

(v) $D = \mathcal{H}$ and $\operatorname{Id} - T$ is invertible.

(vi) T is demicompact [17]: for every bounded sequence $(y_n)_{n\in\mathbb{N}}$ in D such that $(y_n - Ty_n)_{n\in\mathbb{N}}$ converges strongly, $(y_n)_{n\in\mathbb{N}}$ admits a strongly convergent subsequence.

In the special case when $(T_i)_{i\in I}$ is a family of projection operators onto closed convex sets, Theorem 2.8 asserts that De Pierro's conjecture is true under any of conditions (i)–(iii). In particular, we obtain weak convergence of each point in the cycle to the point in $\operatorname{Fix} T$ if this set is a singleton, which can be considered as a generic situation in many practical instances when $\bigcap_{i\in I} \operatorname{Fix} T_i \neq \varnothing$. The following example illustrates a degenerate case in which weak convergence of the cycles can fail.

Example 2.10 Suppose that in Theorem 2.8 we have $\bigcap_{i\in I} \operatorname{Fix} T_i \neq \varnothing$. Then it follows from the results of [5, Section 4.5] that

$$(\forall \varepsilon \in \left]0,1\right[) \quad \operatorname{Fix} R^{\varepsilon} = \bigcap_{i\in I} \operatorname{Fix}\big((1-\varepsilon)\operatorname{Id} + \varepsilon T_i\big) = \bigcap_{i\in I} \operatorname{Fix} T_i = \operatorname{Fix} T. \tag{2.19}$$

Now suppose y and z are two distinct points in $\operatorname{Fix} T$ and set

$$(\forall \varepsilon \in \left]0,1\right[) \quad x_m^{\varepsilon} = \begin{cases} y, & \text{if } \lfloor 1/\varepsilon \rfloor \text{ is even;} \\ z, & \text{if } \lfloor 1/\varepsilon \rfloor \text{ is odd.} \end{cases} \tag{2.20}$$

Then $(x_m^{\varepsilon})_{\varepsilon\in\left]0,1\right[}$ has two distinct weak cluster points and therefore it does not converge weakly, although Assumptions 1.1 and 2.1 are trivially satisfied.

3 Convergence of limit cycles of under-relaxed iterations

As illustrated in Example 2.10, in general one cannot expect every solution cycle $(x_1^{\varepsilon}, \ldots, x_m^{\varepsilon})_{\varepsilon\in\left]0,\eta\right[}$ in (1.8) to converge as there are cases that oscillate. Theorem 2.8 provided conditions that rule out multiple clustering and ensure the weak convergence of the cycles as $\varepsilon \to 0$. An alternative approach, inspired from [14], is to focus on solutions of (1.8) that arise as limit cycles of the under-relaxed periodic iteration (1.9) started from the same initial point $y_0 \in D$ for every $\varepsilon \in \left]0,\eta\right[$. This arbitrary but fixed initial point is intended to act as an anchor that avoids multiple cluster points of the resulting family of limit cycles $(x_1^{\varepsilon}, \ldots, x_m^{\varepsilon})_{\varepsilon\in\left]0,\eta\right[}$.

As mentioned in the Introduction, for convex projection operators De Pierro conjectured that, as $\varepsilon \to 0$, the limit cycles shrink to a least-squares solution, namely $(x_1^\varepsilon, \ldots, x_m^\varepsilon) \rightharpoonup (\overline{x}, \ldots, \overline{x})$, where $\overline{x}$ is a minimizer of the function Φ of (1.3). In [6, Theorem 6.4] the conjecture was proved for closed affine subspaces satisfying a regularity conditions, in which case the limit $\overline{x}$ exists in the strong topology and is in fact the point in $S = \operatorname{Argmin} \Phi = \operatorname{Fix} T$ closest to the initial point y_0, namely $\overline{x} = P_S y_0$. However, for general convex sets the conjecture remains open.

We revisit this question in the general framework delineated by Assumptions 1.1 and 2.1 with a different strategy than that adopted in Section 2. Our approach consists in showing that, for ε small, the iterates (1.9) follow closely the orbit of the semigroup generated by $A = \operatorname{Id} - T$, i.e., the semigroup associated with the autonomous Cauchy problem

$$\begin{cases} x'(t) = -Ax(t) \text{ on }]0, +\infty[\\ x(0) = y_0. \end{cases} \tag{3.1}$$

This allows us to relate the limit cycles $(x_1^\varepsilon, \ldots, x_m^\varepsilon)_{\varepsilon \in]0,\eta[}$ to the limit of $x(t)$ when $t \to +\infty$. Note that, since $y_0 \in D = \operatorname{dom} A$ and A is Lipschitz, (3.1) has a unique solution $x \in \mathcal{C}^1(]0, +\infty[\,; D)$; see, e.g., [8, Theorem I.1.4]. In addition, if there exists $x_\infty \in \mathcal{H}$ such that $x(t) \rightharpoonup x_\infty$ as $t \to +\infty$, then $\overline{x} \in \operatorname{Fix} T$. In the case of convex projections, (3.1) reduces to the gradient flow

$$\begin{cases} x'(t) = -\nabla \Phi(x(t)) \text{ on }]0, +\infty[\\ x(0) = y_0, \end{cases} \tag{3.2}$$

which converges weakly to some point $x_\infty \in S$ as $t \to +\infty$ [9, Theorem 4], and one may therefore expect De Pierro's conjecture to hold with $\overline{x} = x_\infty$ under suitable assumptions. Note, however, that for non-affine convex sets the limit x_∞ might not coincide with the projection $P_S y_0$.

