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► **To cite this version:**

| Omar Aboura. Weak error expansion of the implicit Euler scheme. 2013. hal-00818036

**HAL Id: hal-00818036**

**<https://hal.science/hal-00818036>**

Preprint submitted on 25 Apr 2013

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# WEAK ERROR EXPANSION OF THE IMPLICIT EULER SCHEME

OMAR ABOURA

ABSTRACT. In this paper, we extend the Talay Tubaro theorem to the implicit Euler scheme.

## 1. INTRODUCTION

Let  $(\Omega, \mathcal{F}, P)$  a probability space and  $T > 0$  a fixed time.  $W$  will be a Brownian motion in  $\mathbf{R}$  with respect to his own filtration  $\mathcal{F}_t$ . We will consider the following stochastic differential equation

$$X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s, \quad (1.1)$$

where  $x \in \mathbf{R}$ ,  $b$  and  $\sigma$  are real functions defined on  $\mathbf{R}$ . It is well know that, under Lipschitz conditions on  $b$  and  $\sigma$ , this equation admits a unique strong solution.

For various reasons, including mathematical finance or partial differential equations, the approximation of  $Ef(X_T)$  is of importance. One way to do this is to use an Euler scheme and to study the speed of convergence. There is a vast literature on this subject and one of the pioneering work is the paper of D. Talay and L. Tubaro [7].

Let  $N \in \mathbf{N}^*$  and  $h := T/N$ . Consider  $(t_k)_{0 \leq k \leq N}$  the uniform subdivision of  $[0, T]$  defined by  $t_k := kh$ . In their paper [7] the authors deal with the explicit Euler scheme  $(\bar{X}_{t_k})_{0 \leq k \leq N}$  defined as:  $\bar{X}_{t_0} = x$  and for  $k = 0, \dots, N-1$ ,

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(\bar{X}_{t_k})h + \sigma(\bar{X}_{t_k})\Delta W_{k+1}, \quad (1.2)$$

where  $\Delta W_{k+1} := W_{t_{k+1}} - W_{t_k}$ . They study the weak error  $Ef(\bar{X}_T) - Ef(X_T)$ .

Here, we will use the implicit Euler scheme defined as follow:  $X_{t_0}^N = x$  and for  $k = 0, \dots, N-1$ ,

$$X_{t_{k+1}}^N = X_{t_k}^N + b(X_{t_{k+1}}^N)h + \sigma(X_{t_k}^N)\Delta W_{k+1}. \quad (1.3)$$

Despite the fact that this implicit scheme cannot be implemented in most cases, it has been studied in [5] but, to the best of our knowledge, its weak error expansion has not been given. The main reason of this study is that we believe it would be a step in order to study a weak convergence error for SPDEs. So far in that framework only few cases have been studied in [2]-[4] for the stochastic heat or Schrödinger equation.

**Notations.** Let  $n \in \mathbf{N}$  and  $v, w : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  be smooth functions. We will denote by  $\partial^n v(t, x)$  the  $n^{th}$  derivative of  $v$  with respect to the space variable  $x$ , except for the second derivative denoted  $\Delta v(t, x)$  as usual. Moreover, by an abuse of notation, for a function  $v : \mathbf{R} \rightarrow \mathbf{R}$  and  $w : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$ , we will write  $(vw)(t, x) := v(x)w(t, x)$ .

Given  $p \in \mathbf{N}$ ,  $C_p$  will denote a constant that depends on  $p$ ,  $T$  and the coefficients  $b$  and  $\sigma$ , but does not depend on  $N$ . As usual,  $C_p$  may change from line to line.

For  $h$  small enough, we denote by  $S_h$  the functions defined on  $\mathbf{R}$  by

$$S_h(x) := 1/(1 - hb'(x)). \quad (1.4)$$

It is similar to the map used by Debussche in [3].

## 2. THE MAIN RESULT

Let  $u$  the (classical) solution of the following pde, called the Kolmogorov equation:

$$\begin{cases} \frac{\partial}{\partial t}u(t, x) + b(x)\partial u(t, x) + \frac{1}{2}\sigma^2(x)\Delta u(t, x) = 0, \\ u(T, x) = f(x). \end{cases} \quad (2.1)$$

The properties of  $u$  will be given in the next section. Let us mention that for  $b$  and  $\sigma$  smooth enough,  $u$  is smooth too. We define the function  $\psi_i : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$ , where  $i$  stands for implicit, as follows for a smooth enough function  $u$ :

$$\psi_i := \frac{1}{2}b\partial(b\partial u) + \frac{1}{4}\sigma^2\Delta(b\partial u) - \frac{1}{2}b^2\Delta u + \frac{1}{8}\sigma^4\partial^4 u - \frac{1}{4}b\partial(\sigma^2\Delta u) - \frac{1}{8}\sigma^2\Delta(\sigma^2\Delta u). \quad (2.2)$$

We are now in position to state the main result of this paper.

**Theorem 2.1.** *Let  $b, \sigma, f$  be  $C^\infty$ -functions with bounded derivatives.*

(i) *The implicit Euler scheme (1.3) is of weak order 1, that is, there exists a constant  $C$ , such that for  $h$  small enough  $|Ef(X_T^N) - Ef(X_T)| \leq Ch$ .*

(ii) *The weak error can be expanded as*

$$Ef(X_T^N) - Ef(X_T) = hE \int_0^T \psi_i(t, X_t)dt + O(h^2).$$

We have not given the minimal hypothesis; indeed we want to focus on the ideas and not on the best set of assumptions. The proof of this theorem is quite long; it uses intensively the Kolmogorov equation (2.1), the Itô and Clark-Ocone formulas. It will be proved in the next section. We at first compare our result with that of Talay Tubaro. In their paper [7], the authors introduce the following function

$$\psi_e = \frac{1}{2}b^2\Delta u + \frac{1}{2}b\sigma^2\partial^3 u + \frac{1}{8}\sigma^4\partial^4 u + \frac{1}{2}\frac{\partial^2}{\partial t^2}u + b\frac{\partial}{\partial t}\partial u + \frac{1}{2}\sigma^2\frac{\partial}{\partial t}\Delta u,$$

and prove the following result (see [7] page 489).

**Theorem 2.2.** *Let  $(\bar{X}_{t_k})_{k=0, \dots, N}$  denote the explicit Euler scheme defined by (1.2). Then weak error has the following expansion*

$$Ef(\bar{X}_T) - Ef(X_T) = hE \int_0^T \psi_e(t, X_t)dt + O(h^2).$$

Applying  $\frac{\partial}{\partial t}$ ,  $b\partial$  and finally  $\frac{1}{2}\sigma^2\Delta$  to (2.1) and summing these equations we have

$$\frac{\partial^2}{\partial t^2}u + 2b\partial\frac{\partial}{\partial t}u + \sigma^2\Delta\frac{\partial}{\partial t}u = -b\partial(b\partial u) - \frac{1}{2}b\partial(\sigma^2\Delta u) - \frac{1}{2}\sigma^2\Delta(b\partial u) - \frac{1}{4}\sigma^2\Delta(\sigma^2\Delta u)$$

So we can rewrite the function  $\psi_e$  as

$$\psi_e = \frac{1}{2}b^2\Delta u + \frac{1}{2}b\sigma^2\partial^3 u + \frac{1}{8}\sigma^4\partial^4 u - \frac{1}{2}b\partial(b\partial u) - \frac{1}{4}b\partial(\sigma^2\Delta u) - \frac{1}{4}\sigma^2\Delta(b\partial u) - \frac{1}{8}\sigma^2\Delta(\sigma^2\Delta u)$$

For  $b = 0$ , we have  $\psi_e = \psi_i = \frac{1}{8}\sigma^4\partial^4 u - \frac{1}{8}\sigma^2\Delta(\sigma^2\Delta u)$  as expected since in this case the explicit and the implicit Euler scheme coincide. We can notice that  $\psi_i = \psi_e - b^2\Delta u + \frac{1}{2}\sigma^2\Delta(b\partial u) + b\partial(b\partial u) - \frac{1}{2}b\sigma^2\partial^3 u$ .

## 3. PROOF THEOREM 2.1

Here is a sketch of the proof: After proving some property of the scheme, we introduce a continuous interpolation of this scheme. Finally, after decomposing the weak error, we study a remainder term.

### 3.1. Some tools.

**Proposition 3.1** (Property of  $u$ ). *Let  $(X_s^{t,x})_{s \in [t,T]}$  denote the stochastic flow, that is the solution of (1.1) starting from  $x$  at time  $t$  and let  $u(t,x) = Ef(X_T^{t,x})$ . Then  $u$  belongs to  $C^{\infty,\infty}([0,T] \times \mathbf{R})$  and satisfies the Kolmogorov equation (2.1). Moreover, for any  $n, p \in \mathbf{N}$ , there exists constants  $C$  and  $k$  such that*

$$\left| \frac{\partial^n}{\partial t^n} \partial^p u(t,x) \right| \leq C \left( 1 + |x|^k \right).$$

See for example [7] page 486 Lemma 2.

Now we recall several results from Malliavin Calculus that will be used in the sequel. For a detailed introduction, we send the reader to D.Nualart's book [6].

