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WEAK ERROR IN NEGATIVE SOBOLEV SPACES FOR THE
STOCHASTIC HEAT EQUATION

OMAR ABOURA

ABSTRACT. In this paper, we make another step in the study of weak error of the sto-
chastic heat equation by considering norms as functional.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ a probability space and $T > 0$ a fixed time. $(W(t))_{t \geq 0}$ will be a cylindrical
Brownian motion on $L^2(0, 1)$. We consider the stochastic heat equation, written in abstract
form in $L^2(0, 1)$:
$$X(0) = 0, \text{ for all } t \in [0, T] \ X(t, 0) = X(t, 1) = 0 \text{ and}$$
$$dX(t) = \frac{1}{2} \frac{d^2}{dx^2}X(t)dt + dW(t). \quad (1.1)$$
It is well know that this equation admits a unique weak solution (from the analytical point
of view).

Let $N \in \mathbb{N}^*$ and $h := T/N$. Consider $\left(t_k\right)_{0 \leq k \leq N}$ the uniform subdivision of $[0, T]$ defined
by $t_k := kh$. We consider the implicit Euler scheme defined as follow:
$$X^N(t_{k+1}) = X^N(t_k) + h \frac{1}{2} \frac{d^2}{dx^2}X^N(t_{k+1}) + \Delta W(k + 1), \quad (1.2)$$
where $\Delta W(k + 1) = W(t_{k+1}) - W(t_k)$.

Let $f : L^2(0, 1) \to \mathbb{R}$ be a functional. The stong error is the study of $E \left|X^N(T) - X(T)\right|^2_{L^2(0,1)}$.
The weak error is the study of $\left|Ef\left(X^N(T)\right) - Ef\left(X(T)\right)\right|$ with respect to the time mesh $h$.

In [6], A. Debussche considers a more general stochastic equation and a more general
functional than the one considered here. He obtains a weak error of order $1/2$, which is
the double of that proved by [15] for the strong speed of convergence. The novelty of this
paper his to prove that for the square of the norm the weak error his better than $1/2$ in
negative Sobolev spaces.

2. Preliminaries and main result

Notations. We collect here some of the notations used through the paper. $\langle \,, \, \rangle_{L^2(0,1)}$ is the inner product in $L^2(0, 1)$, $H^1_0(0, 1)$ is the Sobolev space of functions $f$ in $L^2(0, 1)$
vanishing in 0 and 1 with first derivatives in $L^2(0, 1)$, $H^2(0, 1)$ is the Sobolev space of functions $f$ in $L^2(0, 1)$
with first and second derivatives in $L^2(0, 1)$. Finally, for $m = 1, 2, \ldots$, let $(\epsilon_m(x) = \sqrt{2} \sin(m \pi x) \text{ and } \lambda_m = \frac{1}{2}(\pi m)^2$ denote the eigenfunction and eigenvalues of
$-\Delta$ with Dirichlet boundary conditions on $[0, 1)$.

An $L^2(0, 1)$-valued stochastic process $(X(t))_{t \in [0, T]}$ is said to be a solution of (1.1) if:
$X(0) = 0$ and for all $g \in H^1_0(0, 1) \cap H^2(0, 1)$ we have
$$\langle X(t), g \rangle_{L^2(0,1)} = \int_0^t \langle X(s), \frac{1}{2} \frac{d^2}{dx^2}g \rangle_{L^2(0,1)} ds + \langle W(t), g \rangle_{L^2(0,1)}.$$
It is well known that (1.1) admits a unique solution: see [4]. Then \((e_m)_{m \geq 1}\) is a complete orthonormal basis of \(L^2(0,1)\). If we denote by \(\lambda_m := \frac{1}{2}(\pi m)^2\), \(W_{\lambda_m}(t) := \langle W(t), e_m \rangle_H\) and \(X_{\lambda_m}(t)\) denote the solution of the evolution equation: \(X_{\lambda_m}(0) = 0\) and for \(t > 0\):
\[
dX_{\lambda_m}(t) = -\lambda_m X_{\lambda_m}(t) dt + dW_{\lambda_m}(t).
\]
Then the processes \((X_{\lambda_m}(.))_{m \geq 1}\) are independent and \(X(t) = \sum_{m \geq 1} X_{\lambda_m}(t)e_m\) for all \(t \geq 0\).

A sequence of \(L^2(0,1)\)-valued \((X^N(t_k))_{k=0,\ldots,N}\) is said to be a solution of (1.2) if: \(X^N(t_0) = 0\) and for all \(k = 0, \ldots, N - 1\) and for all \(g \in H^1_0(0,1) \cap H^2(0,1)\) we have
\[
<x^N(t_{k+1}), g>_{L^2(0,1)} = <x^N(t_k), g>_{L^2(0,1)} + h <x^N(t_{k+1}), \frac{d^2}{dx^2}g>_{L^2(0,1)} + <\Delta W(k+1), g>_{L^2(0,1)}.
\]

It is well known that (1.2) has a unique solution and there exists a constant \(C > 0\), independent of \(N\), such that \(E \|X^N(T) - X(T)\|^2_{L^2(0,1)} \leq Ch^\frac{1}{2}\) where \(h = T/N\). Now if we denote by \((X^N_{\lambda_m}(t_k))_{k=0,\ldots,N}\) the solution of: \(X^N_{\lambda_m}(t_0) = 0\) and for \(k = 0, \ldots, N - 1\)
\[
X^N_{\lambda_m}(t_{k+1}) = X^N_{\lambda_m}(t_k) - \lambda_m h X^N_{\lambda_m}(t_{k+1}) + W_{\lambda_m}(k+1).
\]
The random vectors \((X^N_{\lambda_m}(t_k), k = 0, \ldots, N)_{m=1,2,\ldots}\) are independent and \(X^N(t_k) = \sum_{m \geq 1} X^N_{\lambda_m}(t_k)e_m\).

Let \(p \geq 0\); define the spaces \(H^{-p}\) as the completion of \(L^2(0,1)\) for the topology induced by the norm \(|u|^2_{H^{-p}} := \sum_{m \geq 1} \lambda^p_m <u, e_m>_H^2\). The following theorem improves the speed of convergence of \(X^N\) to \(X\) for negative Sobolev spaces.