3.1 Under-relaxed cyclic iterations and semigroup flows

In order to study (1.9) for a fixed $\varepsilon \in]0, 1[$, it suffices to consider the iterates modulo m, that is, the sequence $(y_{km}^\varepsilon)_{k \in \mathbb{N}} = ((R^\varepsilon)^k y_0)_{k \in \mathbb{N}}$, which converge weakly towards some point $x_m^\varepsilon \in \operatorname{Fix} R^\varepsilon$ (see Proposition 2.2). The key to establish a formal connection between the iteration (1.9) and the semigroup associated with (3.1), is the following approximation lemma that relates R^ε to $A = \operatorname{Id} - T$.

Lemma 3.1 *Set $A = \operatorname{Id} - T$, fix $z \in D$, and set $\rho = \max_{i \in I} \|T_i z - z\|/2$. Then*

$$(\forall \varepsilon \in [0, 1])(\forall x \in D) \quad \|R^\varepsilon x - x + \varepsilon m A x\| \leqslant \varepsilon^2 (3^m - 2m - 1)(\|x - z\| + \rho). \tag{3.3}$$

Proof. Since the case $\varepsilon = 0$ is trivial, we take $\varepsilon \in]0, 1]$. Define operators on D by

$$(\forall j \in I) \quad R_j^\varepsilon = \big(\operatorname{Id} + \varepsilon(T_j - \operatorname{Id})\big) \circ \cdots \circ \big(\operatorname{Id} + \varepsilon(T_1 - \operatorname{Id})\big) \tag{3.4}$$

and

$$(\forall j \in I) \quad E_j^\varepsilon = \frac{1}{\varepsilon^2}(R_j^\varepsilon - \operatorname{Id}) + \frac{1}{\varepsilon} \sum_{i=1}^{j} (\operatorname{Id} - T_i). \tag{3.5}$$

Then $R^\varepsilon = R_m^\varepsilon$ and therefore $R^\varepsilon - \mathrm{Id} + \varepsilon m A = \varepsilon^2 E_m^\varepsilon$. Thus, the result boils down to showing that $(\forall x \in D)\ \|E_m^\varepsilon x\| \leqslant (3^m - 2m - 1)(\|x - z\| + \rho)$. We derive from (3.5) that

$$(\forall j \in \{1, \ldots, m-1\}) \quad E_{j+1}^\varepsilon = E_j^\varepsilon + \frac{1}{\varepsilon}\Big((\mathrm{Id} - T_{j+1}) - (\mathrm{Id} - T_{j+1}) \circ R_j^\varepsilon\Big). \tag{3.6}$$

Now let $x \in D$. Since the operators $(\mathrm{Id} - T_j)_{1 \leqslant j \leqslant m-1}$ are 2-Lipschitz, we have

$$\begin{aligned}
(\forall j \in \{1, \ldots, m-1\}) \quad \|E_{j+1}^\varepsilon x\| &\leqslant \|E_j^\varepsilon x\| + \frac{2}{\varepsilon}\|x - R_j^\varepsilon x\| \\
&= \|E_j^\varepsilon x\| + 2\bigg\| \sum_{i=1}^{j} (\mathrm{Id} - T_i)x - \varepsilon E_j^\varepsilon x \bigg\| \\
&\leqslant (1 + 2\varepsilon)\|E_j^\varepsilon x\| + 2\sum_{i=1}^{j} (\|x - z\| + \|z - T_i z\| + \|T_i z - T_i x\|) \\
&\leqslant (1 + 2\varepsilon)\|E_j^\varepsilon x\| + 4j(\|x - z\| + \rho).
\end{aligned} \tag{3.7}$$

Using (3.7) recursively, and observing that $E_1^\varepsilon x = 0$, it follows that

$$\|E_m^\varepsilon x\| \leqslant 4(\|x - z\| + \rho) \sum_{j=1}^{m-1} j(1 + 2\varepsilon)^{m-1-j}. \tag{3.8}$$

Upon applying the identity $\sum_{j=1}^{m-1} j\alpha^j = ((m-1)\alpha^{m+1} - m\alpha^m + \alpha)/(1-\alpha)^2$ to $\alpha = (1+2\varepsilon)^{-1} \in]0, 1[$, we see that the sum in (3.8) is equal to $((1 + 2\varepsilon)^m - 1 - 2m\varepsilon)/(4\varepsilon^2)$, which increases with ε attaining its maximum $(3^m - 2m - 1)/4$ at $\varepsilon = 1$. This combined with (3.8) yields the announced bound. □