**Proposition 3.2** (Clark-Ocone formula). *Let  $t \in [0, T]$  and  $F \in L^2(\mathcal{F}_t) \cap D^{1,2}$ ; then we have for all  $s \in [0, t]$*

$$F = E(F|\mathcal{F}_s) + \int_s^t E(D_r F|\mathcal{F}_r) dW_r.$$

**Lemma 3.3.** *Let  $F, G \in D^{1,2}$ .*

(i) *If  $F$  and  $DF$  are bounded, then  $FG \in D^{1,2}$  and  $D(FG) = FDG + GDF$ .*

(ii) *Let  $f \in C^1$  with a bounded derivative; then  $f(F) \in D^{1,2}$  and  $Df(F) = f'(F)DF$ .*

(iii) *Let  $(s, t) \in [0, T]^2$  such that  $s < t$  and let  $F \in D^{1,2} \cap L^2(\mathcal{F}_s)$ . Then  $F(W_t - W_s) \in D^{1,2}$  and*

$$D_r [F(W_t - W_s)] = D_r F(W_t - W_s) + F1_{\{s \leq r \leq t\}}.$$

(iv) *Let  $\{H_n, n \geq 1\}$  be a sequence of random variables in  $D^{1,2}$  that converges to  $H$  in  $L^2(\Omega)$  and such that  $\sup_n E(\|DH_n\|_{L^2(0,T)}^2) < \infty$ . Then  $H$  belongs to  $D^{1,2}$ .*

For a proof of (iii), see [6] Lemma 1.3.4. Now we state some technical lemmas that will be useful in the sequel. The following discrete Gronwall lemma is classical.

**Lemma 3.4** (Gronwall's lemma). *For any nonnegative sequences  $(a_k)_{0 \leq k \leq N}$  and  $(b_k)_{0 \leq k \leq N}$  satisfying  $a_{k+1} \leq (1+Ch)a_k + b_{k+1}$ , with  $C > 0$ . Then we have  $a_k \leq e^{C(T-t_k)} \left( a_0 + \sum_{i=1}^k b_i \right)$ .*

**Lemma 3.5.** *Let  $L > 0$ ; then for  $h^*$  small enough (more precisely  $Lh^* < 1$ ) there exists  $\Gamma := \frac{L}{1-Lh^*} > 0$  such that for all  $h \in (0, h^*)$  we have  $\frac{1}{1-Lh} < 1 + \Gamma h$ .*

*Proof.* Let  $h \in (0, h^*)$ ; then we have  $1 - Lh > 1 - Lh^* > 0$ . Hence  $\frac{L}{1-Lh} < \frac{L}{1-Lh^*} = \Gamma$ , so that  $Lh < \Gamma h(1 - Lh)$ , which yields  $1 + \Gamma h - Lh - \Gamma Lh^2 = (1 + \Gamma h)(1 - Lh) > 1$ . This concludes the proof.  $\square$

**Lemma 3.6** (Generalization of Young's lemma). *For an integer  $p \geq 1$  and for  $\epsilon > 0$ , we have*

$$(a+b)^{2p} \leq (1+\epsilon)^{2p-1} a^{2p} + \left(1 + \frac{1}{\epsilon}\right)^{2p-1} b^{2p}.$$

*Proof.* We use an induction argument. The inequality is true for  $p = 1$ , that is  $(a+b)^2 \leq (1+\epsilon)a^2 + (1+\frac{1}{\epsilon})b^2$ . Now, suppose that it is true until  $p$  and will prove it for  $p+1$ ; indeed the induction hypothesis yields

$$\begin{aligned} (a+b)^{2p+1} &\leq \left| (1+\epsilon)^{2p-1} a^{2p} + \left(1 + \frac{1}{\epsilon}\right)^{2p-1} b^{2p} \right|^2 \\ &\leq (1+\epsilon) \left| (1+\epsilon)^{2p-1} \right|^2 |a^{2p}|^2 + \left(1 + \frac{1}{\epsilon}\right) \left| \left(1 + \frac{1}{\epsilon}\right)^{2p-1} \right|^2 |b^{2p}|^2. \end{aligned}$$

This concludes the proof.  $\square$

### 3.2. Property of the implicit Euler scheme.

**Lemma 3.7** (Existence of the scheme). *For small  $h$ , the implicit Euler scheme (1.3) is well defined. Moreover, for all  $k = 0, \dots, N$ , we have  $X_{t_k}^N \in L^2(\mathcal{F}_{t_k})$ .*

We will denote by  $N_0$  the smallest integer such that the scheme is well defined.

*Proof.* For  $k = 0$ , we have  $X_{t_0}^N = x \in L^2(\mathcal{F}_{t_0})$ . Suppose that for all  $j = 0, \dots, k$ ,  $X_{t_j}^N$  is well defined and belongs to  $L^2(\mathcal{F}_{t_j})$ ; we prove this for  $j = k + 1$ . We define  $\xi_{k+1} := X_{t_k}^N + \sigma(X_{t_k}^N) \Delta W_{k+1}$ . By independence of  $\Delta W_{k+1}$  and  $\mathcal{F}_{t_k}$  and the linear growth of  $\sigma$ , we have that  $\xi_{k+1} \in L^2(\Omega)$ . Let  $F_{k+1} : L^2(\Omega) \rightarrow L^2(\Omega)$  be defined by

$$F_{k+1}(X) := \xi_{k+1} + b(X)h, \quad (3.1)$$

for all  $X \in L^2(\Omega)$ . Using the Lipschitz property of  $b$  we have  $E|F_{k+1}(X) - F_{k+1}(Y)|^2 \leq \|b'\|_\infty h^2 E|X - Y|^2$ . So by the fixed point theorem, if  $\|b'\|_\infty h < 1$  there exist an unique element of  $L^2(\Omega)$ , noted  $X_{t_{k+1}}^N$ , such that  $X_{t_{k+1}}^N = F_{k+1}(X_{t_{k+1}}^N)$ . The measurability of  $X_{t_{k+1}}^N$  with respect to  $\mathcal{F}_{t_{k+1}}$  is obvious.  $\square$

**Lemma 3.8** (Malliavin derivability). *Let  $h > 0$  small enough; then for all  $k = 0, \dots, N$ , we have  $X_{t_k}^N \in D^{1,2}$ . Moreover, for all  $t \in [t_k, t_{k+1})$ , we have  $D_t X_{t_{k+1}}^N = S_h(X_{t_{k+1}}^N) \sigma(X_{t_k}^N)$ , where  $S_h$  is defined by (1.4).*

*Proof.* It is true for  $k = 0$ , since  $X_0^N = x$ . Now suppose that for all  $j = 1, \dots, k$ ,  $X_{t_j}^N \in D^{1,2}$  and prove that  $X_{t_{k+1}}^N \in D^{1,2}$ . First, we define the following sequence in  $L^2(\Omega)$ :  $X_{k+1}^N(0) = 0$  and for  $i \geq 0$ ,  $X_{k+1}^N(i+1) = F_{k+1}(X_{k+1}^N(i))$  where  $F_{k+1}$  is defined by (3.1). Using the Lipschitz property of  $F_{k+1}$ , since  $X_{t_{k+1}}^N$  is a fixed point of  $F_{k+1}$ , we have

$$E \left| X_{t_{k+1}}^N - X_{k+1}^N(i+1) \right|^2 \leq \|b'\|_\infty h^2 E \left| X_{t_{k+1}}^N - X_{k+1}^N(i) \right|^2 \leq \left( \|b'\|_\infty h \right)^{i+1} E \left| X_{t_{k+1}}^N \right|^2.$$

So  $X_{k+1}^N(i)$  converge to  $X_{t_{k+1}}^N$  in  $L^2(\Omega)$  if  $\|b'\|_\infty h < 1$ . Using the induction hypothesis, the assumptions on  $\sigma$  and Lemma 3.3 (ii) and (iii), we deduce that  $\xi_{k+1} = X_{t_k}^N + \sigma(X_{t_k}^N) \Delta W_{k+1}$  belongs to  $D^{1,2}$ . Finally, since  $b$  is Lipschitz, we deduce by induction that for all  $i \geq 0$   $X_k^N(i) \in D^{1,2}$ . Moreover we have  $DX_k^N(i+1) = D\xi_{k+1} + hb'(X_k^N(i))DX_k^N(i)$  and  $DX_{k+1}^N(0) = 0$ , so that

$$\|DX_{t_k}^N(i+1)\|_{L^2(0,T)}^2 \leq 2\|D\xi_{k+1}\|_{L^2(0,T)}^2 + 2h^2\|b'\|_\infty^2 \|DX_{t_k}^N(i)\|_{L^2(0,T)}^2.$$

An induction argument yields for  $i \geq 1$  and  $2h^2\|b'\|_\infty^2 < 1$ ,

$$\|DX_{t_k}^N(i)\|_{L^2(0,T)}^2 \leq 2 \frac{\|D\xi_{k+1}\|_{L^2(0,T)}^2}{1 - 2h^2\|b'\|_\infty^2}.$$

Finally, we have  $\sup_i \|DX_k^N(i)\| < \infty$ . Lemma 3.3 (iv) proves that  $X_{t_{k+1}}^N \in D^{1,2}$ .

Finally, let  $t \in [t_k, t_{k+1})$ ; applying the Malliavin derivative  $D$  to (1.3) and using Lemma 3.3 we have

$$D_t X_{t_{k+1}}^N = hb'(X_{t_{k+1}}^N) D_t X_{t_{k+1}}^N + \sigma(X_{t_k}^N);$$

which concludes the proof.  $\square$

The following result gives a bound of  $p^{\text{th}}$  moments of the implicit scheme.