**Theorem 2.1.** Suppose that \(h < 1\) and let \(p \in [0, \frac{1}{2})\). There exists a constant \(C > 0\), independent of \(N\), such that
\[
|E\|X^N(T)\|^2_{H^{-p}} - E\|X(T)\|^2_{H^{-p}}| \leq Ch^{p+\frac{1}{2}}.
\]

### 3. Proof of the Theorem 2.1

The proof of the theorem will be done in several steps. First we recall the weak error of the Ornstein-Uhlenbeck process. Secondly we prove some technical lemmas. Then we decompose the weak error and analyse each term of these decomposition.

#### 3.1. Weak error of the Ornstein-Uhlenbeck process.

Let \(\lambda > 0\), \((W_{\lambda}(t))_{t \geq 0}\) be a one dimensional Brownian motion and \((X_{\lambda}(t))_{t \geq 0}\) be the Ornstein Uhlenbeck process solution of the following stochastic differential equation: \(X_{\lambda}(0) = x \in \mathbb{R}\) and
\[
dX_{\lambda}(t) = -\lambda X_{\lambda}(t) dt + dW_{\lambda}(t). \tag{3.1}
\]

In this step, we study two properties associated with this process: the Kolmogorov equation and the implicit Euler scheme.

Let \(\left(X^{t,x}_{\lambda}(s)\right)_{t \leq s \leq T}\) be the solution of (3.1) starting from \(x\) at time \(t\). It is well known that \(X^{t,x}_{\lambda}(T)\) is a normal random variable:
\[
X^{t,x}_{\lambda}(T) \sim \mathcal{N}\left(e^{-\lambda(T-t)}x, \frac{1-e^{-2\lambda(T-t)}}{2\lambda}\right).
\]
For \( t \in [0, T] \) and \( x \in \mathbb{R} \) set \( u_\lambda(t, x) := E \left| X_\lambda^{t,x}(T) \right|^2 \). Then \( u_\lambda \) is the solution of the following partial differential equation, called Kolmogorov equation: for all \( x \in \mathbb{R} \), \( u_\lambda(T, x) = |x|^2 \) and for all \((t, x) \in [0, T] \times \mathbb{R}\)

\[
-\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x) - \lambda x \frac{\partial}{\partial x} u(t, x). \tag{3.2}
\]

Since \( X_\lambda^{t,x}(T) \) has a normal law, we can write \( u_\lambda \) explicitly:

\[
u_\lambda(t, x) = \frac{1 - e^{-2\lambda(T-t)}}{2\lambda} + e^{-2\lambda(T-t)} x^2. \tag{3.3}\]

With this expression we see that \( u_\lambda \in C^{1,2}([0, T] \times \mathbb{R}) \) and we have the following derivatives:

\[
\frac{\partial}{\partial x} u_\lambda(t, x) = 2e^{-2\lambda(T-t)}x, \tag{3.4}
\]

\[
\frac{\partial^2}{\partial x^2} u_\lambda(t, x) = 2e^{-2\lambda(T-t)} \tag{3.5}
\]

\[
\frac{\partial}{\partial t} u_\lambda(t, x) = -e^{-2\lambda(T-t)} + 2\lambda e^{-2\lambda(T-t)}x^2, \tag{3.6}
\]

\[
\frac{\partial^2}{\partial t \partial x} u_\lambda(t, x) = 4\lambda e^{-2\lambda(T-t)}x. \tag{3.7}
\]

The implicit Euler scheme for the Ornstein-Uhlenbeck equation (3.1) starting from 0 at time \( t_0 \), is defined as follow: \( X_\lambda^{N}(t_0) = 0 \) and for \( k = 0, \ldots, N - 1 \)

\[
X_\lambda^{N}(t_{k+1}) = X_\lambda^{N}(t_k) - \lambda h X_\lambda^{N}(t_{k+1}) + \Delta W_\lambda(k+1), \tag{3.8}
\]

where \( \Delta W_\lambda(k+1) = W_\lambda(t_{k+1}) - W_\lambda(t_k) \). Since we have the following equation

\[
X_\lambda^{N}(t_{k+1}) = \frac{1}{1 + \lambda h} X_\lambda^{N}(t_k) + \frac{1}{1 + \lambda h} \Delta W_\lambda(k+1), \tag{3.9}
\]

we see that the scheme is well defined.

**Lemma 3.1.** For \( k = 1, \ldots, N \) we have \( X_\lambda^{N}(t_k) = \sum_{j=0}^{k-1} \frac{\Delta W_\lambda(k-j)}{(1 + \lambda h)^{j+1}}. \)

**Proof.** We proceed by induction. If \( k = 1 \), we have \( X_\lambda^{N}(t_1) = \frac{1}{1 + \lambda h} \Delta W_\lambda(1) \). Suppose the result true until \( k \). Using (3.9), we have

\[
X_\lambda^{N}(t_{k+1}) = \sum_{j=0}^{k-1} \frac{\Delta W_\lambda(k-j)}{(1 + \lambda h)^{j+2}} + \frac{1}{1 + \lambda h} \Delta W_\lambda(k+1)
\]

\[
= \sum_{l=1}^{k} \frac{\Delta W_\lambda(k+1-l)}{(1 + \lambda h)^{l+1}} + \frac{1}{(1 + \lambda h)^{0+1}} \Delta W_\lambda(k+1-0),
\]

which concludes the proof. \( \square \)

**Lemma 3.2.** For all \( k = 0, \ldots, N \), we have the following bound \( E \left| X_\lambda^{N}(t_k) \right|^2 \leq \frac{1}{2\lambda} \).

**Proof.** Using the independence of the increments of the Brownian motion and Lemma 3.1, we have

\[
E \left| X_\lambda^{N}(t_k) \right|^2 = \sum_{j=0}^{k-1} \frac{1}{(1 + \lambda h)^{2(j+1)}} E \left| \Delta W_\lambda(k-j) \right|^2 = h \sum_{j=0}^{k-1} \frac{1}{(1 + \lambda h)^{2(j+1)}}.
\]
Let $a := 1/(1 + \lambda h)^2$; we deduce that $E \left| X^N_\lambda(t_k) \right|^2 = ha \frac{1-a^k}{1-a}$. Simple computations yield $ha/(1 - a) = 1/(2\lambda + \lambda^2 h)$, which implies
\[
E \left| X^N_\lambda(t_k) \right|^2 = \frac{1}{2\lambda + \lambda^2 h} \left( 1 - \frac{1}{(1 + \lambda h)^2} \right).
\]
This concludes the proof. \hfill \Box

For $t \geq 0$, we denote $\mathcal{F}_t^\lambda := \sigma(W_\lambda(s), s \leq t)$ and $D^1_\lambda$ the Malliavin Sobolev space with respect to $W_\lambda$.