Remark 3.2 For firmly nonexpansive operators, such as projection operators onto closed convex sets, the operators $(\mathrm{Id} - T_i)_{i \in I}$ are nonexpansive and the previous proof can be modified to derive a tighter bound in (3.3), namely

$$(\forall \varepsilon \in [0, 1])(\forall x \in D) \quad \|R^\varepsilon x - x + \varepsilon m A x\| \leqslant \varepsilon^2(2^m - m - 1)(\|x - z\| + 2\rho). \tag{3.9}$$

We proceed with the announced connection between (1.9) and (3.1). This will be used later to establish De Pierro's conjecture in several alternative settings.

Proposition 3.3 *Let $y_0 \in D$, let x be the solution of (3.1), suppose that Assumptions 1.1 and 2.1 are satisfied. For every $\varepsilon \in]0, \eta[$, set $(z_k^\varepsilon)_{k \in \mathbb{N}} = ((R^\varepsilon)^k y_0)_{k \in \mathbb{N}}$ and let ψ^ε be the linear interpolation of $(z_k^\varepsilon)_{k \in \mathbb{N}}$ given by*

$$\big(\forall k \in \mathbb{N}\big)\big(\forall t \in [km\varepsilon, (k+1)m\varepsilon[\big) \quad \psi^\varepsilon(t) = z_k^\varepsilon + \frac{t - km\varepsilon}{m\varepsilon}(z_{k+1}^\varepsilon - z_k^\varepsilon). \tag{3.10}$$

Then $(\forall \bar{t} \in]0, +\infty[)\ \sup_{0 \leqslant t \leqslant \bar{t}} \|\psi^\varepsilon(t) - x(t)\| \to 0$ when $\varepsilon \to 0$.

Proof. Set $A = \mathrm{Id} - T$, let $\varepsilon \in]0, \eta[$, and fix $z \in D$. The function ψ^ε is differentiable except at the breakpoints $\big\{km\varepsilon \;\big|\; k \in \mathbb{N}\big\}$. Now set $(\forall k \in \mathbb{N})\ J_k =]km\varepsilon, (k+1)m\varepsilon[$. According to Lemma 3.1, we have

$$(\forall k \in \mathbb{N})(\forall t \in J_k) \quad (\psi^\varepsilon)'(t) = \frac{1}{m\varepsilon}(z_{k+1}^\varepsilon - z_k^\varepsilon) = \frac{1}{m\varepsilon}(R^\varepsilon z_k^\varepsilon - z_k^\varepsilon) = -Az_k^\varepsilon + \varepsilon h_k^\varepsilon, \tag{3.11}$$

where $\|h_k^\varepsilon\| \leqslant (3^m - 2m - 1)(\|z_k^\varepsilon - z\| + \rho)/m$. Now set

$$(\forall k \in \mathbb{N})(\forall t \in J_k) \quad h^\varepsilon(t) = A\psi^\varepsilon(t) - Az_k^\varepsilon + \varepsilon h_k^\varepsilon. \tag{3.12}$$

Then

$$(\forall k \in \mathbb{N})(\forall t \in J_k) \quad (\psi^\varepsilon)'(t) = -A\psi^\varepsilon(t) + h^\varepsilon(t). \tag{3.13}$$

Moreover, it follows from (2.1) that there exists a constant $\alpha \in]0, +\infty[$ independent from ε such that $(\forall k \in \mathbb{N})\ \|h_k^\varepsilon\| \leqslant \alpha$. Hence, since A is 2-Lipschitz, there exists $\gamma \in]0, +\infty[$ such that

$$\begin{aligned}(\forall k \in \mathbb{N})(\forall t \in J_k) \quad \|h^\varepsilon(t)\| &\leqslant 2\|\psi^\varepsilon(t) - z_k^\varepsilon\| + \varepsilon\|h_k^\varepsilon\| \\ &\leqslant 2\|z_{k+1}^\varepsilon - z_k^\varepsilon\| + \varepsilon\|h_k^\varepsilon\| \\ &= 2\varepsilon m\| - Az_k^\varepsilon + \varepsilon h_k^\varepsilon\| + \varepsilon\|h_k^\varepsilon\| \\ &\leqslant \varepsilon\gamma. \end{aligned} \tag{3.14}$$