**Lemma 3.9.** *Fix  $p \geq 1$ ; then for  $N_0$  large enough, there exists a constant  $C(p) > 0$  such that*

$$\sup_{N \geq N_0} \max_{k=0, \dots, N} E |X_{t_k}^N|^p \leq C(p). \quad (3.2)$$

*Proof.* Holder's inequality shows that it suffices to consider moments which are power of 2, that is to check  $\sup_{N \geq N_0} \max_{k=0, \dots, N} E |X_{t_k}^N|^{2^p} \leq C_p$ , for every integer  $p \geq 1$ . Using the generalized Young Lemma 3.6 the independence between  $\Delta W_{k+1}$  and  $\mathcal{F}_{t_k}$ , and the fact that for all  $j \in \mathbf{N}$ ,  $E(\Delta W_{k+1})^{2j+1} = 0$ , we have for  $h \in (0, h^*)$  and some constant  $C_p$  depending on  $h^*$

$$\begin{aligned} E |X_{t_{k+1}}^N|^{2^p} &\leq (1+h)^{2^p-1} E |X_{t_k}^N + \sigma(X_{t_k}^N) \Delta W_{k+1}|^{2^p} \\ &\quad + \left(1 + \frac{1}{h}\right)^{2^p-1} h^{2^p} E \left| b(X_{t_{k+1}}^N) \right|^{2^p} \\ &\leq (1+C_p h) E |X_{t_k}^N|^{2^p} + (1+C_p h) \sum_{j=1, \dots, 2^p-1} \binom{2^p}{2j} E \left( |X_{t_k}^N|^{2^p-2j} |\sigma(X_{t_k}^N)|^{2j} \right) E |\Delta W_{k+1}|^{2j} \\ &\quad + C_p h \left( 1 + E |X_{t_{k+1}}^N|^{2^p} \right) \end{aligned}$$

Using the identity  $E |\Delta W_{k+1}|^{2j} = C(2j)h^j$  and the linear growth of  $\sigma$  we deduce for  $h < 1$

$$\begin{aligned} E |X_{t_{k+1}}^N|^{2^p} &\leq (1+C_p h) E |X_{t_k}^N|^{2^p} + C_p h + C_p h E |X_{t_{k+1}}^N|^{2^p} \\ &\quad + (1+C_p h) C_p h \sum_{j=1, \dots, 2^p-1} E \left( |X_{t_k}^N|^{2^p-2j} \left( 1 + |X_{t_k}^N|^{2j} \right) \right). \end{aligned}$$

Using the inequality:  $a^{2^{p+1}-2j} \leq a^{2^{p+1}} + 1$  valid for any  $a > 0$ , we get for some constant  $C_p > 0$  and  $h < 1$

$$E |X_{t_{k+1}}^N|^{2^p} \leq (1+C_p h) E |X_{t_k}^N|^{2^p} + C_p h + C_p h E |X_{t_{k+1}}^N|^{2^p}$$

Provided that  $h$  is small enough, the Gronwall Lemma 3.4 and Lemma 3.5 conclude the proof.  $\square$

**3.3. Some martingales and related process:**  $\beta_t^{k,N}, z_t^{k,N}, \gamma_t^{k,N}$  and  $\eta_t^{k,N}$ . Let  $k \in \{0, \dots, N-1\}$  be fixed; in the sequel, we will use the following processes defined for  $t \in [t_k, t_{k+1}]$

$$\begin{aligned} \beta_t^{k,N} &:= E \left( b(X_{t_{k+1}}^N) \middle| \mathcal{F}_t \right), \quad z_t^{k,N} := E \left( D_t b(X_{t_{k+1}}^N) \middle| \mathcal{F}_t \right), \\ \gamma_t^{k,N} &:= \sigma(X_{t_k}^N) + (t-t_k) z_t^{k,N}, \quad \eta_t^{k,N} := \sigma(X_{t_k}^N) E \left( D_t (S_h b') (X_{t_{k+1}}^N) \middle| \mathcal{F}_t \right). \end{aligned} \quad (3.3)$$

The following lemma describes the time evolution of these processes.

**Lemma 3.10.** *For all  $k = 0, \dots, N-1$ , and for  $t \in [t_k, t_{k+1}]$ , we have the following relation*

$$\begin{aligned} d\beta_t^{k,N} &= z_t^{k,N} dW_t, \quad dz_t^{k,N} = \eta_t^{k,N} dW_t, \quad d\gamma_t^{k,N} = z_t^{k,N} dt + \eta_t^{k,N} (t-t_k) dW_t, \\ d\eta_t^{k,N} &= |\sigma(X_{t_k}^N)|^2 E \left( D_t (S_h^3 b'') (X_{t_{k+1}}^N) \middle| \mathcal{F}_t \right) dW_t. \end{aligned}$$

*Proof.* Let  $k = 0, \dots, N-1$  and let  $t \in [t_k, t_{k+1}]$ . Lemmas 3.3 (ii), 3.7 and 3.8 and the bounds of  $\|b''\|_\infty$  imply that  $b(X_{t_{k+1}}^N) \in L^2(\mathcal{F}_{t_{k+1}}) \cap D^{1,2}$  and

$$D_t b(X_{t_{k+1}}^N) = b'(X_{t_{k+1}}^N) D_t X_{t_{k+1}}^N = (S_h b')(X_{t_{k+1}}^N) \sigma(X_{t_k}^N). \quad (3.4)$$

So the Clark-Ocone formula in Proposition 3.2 yields  $\beta_t^{k,N} = b(X_{t_{k+1}}^N) - \int_t^{t_{k+1}} z_s^{k,N} dW_s$  and hence  $d\beta_t^{k,N} = z_t^{k,N} dW_t$ ; where  $z_s^{k,N} = E(D_s b(X_{t_{k+1}}^N) | \mathcal{F}_s)$ ; using (3.4)

$$z_s^{k,N} = \sigma(X_{t_k}^N) E((S_h b')(X_{t_{k+1}}^N) | \mathcal{F}_s). \quad (3.5)$$

So taking conditionnal expectation with respect to  $\mathcal{F}_t$ , we have (3.5). Since  $b''$  is bounded and  $b'$  Lipschitz we have that for  $h$  small enough,  $S_h b' = \frac{b'}{1-hb'} \in C_b^1$ . So we can conclude that  $(S_h b')(X_{t_{k+1}}^N) \in D^{1,2}$  and using the Clark-Ocone formula we deduce that for  $s \in [t_k, t_{k+1})$ ,

$$(S_h b')(X_{t_{k+1}}^N) = E(S_h b'(X_{t_{k+1}}^N) | \mathcal{F}_s) + \int_s^{t_{k+1}} E(D_u [S_h b'(X_{t_{k+1}}^N)] | \mathcal{F}_u) dW_u,$$

and hence

$$dz_s^{k,N} = \sigma(X_{t_k}^N) E\left(D_s \left[\frac{b'}{1-hb'}(X_{t_{k+1}}^N)\right] | \mathcal{F}_s\right) dW_s = \eta_s^{k,N} dW_s.$$

The differential of  $\gamma_t^{k,N}$  is a consequence of the previous result and Itô's formula. Finally, since  $S_h b' \in C_b^1$  and  $(S_h b')' = S_h^2 b''$ , Lemma 3.8 implies that  $D_t(S_h b')(X_{t_{k+1}}^N) = \sigma(X_{t_k}^N) (S_h^3 b'')(X_{t_{k+1}}^N)$  and then

$$\eta_t^{k,N} = |\sigma(X_{t_k}^N)|^2 E((S_h^3 b'')(X_{t_{k+1}}^N) | \mathcal{F}_t). \quad (3.6)$$

Applying once more the Clark-Ocone formula in Proposition 3.2, we deduce

$$E(S_h^3 b''(X_{t_{k+1}}^N) | \mathcal{F}_t) = (S_h^3 b'')(X_{t_{k+1}}^N) - \int_t^{t_{k+1}} E(D_s(S_h^3 b'')(X_{t_{k+1}}^N) | \mathcal{F}_s) dW_s.$$

Multiplying this by  $|\sigma(X_{t_k}^N)|^2$  and using (3.6), we conclude the proof.  $\square$

The next lemma provides uniform moment estimates of the above processes.

**Lemma 3.11.** *Let  $p \in \mathbf{N}$ ; then for  $N_0$  large enough there exists a constant  $C_p$  such that for  $N \geq N_0$ ,*

$$\max_{k=0, \dots, N-1} \sup_{t_k \leq t \leq t_{k+1}} E \left\{ |\beta_t^{k,N}|^p + |z_t^{k,N}|^p + |\gamma_t^{k,N}|^p + |\eta_t^{k,N}|^p \right\} \leq C_p.$$

*Proof.* Using Jensen's inequality, the Lipschitz property of  $b$  and Lemma 3.9 we have

$$E|\beta_t^{k,N}|^p \leq E|b(X_{t_{k+1}}^N)|^p \leq C_p \left(1 + E|X_{t_{k+1}}^N|^p\right) \leq C_p.$$

The identity (3.5), Jensen's inequality, the growth property of  $\sigma$  and the upper estimate  $b'/(1-hb') \geq C$  for small  $h$ , Schwarz's inequality and Lemma 3.9 yield

$$E|z_t^{k,N}|^p \leq E|\sigma(X_{t_k}^N) (S_h b')(X_{t_{k+1}}^N)|^p \leq C_p.$$

Using the definition of  $\gamma_t^{k,N}$  in (3.3) and the previous upper estimates we deduce

$$E|\gamma_t^{k,N}|^p \leq C_p E|\sigma(X_{t_k}^N)|^p + C_p h^p E|z_t^{k,N}|^p \leq C_p.$$

Finally (3.6), the Jensen inequality, the growth condition on  $\sigma$ , the upper bounds of  $b'$  and  $b''$ , Lemma 3.9 and Schwarz's inequality yield

$$E \left| \eta_t^{k,N} \right|^p \leq E \left[ \left| \sigma(X_{t_k}^N) \right|^{2p} \left| (S_h^3 b'')(X_{t_{k+1}}^N) \right|^p \right] \leq C_p$$

This concludes the proof.  $\square$

**3.4. Continuous interpolation.** As usual we need to introduce a continuous process that interpolates the implicit Euler scheme (1.3). With an abuse of notation, let  $(X_t^N)_{t \in [0, T]}$  be the process defined as follow:  $X_0^N = x_0$  and for  $k = 0, \dots, N-1$  and  $t_k \leq t \leq t_{k+1}$

$$X_t^N := X_{t_k}^N + E \left( b \left( X_{t_{k+1}}^N \right) \middle| \mathcal{F}_t \right) (t - t_k) + \sigma(X_{t_k}^N) (W_t - W_{t_k}). \quad (3.7)$$

This process satisfies the following

**Lemma 3.12.** *The process  $(X_t^N)_{t \in [0, T]}$  is continuous  $\mathcal{F}_t$ -adapted and is an interpolation of the scheme (1.3). Moreover, for  $k \in \{0, \dots, N-1\}$  and  $t \in [t_k, t_{k+1}]$  we have  $dX_t^N = \beta_t^{k,N} dt + \gamma_t^{k,N} dW_t$ , where the process  $(\beta_t^{k,N})$  and  $(\gamma_t^{k,N})$  are defined by (3.3).*

*Remark 3.13.* (1) If  $b = 0$ , (3.7) corresponds to the classical interpolation given by Talay Tubaro [7], since the explicit and implicit Euler scheme are the same.