**Lemma 3.3.** For all $k = 1, \ldots, N$, we have $X^N_\lambda(t_k) \in D^{1,2}_\lambda \cap L^2(\mathcal{F}_{t_k}^\lambda)$.

**Proof.** This is a consequence of Lemma 3.1, the fact that $L^2(\mathcal{F}_{t_k}^\lambda)$ and $D^{1,2}_\lambda$ are linear space and for all $j = 0, \ldots, k - 1$, $\Delta W_\lambda(k - j) \in D^{1,2}_\lambda \cap L^2(\mathcal{F}_{t_k}^\lambda)$. \hfill \Box

As usual in the study of weak error, we need to use a continuous process that interpolates the Euler scheme. The interpolation process that we used was introduced in [1]. We recall its construction and prove some of its properties.

Let $k \in \{0, \ldots, N - 1\}$ be fixed. In order to interpolate the scheme between the points $(t_k, X^N_\lambda(t_k))$ and $(t_{k+1}, X^N_\lambda(t_{k+1}))$, we define the process as follows: for $t \in [t_k, t_{k+1}]$, set
\[
X^N_\lambda(t) := X^N_\lambda(t_k) - \lambda E \left( X^N_\lambda(t_{k+1})|\mathcal{F}_t \right) (t - t_k) + W_\lambda(t) - W_\lambda(t_k).
\]
In the sequel, we will use the following processes: for $t \in [t_k, t_{k+1}]
\[
\beta^k_N(t) := -\lambda E \left( X^N_\lambda(t_{k+1})|\mathcal{F}_t \right),
\]
\[
z^k_N(t) := -\lambda E \left( D_t X^N_\lambda(t_{k+1})|\mathcal{F}_t \right),
\]
\[
\gamma^k_N(t) := 1 + (t - t_k)z^k_N(t).
\]

The next lemma relates the above processes.

**Lemma 3.4.** Let $k = 0, \ldots, N - 1$. For $t \in [0, T]$, we have
\[
d\beta^k_N(t) = z^k_N(t) dW_\lambda(t), \quad z^k_N(t) = -\frac{\lambda}{1 + \lambda h},
\]
\[
\gamma^k_N(t) = 1 - (t - t_k)\frac{\lambda}{1 + \lambda h}, \quad dX^N_\lambda(t) = \beta^k_N(t) dt + \gamma^k_N(t) dW_\lambda(t).
\]

**Proof.** Using the Clark-Ocone formula and Lemma 3.3, we have
\[
X^N_\lambda(t_{k+1}) = E \left( X^N_\lambda(t_{k+1})|\mathcal{F}_t \right) + \int_t^{t_{k+1}} E \left( D_s X^N_\lambda(t_{k+1})|\mathcal{F}_s \right) dW_\lambda(s).
\]
Multiplying by $(-\lambda)$, we deduce
\[
-\lambda X^N_\lambda(t_{k+1}) = \beta^k_N(t) + \int_t^{t_{k+1}} z^k_N(s) dW_\lambda(s),
\]
which gives the first identity. Applying the Malliavin derivative to (3.9), we have for $s \in [t_k, t_{k+1}] D_s X^N_\lambda(t_{k+1}) = \frac{1}{1 + X^N_\lambda(t_{k+1})}$. Multiplying by $(-\lambda)$, we deduce the second and third equalities.

Finally, Itô’s formula gives us
\[
d \left( (t - t_k)\beta^k_N(t) \right) = (t - t_k)z^k_N(t) dW_\lambda(t) + \beta^k_N(t) dt,
\]
which concludes the proof. \hfill \Box
Lemma 3.5. Let \( k \in \{0, \ldots, N-1\} \). For any \( s \in [t_k, t_{k+1}] \), we have
\[
E \left| \beta^{k,N}_\lambda(s) \right|^2 \leq 2\lambda, \quad E \left| X^N_\lambda(s) \right|^2 \leq \frac{1}{2\lambda} + h, \quad E \beta^{k,N}_\lambda(s) X^N_\lambda(s) \leq 1.
\]

Proof. Applying the conditional expectation with respect to \( \mathcal{F}_s \) on both sides of (3.9) for \( s \in [t_k, t_{k+1}] \) we have
\[
E \left( X^N_\lambda(t_{k+1}) | \mathcal{F}_s \right) = \frac{1}{1 + \lambda h} \left[ X^N_\lambda(t_k) + (W_\lambda(s) - W_\lambda(t_k)) \right].
\]
Multiplying by \((-\lambda)\) and using (3.11), we obtain
\[
\beta^{k,N}_\lambda(s) = -\frac{\lambda}{1 + \lambda h} X^N_\lambda(t_k) - \frac{\lambda}{1 + \lambda h} (W_\lambda(s) - W_\lambda(t_k)). \tag{3.14}
\]
The independence of \( \mathcal{F}_{t_k} \) and \( W_\lambda(s) - W_\lambda(t_k) \) yields
\[
E \left| \beta^{k,N}_\lambda(s) \right|^2 \leq \frac{\lambda^2}{(1 + \lambda h)^2} E \left| X^N_\lambda(t_k) \right|^2 + \frac{\lambda^2}{(1 + \lambda h)^2} (s - t_k).
\]
Using Lemma 3.2, we deduce
\[
E \left| \beta^{k,N}_\lambda(s) \right|^2 \leq \frac{\lambda}{2(1 + \lambda h)^2} + \frac{\lambda^2 h}{(1 + \lambda h)^2},
\]
which proves the first upper estimate.