Next, consider the function $\theta\colon [0, +\infty[\to [0, +\infty[$ defined by $\theta(t) = \|x(t) - \psi^\varepsilon(t)\|^2$. Then it follows from the monotonicity of A that

$$\begin{aligned}(\forall t \in [0, +\infty[\smallsetminus \big\{km\varepsilon \;\big|\; k \in \mathbb{N}\big\}) \quad \theta'(t) &= 2\big\langle x(t) - \psi^\varepsilon(t) \mid x'(t) - (\psi^\varepsilon)'(t)\big\rangle \\ &= 2\langle x(t) - \psi^\varepsilon(t) \mid A\psi^\varepsilon(t) - h^\varepsilon(t) - Ax(t)\rangle \\ &\leqslant 2\langle x(t) - \psi^\varepsilon(t) \mid -h^\varepsilon(t)\rangle \\ &\leqslant 2\|x(t) - \psi^\varepsilon(t)\|\,\|h^\varepsilon(t)\| \\ &\leqslant 2\varepsilon\gamma\sqrt{\theta(t)}. \end{aligned} \tag{3.15}$$

Integrating this inequality and noting that $\theta(0) = 0$, we obtain $(\forall t \in [0, +\infty[)\ \|\psi^\varepsilon(t) - x(t)\| = \sqrt{\theta(t)} \leqslant \varepsilon\gamma t$. Now let $\bar{t} \in]0, +\infty[$. Then $\sup_{0 \leqslant t \leqslant \bar{t}} \|\psi^\varepsilon(t) - x(t)\| \leqslant \varepsilon\gamma\bar{t} \to 0$ as $\varepsilon \to 0$. □

3.2 Strong convergence under stability of approximate cycles

In this section, we investigate the strong convergence of the cycles defined in (1.8) when a stability condition holds.

Theorem 3.4 *Suppose that Assumptions 1.1 and 2.1 are satisfied, and that*

$$(\forall z \in \operatorname{Fix} T) \quad \lim_{\varepsilon \to 0} d_{\operatorname{Fix} R^\varepsilon}(z) = 0. \tag{3.16}$$

In addition, let $y_0 \in D$, and suppose that the orbit of y_0 in the Cauchy problem (3.1) converges strongly, say $x(t) \to \overline{x} \in D$ as $t \to +\infty$. For every $\varepsilon \in]0, \eta[$, let $(x_i^\varepsilon)_{i \in I}$ be the cycle obtained as the weak limit of (1.9) in Proposition 2.2. Then $\overline{x} \in \operatorname{Fix} T$ and $(\forall i \in I)\ x_i^\varepsilon \to \overline{x}$ when $\varepsilon \to 0$.

Proof. Since $x(t) \to \overline{x}$, (3.1) implies that $x'(t)$ converges to $A\overline{x}$ and therefore $A\overline{x} = 0$ since $x'(t) \to 0$. Hence, $\overline{x} \in \operatorname{Fix} T$. Now fix $\delta \in]0, +\infty[$ and $\bar{t} \in]0, +\infty[$ such that $(\forall t \in [\bar{t}, +\infty[)$ $\|x(t) - \overline{x}\| \leqslant \delta$. For every $\varepsilon \in]0, \eta[$, set $(z_k^\varepsilon)_{k \in \mathbb{N}} = (y_{km}^\varepsilon)_{k \in \mathbb{N}} = ((R^\varepsilon)^k y_0)_{k \in \mathbb{N}}$ and define the function ψ^ε as in (3.10). By Proposition 3.3, there exists $\varepsilon_0 \in]0, \eta[$ such that

$$(\forall \varepsilon \in]0, \varepsilon_0[)(\forall t \in [0, \bar{t} + m]) \quad \|\psi^\varepsilon(t) - x(t)\| \leqslant \delta. \tag{3.17}$$

Now let $\varepsilon \in]0,\varepsilon_0[$, choose $k_0 \in \mathbb{N}$ such that $k_0 m\varepsilon \in [\bar{t}, \bar{t}+m]$, and set $\bar{x}^\varepsilon = P_{\operatorname{Fix} R^\varepsilon}\overline{x}$ (recall that, since D is closed and convex and R^ε is nonexpansive, $\operatorname{Fix} R^\varepsilon$ is closed and convex [5, Corollary 4.15]). Then $\|z^\varepsilon_{k_0} - x(k_0 m\varepsilon)\| = \|\psi^\varepsilon(k_0 m\varepsilon) - x(k_0 m\varepsilon)\| \leqslant \delta$ and therefore $\|z^\varepsilon_{k_0} - \overline{x}\| \leqslant 2\delta$. Since R^ε is nonexpansive, we have $(\forall k \in \mathbb{N})\ \|z^\varepsilon_{k+1} - \bar{x}^\varepsilon\| \leqslant \|z^\varepsilon_k - \bar{x}^\varepsilon\|$. Hence, for every integer $k \geqslant k_0$, we have

$$\|z^\varepsilon_k - \bar{x}^\varepsilon\| \leqslant \|z^\varepsilon_{k_0} - \bar{x}^\varepsilon\| \leqslant \|z^\varepsilon_{k_0} - \overline{x}\| + \|\overline{x} - \bar{x}^\varepsilon\| \leqslant 2\delta + d_{\operatorname{Fix} R^\varepsilon}(\overline{x}) \tag{3.18}$$

and therefore

$$\|y^\varepsilon_{km} - \overline{x}\| = \|z^\varepsilon_k - \overline{x}\| \leqslant 2\delta + 2d_{\operatorname{Fix} R^\varepsilon}(\overline{x}). \tag{3.19}$$