(2) If  $b$  is linear, this continuous process differs from that used by Debussche in [3]. Indeed, the finite dimensional analog of the interpolation corresponding to the process  $dX_t = -\beta X_t dt + \sigma(X_t) dW_t$ , is defined by

$$X_t^D = X_{t_k}^N + \int_{t_k}^t \frac{-\beta X_{t_k}^N}{1 + h\beta} ds + \int_{t_k}^t \frac{\sigma(X_{t_k}^N)}{1 + h\beta} dW_s;$$

for  $t \in [t_k, t_{k+1}]$  (see [3] page 96 equation (3.2)). In this particular case, our interpolation is given by

$$X_t^N = X_{t_k}^N + \int_{t_k}^t E \left( -\beta X_{t_{k+1}}^N \middle| \mathcal{F}_s \right) ds + \int_{t_k}^t \left\{ \sigma(X_{t_k}^N) + (s - t_k) E \left( -D_s \beta X_{t_{k+1}}^N \middle| \mathcal{F}_s \right) \right\} dW_s.$$

*Proof of Lemma 3.12.* The fact that  $(X_t^N)$  is an  $(\mathcal{F}_t)$ -adapted process which interpolates the scheme (1.3) is a consequence of (3.7). The continuity is a consequence of the fact that the map  $(t \mapsto E(X|\mathcal{F}_t))$  has a continuous modification. So, applying Itô's formula and Lemma 3.10, we obtain  $d(\beta_t^{k,N}(t - t_k)) = (t - t_k) z_t^{k,N} dW_t + \beta_t^{k,N} dt$ , and hence

$$dX_t^N = \beta_t^{k,N} dt + \left( \sigma(X_{t_k}^N) + (t - t_k) z_t^{k,N} \right) dW_t.$$

This concludes the proof.  $\square$

We next give moment estimates of the interpolation process  $X_t^N$ .

**Lemma 3.14.** *Let  $p \geq 1$  and  $h^* > 0$  be small enough. There exists a constant  $C_p > 0$  depending on  $h^*$  such that*

$$\sup_{N \geq N_0} \sup_{t \in [0, T]} E |X_t^N|^p < C_p.$$

*Proof.* Using Lemma 3.9, Jensen's inequality and the independence of  $W_t - W_{t_k}$  and  $X_{t_k}^N$ , we have for  $t \in [t_k, t_{k+1}]$ :

$$\begin{aligned} E |X_t^N|^p &\leq C_p E |X_{t_k}^N|^p + C_p E \left| E \left( b \left( X_{t_{k+1}}^N \right) \middle| \mathcal{F}_t \right) \right|^p |t - t_k|^p + C_p E \left| \sigma(X_{t_k}^N) \right|^p |W_t - W_{t_k}|^p \\ &\leq C_p + C_p h^p E \left| b \left( X_{t_{k+1}}^N \right) \right|^p + E \left| \sigma(X_{t_k}^N) \right|^p E |W_t - W_{t_k}|^p. \end{aligned}$$

Using the growth condition on  $b$  and  $\sigma$ , moments of the normal law and Lemma 3.9, we deduce the result.  $\square$



The following is a straightforward consequence of Lemmas 3.11 and 3.14

**Corollary 3.15.** *Let  $v : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  be a function with polynomial growth, and let  $n_1, \dots, n_6$  non negative integers. Then there exists a constant  $C$  independent of  $h^*$  such that for  $r \in [t_k, t_{k+1}]$  and  $h \in (0, h^*)$*

$$E \left( \left| \beta_r^{k,N} \right|^{n_1} \left| z_r^{k,N} \right|^{n_2} \left| \gamma_r^{k,N} \right|^{n_3} \left| \eta_r^{k,N} \right|^{n_4} |r - t_k|^{n_5} |v(r, X_r^N)|^{n_6} \right) \leq C.$$

**3.5. Local decomposition.** Now we return to the proof of the main theorem. Let  $u$  be the solution to the Kolmogorov equation (2.1). Using (2.1), we decompose the weak error into a sum of local errors. Let  $\delta_k^N := Eu(t_{k+1}, X_{t_{k+1}}^N) - Eu(t_k, X_{t_k}^N)$ ; we deduce

$$Ef(X_T^N) - Ef(X_T) = Eu(T, X_T^N) - Eu(0, x) = \sum_{k=0}^{N-1} \delta_k^N. \quad (3.8)$$

We introduce, for  $t_k \leq t \leq t_{k+1}$ ,

$$\mathcal{I}_k^N(t) := E \left[ \left( \beta_t^{k,N} - b(X_t^N) \right) \partial u(t, X_t^N) \right], \quad \mathcal{J}_k^N(t) := E \left[ \left( \left| \gamma_t^{k,N} \right|^2 - \sigma^2(X_t^N) \right) \Delta u(t, X_t^N) \right].$$

Since  $u \in C^{1,2}$ , using Itô's formula, Lemma 3.11 and the Kolmogorov equation (2.1) at the point  $(t, X_t^N)$ , we obtain

$$\delta_k^N = E \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} u + \beta_t^{k,N} \partial u + \frac{1}{2} \left| \gamma_t^{k,N} \right|^2 \Delta u \right\} (t, X_t^N) dt \quad (3.9)$$

$$= E \int_{t_k}^{t_{k+1}} \left\{ \mathcal{I}_k^N(t) + \frac{1}{2} \mathcal{J}_k^N(t) \right\} dt. \quad (3.10)$$

Now for  $k = 0, \dots, N-1$ , we introduce the following quantities for  $s \in [t_k, t_{k+1}]$ :

$$\begin{aligned} i_k^N(s) &:= \frac{\partial}{\partial s} (b \partial u)(s, X_s^N) + \beta_s^{k,N} \partial (b \partial u)(s, X_s^N) \\ &\quad + \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta (b \partial u)(s, X_s^N) - \beta_s^{k,N} \frac{\partial}{\partial s} \partial u(s, X_s^N) \\ &\quad - \left( \left| \beta_s^{k,N} \right|^2 + z_s^{k,N} \gamma_s^{k,N} \right) \Delta u(s, X_s^N) - \frac{1}{2} \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^3 u(s, X_s^N), \end{aligned} \quad (3.11)$$

$$\begin{aligned} j_k^N(s) &:= \left| \gamma_s^{k,N} \right|^2 \frac{\partial}{\partial s} \Delta u(s, X_s^N) + \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^3 u(s, X_s^N) \\ &\quad + \frac{1}{2} \left| \gamma_s^{k,N} \right|^4 \partial^4 u(s, X_s^N) + 2 \gamma_s^{k,N} z_s^{k,N} \Delta u(s, X_s^N) \\ &\quad + |s - t_k|^2 \left| \eta_s^{k,N} \right|^2 \Delta u(s, X_s^N) + 2(s - t_k) \left| \gamma_s^{k,N} \right|^2 \eta_s^{k,N} \partial^3 u(s, X_s^N) \\ &\quad - \frac{\partial}{\partial s} (\sigma^2 \Delta u)(s, X_s^N) - \beta_s^{k,N} \partial (\sigma^2 \Delta u)(s, X_s^N) - \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta (\sigma^2 \Delta u)(s, X_s^N). \end{aligned} \quad (3.12)$$

The next two lemmas explain that, up to some sign,  $\mathcal{I}_k^N$  (resp.  $\mathcal{J}_k^N$ ) can be viewed as an antiderivative of  $i_k^N$  (resp.  $j_k^N$ ).

**Lemma 3.16.** *For all  $k = 0, \dots, N-1$ , we have  $\mathcal{I}_k^N(t) = E \int_t^{t_{k+1}} i_k^N(s) ds$  for  $t \in [t_k, t_{k+1}]$ .*

*Proof.* If we denote by  $A := E \left[ \beta_t^{k,N} \partial u(t, X_t^N) \right] - E \left[ b(X_{t_{k+1}}^N) \partial u(t_{k+1}, X_{t_{k+1}}^N) \right]$  and by  $B := E \left[ b(X_{t_{k+1}}^N) \partial u(t_{k+1}, X_{t_{k+1}}^N) \right] - E \left[ b(X_t^N) \partial u(t, X_t^N) \right]$  we can write  $\mathcal{I}_k^N(t) = A + B$ .