Using (3.10) and (3.14), we have for \( s \in [t_k, t_{k+1}] \)
\[
X^N_\lambda(s) = \left( 1 - \frac{\lambda(s - t_k)}{1 + \lambda h} \right) \left[ X^N_\lambda(t_k) + (W_\lambda(s) - W_\lambda(t_k)) \right]. \tag{3.15}
\]
Taking the expectation of the square and using the independence of \( \mathcal{F}_{t_k} \) and \( W_\lambda(s) - W_\lambda(t_k) \), we have
\[
E \left| X^N_\lambda(s) \right|^2 = \left( 1 - \frac{\lambda(s - t_k)}{1 + \lambda h} \right)^2 \left( E \left| X^N_\lambda(t_k) \right|^2 + (s - t_k) \right) \leq E \left| X^N_\lambda(t_k) \right|^2 + h \leq \frac{1}{2\lambda} + h,
\]
where the last upper estimates follows from Lemma 3.2.

Multiplying (3.14) and (3.15), taking expectation we obtain
\[
E \left( X^N_\lambda(s) \beta^{k,N}_\lambda(s) \right) = -\frac{\lambda}{1 + \lambda h} \left( 1 - \frac{\lambda(s - t_k)}{1 + \lambda h} \right) \left[ E \left| X^N_\lambda(t_k) \right|^2 + (s - t_k) \right].
\]
Using Lemma 3.2, we deduce
\[
\left| E \left( X^N_\lambda(s) \beta^{k,N}_\lambda(s) \right) \right| \leq \frac{\lambda}{1 + \lambda h} \frac{1}{2\lambda} + \frac{\lambda h}{1 + \lambda h}.
\]
This concludes the proof. \( \square \)

3.2. Some useful analytical lemmas. We at first give a precise upper bound of a series defined in terms of the eigenvalues of the Laplace operator with Dirichlet boundary conditions.

Lemma 3.6. Let \( p \in \left[ 0, \frac{1}{2} \right) \). There exists a constant \( C > 0 \), such that for all \( \alpha > 0 \), we have
\[
\sum_{m \geq 1} \lambda_m^p e^{-2\lambda_m \alpha} \leq C \alpha^{p - \frac{1}{2}}
\]
Proof. The function \((x \in \mathbb{R}_+ \mapsto x^{-2p}e^{-2x^2\alpha})\) is decreasing. So by comparaison, we obtain
\[
\sum_{m \geq 1} m^{-2p}e^{-2m^2\alpha} \leq \int_0^\infty x^{-2p}e^{-2x^2\alpha}dx \leq \alpha^{p-\frac{1}{2}} \int_0^\infty y^{-2p}e^{-2y^2}dy = C\alpha^{p-\frac{1}{2}}.
\]
Since \(\lambda_m = \frac{1}{2}(\pi m)^2\), we deduce the desired upper estimate. \(\square\)

**Lemma 3.7.** Let \(q > 0\). There exists a constant \(C > 0\), such that for all \(\alpha > 0\)
\[
\sum_{m \geq 1} \lambda_m^\alpha e^{-\lambda_m} \leq C \left(1 + \frac{1}{\alpha^{q+2}}\right).
\]

**Proof.** Let \(f(x) = x^{2q}e^{-x^2\alpha}\). His derivatives is given by \(f'(x) = 2x^{2q-1}e^{-x^2\alpha}(q - \alpha x^2)\).

**Case 1:** \(\alpha > q/4\). Then \(f\) is decreasing on \([2, \infty)\) and a standard comparaison argument yields
\[
\sum_{m \geq 1} m^{2q}e^{-m^2\alpha} \leq e^{-\alpha} + 4^q e^{-4\alpha} + \sum_{m \geq 3} \int_{m-1}^m x^{2q}e^{-x^2\alpha}dx
\]
\[
\leq C + \int_0^\infty x^{2q}e^{-x^2\alpha}dx
\]
\[
\leq C + \alpha^{-q-\frac{1}{2}} \int_0^\infty y^{2q}e^{-y^2}dy
\]
\[
\leq C(1 + \alpha^{-q-\frac{3}{2}}).
\]

**Case 2:** \(\alpha \leq q/4\). The function \(f\) is increasing on \([0, \sqrt{q/\alpha}]\). So for each \(m = 1, \ldots, \lfloor\sqrt{\frac{q}{\alpha}}\rfloor - 1\), we have
\[
m^{2q}e^{-m^2\alpha} \leq \int_{m}^{m+1} x^{2q}e^{-x^2\alpha}dx.
\]
On the interval \([\sqrt{\frac{q}{\alpha}}, \infty)\), \(f\) is decreasing. So for each integer \(m \geq \lfloor\sqrt{\frac{q}{\alpha}}\rfloor + 2\), we have
\[
m^{2q}e^{-m^2\alpha} \leq \int_{m-1}^{m} x^{2q}e^{-x^2\alpha}dx.
\]
The above upper estimates yield
\[
\sum_{m \geq 1} m^{2q}e^{-m^2\alpha} \leq \sum_{m \leq \lfloor\sqrt{\frac{q}{\alpha}}\rfloor - 1} \int_m^{m+1} x^{2q}e^{-x^2\alpha}dx + \sum_{m \geq \lfloor\sqrt{\frac{q}{\alpha}}\rfloor + 2} \int_{m-1}^{m} x^{2q}e^{-x^2\alpha}dx
\]
\[
+ \sum_{m \in (\lfloor\sqrt{\frac{q}{\alpha}}\rfloor, \lfloor\sqrt{\frac{q}{\alpha}}\rfloor + 1]} m^{2q}e^{-m^2\alpha}
\]
\[
\leq \int_0^\infty x^{2q}e^{-x^2\alpha}dx + \sum_{m \in (\lfloor\sqrt{\frac{q}{\alpha}}\rfloor, \lfloor\sqrt{\frac{q}{\alpha}}\rfloor + 1]} m^{2q}e^{-m^2\alpha}
\]
\[
\leq C\alpha^{-q-\frac{1}{2}} + \sum_{m \in (\lfloor\sqrt{\frac{q}{\alpha}}\rfloor, \lfloor\sqrt{\frac{q}{\alpha}}\rfloor + 1]} m^{2q}e^{-m^2\alpha}
\]
Now we study each term of the sum in the right hand side. Since \(q \geq \alpha\), we have
\[
\left(\sqrt{\frac{q}{\alpha}}\right)^{2q}e^{-\left(\sqrt{\frac{q}{\alpha}}\right)^2} \leq \left(\frac{q}{\alpha}\right)^q \leq \left(\frac{q}{\alpha}\right)^{q+\frac{1}{2}} \leq C\alpha^{-q-\frac{1}{2}}.
\]
For the second term, we remark that since $q \geq \alpha \left[ \sqrt{\frac{q}{\alpha}} \right] + 1 \leq 2 \left[ \sqrt{\frac{q}{\alpha}} \right] \leq 2 \sqrt{\frac{q}{\alpha}}$. This implies
\[ \left( \left[ \sqrt{\frac{q}{\alpha}} \right] + 1 \right)^{2q} e^{-\left( \sqrt{\frac{q}{\alpha}} \right) + 1} \leq \left( 2 \sqrt{\frac{q}{\alpha}} \right)^{2q} \leq C \alpha^{-q - \frac{3}{2}}. \]
Therefore, in both cases we obtain
\[ \sum_{m \geq 1} m^{2q} e^{-m^2 \alpha} \leq C \left( 1 + \frac{1}{\alpha^{q + \frac{3}{2}}} \right). \]
Since $\lambda_m = \frac{1}{2} (\pi m)^2$, the proof is complete.