Since Proposition 2.2 asserts that $y^\varepsilon_{km} \rightharpoonup x^\varepsilon_m$, we get

$$\|x^\varepsilon_m - \overline{x}\| \leqslant \varliminf_{k\to+\infty} \|y^\varepsilon_{km} - \overline{x}\| \leqslant 2\delta + 2d_{\operatorname{Fix} R^\varepsilon}(\overline{x}), \tag{3.20}$$

and (3.16) yields

$$\varlimsup_{\varepsilon\to 0} \|x^\varepsilon_m - \overline{x}\| \leqslant 2\delta. \tag{3.21}$$

Letting $\delta \to 0$, we deduce that $x^\varepsilon_m \to \overline{x}$ as $\varepsilon \to 0$. In turn, it follows from (2.9) that $(\forall i \in I)$ $x^\varepsilon_i \to \overline{x}$ as $\varepsilon \to 0$. □

The following corollary settles entirely De Pierro's conjecture in the case of $m = 2$ closed convex sets in Euclidean spaces.

Corollary 3.5 *In Assumption 1.1, suppose that $\mathcal{H}$ is finite-dimensional, $D = \mathcal{H}$, and $m = 2$, and let $T_1 = P_1$ and $T_2 = P_2$ be the projection operators onto nonempty closed convex sets such that*

$$\operatorname{Fix} T = S = \operatorname{Argmin} \Phi \neq \varnothing, \quad \text{where} \quad \Phi = \frac{1}{4}\big(d^2_{C_1} + d^2_{C_2}\big). \tag{3.22}$$

Let $y_0 \in \mathcal{H}$ and let $\overline{x} \in S$ be the limit of the the solution x of Cauchy problem

$$\begin{cases} x'(t) + x(t) = \frac{1}{2}\big(P_1 x(t) + P_2 x(t)\big) \ \text{ on }]0,+\infty[\\ x(0) = y_0. \end{cases} \tag{3.23}$$

For for every $\varepsilon \in]0,1[$, let $x^\varepsilon_1 = \lim_{k\to+\infty} y^\varepsilon_{2k+1}$ and $x^\varepsilon_2 = \lim_{k\to+\infty} y^\varepsilon_{2k+2}$, where

$$(\forall k \in \mathbb{N}) \quad \left\lfloor \begin{aligned} y^\varepsilon_{2k+1} &= \big(\operatorname{Id} + \varepsilon(P_1 - \operatorname{Id})\big) y^\varepsilon_{2k} \\ y^\varepsilon_{2k+2} &= \big(\operatorname{Id} + \varepsilon(P_2 - \operatorname{Id})\big) y^\varepsilon_{2k+1}. \end{aligned} \right. \tag{3.24}$$

Then $x^\varepsilon_1 \to \overline{x}$ and $x^\varepsilon_2 \to \overline{x}$ when $\varepsilon \to 0$.

Proof. Fix $z \in S$, and set $a = P_1 z$ and $b = P_2 z$. Then $z = (a+b)/2$ and $(\forall \varepsilon \in]0,1[)$ $z^\varepsilon = ((1-\varepsilon)a + b)/(2-\varepsilon) \in \operatorname{Fix} R^\varepsilon$. Thus

$$d_{\operatorname{Fix} R^\varepsilon}(z) \leqslant \|z - z^\varepsilon\| = \frac{\varepsilon\|b-a\|}{2(2-\varepsilon)} \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{3.25}$$

and the conclusion follows from Theorem 3.4. □

We conclude this section by showing that, in contrast with (3.25), the condition (3.16) can fail in the case of projection operators in the presence of $m = 3$ sets.

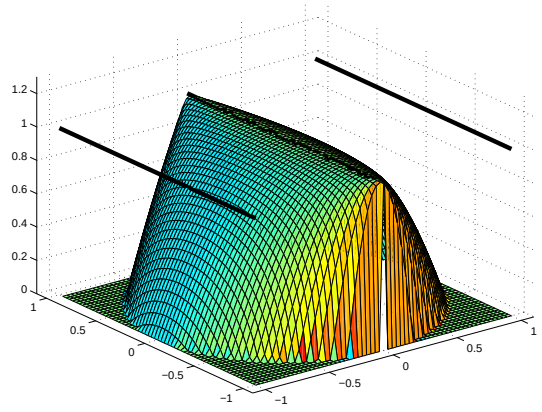

Figure 2: An example in which the condition (3.16) fails.