Lemma 3.11 enables us to apply Itô's formula: Let  $v : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  be of class  $C^{1,2}$ ; Itô's formula yields

$$dv(t, X_t^N) = \left\{ \frac{\partial}{\partial t} v + \beta_t^{k,N} \partial v + \frac{1}{2} \left| \gamma_t^{k,N} \right|^2 \Delta v \right\} (t, X_t^N) dt + \gamma_t^{k,N} \partial v(t, X_t^N) dW_t. \quad (3.13)$$

Using this equation with Lemma 3.10 we have for  $v \in C^{1,2}$

$$\begin{aligned} d \left[ \beta_r^{k,N} v(r, X_r^N) \right] &= \left\{ \beta_r^{k,N} \frac{\partial}{\partial r} v + \left| \beta_r^{k,N} \right|^2 \partial v + \frac{1}{2} \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 \Delta v + z_r^{k,N} \gamma_r^{k,N} \partial v \right\} (r, X_r^N) dr \\ &\quad + \left\{ \beta_r^{k,N} \gamma_r^{k,N} \partial v + z_r^{k,N} v \right\} (r, X_r^N) dW_r. \end{aligned} \quad (3.14)$$

The function  $\Delta u$  has polynomial growth; hence corollary 3.15 implies that

$E \int_t^{t_{k+1}} \left\{ \beta_s^{k,N} \gamma_s^{k,N} \Delta u + z_s^{k,N} \partial u \right\} (s, X_s^N) dW_s = 0$ . Using equation (3.14) with  $v = \partial u$ , integrating between  $t$  and  $t_{k+1}$ , using the fact that  $\beta_{t_{k+1}}^{k,N} = b(X_{t_{k+1}}^N)$  and taking expectation we obtain

$$A = -E \int_t^{t_{k+1}} \left\{ \beta_s^{k,N} \frac{\partial}{\partial s} \partial u + \left| \beta_s^{k,N} \right|^2 \Delta u + \frac{1}{2} \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^3 u + z_s^{k,N} \gamma_s^{k,N} \Delta u \right\} (s, X_s^N) ds. \quad (3.15)$$

Similarly, Corollary 3.15 implies that  $E \int_t^{t_{k+1}} \gamma_s^{k,N} \partial(b\partial u)(s, X_s^N) dW_s = 0$ . Using (3.13) with  $v = b\partial u$ , integrating between  $t$  and  $t_{k+1}$  and taking expectation yields

$$B = E \int_t^{t_{k+1}} \left\{ \frac{\partial}{\partial s} (b\partial u) + \beta_s^{k,N} \partial(b\partial u) + \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta(b\partial u) \right\} (s, X_s^N) ds.$$

The stochastic integral is centered by Corollary 3.15. This identity combined with (3.15) concludes the proof.  $\square$

**Lemma 3.17.** *For all  $k = 0, \dots, N-1$ , we have  $\mathcal{J}_k^N(t) = E \int_{t_k}^t j_k^N(s) ds$  for  $t \in [t_k, t_{k+1}]$ .*

*Proof.* Using (3.3) we clearly deduce that  $\mathcal{J}_k^N(t) = C + D$  where

$$\begin{aligned} C &:= E \left[ \left| \sigma(X_{t_k}^N) + (t - t_k) z_t^{k,N} \right|^2 \Delta u(t, X_t^N) \right] - E \left[ \sigma^2(X_{t_k}^N) \Delta u(t_k, X_{t_k}^N) \right], \\ D &:= E \left[ \sigma^2(X_{t_k}^N) \Delta u(t_k, X_{t_k}^N) \right] - E \left[ \sigma^2(X_t^N) \Delta u(t, X_t^N) \right]. \end{aligned}$$

We at first rewrite the term  $D$ : using (3.13) with  $v = \sigma^2 \Delta u$ , integrating between  $t_k$  and  $t$  and taking expectation, we obtain:

$$D = -E \int_{t_k}^t \left\{ \frac{\partial}{\partial t} (\sigma^2 \Delta u) + \beta_s^{k,N} \partial(\sigma^2 \Delta u) + \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta(\sigma^2 \Delta u) \right\} (s, X_s^N) ds,$$

since  $\sigma^2 \Delta u$  has polynomial growth which implies that the stochastic integral is centered using Corollary 3.15. Itô's formula and Lemma 3.10 yield for  $r \in [t_k, t_{k+1}]$

$$d \left| \gamma_r^{k,N} \right|^2 = \left\{ 2\gamma_r^{k,N} z_r^{k,N} + \left| \eta_r^{k,N} \right|^2 |r - t_k|^2 \right\} dr + 2\gamma_r^{k,N} \eta_r^{k,N} (r - t_k) dW_r. \quad (3.16)$$

Using this equation and (3.13), we have for  $v$  of class  $C^{1,2}$  and  $r \in [t_k, t_{k+1}]$

$$\begin{aligned} d \left| \gamma_r^{k,N} \right|^2 v(r, X_r^N) &= \left\{ \left| \gamma_r^{k,N} \right|^2 \frac{\partial}{\partial r} v + \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 \partial v + \frac{1}{2} \left| \gamma_r^{k,N} \right|^4 \Delta v \right. \\ &\quad \left. + 2\gamma_r^{k,N} z_r^{k,N} v + \left| \eta_r^{k,N} \right|^2 |r - t_k|^2 v + 2 \left| \gamma_r^{k,N} \right|^2 \eta_r^{k,N} (r - t_k) \partial v \right\} (r, X_r^N) dr \\ &\quad + \left\{ \left( \gamma_r^{k,N} \right)^3 \partial v + 2\gamma_r^{k,N} \eta_r^{k,N} (r - t_k) v \right\} (r, X_r^N) dW_r. \end{aligned} \quad (3.17)$$

Using equation (3.17) with  $v = \Delta u$ , integrating between  $t_k$  and  $t$ , using the identity  $\gamma_{t_k}^{k,N} = \sigma(X_{t_k}^N)$  and taking expectation, we deduce

$$\begin{aligned} C &= E \int_{t_k}^t \left\{ \left| \gamma_s^{k,N} \right|^2 \frac{\partial}{\partial s} \Delta u + \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^3 u + \frac{1}{2} \left| \gamma_s^{k,N} \right|^4 \partial^4 u \right. \\ &\quad \left. + 2\gamma_s^{k,N} z_s^{k,N} \Delta u + |s - t_k|^2 \left| \eta_s^{k,N} \right|^2 \Delta u + 2(s - t_k) \left| \gamma_s^{k,N} \right|^2 \eta_s^{k,N} \partial^3 u \right\} (s, X_s^N) ds. \end{aligned}$$

Indeed, once more Corollary 3.15 and the polynomial growth of  $\partial \Delta u$  and  $\Delta u$  implies that the corresponding stochastic integral is centered. This concludes the proof.  $\square$

Plugging the results of Lemmas 3.16 and 3.17 into (3.10) we obtain

$$Ef(X_T^N) - Ef(X_T) = \sum_{k=0}^{N-1} E \int_{t_k}^{t_{k+1}} \left\{ \int_{t_k}^{t_{k+1}} i_k^N(s) ds + \frac{1}{2} \int_{t_k}^t j_k^N(s) ds \right\} dt. \quad (3.18)$$

*Note:* Thanks to Corollary 3.15 and the assumptions growth or boundness on the coefficients, all the stochastic integrals appearing in the next section, are centered.

**3.6. Upper estimate of  $\mathcal{I}_k^N(t)$ .** We next upper estimate the difference  $\phi_i(s) - \phi_i(t_{k+1})$ , where  $\phi_i$  is one of the seven terms in the right hand side of (3.11)

**3.6.1.** *The term  $\phi_1(s) = \frac{\partial}{\partial s}(b\partial u)(s, X_s^N)$ .* Using (3.13) with  $v = \frac{\partial}{\partial t}(b\partial u)$ , integrating from  $s$  to  $t_{k+1}$  and taking expected value we deduce

$$E \frac{\partial}{\partial s}(b\partial u)(s, X_s^N) = E \frac{\partial}{\partial s}(b\partial u)(t_{k+1}, X_{t_{k+1}}^N) + R_1(s),$$

where

$$R_1(s) := -E \int_s^{t_{k+1}} \left\{ \frac{\partial^2}{\partial s^2}(b\partial u) + \beta_r^{k,N} \partial \frac{\partial}{\partial s}(b\partial u) + \frac{1}{2} \left| \gamma_r^{k,N} \right|^2 \Delta \frac{\partial}{\partial s}(b\partial u) \right\} (r, X_r^N) dr.$$

Futhermore, Lemmas 3.14 and 3.11 and the polynomial growth of the functions involved imply that  $|R_1(s)| \leq Ch$ .

**3.6.2.** *The term  $\phi_2(s) = \beta_s^{k,N} \partial(b\partial u)(s, X_s^N)$ .* Using (3.14) with  $v = \partial(b\partial u)$ , integrating between  $s$  and  $t_{k+1}$  and taking expectation we obtain

$$E \left[ \beta_s^{k,N} \partial(b\partial u)(s, X_s^N) \right] = E \left[ b(X_{t_{k+1}}^N) \partial(b\partial u)(t_{k+1}, X_{t_{k+1}}^N) \right] + R_2(s),$$

where

$$\begin{aligned} R_2(s) &:= -E \int_s^{t_{k+1}} \left[ \beta_r^{k,N} \left\{ \frac{\partial}{\partial s} \partial(b\partial u) + \beta_r^{k,N} \Delta(b\partial u) + \frac{1}{2} \left| \gamma_r^{k,N} \right|^2 \partial^3(b\partial u) \right\} \right. \\ &\quad \left. + \gamma_r^{k,N} z_r^{k,N} \Delta(b\partial u) \right] (r, X_r^N) dr. \end{aligned}$$

The polynomial growth of the functions and Lemmas 3.14 and 3.11 imply that  $|R_2(s)| \leq Ch$ .

3.6.3. The term  $\phi_3(s) = \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta(b\partial u)(s, X_s^N)$ . Let

$$R_3(s) := -\frac{1}{2} E \int_s^{t_{k+1}} \left\{ \left| \gamma_r^{k,N} \right|^2 \frac{\partial}{\partial t} \Delta(b\partial u) + \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 \partial^3(b\partial u) + \frac{1}{2} \left| \gamma_r^{k,N} \right|^4 \partial^4(b\partial u) \right. \\ \left. + 2\gamma_r^{k,N} z_r^{k,N} \Delta(b\partial u) + \left| \eta_r^{k,N} \right|^2 |r - t_k|^2 \Delta(b\partial u) + 2 \left| \gamma_r^{k,N} \right|^2 \eta_r^{k,N} (r - t_k) \partial^3(b\partial u) \right\} (r, X_r^N) dr.$$

Using (3.17) with  $v = \frac{1}{2} \Delta(b\partial u)$ , integrating between  $s$  and  $t_{k+1}$ , and taking expectation give us

$$\frac{1}{2} E \left[ \left| \gamma_s^{k,N} \right|^2 \Delta(b\partial u)(s, X_s^N) \right] = \frac{1}{2} E \left[ \left| \gamma_{t_{k+1}}^{k,N} \right|^2 \Delta(b\partial u)(t_{k+1}, X_{t_{k+1}}^N) \right] + R_3(s),$$

with  $|R_3(s)| \leq Ch$ .