\[ \square \]

**Lemma 3.8.** Let $p \in [0, \frac{1}{2})$ and $n \in \mathbb{N}^*$. Let $(v(k,m))_{(k,m) \in \{0, \ldots, N-2\} \times \mathbb{N}^*}$ be a sequence such that for all $k \in \{0, \ldots, N-2\}$ and $m \geq 1$, we have
\[ 0 \leq v(k,m) \leq \lambda_m^{n-p} p \lambda_m^{n+1} e^{-2 \lambda_m (T-t_{k+1})}. \]
Then, there exists a constant $C > 0$, independent of $N$, such that
\[ \sum_{m \geq 1} \sum_{k=0}^{N-2} v(k,m) \leq Ch^{p+\frac{1}{2}}. \]

**Proof.** First we remark that $T - t_{k+1} = h(N - k - 1)$. Using Lemma 3.7, we deduce the existence of $C$ depending on $n$ and $p$, but independent of $N$, such that for $k = 0, \ldots, N-2$ :
\[ \sum_{m \geq 1} v(k,m) \leq C h^{n+1} \left( 1 + \frac{1}{h^{n-p+\frac{1}{2}} (N - k - 1)^{n-p+\frac{1}{2}}} \right), \]
\[ \leq C \left( h^{n+1} + \frac{h^{p+\frac{1}{2}}}{(N - k - 1)^{n-p+\frac{1}{2}}} \right). \]
Therefore, there exists a constant $C$ as above such that
\[ \sum_{m \geq 1} \sum_{k=0}^{N-2} v(k,m) \leq C \left( h^n + h^{p+\frac{1}{2}} \sum_{k=0}^{N-2} \frac{1}{(N - k - 1)^{n-p+\frac{1}{2}}} \right) \]
\[ \leq C \left( h^n + h^{p+\frac{1}{2}} \sum_{l=1}^{N-1} \frac{1}{l^{n-p+\frac{1}{2}}} \right) \leq Ch^{p+\frac{1}{2}}, \]
which concludes the proof. \[ \square \]

**3.3. Decomposition of the weak error.** We follow the classical decomposition introduced in [16]. The definition of $u_\alpha(t,x)$ in section 3.1 yields
\[ E \left| X^N(T) \right|^2_{H^{-p}} - E \left| X(T) \right|^2_{H^{-p}} = \sum_{m \geq 1} \lambda_m^{-p} \left( E \left| X^N_\lambda(T) \right|^2 - E \left| X_\lambda(T) \right|^2 \right), \]
\[ = \sum_{m \geq 1} \lambda_m^{-p} \left( Eu_\lambda(T, X^N_\lambda(T)) - u_\lambda(0, X^N_\lambda(0)) \right). \]
Let $\delta^N(k,m) := \lambda_m^{-p} \left( Eu_\lambda(T, X^N_\lambda(t_{k+1})) - Eu_\lambda(T, X^N_\lambda(t_k)) \right)$; then
\[ E \left| X^N(T) \right|^2_{H^{-p}} - E \left| X(T) \right|^2_{H^{-p}} = \sum_{m \geq 1} \sum_{k=0}^{N-1} \delta^N(k,m). \]
Note that using Lemmas 3.3, 3.4 and (3.4) we deduce that for any \( k = 0, \ldots, N - 1 \)
\[
E \int_{t_k}^{t_{k+1}} |\gamma^{k,N}_\lambda(t) \frac{\partial u}{\partial x}(t, X^N_\lambda(t))|^2 \, dt < \infty.
\]

From now, we do not justify that the stochastic integral are centered. Itô’s formula and Lemma 3.4, we imply that for \( k = 0, \ldots, N - 1 \)
\[
\delta^N(k, m) = \lambda^p_m E \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} u_{\lambda m} + \beta^{k,N}_\lambda(t) \frac{\partial}{\partial x} u_{\lambda m} + \frac{1}{2} \left| \gamma^{k,N}_\lambda(t) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda m} \right\} (t, X^N_{\lambda m}(t)) \, dt
\]
\[
= \lambda^p_m E \int_{t_k}^{t_{k+1}} \left\{ I^{k,N}_{\lambda m}(t) + \frac{1}{2} J^{k,N}_{\lambda m}(t) \right\} dt,
\]
where
\[
I^{k,N}_{\lambda m}(t) := \left( \beta^{k,N}_\lambda(t) + \lambda_m X^N_{\lambda m}(t) \right) \frac{\partial}{\partial x} u_{\lambda m}(t, X^N_{\lambda m}(t)),
\]
\[
J^{k,N}_{\lambda m}(t) := \left( \left| \gamma^{k,N}_\lambda(t) \right|^2 - 1 \right) \frac{\partial^2}{\partial x^2} u_{\lambda m}(t, X^N_{\lambda m}(t)).
\]

This yields the following decomposition:
\[
E \left| X^N(T) \right|^2_{H^{-p}} - E \left| X(T) \right|^2_{H^{-p}} = \sum_{m \geq 1} \delta^N(N - 1, m) + \sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda^p_m E \int_{t_k}^{t_{k+1}} I^{k,N}_{\lambda m}(t) \, dt
\]
\[
+ \frac{1}{2} \sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda^p_m E \int_{t_k}^{t_{k+1}} J^{k,N}_{\lambda m}(t) \, dt.
\]

Now we study each term of this decomposition.