Example 3.6 Suppose that $\mathcal{H} = \mathbb{R}^3$ and $m = 3$, and let T_1, T_2, and T_3 be, respectively, the projection operators onto the bounded closed convex sets (see Fig. 2)

$$\begin{cases} C_1 = [-1,1] \times \{-1\} \times \{1\} \\ C_2 = [-1,1] \times \{1\} \times \{1\} \\ C_3 = \big\{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 \;\big|\; \xi_1 \in [-1,1],\, \xi_3 \in [0,1],\, (1-\xi_3)(\xi_1^2-1)+\xi_2^2 \leqslant 0\big\}. \end{cases} \tag{3.26}$$

Then the set of least-squares solutions is $S = \operatorname{Fix} T = [-1,1] \times \{0\} \times \{1\} \subset C_3$. Moreover,

$$(\forall \varepsilon \in \left]0,1\right[) \quad \operatorname{Fix} R^\varepsilon = \{z^\varepsilon\} = \left\{\left(0, \frac{w_\varepsilon + \varepsilon(1-\varepsilon)}{3(1-\varepsilon)+\varepsilon^2}, 1 - \frac{w_\varepsilon^2}{3(1-\varepsilon)+\varepsilon^2}\right)\right\}, \tag{3.27}$$

where w_ε is the unique real solution of $2w^3 + w = \varepsilon/(2-\varepsilon)$. Clearly $z^\varepsilon \to (0,0,1) \in S$ as $\varepsilon \to 0$, but $(\forall z \in S \smallsetminus \{(0,0,1)\})$ $d_{\operatorname{Fix} R^\varepsilon}(z) \not\to 0$ as $\varepsilon \to 0$.

3.3 Strong convergence under local strong monotonicity

Another situation covered by Theorem 3.4 is when the operator T has a unique fixed point $\overline{x}$ and $A = \operatorname{Id} - T$ is locally strongly monotone around $\overline{x}$, namely

$$(\exists\, \alpha \in \left]0,+\infty\right[)(\exists\, \delta \in \left]0,+\infty\right[)(\forall x \in D \cap B(\overline{x};\delta)) \quad \langle x - \overline{x} \mid x - Tx\rangle \geqslant \alpha \|x - \overline{x}\|^2. \tag{3.28}$$

In the case of convex projections operators, then $A = \nabla\Phi$ and, if Φ is twice differentiable at $\overline{x}$, then (3.28) is equivalent to the positive-definiteness of $\nabla^2\Phi(\overline{x})$. Another case in which (3.28) is satisfied, with $\alpha = 1-\rho$, is when T is a local strict contraction with constant $\rho \in \left]0,1\right[$ at the fixed point $\overline{x}$, namely, for all x in some ball $B(\overline{x};\delta)$, $\|Tx - T\overline{x}\| \leqslant \rho\|x - \overline{x}\|$. If T is differentiable at $\overline{x}$ this amounts to $\|T'(\overline{x})\| < 1$.

Theorem 3.7 *Suppose that Assumptions 1.1 and 2.1 are satisfied, together with* (3.28)*, and let* $\operatorname{Fix} T = \{\bar{x}\}$*. In addition, let* $y_0 \in D$ *and, for every* $\varepsilon \in \left]0,\eta\right[$*, let* $(x_i^\varepsilon)_{i\in I}$ *be the cycle obtained as the weak limit of* (1.9) *in Proposition 2.2. Then* $(\forall i \in I)$ $x_i^\varepsilon \to \overline{x}$ *as* $\varepsilon \to 0$*.*

Proof. It suffices to check the assumptions of Theorem 3.4. Set $A = \mathrm{Id} - T$ and let x be the solution to (3.1).

- $d_{\mathrm{Fix}\, R^\varepsilon}(\overline{x}) \to 0$ as $\varepsilon \to 0$:
  Let $\varepsilon \in \left]0, \min\{\eta, \alpha/(2m)\}\right[$, set $Q^\varepsilon = \mathrm{Id} - m\varepsilon A$ and $\gamma(\varepsilon) = 1 - m\varepsilon(\alpha - 2m\varepsilon)$, and let $y \in D \cap B(\overline{x}; \delta)$. Since $A\overline{x} = 0$ and A is 2-Lipschitz, we have

  $$\begin{aligned} \|Q^\varepsilon y - \overline{x}\|^2 &= \|y - \overline{x}\|^2 - 2m\varepsilon\langle y - \overline{x} \mid Ay - A\overline{x}\rangle + (m\varepsilon)^2\|Ay - A\overline{x}\|^2 \\ &\leqslant (1 - 2m\varepsilon(\alpha - 2m\varepsilon))\|y - \overline{x}\|^2 \\ &\leqslant \gamma(\varepsilon)^2\|y - \overline{x}\|^2. \end{aligned} \tag{3.29}$$