3.6.4. The term  $\phi_4(s) = \beta_s^{k,N} \frac{\partial}{\partial s} \partial u(s, X_s^N)$ . Let

$$R_4(s) := E \int_s^{t_{k+1}} \left[ \beta_r^{k,N} \frac{\partial^2}{\partial s^2} \partial u + \left| \beta_r^{k,N} \right|^2 \partial \frac{\partial}{\partial s} \partial u + \frac{1}{2} \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 \Delta \frac{\partial}{\partial s} \partial u \right. \\ \left. + \gamma_r^{k,N} z_r^{k,N} \partial \frac{\partial}{\partial s} \partial u \right] (r, X_r^N) dr.$$

Using (3.14) for  $v = \frac{\partial}{\partial s} \partial u$  and integrating between  $s$  and  $t_{k+1}$ , we obtain

$$-E \left[ \beta_s^{k,N} \frac{\partial}{\partial t} \partial u(s, X_s^N) \right] = -E \left[ b(X_{t_{k+1}}^N) \frac{\partial}{\partial t} \partial u(t_{k+1}, X_{t_{k+1}}^N) \right] + R_4(s),$$

with  $|R_4(s)| \leq Ch$ .

3.6.5. The term  $\phi_5(s) = \left| \beta_s^{k,N} \right|^2 \Delta u(s, X_s^N)$ . Using Itô's formula and Lemma 3.10 we have

$$d \left| \beta_r^{k,N} \right|^2 = \left| z_r^{k,N} \right|^2 dr + 2\beta_r^{k,N} z_r^{k,N} dW_r.$$

Using this equation, (3.13) and Itô's formula we obtain

$$d \left[ \left| \beta_r^{k,N} \right|^2 \Delta u(r, X_r^N) \right] = \left[ \left| \beta_r^{k,N} \right|^2 \frac{\partial}{\partial t} \Delta u + \left( \beta_r^{k,N} \right)^3 \partial^3 u + \frac{1}{2} \left| \beta_r^{k,N} \right|^2 \left| \gamma_r^{k,N} \right|^2 \partial^4 u \right. \\ \left. + \left| z_r^{k,N} \right|^2 \Delta u + 2\beta_r^{k,N} z_r^{k,N} \gamma_r^{k,N} \partial^3 u \right] (r, X_r^N) dr + dM_r,$$

where  $dM_r = \left\{ 2\beta_r^{k,N} z_r^{k,N} \Delta u + \left| \beta_r^{k,N} \right|^2 \gamma_r^{k,N} \partial^3 u \right\} (r, X_r^N) dW_r$  and  $M_t$  is a square integrable martingale. Let

$$R_5(s) := E \int_s^{t_{k+1}} \left[ \left| \beta_r^{k,N} \right|^2 \frac{\partial}{\partial r} \Delta u + \left( \beta_r^{k,N} \right)^3 \partial^3 u + \frac{1}{2} \left| \beta_r^{k,N} \right|^2 \left| \gamma_r^{k,N} \right|^2 \partial^4 u \right. \\ \left. + z_r^{k,N} \Delta u + 2\beta_r^{k,N} z_r^{k,N} \gamma_r^{k,N} \partial^3 u \right] (r, X_r^N) dr.$$

Integrating between  $s$  and  $t_{k+1}$  and taking expectation we have

$$-E \left[ \left| \beta_s^{k,N} \right|^2 \Delta u(s, X_s^N) \right] = -E \left[ \left| \beta_{t_{k+1}}^{k,N} \right|^2 \Delta u(t_{k+1}, X_{t_{k+1}}^N) \right] + R_5(s),$$

with  $|R_5(s)| \leq Ch$ .

3.6.6. *The term  $\phi_6(s) = z_s^{k,N} \gamma_s^{k,N} \Delta u(s, X_s^N)$ .* Applying Itô's formula to the product of  $z_r^{k,N} \gamma_r^{k,N}$  and (3.13), and Lemma 3.10, we obtain for  $r \in [t_k, t_{k+1}]$

$$\begin{aligned} d \left[ z_r^{k,N} \gamma_r^{k,N} v(r, X_r^N) \right] &= \left\{ z_r^{k,N} \gamma_r^{k,N} \frac{\partial}{\partial t} v + z_r^{k,N} \gamma_r^{k,N} \beta_r^{k,N} \partial v + \frac{1}{2} z_r^{k,N} \left( \gamma_r^{k,N} \right)^3 \Delta v + \left| z_r^{k,N} \right|^2 v \right. \\ &\quad \left. + \left| \eta_r^{k,N} \right|^2 (r - t_k) v + z_r^{k,N} \eta_r^{k,N} (r - t_k) \gamma_r^{k,N} \partial v + \left| \gamma_r^{k,N} \right|^2 \eta_r^{k,N} \partial v \right\} (r, X_r^N) dr + dM_r \end{aligned} \quad (3.19)$$

where  $dM_r = \left\{ z_r^{k,N} \left| \gamma_r^{k,N} \right|^2 \partial v + z_r^{k,N} \eta_r^{k,N} (r - t_k) v + \gamma_r^{k,N} \eta_r^{k,N} v \right\} (r, X_r^N) dW_r$  and  $M_r$  is a square integrable martingale. Using equation (3.19) with  $v = \Delta u$ , integrating between  $s$  and  $t_{k+1}$  and taking expectation give us

$$-E \left[ z_s^{k,N} \gamma_s^{k,N} \Delta u(s, X_s^N) \right] = -E \left[ z_{t_{k+1}}^{k,N} \gamma_{t_{k+1}}^{k,N} \Delta u(t_{k+1}, X_{t_{k+1}}^N) \right] + R_6(s),$$

with  $|R_6(s)| \leq Ch$ .

3.6.7. *The term  $\phi_7(s) = \frac{1}{2} \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^3 u(s, X_s^N)$ .* Using Lemma 3.10 and equation (3.17), Itô's formula give us for  $v$  of class  $C^{1,2}$

$$\begin{aligned} d \left[ \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 v \right] (r, X_r^N) &= \left\{ \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 \frac{\partial}{\partial r} v + \left| \beta_r^{k,N} \right|^2 \left| \gamma_r^{k,N} \right|^2 \partial v + \frac{1}{2} \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^4 \Delta v \right. \\ &\quad \left. + 2\beta_r^{k,N} \gamma_r^{k,N} z_r^{k,N} v + \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 |r - r_k|^2 v + 2\beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 \eta_r^{k,N} (r - r_k) \partial v + z_r^{k,N} \left( \gamma_r^{k,N} \right)^3 \partial v \right. \\ &\quad \left. + 2\gamma_r^{k,N} \eta_r^{k,N} z_r^{k,N} (r - r_k) v \right\} (r, X_r^N) dr \\ &\quad + \left\{ \left| \gamma_r^{k,N} \right|^2 z_r^{k,N} v + \beta_r^{k,N} \left( \gamma_r^{k,N} \right)^3 \partial v + 2\beta_r^{k,N} \gamma_r^{k,N} \eta_r^{k,N} (r - r_k) v \right\} (r, X_r^N) dW_r. \end{aligned} \quad (3.20)$$

Using this equation with  $v = \frac{1}{2} \partial^3 u$ , integrating between  $s$  and  $t_{k+1}$  and taking expectation we have

$$-\frac{1}{2} E \left[ \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^3 u(s, X_s^N) \right] = -\frac{1}{2} E \left[ \beta_{t_{k+1}}^{k,N} \left| \gamma_{t_{k+1}}^{k,N} \right|^2 \partial^3 u(t_{k+1}, X_{t_{k+1}}^N) \right] + R_7(s),$$

with  $|R_7(s)| \leq Ch$ .

3.7. **Upper estimate of  $\mathcal{J}_k^N(t)$ .** We upper estimate the error  $\tilde{\phi}_i(s) - \tilde{\phi}_i(t_k)$  where  $\tilde{\phi}_i$  is one of the nine terms in the right hand side of (3.12)

3.7.1. *The term  $\tilde{\phi}_1(s) = \left| \gamma_s^{k,N} \right|^2 \frac{\partial}{\partial t} \Delta u(s, X_s^N)$ .* Using (3.17) with  $v = \frac{\partial}{\partial t} \Delta u$ , integrating between  $t_k$  and  $s$ , taking expectation and using the fact that  $\gamma_{t_k}^{k,N} = \sigma(X_{t_k}^N)$  we have

$$E \left[ \left| \gamma_s^{k,N} \right|^2 \frac{\partial}{\partial t} \Delta u(s, X_s^N) \right] = E \left[ \left| \sigma(X_{t_k}^N) \right|^2 \frac{\partial}{\partial t} \Delta u(t_k, X_{t_k}^N) \right] + \tilde{R}_1(s),$$

with

$$\begin{aligned} \tilde{R}_1(s) &= E \int_{t_k}^s \left[ \left| \gamma_r^{k,N} \right|^2 \frac{\partial^2}{\partial r^2} \Delta u + \{2\gamma_r^{k,N} z_r^{k,N} + \left| \eta_r^{k,N} \right|^2 |r - t_k|^2\} \frac{\partial}{\partial r} \Delta u \right. \\ &\quad \left. + \{\beta_r^{k,N} + 2\eta_r^{k,N} (r - t_k)\} \left| \gamma_r^{k,N} \right|^2 \frac{\partial}{\partial r} \partial^3 u + \frac{1}{2} \left| \gamma_r^{k,N} \right|^4 \frac{\partial}{\partial r} \partial^4 u \right] (r, X_r^N) dr. \end{aligned}$$

Corollary 3.15 implies that  $|\tilde{R}_1(s)| \leq Ch$ .