**Lemma 3.9.** There exists a constant \( C \), independant of \( N \), such that
\[
\sum_{m \geq 1} |\delta^N(N - 1, m)| \leq C h^{p+\frac{1}{2}}.
\]

This study is similar to the third step of [6], page 97.

**Proof.** Using the definition of \( u_{\lambda m}(t, x) \) (3.3) and (3.9), we have
\[
u_{\lambda m}(t_N, X^N_{\lambda m}(t_N)) = \left| X^N_{\lambda m}(t_N) \right|^2 = \frac{1}{(1 + \lambda_m h)^2} \left| X^N_{\lambda m}(t_{N-1}) + \Delta W_m(N) \right|^2,
\]
\[
u_{\lambda m}(t_{N-1}, X^N_{\lambda m}(t_{N-1})) = \frac{1 - e^{-2\lambda_m h}}{2\lambda_m} + e^{-2\lambda_m h} \left| X^N_{\lambda m}(t_{N-1}) \right|^2.
\]

By independence between \( \Delta W_m(N) \) and \( X^N_{\lambda m}(t_{N-1}) \), we have
\[
\delta^N(N - 1, m) = \lambda^p_m \left\{ \frac{1}{(1 + \lambda_m h)^2} - e^{-2\lambda_m h} \right\} E \left| X^N_{\lambda m}(t_{N-1}) \right|^2
\]
\[
+ \frac{h}{\lambda^p_m (1 + \lambda_m h)^2} - \frac{1 - e^{-2\lambda_m h}}{2\lambda^{p+1}_m}.
\]

Let \( \delta_1(\lambda_m) := \frac{1 - e^{-2\lambda_m h}}{2\lambda^{p+1}_m} \), \( \delta_2(\lambda_m) := \frac{h}{\lambda^{p+1}_m (1 + \lambda_m h)^2} \), and
\[
\delta_3(\lambda_m) := \lambda^p_m \left\{ \frac{1}{(1 + \lambda_m h)^2} - e^{-2\lambda_m h} \right\} E \left| X^N_{\lambda m}(t_{N-1}) \right|^2.
\]

With these notations we have
\[ \delta^N(N - 1, m) \leq \delta_1(\lambda_m) + \delta_2(\lambda_m) + \delta_3(\lambda_m). \]

First, we study \( \delta_1(\lambda_m) \). Since \( \frac{1-e^{-2\lambda h}}{2h} = \int_0^h e^{-2\lambda x} \, dx \), using Lemma 3.6, we obtain
\[
\sum_{m \geq 1} \delta_1(\lambda_m) = \int_0^h \sum_{m \geq 1} \lambda_m^{-p} e^{-2\lambda_m x} \, dx \leq C \int_0^h x^{p-\frac{1}{2}} \, dx = Ch^{p+\frac{1}{2}}.
\]

(3.19)

Now we study \( \delta_2(\lambda_m) \). Since \( (x \in [0, \infty) \mapsto x^{-2p}(1 + x^2 h)^2) \) is decreasing, we have for \( p \in [0, \frac{1}{2}) \)
\[
\sum_{m \geq 1} \delta_2(\lambda_m) \leq Ch \int_0^\infty \frac{1}{x^{2p}(1 + x^2 h)^2} \, dx \leq Ch^{p+\frac{1}{2}} \int_0^\infty \frac{y^{-2p}}{(1 + y^2)^2} \, dy \leq Ch^{p+\frac{1}{2}}.
\]

(3.20)

Finally, we study \( \delta_3(\lambda_m) \). Using Lemma 3.2, we have
\[
\delta_3(\lambda_m) \leq \lambda_m^{-p} \left\{ \frac{1}{(1 + \lambda_m h)^2} - e^{-2\lambda_m h} \right\} \frac{1}{2\lambda_m}.
\]

Since \( \frac{1}{(1 + \lambda h)^2} - e^{-2\lambda h} = 2\lambda \int_0^h \left\{ e^{-2\lambda x} - \frac{1}{(1 + \lambda x)^2} \right\} \, dx \), we have
\[
\delta_3(\lambda_m) \leq \lambda_m^{-p} \int_0^h \left\{ e^{-2\lambda_m x} + \frac{1}{(1 + \lambda_m x)^3} \right\} \, dx.
\]

Using Lemma 3.6, we have for \( p \in [0, \frac{1}{2}) \)
\[
\sum_{m \geq 1} \lambda_m^{-p} \int_0^h e^{-2\lambda_m x} \, dx \leq C \int_0^h x^{p-\frac{1}{2}} \, dx \leq Ch^{p+\frac{1}{2}}.
\]

Now since for \( x \geq 0 \) the map \( (y \in \mathbb{R}_+ \mapsto y^{-2p}(1 + y^2 x)^{-3}) \) is decreasing, we have for \( p \in [0, \frac{1}{2}) \)
\[
\sum_{m \geq 1} \frac{\lambda_m^{-p}}{(1 + \lambda_m x)^3} \leq C \int_0^\infty \frac{1}{y^{2p}(1 + y^2 x)} \, dy \leq C x^{p-\frac{1}{2}} \int_0^\infty \frac{1}{z^{2p}(1 + z^2)^3} \, dz \leq C x^{p-\frac{1}{2}},
\]

and hence Fubini’s theorem yields
\[
\sum_{m \geq 1} \int_0^h \frac{\lambda_m^{-p}}{(1 + \lambda_m x)^3} \, dx \leq C \int_0^h x^{p-\frac{1}{2}} \, dx \leq Ch^{p+\frac{1}{2}}.
\]

The above inequalities imply \( \sum_{m \geq 1} \delta_3(\lambda_m) \leq Ch^{p+\frac{1}{2}} \). This inequality, (3.19) and (3.20) give the stated upper estimate.