  On the other hand, setting $\rho = \max_{i\in I} \|T_i\overline{x} - \overline{x}\|/2$ and $\beta = 3^m - 2m - 1$, Lemma 3.1 gives

  $$\|R^\varepsilon y - Q^\varepsilon y\| \leqslant \varepsilon^2\beta(\|y - \overline{x}\| + \rho) \tag{3.30}$$

  which, combined with (3.29), yields

  $$\|R^\varepsilon y - \overline{x}\| \leqslant \|R^\varepsilon y - Q^\varepsilon y\| + \|Q^\varepsilon y - \overline{x}\| \leqslant \varepsilon^2\beta(\|y - \overline{x}\| + \rho) + \gamma(\varepsilon)\|y - \overline{x}\|. \tag{3.31}$$

  From this estimate it follows that given $\delta' \in \left]0, \delta\right]$, for every $\varepsilon \leqslant m\alpha\delta'/(\beta(\delta' + \rho) + 2m^2\delta')$, we have $R^\varepsilon(D \cap B(\overline{x}; \delta')) \subset D \cap B(\overline{x}; \delta')$. Therefore R^ε has a fixed point in $B(\overline{x}; \delta')$ and hence $d_{\mathrm{Fix}\, R^\varepsilon}(\overline{x}) \leqslant \delta'$. Since δ' can be arbitrarily small, this proves that $d_{\mathrm{Fix}\, R^\varepsilon}(\overline{x}) \to 0$ as $\varepsilon \to 0$.

- $x(t) \to \overline{x}$ as $t \to +\infty$:
  Let $\theta\colon [0, +\infty[\to [0, +\infty[$ be defined by $\theta(t) = \|x(t) - \overline{x}\|^2/2$, and let us show that $\lim_{t\to+\infty} \theta(t) = 0$. We note that this holds whenever the orbit enters the ball $B(\overline{x}; \delta)$ at some instant t_0. Indeed, the monotonicity of A implies that θ is decreasing so that, for every $t \in [t_0, +\infty[$, $x(t) \in D \cap B(\overline{x}; \delta)$ and hence (3.1) and (3.28) give

  $$\theta'(t) = \big\langle x(t) - \overline{x} \mid x'(t)\big\rangle = \langle \overline{x} - x(t) \mid x(t) - Tx(t)\rangle \leqslant -\alpha\|x(t) - \overline{x}\|^2 = -2\alpha\theta(t). \tag{3.32}$$

  Consequently, $\theta(t) \leqslant \theta(t_0)\exp(-2\alpha(t - t_0)) \to 0$ as $t \to +\infty$. It remains to prove that $x(t)$ enters the ball $B(\overline{x}; \delta)$. If this was not the case we would have $\mu = \lim_{t\to+\infty} \sqrt{\theta(t)} \geqslant \delta$. Choose t_0 large enough so that $\sqrt{\theta(t_0)} \leqslant \mu + \delta/2$ and let $\tilde{x}$ be the solution to the Cauchy problem

  $$\begin{cases} \tilde{x}'(t) = -A\tilde{x}(t) \text{ on } [t_0, +\infty[\\ \tilde{x}(t_0) = \tilde{x}_0, \end{cases} \tag{3.33}$$

  where $\tilde{x}_0 = \overline{x} + \delta(x(t_0) - \overline{x})/\|x(t_0) - \overline{x}\| \in D \cap B(\overline{x}; \delta)$. By monotonicity of A, $t \mapsto \|x(t) - \tilde{x}(t)\|$ is decreasing and hence

  $$\begin{aligned} (\forall t \in [t_0, +\infty[) \quad \|x(t) - \overline{x}\| &\leqslant \|x(t) - \tilde{x}(t)\| + \|\tilde{x}(t) - \overline{x}\| \\ &\leqslant \|x(t_0) - \tilde{x}(t_0)\| + \|\tilde{x}(t) - \overline{x}\| \\ &\leqslant (\mu - \delta/2) + \|\tilde{x}(t) - \overline{x}\|. \end{aligned} \tag{3.34}$$

  Since by the previous argument $\|\tilde{x}(t) - \overline{x}\| \to 0$, we reach a contradiction with the fact that $(\forall t \in [0, +\infty[)\ \|x(t) - \overline{x}\| \geqslant \mu$.

Altogether, the conclusion follows from Theorem 3.4. □

Remark 3.8 If $\mathrm{Id}-T$ were globally (rather than just locally as in (3.28)) strongly monotone at every point in $\operatorname{Fix} T$, we could derive Theorem 3.7 directly from Theorem 2.8 and Remark 2.9(ii).

Theorem 3.7 can also be applied when the local strong monotonicity or the local contraction properties hold up to an affine subspace (see (3.35) below). This is relevant in the case studied in [6] when $(T_i)_{i\in I}$ is a family of projection operators onto closed affine subspaces $(x_i+E_i)_{i\in I}$, where $(E_i)_{i\in I}$ is a family of closed vector subspaces of $\mathcal{H}$, and more generally for unbounded closed convex cylinders of the form $(B_i+E_i)_{i\in I}$, where B_i is a nonempty bounded closed convex subset of $E_i^\perp$.