3.7.2. The term  $\tilde{\phi}_2(s) = \beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^3 u(s, X_s^N)$ . For an  $\mathcal{F}_s$ -measurable random variable  $Z$ , we have  $E\left(Z \beta_{t_k}^{k,N}\right) = E\left(Z b\left(X_{t_{k+1}}^N\right)\right)$ . Using (3.20) with  $v = \partial^3 u$ , integrating between  $t_k$  and  $s$  and taking expectation we have

$$E\left[\beta_s^{k,N} \left| \gamma_s^{k,N} \right|^2 \partial^3 u(s, X_s^N)\right] = E\left[b\left(X_{t_{k+1}}^N\right) \left| \sigma\left(X_{t_k}^N\right) \right|^2 \partial^3 u(t_k, X_{t_k}^N)\right] + \tilde{R}_2(s),$$

where

$$\begin{aligned} \tilde{R}_2(s) := E \int_{t_k}^s & \left\{ 2\gamma_r^{k,N} z_r^{k,N} \beta_r^{k,N} + \beta_r^{k,N} \left| \eta_r^{k,N} \right|^2 |r - t_k|^2 + 2\gamma_r^{k,N} \eta_r^{k,N} z_r^{k,N} (r - t_k) \right\} \partial^3 u \\ & + \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^2 \frac{\partial}{\partial r} \partial^3 u + \frac{1}{2} \beta_r^{k,N} \left| \gamma_r^{k,N} \right|^4 \partial^5 u \\ & + \left\{ \left| \beta_r^{k,N} \right|^2 + 2\beta_r^{k,N} \eta_r^{k,N} (r - t_k) + \gamma_r^{k,N} z_r^{k,N} \right\} \left| \gamma_r^{k,N} \right|^2 \partial^4 u \right\} (r, X_r^N) dr. \end{aligned}$$

Corollary 3.15 implies that  $|\tilde{R}_2(s)| \leq Ch$ .

3.7.3. The term  $\tilde{\phi}_3(s) = \frac{1}{2} \left| \gamma_s^{k,N} \right|^4 \partial^4 u(s, X_s^N)$ . Using Lemma 3.10 and Itô's formula we deduce

$$d \left| \gamma_r^{k,N} \right|^4 = \left| \gamma_r^{k,N} \right|^2 \left\{ 4\gamma_r^{k,N} z_t^{k,N} + 6 \left| \eta_r^{k,N} \right|^2 |r - t_k|^2 \right\} dr + 4 \left( \gamma_r^{k,N} \right)^3 \eta_r^{k,N} (r - t_k) dW_r.$$

Using this equation and (3.13) with  $v = \frac{1}{2} \partial^4 u$  and applying Itô formula, we have

$$\frac{1}{2} E \left[ \left| \gamma_s^{k,N} \right|^4 \partial^4 u(s, X_s^N) \right] = \frac{1}{2} E \left[ \left| \sigma\left(X_{t_k}^N\right) \right|^4 \partial^4 u(t_k, X_{t_k}^N) \right] + \tilde{R}_3(s),$$

where  $\tilde{R}_3(s) \leq Ch$  by Corollary 3.15.

3.7.4. The term  $\tilde{\phi}_4(s) = 2\gamma_s^{k,N} z_s^{k,N} \Delta u(s, X_s^N)$ . Using (3.19) with  $v = 2\Delta u$  we have  $E\left[2\gamma_s^{k,N} z_s^{k,N} \Delta u(s, X_s^N)\right] = E\left[2\gamma_{t_k}^{k,N} z_{t_k}^{k,N} \Delta u(t_k, X_{t_k}^N)\right] + \tilde{R}_4(s)$ , and Corollary 3.15 implies  $|\tilde{R}_4(s)| \leq Ch$ .

3.7.5. The term  $\tilde{\phi}_5(s) := \tilde{R}_5(s) := |s - t_k|^2 \left| \eta_s^{k,N} \right|^2 \Delta u(s, X_s^N) + 2(s - t_k) \left| \gamma_s^{k,N} \right|^2 \eta_s^{k,N} \partial^3 u(s, X_s^N)$ .

Using Corollary 3.15, we have  $|\tilde{R}_5(s)| \leq Ch$ .

3.7.6. The term  $\tilde{\phi}_6(s) = \frac{\partial}{\partial t} (\sigma^2 \Delta u)(s, X_s^N)$ . Using (3.13) with  $v = \frac{\partial}{\partial t} (\sigma^2 \Delta u)$ , integrating between  $t_k$  and  $s$  and taking expectation, we have

$$-E \left[ \frac{\partial}{\partial t} (\sigma^2 \Delta u)(s, X_s^N) \right] = -E \left[ \frac{\partial}{\partial t} (\sigma^2 \Delta u)(t_k, X_{t_k}^N) \right] + \tilde{R}_6(s),$$

with  $|\tilde{R}_6(s)| \leq Ch$  by Corollary 3.15.

3.7.7. The term  $\tilde{\phi}_7(s) = \beta_s^{k,N} \partial (\sigma^2 \Delta u)(s, X_s^N)$ . Using (3.14) with  $v = \partial (\sigma^2 \Delta u)$ , integrating between  $t_k$  and  $s$ , taking expectation we have

$$-E \left[ \beta_s^{k,N} \partial (\sigma^2 \Delta u)(s, X_s^N) \right] = -E \left[ b\left(X_{t_k}^N\right) \partial (\sigma^2 \Delta u)(t_k, X_{t_k}^N) \right] + \tilde{R}_7(s),$$

with  $|\tilde{R}_7(s)| \leq Ch$  by Corollary 3.15.

3.7.8. The term  $\tilde{\phi}_8(s) = \frac{1}{2} \left| \gamma_s^{k,N} \right|^2 \Delta(\sigma^2 \Delta u)(s, X_s^N)$ . Using (3.17) with  $v = \frac{1}{2} \Delta(\sigma^2 \Delta u)$ , integrating between  $t_k$  and  $s$ , and finally taking expectation we have

$$-\frac{1}{2} E \left[ \left| \gamma_s^{k,N} \right|^2 \Delta(\sigma^2 \Delta u)(s, X_s^N) \right] = -\frac{1}{2} E \left[ \left| \sigma(X_{t_k}^N) \right|^2 \Delta(\sigma^2 \Delta u)(t_k, X_{t_k}^N) \right] + \tilde{R}_8(s),$$

with  $|\tilde{R}_8(s)| \leq Ch$  by Corollary 3.15.

3.8. **Proof Theorem 2.1 (i).** The identity (3.18) and the upper estimate in section 3.6 and 3.7 imply that

$$\begin{aligned} Ef(X_T^N) - Ef(X_T) &= \sum_{k=0}^{N-1} E \int_{t_k}^{t_{k+1}} \left\{ \int_t^{t_{k+1}} i_k^N(t_{k+1}) ds + \frac{1}{2} \int_{t_k}^t j_k^N(t_k) ds \right\} dt + R \\ &= \frac{1}{2} h^2 \sum_{k=0}^{N-1} E i_k^N(t_{k+1}) + \frac{1}{4} h^2 \sum_{k=0}^{N-1} E j_k^N(t_k) + R, \end{aligned} \quad (3.21)$$

where

$$R := \sum_{k=0}^{N-1} \sum_{j=1}^7 \int_{t_k}^{t_{k+1}} \int_t^{t_{k+1}} R_j(s) ds dt + \sum_{k=0}^{N-1} \sum_{j=1}^8 \int_{t_k}^{t_{k+1}} \int_{t_k}^t \tilde{R}_j(s) ds dt.$$

Hence  $|R| \leq Ch^2$ . Note that  $\beta_{t_{k+1}}^{k,N} = b(X_{t_{k+1}}^N)$ . Using (3.3) and (3.5) we deduce that  $z_{t_{k+1}}^{k,N} = \sigma(X_{t_k}^N) (S_h b') (X_{t_{k+1}}^N)$  and  $\gamma_{t_{k+1}}^{k,N} = \sigma(X_{t_k}^N) [1 + h(S_h b') (X_{t_{k+1}}^N)] = \sigma(X_{t_k}^N) S_h (X_{t_{k+1}}^N)$ . Therefore, we deduce that

$$\begin{aligned} i_k^N(t_{k+1}) &= \frac{\partial}{\partial t} (b \partial u) (t_{k+1}, X_{t_{k+1}}^N) + [b \partial (b \partial u)] (t_{k+1}, X_{t_{k+1}}^N) + \frac{1}{2} \sigma^2 (X_{t_k}^N) [S_h^2 \Delta (b \partial u)] (t_{k+1}, X_{t_{k+1}}^N) \\ &\quad - \left[ b \frac{\partial}{\partial t} \partial u \right] (t_{k+1}, X_{t_{k+1}}^N) - [b^2 \Delta u] (t_{k+1}, X_{t_{k+1}}^N) - \sigma^2 (X_{t_k}^N) [b' S_h^2 \Delta u] (t_{k+1}, X_{t_{k+1}}^N) \\ &\quad - \frac{1}{2} \sigma^2 (X_{t_k}^N) [b S_h^2 \partial^3 u] (t_{k+1}, X_{t_{k+1}}^N). \end{aligned}$$

Similarly,  $\gamma_{t_k}^{k,N} = \sigma(X_{t_k}^N)$ . So we have

$$\begin{aligned} E j_k^N(t_k) &= E \sigma^2 \frac{\partial}{\partial t} \Delta u (t_k, X_{t_k}^N) + E b (X_{t_{k+1}}^N) \sigma^2 \partial^3 u (t_k, X_{t_k}^N) + \frac{1}{2} E \sigma^4 \partial^4 u (t_k, X_{t_k}^N) \\ &\quad + 2 E S_h b' (X_{t_{k+1}}^N) \sigma^2 \Delta u (t_k, X_{t_k}^N) - E \frac{\partial}{\partial t} (\sigma^2 \Delta u) (t_k, X_{t_k}^N) \\ &\quad - E b (X_{t_{k+1}}^N) \partial (\sigma^2 \Delta u) (t_k, X_{t_k}^N) - \frac{1}{2} E \sigma^2 \Delta (\sigma^2 \Delta u) (t_k, X_{t_k}^N). \end{aligned}$$