\( \square \)

**Lemma 3.10.** There exists a constant \( C > 0 \), independent of \( N \), such that
\[
\sum_{m \geq 1} \sum_{k=0}^{N-1} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| \gamma_{\lambda_m}^{k,N}(t) \right| \, dt \leq Ch^{p+\frac{1}{2}}.
\]

**Proof.** Using Lemma 3.4, we have
\[
\left| \gamma_{\lambda_m}^{k,N}(t) \right|^2 = 1 = \frac{2(t - t_k)\lambda_m}{1 + \lambda_m h} + \frac{|t - t_k|^2 \lambda_m^2}{(1 + \lambda_m h)^2}.
\]

Using (3.5) and (3.17), we have
\[
\lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| \gamma_{\lambda_m}^{k,N}(t) \right| \, dt \leq C \left( \lambda_m^{1-p} h^2 + \lambda_m^{2-p} h^3 \right) e^{-2\lambda_m(T-t_{k+1})}.
\]

Lemma 3.8 concludes the proof.

\( \square \)
**Lemma 3.11.** There exists a constant $C > 0$, independant of $N$, such that
\[ \sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| \int_{t_k}^{t_{k+1}} (s, X_{s}^{N}(s)) \right| \, ds \leq C h^{p+\frac{1}{2}}. \]

**Proof.** Let $I_{1,\lambda_m}^{k,N}(t) := E \lambda_m^{N} (t) \frac{\partial}{\partial t} u_{\lambda_m} (t, X_{\lambda_m}(t))$ and $I_{2,\lambda_m}^{k,N}(t) := \lambda_m E \lambda_m^{N} (t) \frac{\partial}{\partial t} u_{\lambda_m} (t, X_{\lambda_m}(t))$.

Using (3.16), we have
\[ E I_{1,\lambda_m}^{k,N}(t) = I_{2,\lambda_m}^{k,N}(t). \tag{3.21} \]

First we study $I_{1,\lambda_m}^{k,N}(t)$. Using (3.4), we know that $\frac{\partial}{\partial x} u_{\lambda_m} \in C^{1,2}$. So using Itô’s formula and Lemma 3.4, we have
\[ d \left[ \beta_{\lambda_m}^{N}(s) \frac{\partial}{\partial x} u_{\lambda_m} (s, X_{\lambda_m}(s)) \right] = \left\{ \beta_{\lambda_m}^{N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \left[ \beta_{\lambda_m}^{N}(s) \right] ^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}(s)) \, ds \]
\[ + \gamma_{\lambda_m}^{N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} (s, X_{\lambda_m}(s)) \, dW_{\lambda_m}(s). \tag{3.22} \]

Using this equation, Lemma 3.4 and the Itô formula we deduce
\[ d \left[ \beta_{\lambda_m}^{N}(s) \frac{\partial}{\partial x} u_{\lambda_m} (s, X_{\lambda_m}(s)) \right] = \left\{ \beta_{\lambda_m}^{N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \left[ \beta_{\lambda_m}^{N}(s) \right] ^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}(s)) \, ds \]
\[ + \left\{ \beta_{\lambda_m}^{N}(s) \gamma_{\lambda_m}^{N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \gamma_{\lambda_m}^{N}(s) \frac{\partial}{\partial x} u_{\lambda_m} \right\} (s, X_{\lambda_m}(s)) \, dW_{\lambda_m}(s). \]

Integrating between $t$ and $t_{k+1}$, taking expectation, and using the fact that $\beta_{\lambda_m}^{N}(t_{k+1}) = -\lambda_m X_{\lambda_m}(t_{k+1})$, so that $I_{1,\lambda_m}^{k,N}(t_{k+1}) = 0$, we obtain
\[ I_{1,\lambda_m}^{k,N}(t) = -E \int_t^{t_{k+1}} \left\{ \beta_{\lambda_m}^{N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \left[ \beta_{\lambda_m}^{N}(s) \right] ^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} (s, X_{\lambda_m}(s)) \, ds. \tag{3.23} \]

Using (3.7) and Lemma 3.5, we have for $s \in [t, t_{k+1}]
\[ E \beta_{\lambda_m}^{N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} (s, X_{\lambda_m}(s)) = 4 \lambda_m e^{-2\lambda_m(T-t_{k+1})} E \beta_{\lambda_m}^{N}(s) X_{\lambda_m}(s) \leq C \lambda_m e^{-2\lambda_m(T-t_{k+1})}, \]
and hence
\[ \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \beta_{\lambda_m}^{N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} (s, X_{\lambda_m}(s)) \leq C \lambda_m^{-p} h^2 e^{-\lambda_m(T-t_{k+1})}. \]

Using Lemma 3.8, and the above inequality, we deduce
\[ \sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \beta_{\lambda_m}^{N}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} (s, X_{\lambda_m}(s)) \leq C h^{p+\frac{1}{2}}. \tag{3.24} \]

Using (3.5) and Lemma 3.5, we have for $s \in [t_k, t_{k+1}]
\[ E \left[ \beta_{\lambda_m}^{N}(s) \right] ^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} (s, X_{\lambda_m}(s)) = 4 \lambda_m e^{-2\lambda_m(T-t_{k+1})} \leq 4 \lambda_m e^{-2\lambda_m(T-t_{k+1})}, \]
Therefore,

\[ \lambda^{-p}_m \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \left| \beta^{k,N}_{\lambda_m}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X^N_{\lambda_m}(s)) \leq C \lambda^{-1-p}_m h^2 e^{-2\lambda(T-t_{k+1})}. \]

Thus, Lemma 3.8 yields

\[ \sum_{m=1}^{N-2} \lambda^{-p}_m \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \left| \beta^{k,N}_{\lambda_m}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X^N_{\lambda_m}(s)) \leq Ch^{p+\frac{1}{2}}. \tag{3.25} \]

Using equations (3.5) and Lemma 3.4 we have for all \( s \in [t, t_{k+1}] \)

\[ E \left| \beta^{k,N}_{\lambda_m}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X^N_{\lambda_m}(s)) \leq C \lambda^{-1-p}_m h^2 e^{-2\lambda(T-t_{k+1})}. \]

Therefore, we obtain

\[ \lambda^{-p}_m \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \left| \beta^{k,N}_{\lambda_m}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X^N_{\lambda_m}(s)) \right| \leq C \lambda^{-1-p}_m h^2 e^{-2\lambda(T-t_{k+1})}. \]

Using once more Lemma 3.8, we deduce

\[ \sum_{m=1}^{N-2} \lambda^{-p}_m \int_{t_k}^{t_{k+1}} dt \int_t^{t_{k+1}} ds E \left| \beta^{k,N}_{\lambda_m}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X^N_{\lambda_m}(s)) \right| \leq Ch^{p+\frac{1}{2}}. \]