Corollary 3.9 *Suppose that Assumptions 1.1 and 2.1 are satisfied, that $D=\mathcal{H}$, and that $(T_i)_{i\in I}$ is a family of projection operators onto nonempty closed convex subsets $(C_i)_{i\in I}$ of $\mathcal{H}$. In addition, suppose that the set S of minimizers of Φ in (1.3) is a closed affine subspace, say $S=z+E$, where $z\in\mathcal{H}$ and E is a closed vector subspace of $\mathcal{H}$. Let $y_0\in D$, set $\overline{x}=P_S y_0$, and, for every $\varepsilon\in\left]0,\eta\right[$, let $(x_i^\varepsilon)_{i\in I}$ be the cycle obtained as the weak limit of (1.9) in Proposition 2.2. Then the following hold.*

(i) $(\forall i\in I)\ x_i^\varepsilon \rightharpoonup \overline{x}$ *as* $\varepsilon\to 0$.

(ii) *Suppose that*

$$(\forall y\in S)(\exists\,\rho\in[0,1[)(\exists\,\delta\in\left]0,+\infty\right[)(\forall x\in B(0;\delta)\cap E^\perp)\ \ \|T(x+y)-Ty\|\leqslant\rho\|x\|. \tag{3.35}$$

Then $(\forall i\in I)\ x_i^\varepsilon\to\overline{x}$ *as* $\varepsilon\to 0$.

Proof. Let $i\in I$. Since $S=z+E$, we have $C_i+E\subset C_i$ and the iterates $(y_k^\varepsilon)_{k\in\mathbb{N}}$ in (1.9) move parallel to $E^\perp$ and remain in $y_0+E^\perp$. Hence, since $\{\overline{x}\}=S\cap(y_0+E^\perp)$, (i) follows by applying Theorem 2.8 in the space $y_0+E^\perp$, while (ii) follows by applying Theorem 3.7 in this same space. □

We conclude the paper by revisiting De Pierro's conjecture in the affine setting investigated in [6]. More precisely, we shall derive an alternative proof of the main result of [6] from Corollary 3.9. For this purpose, we need the following notion of regularity.

Definition 3.10 A finite family $(E_i)_{i\in I}$ of closed vector subspaces of $\mathcal{H}$ with intersection E is regular if

$$\big(\forall(y_k)_{n\in\mathbb{N}}\in\mathcal{H}^{\mathbb{N}}\big)\quad \max_{i\in I} d_{E_i}(y_k)\to 0\quad\Rightarrow\quad d_E(y_k)\to 0. \tag{3.36}$$

Theorem 3.11 *Let $(E_i)_{i\in I}$ be a regular family of closed vector subspaces of $\mathcal{H}$ with intersection E and for, every $i\in I$, let $\overline{x}_i\in\mathcal{H}$ and let P_i be the projection operator onto the affine subspace $C_i=\overline{x}_i+E_i$. Let $y_0\in\mathcal{H}$ and set $S=\operatorname{Argmin}\sum_{i\in I} d_{C_i}^2$. Then there exists $z\in\mathcal{H}$ such that $S=z+E$. Moreover, for every $\varepsilon\in\left]0,1\right]$, the cycle $(x_i^\varepsilon)_{i\in I}$ obtained as the weak limit of (1.9) in Proposition 2.2 exists, and $(\forall i\in I)\ x_i^\varepsilon\to P_S y_0$ as $\varepsilon\to 0$.*

Proof. We have $(\forall i \in I)\ P_i\colon x \mapsto \overline{x}_i + P_{E_i}(x - \overline{x}_i)$. Hence $Tx = a + Lx$, where $a = (1/m)\sum_{i\in I}(\overline{x}_i - P_{E_i}\overline{x}_i)$ and $L = (1/m)\sum_{i\in I} P_{E_i}$. According to [6, Theorem 5.4], the subspaces $(E_i)_{i\in I}$ are regular if and only if $\rho = \|L \circ P_{E^\perp}\| < 1$, which implies that T is a strict contraction on $y_0 + E^\perp$. From this we get simultaneously that T has a fixed point z, that the least-squares solution set is of the form $S = z + E$, and that (3.35) holds. Hence, the result will follow from Corollary 3.9 provided that $(x_i^\varepsilon)_{i\in I}$ exists for every $\varepsilon \in]0, 1]$. This was proved in [6, Theorem 5.6] by noting that $R^\varepsilon|_{y_0+E^\perp}$ is a strict contraction. Indeed, R^ε is a composition of affine maps and an inductive calculation reveals that it can be written as $R^\varepsilon x = a^\varepsilon + L^\varepsilon x$, where $a^\varepsilon \in \mathcal{H}$ and L^ε a linear operator which is a convex combination of nonexpansive linear maps, one of which is the strict contraction $L \circ P_{E^\perp}$. □

Remark 3.12 Corollary 3.9(i) seems to be new even for affine subspaces $(C_i)_{i\in I}$. Also new in Corollary 3.9(ii) is the fact that strong convergence holds for more general convex sets than just translates of regular subspaces.

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