Notice that  $b$  and  $\sigma$  do not depend upon  $t$ ; hence after simplification we have

$$i_k^N(t_{k+1}) = b \partial (b \partial u) (t_{k+1}, X_{t_{k+1}}^N) - b^2 \Delta u (t_{k+1}, X_{t_{k+1}}^N) + \frac{1}{2} \sigma^2 (X_{t_k}^N) (S_h^2 b'' \partial u) (t_{k+1}, X_{t_{k+1}}^N), \quad (3.22)$$

$$\begin{aligned} E j_k^N(t_k) &= E b (X_{t_{k+1}}^N) \sigma^2 \partial^3 u (t_k, X_{t_k}^N) + \frac{1}{2} E \sigma^4 \partial^4 u (t_k, X_{t_k}^N) - E b (X_{t_{k+1}}^N) \partial (\sigma^2 \Delta u) (t_k, X_{t_k}^N) \\ &\quad - \frac{1}{2} E \sigma^2 \Delta (\sigma^2 \Delta u) (t_k, X_{t_k}^N) + 2 E b' S_h (X_{t_{k+1}}^N) \sigma^2 \Delta u (t_k, X_{t_k}^N). \end{aligned} \quad (3.23)$$

Corollary 3.15 implies the existence of a constant  $C$ , such that for all  $k = 0, \dots, N-1$ :

$$|E (i_k^N(t_{k+1}) + j_k^N(t_k))| \leq C.$$

Using this bound with (3.21) proves the first part of Theorem 2.1.

**3.9. Proof Theorem 2.1 (ii).** We at first prove the following lemma, which upper estimates the error in the approximation of an integral by a Riemann sum.

**Lemma 3.18.** *Let  $v$  and  $w$  in  $C_b^{\infty, \infty}([0, T] \times \mathbf{R})$ . Then there exists a constant  $C$  independent of  $h$  such that*

$$\left| h \sum_{k=0}^{N-1} Ev \left( t_{k+1}, X_{t_{k+1}}^N \right) w \left( t_k, X_{t_k}^N \right) - E \int_0^T vw(t, X_t) dt \right| \leq Ch.$$

*Proof.* Using (3.13) multiply by  $w(t_k, X_{t_k}^N)$  and taking expected value, we deduce for  $v \in C^{1,2}$ ,  $Ev \left( t_{k+1}, X_{t_{k+1}}^N \right) w \left( t_k, X_{t_k}^N \right) = Evw \left( t_k, X_{t_k}^N \right) + A_k$ , where

$$A_k := Ew \left( t_k, X_{t_k}^N \right) \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} v + \beta_t^{k,N} \partial v + \frac{1}{2} \left| \gamma_t^{k,N} \right|^2 \Delta v \right\} \left( t, X_t^N \right) dt.$$

This yields

$$h \sum_{k=0}^{N-1} Ev \left( t_{k+1}, X_{t_{k+1}}^N \right) w \left( t_k, X_{t_k}^N \right) - E \int_0^T vw(t, X_t) dt = \sum_{k=0}^{N-1} (hA_k + hB_k + C_k).$$

where

$$B_k := E(vw) \left( t_k, X_{t_k}^N \right) - E(vw) \left( t_k, X_{t_k} \right)$$

$$C_k := hE(vw) \left( t_k, X_{t_k} \right) - E \int_{t_k}^{t_{k+1}} (vw)(t, X_t) dt.$$

Using the Cauchy-Schwarz inequality, the fact that  $\frac{\partial}{\partial t} v$ ,  $\partial v$  and  $\Delta v$  have polynomial grow so that Corollary 3.15 can be applied, we deduce

$$\begin{aligned} |A_k|^2 &\leq E \left[ \left| w \left( t_k, X_{t_k}^N \right) \right|^2 \right] E \left[ \left| \int_{t_k}^{t_{k+1}} \left\{ \frac{\partial}{\partial t} v + \beta_t^{k,N} \partial v + \frac{1}{2} \left| \gamma_t^{k,N} \right|^2 \Delta v \right\} \left( t, X_t^N \right) dt \right|^2 \right] \\ &\leq ChE \int_{t_k}^{t_{k+1}} \left| \left\{ \frac{\partial}{\partial t} v + \beta_t^{k,N} \partial v + \frac{1}{2} \left| \gamma_t^{k,N} \right|^2 \Delta v \right\} \left( t, X_t^N \right) \right|^2 dt \\ &\leq Ch^2, \end{aligned}$$

Hence,  $|A_k| \leq Ch$  which implies  $h \sum_{0 \leq k \leq N-1} |A_k| \leq Ch$ .

Since  $(vw)(t_k, \cdot)$  is in  $C_b^\infty$ , we use Theorem 2.1 (i), changing  $T$  by  $t_k$ , which yields  $|B_k| \leq Ch$  and then  $h \sum_{0 \leq k \leq N-1} |B_k| \leq Ch$ . Finally, Itô's formula implies

$$\begin{aligned} C_k &= E \int_{t_k}^{t_{k+1}} \{ (vw) \left( t_k, X_{t_k} \right) - (vw) \left( t, X_t \right) \} dt \\ &= - E \int_{t_k}^{t_{k+1}} \int_{t_k}^t \left\{ \frac{\partial}{\partial t} (vw) + b \partial (vw) + \frac{1}{2} \sigma^2 \Delta (vw) \right\} \left( s, X_s \right) ds dt. \end{aligned}$$

Once more the polynomial growth imposed on  $v$ ,  $w$  and their partial derivatives implies that  $|C_k| \leq Ch^2$  and then  $\sum_{0 \leq k \leq N-1} |C_k| \leq Ch$ . This concludes the proof.  $\square$

Now we introduce the function  $\psi_{ih} : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  defined by

$$\begin{aligned} \psi_{ih}(t, x) &:= \frac{1}{2} b \partial (b \partial u)(t, x) - \frac{1}{2} b^2 \Delta u(t, x) + \frac{1}{4} \sigma^2 S_h^2 b'' \partial u(t, x) + \frac{1}{4} b \sigma^2 \partial^3 u(t, x) \\ &\quad + \frac{1}{8} \sigma^4 \partial^4 u(t, x) + \frac{1}{2} b' S_h \sigma^2 \Delta u(t, x) - \frac{1}{4} b \partial (\sigma^2 \Delta u)(t, x) - \frac{1}{8} \sigma^2 \Delta (\sigma^2 \Delta u)(t, x) \end{aligned} \quad (3.24)$$



Using the expression of  $i_k^N$  in (3.22) (resp.  $j_k^N$  in (3.23)) and the previous lemma, we deduce

$$\left| \frac{1}{2}h \sum_{k=0}^{N-1} E i_k^N(t_{k+1}) + \frac{1}{4}h \sum_{k=0}^{N_1} E j_k^N(t_k) - \int_0^T E \psi_{ih}(t, X_t) dt \right| \leq Ch. \quad (3.25)$$

Using the definitions of  $\psi_i$  and  $\psi_{ih}$  given in (2.2) and (3.24) respectively, we have

$$\begin{aligned} \psi_{ih}(t, x) - \psi_i(t, x) &= \frac{1}{4} \{ \sigma^2 S_h^2 b'' \partial u + b \sigma^2 \partial^3 u + 2b' S_h \sigma^2 \Delta u - \sigma^2 \Delta (b \partial u) \} (t, x) \\ &= \frac{1}{4} \sigma^2 (S_h^2 - 1) b'' \partial u(t, x) + \frac{1}{2} b' (S_h - 1) \sigma^2 \Delta u(t, x). \end{aligned}$$

Since  $(S_h - 1)(x) = \frac{hb'}{1-hb'}(x)$  and  $|S_h(x) + 1| \leq C$  for  $h \in (0, h^*)$ , we have  $|(S_h - 1)(x)| + |(S_h^2 - 1)(x)| \leq Ch$ , where as usually  $C$  does not depend on  $N$  and  $h$ . This yields

$$\left| \int_0^T E \{ \psi_{ih}(t, X_t) - \psi_i(t, X_t) \} dt \right| \leq Ch.$$

This last equation with (3.21) and (3.25) concludes the proof.

**Acknowledgments:** The author wishes to thank Annie Millet for many helpful comments.

#### REFERENCES

- [1] Bally V., Talay D., The law of the Euler scheme for stochastic differential equations: I. Convergence rate of the distribution function. *Prob. Th. Rel. Fields*, 104-1:43–60, 1996.
- [2] de Bouard A., Debussche A., Weak and strong order of convergence of a semi discrete scheme for the stochastic Nonlinear Schrodinger equation, *Applied Mathematics and Optimization journal*, 54, 3, pp. 369–399 2006.
- [3] Debussche A., Weak approximation of stochastic partial differential equations: the nonlinear case, *Math of Comp*, 80, pp. 89–117 2011.
- [4] Debussche A., Printems J., Weak order for the discretization of the stochastic heat equation, *Math of Comp*, 78, 266, pp. 845–863 2009.
- [5] Kloeden P. E., Platen E., *Numerical solution of stochastic differential equations*. Applications of Mathematics (New York), 23. Springer-Verlag, Berlin, 1992.
- [6] Nualart D., *The Malliavin Calculus and Related Topics. Second Edition*. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006.
- [7] Talay D., Tubaro L., Expansion of the global error for numerical schemes solving SDEs *Stoch. Anal. and App.*, 8(4),483–509, 1990.

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