Plugging this inequality together with (3.24) and (3.25) into (3.23) gives us

\[ \sum_{m=1}^{N-2} \sum_{k=0}^{t_{k+1}} \lambda^{-p}_m E \int_{t_k}^{t_{k+1}} \left| I^{k,N}_{2,\lambda_m}(t) \right| dt \leq Ch^{p+\frac{1}{2}}. \tag{3.26} \]

Now we study \( I^{k,N}_{2,\lambda_m}(t) \). Using Lemma 3.4, equation (3.22) and the Itô formula we have

\[ dX^N_{\lambda_m}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X^N_{\lambda_m}(s)) = \left\{ X^N_{\lambda_m}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + X^N_{\lambda_m}(s) \beta^{k,N}_{\lambda_m}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right. \]

\[ + \left. \beta^{k,N}_{\lambda_m}(s) \frac{\partial}{\partial x} u_{\lambda_m} + \left| \gamma^{k,N}_{\lambda_m}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} \left. (s, X^N_{\lambda_m}(s)) \right| ds \]

\[ + \left\{ \gamma^{k,N}_{\lambda_m}(s) \frac{\partial}{\partial x} u_{\lambda_m} + X^N_{\lambda_m}(s) \gamma^{k,N}_{\lambda_m}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} \left. (s, X^N_{\lambda_m}(s)) \right| dW_{\lambda_m}(s) \]

So integrating between \( t \) and \( t_{k+1} \) and taking expectation, we obtain

\[ I^{k,N}_{2,\lambda_m}(t) = -\lambda_m E \int_t^{t_{k+1}} \left\{ X^N_{\lambda_m}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} + \beta^{k,N}_{\lambda_m}(s) \frac{\partial}{\partial x} u_{\lambda_m} + X^N_{\lambda_m}(s) \beta^{k,N}_{\lambda_m}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right. \]

\[ + \left. \left| \gamma^{k,N}_{\lambda_m}(s) \right|^2 \frac{\partial^2}{\partial x^2} u_{\lambda_m} \right\} \left. (s, X^N_{\lambda_m}(s)) \right| ds. \tag{3.27} \]

Using equation (3.7) and Lemma 3.5, we have for all \( s \in [t, t_{k+1}] \)

\[ \lambda_m E X^N_{\lambda_m}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} \left( s, X^N_{\lambda_m}(s) \right) = 4\lambda^2_m e^{-2\lambda(T-t)} E \left| X^N_{\lambda_m}(s) \right|^2 \]

\[ \leq C \lambda^2_m \left( \frac{1}{\lambda_m} + h \right) e^{-2\lambda(T-t_{k+1})}. \]

Therefore,

\[ \lambda^{-p}_m \int_{t_k}^{t_{k+1}} \int_t^{t_{k+1}} \lambda_m E X^N_{\lambda_m}(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m} \left( s, X^N_{\lambda_m}(s) \right) \leq C \left( \lambda^{-1-p}_m h^2 + \lambda^{-2-p}_m h^3 \right) e^{-2\lambda(T-t_{k+1})}, \]
and using Lemma 3.8, we deduce
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_m E X_m^N(s) \frac{\partial^2}{\partial t \partial x} u_{\lambda_m}(s, X_m^N(s)) \leq C h^{p+\frac{1}{2}}. \tag{3.28}
\]
The equation (3.4) and Lemma 3.5 yield for all \( s \in [t, t_{k+1}] \)
\[
\lambda_m E \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_m^N(s)) = 2 \lambda_m e^{-2 \lambda_m(T-s)} E \beta_{\lambda_m}^{k,N}(s) X_m^N(s) \leq C \lambda_m e^{-2 \lambda_m(T-t_{k+1})},
\]
This upper estimate implies
\[
\lambda_m^{-1} \int_{t_k}^{t_{k+1}} ds \lambda_m E \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_m^N(s)) \leq C \lambda_{m}^{-p} h^2 e^{-2 \lambda_m(T-t_{k+1})},
\]
and Lemma 3.8 yields
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_m E \beta_{\lambda_m}^{k,N}(s) \frac{\partial}{\partial x} u_{\lambda_m}(s, X_m^N(s)) \leq C h^{p+\frac{1}{2}}. \tag{3.29}
\]
Using equation (3.5) and Lemma 3.5, we have for all \( s \in [t, t_{k+1}] \)
\[
\lambda_m E X_m^N(s) \beta_{\lambda_m}^{k,N}(s) \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_m^N(s)) \leq C \lambda_m e^{-2 \lambda_m(T-t_{k+1})},
\]
Therefore, we obtain
\[
\lambda_m E \left| \gamma_{\lambda_m}^{k,N}(s) \right| \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_m^N(s)) \leq C \lambda_m e^{-2 \lambda_m(T-t_{k+1})},
\]
This yields
\[
\lambda_m^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_m E \left| \gamma_{\lambda_m}^{k,N}(s) \right| \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_m^N(s)) \leq C \lambda_{m}^{-p} h^2 e^{-2 \lambda_m(T-t_{k+1})},
\]
and Lemma 3.8 implies
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} ds \lambda_m E \left| \gamma_{\lambda_m}^{k,N}(s) \right| \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_m^N(s)) \leq C h^{p+\frac{1}{2}}. \tag{3.30}
\]
Finally, (3.5) and Lemma 3.4 imply that for all \( s \in [t, t_{k+1}] \)
\[
\lambda_m E \left| \gamma_{\lambda_m}^{k,N}(s) \right| \frac{\partial^2}{\partial x^2} u_{\lambda_m}(s, X_m^N(s)) \leq C \lambda_m e^{-2 \lambda_m(T-t_{k+1})},
\]
This equation together with (3.28) - (3.30) into (3.27), we deduce
\[
\sum_{m \geq 1} \sum_{k=0}^{N-2} \lambda_m^{-p} E \int_{t_k}^{t_{k+1}} \left| \gamma_{\lambda_m}^{k,N}(t) \right| dt \leq C h^{p+\frac{1}{2}}.
\]
This equation together with (3.21) and (3.26) conclude the proof. \( \square \)

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